Regular central structures in topologically nontrivial anti–de Sitter spacetimes

W. L. Smith and R. B. Mann
Department of Physics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
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We investigate regular central structures in multiply connected, anti–de Sitter spacetimes with spherical, planar and hyperbolic geometries. We obtain an exact solution for the pressure in terms of the radius when the density is constant. We find that, apart from the usual simply connected spherically symmetric star with a well-behaved metric at $r=0$, the only solutions with non-singular pressure and density have a wormhole topology. However these wormhole solutions must be composed of matter which violates the weak energy condition. Admitting this type of matter, we obtain a structure which is maintained via a balance between its cohesive tension and its repulsive negative matter density. If the tension is insufficiently large, this structure can collapse to a black hole of negative mass. [S0556-2821(98)03724-2]

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I. INTRODUCTION

Multiply connected spacetimes are attracting an increasing amount of attention among gravitational physicists and cosmologists. Although the idea that our universe could be topologically non-trivial has been around for quite some time [1], the possibility of setting observational constraints on this topology by performing a careful search for particular correlations in the cosmic microwave background is a recent development [2]. In a topologically non-trivial universe, a central structure with an edge at a constant radius will also be topologically non-trivial. It is with the formation of these objects that we are concerned in this work. Further interest in topological structures has been spurred by the realization that domain walls in the early universe can give rise to pair-production of black holes with event horizons whose topology is non-trivial [4]. Such objects have been referred to as topological black holes (TBHs).

We demonstrated that a dust cloud in a multiply connected, anti–de Sitter spacetime could collapse to a TBH in a process that is analogous to the usual Oppenheimer-Snyder collapse in an earlier paper [3]. The solutions obtained matched a static exterior spacetime to a dynamic collapsing cloud. The question of whether corresponding static solutions exist was left unresolved.

In this paper, we investigate the existence of regular perfect fluid solutions in multiply-connected, anti–de Sitter spacetime. The spatial sections of such spacetimes have the topology $R\times H_g$ where H_g is a two-dimensional compact space of genus g. Such a space may be described by a metric which is either flat or spherically or hyperbolically symmetric. The spherically symmetric case is simply connected and has genus $g=0$. The flat and hyperbolic cases are made compact via appropriate identifications in those two dimensions. Perfect fluid solutions in such spacetimes are centrally located, separated from the exterior spacetime at a constant radius. In the $g=0$ case they correspond to a ball of fluid surrounded by a cosmological vacuum spacetime; in the $g\neq 0$ case the analogous objects may be referred to as topological stars. For instance, when $g=1$ the universe has a toroidal topology, and the central structure is a torus inside it with both holes aligned.

We find that a necessary condition for regular perfect fluid solutions for $g>0$ is that a wormhole be present inside the star. However this condition is not sufficient. A constant positive matter density throughout the star necessarily implies an infinite pressure somewhere in its interior, excluding such objects as regular solutions.

The situation is considerably different if we consider matter which violates the weak energy condition. It has been shown that negative concentrations of stress-energy can collapse to black holes of negative mass provided the cosmological constant is sufficiently large in magnitude [5]. We find that it is possible to construct topological (wormhole) stars with constant negative matter density and finite negative pressure throughout. The negative pressure is a tension which acts to hold together the negative density, which is gravitationally self-repulsive.

Section II discusses the three spacetimes of interest, as well as their topologies. The spherically symmetric case has much in common with earlier studies, but we drop the demand that the metric be well-behaved at the origin. Section III introduces the interior metrics and their Einstein equations. In Sec. IV, the structure of the solutions for a variable matter density is found using perturbative techniques. Section V produces a solution for the pressure within wormholes of constant density as a function of radius. It is shown in Sec. VI that this pressure must necessarily become infinite somewhere within a wormhole with constant positive density. The question of wormholes with negative matter density is addressed in Sec. VII.

II. FLUID TOPOLOGY

The universe described by the static solution presented here is an asymptotically anti–de Sitter spacetime with a nontrivial topology. The exterior metric in this universe, adapted from [6]

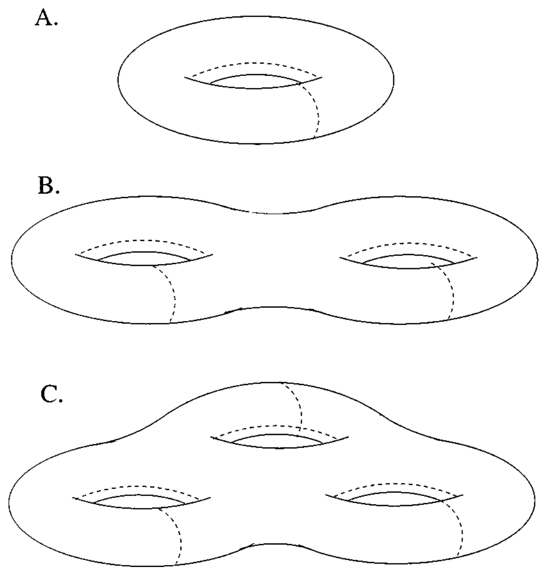

FIG. 1. Examples of higher genus two-surfaces. The dotted lines represent the identifications of edges. A. The torus, $g=1$, is a representation of a square or rectangle in flat space identified in two dimensions. B. The pseudosphere, $g=2$, is the simplest appropriate hyperbolic two-surface, an octagon, after identification. C. This $g=3$ surface is a slightly more complicated possibility.

$$ds^2=-\left(-\frac{\Lambda}{3G}R^2+b-\frac{2M}{R}\right)dT^2+\frac{dR^2}{-(\Lambda/3)R^2+b-2M/R}+R^2[d\theta^2+s(b,\theta)^2d\phi^2]s(b,\theta)=\begin{cases}\sin(\theta) & \text{if } b=+1,\\ 1 & \text{if } b=0,\\ \sinh(\theta) & \text{if } b=-1,\end{cases} \tag{1}$$

where T and R are the time and radial coordinates. M represents the mass of the star and Λ is the cosmological constant (where $\Lambda>0$ corresponds to the de Sitter case). ϕ assumes values between 0 and 2π.

When $b=+1$, the universe takes on the familiar spherically symmetric form, and the (θ,ϕ) sector has constant positive curvature. The cosmological constant may assume either sign, but we will only examine behavior resulting from negative values in the models explored here. When $b=0$, the space is flat, with Λ less than zero. In order to produce the central star to be examined, a space with two sides identified is considered. The outer edge of the star will be an identified flat plane of constant ''radius,'' R. Identification requires the edges of this plane to be geodesics, which are in this case straight lines. The sum of the angles must be 2π, yielding a parallelogram, or in the simplest case, a square or rectangle with opposite sides identified, creating a toroidal topology, as in Fig. 1(A). The exterior universe will maintain this topology with two compact and one infinite spatial dimensions.

When $b=-1$, the (θ,ϕ) sector is a space with constant negative curvature, also known as a hyperbolic plane, a discussion of which is available in Balasz and Voros [7]. Geodesics are intersections between the hyperbolic plane and planes through the origin. A compact surface is formed from the hyperbolic plane by identifying opposite sides of a suitable polygon whose edges are geodesics. The polygon must have a minimum of eight sides, and the number of sides must be a multiple of four to avoid conical singularities. An identified polygon with $4g$ sides is of genus g. The genus determines the topology of the compact space. The surface genus $g=2$ has two ''holes,'' and so is a double-holed doughnut or a pacifier. The surface with $g=3$ has three ''holes'' and has a pretzel shape, and so on, as shown in Fig. 1. This type of identification of hyperbolic surfaces is described in more detail in [3].

The fluid cloud in these identified spacetimes is located in a central position, analogous to the central sphere in a spherically symmetric spacetime. The boundary between the fluid and the exterior universe is an identified flat or hyperbolic plane. The universe outside the fluid maintains the same topology of the cloud boundary itself. The fluid forms a pacifier within a pacifier with the holes lined up, or the banana cream in our doughnut, if you will. The situation is analogous for higher genus topologies. Beyond the cloud, the radial coordinate may range to infinity.

III. CALCULATIONS OF THE EINSTEIN EQUATIONS

The standard metric of an arbitrary, static spacetime with spherical, toroidal or hyperbolic symmetry was used in conjunction with the Einstein equations to generate interior solutions. The metric is given by:

$$ds^2=-F(r,b)dt^2+H(r,b)dr^2+r^2(d\theta^2+s(b,\theta)^2d\phi^2), \tag{2}$$

with $s(b,\theta)$ as given in Eq. (1) above. Here, t is the time coordinate, r is the radial coordinate, and θ and ϕ are coordinates on a two-surface of constant positive, zero or negative curvature, where ϕ has a range of 0 to 2π. The matching of the metrics is analogous to that in [3], and is carried out under the conditions that the metrics and the extrinsic curvatures match smoothly across the boundary.

The cosmological constant will be absorbed into definitions of the density and pressure, so that

$$\rho=\rho_m-\frac{|\Lambda|}{8\pi G},\quad P=P_m+\frac{|\Lambda|}{8\pi G}, \tag{3}$$

where ρ_m and P_m are the density and pressure due to matter, respectively. Since the $\Lambda<0$ case is of interest here, the absolute value of Λ is used for the remainder of this paper. Einstein equations are therefore simply

$$G_{\mu\nu}=-8\pi G T_{\mu\nu} \tag{4}$$

$$T_{\mu\nu}=Pg_{\mu\nu}+(\rho+P)u_\mu u_\nu \tag{5}$$

TABLE I. A summary of small r behavior for stars with variable ρ_m. In every case, the relevant nonzero constant A, B, C, or D must be positive, restricting the possible values of ρ_m and $|\Lambda|$. For more detail on the solution types, see Fig. 2. (i) For $A=0$, $B\neq 0$, when $r_m=0$ a spherically symmetric ($b=+1$) star in an anti–de Sitter spacetime may form, with the Ricci scalar vanishing at the center. (ii) For the infinite hourglass, only $b=0$ is allowed.

	$A\neq 0$	$A=0$, $B\neq 0$	$A=B=0$, $C\neq 0$	$A=B=C=0$, $D\neq 0$
$r_m=0$	cusp	no solution[i]	no solution	infinite hourglass[ii]
Ricci scalar	$2b(\frac{3}{2}c\mathscr{l})^{-4/3}$	N/A[i]	N/A	$-6D$
$r_m\neq 0$	finite throat	finite throat	infinite trumpet	infinite trumpet
Ricci scalar	$2b/r_m^2$	$2b/r_m^2$	$2b/r_m^2$	$2b/r_m^2$

with the fluid four-velocity given by $u^\alpha=F^{-1/2}(\partial/\partial t)^\alpha$.

This means that the components of the Einstein tensor are

$$G_{00}=-F\left(\frac{H'}{H^2r}-\frac{1}{r^2}(1/H-b)\right)=-8\pi\rho F \tag{6}$$

$$G_{11}=-H\left(\frac{F'}{FHr}+\frac{1}{r^2}(1/H-b)\right)=-8\pi PH \tag{7}$$

$$G_{22}=-\frac{r}{4F^2H^2}(2F'FH-2H'F^2+2rF''FH$$
$$-rF'^2H-rF'H'F)$$
$$=-8\pi Pr^2=\frac{G_{33}}{\sinh^2\theta} \tag{8}$$

where the primed variables refer to the derivative, d/dr.

IV. VARIABLE DENSITY SOLUTIONS

Equation (6) produces the following solution for $H(r)$:

$$H=\frac{r}{(|\Lambda|/3G)\,r^3+br+c-\int_{r_m}^{r}8\pi\rho_m r^2} \tag{9}$$

where c is a constant of undetermined sign, and r_m is the minimum value of the radius. In order to examine behavior of the metric, expand about the minimum radius with the infinitesimal, $r=r_m+\epsilon$, and use a Taylor series approximation for the integral. The spatial metric then becomes

$$ds_s^2=\frac{(r_m+\epsilon)d\epsilon^2}{A+B\epsilon+C\epsilon^2+D\epsilon^3+\ldots}+(r_m+\epsilon)^2d\Omega^2, \tag{10}$$

where $d\Omega^2$ is the appropriate angular spatial section. The behavior at small r will depend on the parameters, A, B, C and D, as well as the value of r_m. The parameters are given by

$$A=\frac{|\Lambda|}{3G}r_m^3+br_m+c, \tag{11}$$

$$B=\frac{|\Lambda|}{G}r_m^2+b-8\pi\rho_m r_m^2, \tag{12}$$

$$C=\frac{|\Lambda|}{G}r_m-8\pi\rho_m r_m-4\pi\rho_m' r_m^2, \tag{13}$$

$$D=\frac{|\Lambda|}{3G}-\frac{8}{3}\pi\rho_m-\frac{16}{3}\pi\rho_m' r_m-\frac{4}{3}\pi\rho_m'' r_m^2. \tag{14}$$

Small r behavior is tabulated in Table I. A finite throat refers to the situation in which the radial coordinate reaches a minimal value at some finite distance from the outer edge of the fluid, and then expands again into another universe. In the genus $g=0$ case a series of spheres of smaller and smaller proper radii are encountered, as an observer travels into the star. Eventually, a sphere of minimum size is encountered, beyond which the spheres begin to grow once more. In the genus $g>0$ case, the situation is the same, except that the spheres are replaced with pacifiers ($g\geqslant 2$) or tori ($g=1$). Hence, the spacetime within the fluid has a wormhole structure, and may be matched at each end of the wormhole (where $r=R$) to an exterior spacetime whose metric is given by Eq. (1).

From Table I, it is readily apparent that the interesting cases, the finite throats, are those for which the behavior of the metric at small r is independent of ρ'. We will next examine the constant density case in more detail to see if the solutions are indeed regular.

V. CONSTANT DENSITY STARS AND THE BUCHDAHL IDENTITY

Equation (6) leads to a solution for the function H, if the density is taken as constant,

$$H=\left(\beta r^2+b+\frac{\alpha}{r}\right)^{-1} \tag{15}$$

in which α an arbitrary constant and

$$\beta=-\frac{8}{3}\pi\rho=-\frac{8}{3}\pi\rho_m+\frac{|\Lambda|}{3G}. \tag{16}$$

Here, ρ is the net density. For now, we assume the matter density is positive or zero. H must always be greater than zero, to preserve the signature of the metric.

If we let $F=e^{2\Phi}$, then Eq. (7) will be

$$\frac{d\Phi}{dr}=\frac{8\pi Pr^3-\beta r^3-\alpha}{2r(\beta r^3+br+\alpha)}. \tag{17}$$

The final Einstein tensor equation, (8), along with Eq. (15) leads to a solution for the change in pressure as a function of r:

$$\frac{dP}{dr}=-(P+\rho)\frac{d\Phi}{dr}=-\frac{(8\pi P-3\beta)(8\pi Pr^3-\beta r^3-\alpha)}{16\pi r(\beta r^3+br+\alpha)}. \tag{18}$$

In order to solve this equation, note that it can be put in the form

$$\left(\frac{16\pi}{r}\frac{(\beta r^3+b+\alpha)}{8\pi P-3\beta}\right)dP+\left(\frac{1}{r^2}(8\pi Pr^3-\beta r^3-\alpha)\right)dr=0 \tag{19}$$

and the integrating factor

$$\mu=\left(\frac{r}{\beta r^3+br+\alpha}\right)^{3/2}\frac{-1}{8\pi P-3\beta} \tag{20}$$

applied. By integrating from the outer edge of the star, where the pressure is $|\Lambda|/8\pi G$ to an arbitrary radius within the star, we find that

$$P=\frac{\rho_m}{\sqrt{(\beta r^2+b+\alpha/r)/(\beta R^2+b+\alpha/R)}-4\pi\rho_m\sqrt{(\beta r^2+b+\alpha/r)}\int_r^R(\tilde{r}d\tilde{r}/(\beta\tilde{r}^2+b+\alpha/\tilde{r})^{3/2})}+\frac{3\beta}{8\pi}. \tag{21}$$

The condition that H be real for all radii, forces the pressure to be real everywhere.

The Buchdahl identity is normally found by demanding the central pressure be finite in a simply connected spacetime, where $g=0$. Furthermore, α is assumed to vanish, allowing the metric to be well-behaved at $r=0$. In this case it is straightforward to explicitly carry out the integral in Eq. (21) to obtain

$$P=\rho\left[\frac{K\sqrt{1-(8/3)\pi\rho R^2}-\sqrt{1-(8/3)\pi\rho r^2}}{\sqrt{1-(8/3)\pi\rho r^2}-3K\sqrt{1-(8/3)\pi\rho R^2}}\right],$$

$$K=\frac{P(R)+\rho}{3P(R)+\rho}=\frac{4\pi\rho_m}{4\pi\rho_m+|\Lambda|/G}. \tag{22}$$

The pressure should be positive definite, meaning

$$\frac{1}{3}\leqslant K\sqrt{\frac{1-(8/3)\pi\rho R^2}{1-(8/3)\pi\rho r^2}}<1. \tag{23}$$

The right hand inequality constrains K most strongly at the edge of the cloud, and the left at the center. This second constraint may be translated as a limit on the mass in terms of the radius,

$$M=\int_0^R 4\pi\rho r^2dr<\frac{9K^2-1}{18K^2}R \tag{24}$$

which reduces to the familiar $M<4R/9$ limit when $\Lambda=0$ [8]. By demanding that $(dP_m/d\rho_m)\geqslant 0$, that $M(0)=0$ and that the pressure was non-negative and bounded everywhere, Hiscock was able to obtain the stronger constraint

$$M/R\leqslant\frac{2}{9}\left[1-\frac{3|\Lambda|}{4G}R^2+\left(1+\frac{3|\Lambda|}{4G}R^2\right)^{1/2}\right] \tag{25}$$

for the $b=+1$ case [9].

We will not demand $\alpha=0$. The $b=+1$ case has already been examined for nonzero α, in which regular stars with $g=0$ may form, by Hiscock. Consider now the higher-genus cases. When $g=0$, $b=0$ and the parameter β is forced to be positive. When $g\geqslant 2$, $b=-1$, implying $\beta>1/R^2$. The equation for the pressure becomes, for any genus,

$$P=\rho\left(\frac{\sqrt{\beta r^2+b}-K\sqrt{\beta R^2+b}}{3K\sqrt{\beta R^2+b}-\sqrt{\beta r^2+b}}\right), \tag{26}$$

provided $\alpha=0$. The analogous Buchdahl identity, found by demanding that the pressure is a definite positive, is then

$$1/3<K\sqrt{\frac{\beta R^2+b}{\beta r^2+b}}\leqslant 1 \tag{27}$$

for all r.

When $R=r$, the left hand equality demands that $|\Lambda|/G<8\pi\rho_m$ but maintaining a metric with the correct signature throughout requires $|\Lambda|/G>8\pi\rho_m$. This contradiction rules out the possibility of a genus $g>0$ regular star with $\alpha=0$ and positive pressure everywhere.

VI. CONSTANT DENSITY SOLUTIONS

When ρ is constant, the parameters A, B, C and D from Sec. IV may still be nonzero. The behavior of the solution will depend on the character of the lower cutoff of the positive region of $\beta r^3+br+\alpha$ being examined. If that function is cut off by a double or triple root, an infinite throat will result. If the largest root is negative, a cusp will be formed. If, however, the function has a non-degenerate, positive root as the lower bound to its positive region, the star will have a finite throat, with a minimum radius given by that root.

None of these wormholes will have a well behaved pressure for a positive matter density. Note that we can rewrite Eq. (21) as

$$P_m = \frac{\rho_m}{\sqrt{(\beta r^2+b+\alpha/r)/(\beta R^2+b+\alpha/R)} - 4\pi\rho_m\sqrt{(\beta r^2+b+\alpha/r)}\int_r^R(\tilde{r}d\tilde{r}/(\beta\tilde{r}^2+b+\alpha/\tilde{r})^{3/2})} - \rho_m . \tag{28}$$

The first term in the denominator varies from a value of unity at the outer radius to zero at the throat. The integral in the second term is a well-behaved positive function, so the second term will be zero at the outer radius and some positive number at the throat. To see this, approximate the behavior of the term near a single root, r_0.

$$\sqrt{\beta r^2+b+\alpha/r}\int_r^R \frac{r dr}{(\beta r^2+b+\alpha/r)^{3/2}}$$
$$\approx \frac{r_0^2\sqrt{r-r_0}}{q r_0^2+s r_0+t}\int_r^R \frac{dr}{(r-r_0)^{3/2}}$$
$$\approx \frac{2 r_0^2}{(q r_0^2+s r_0+t)}. \tag{29}$$

It is therefore unavoidable that the two terms in the denominator will become equal at some value of r, at which point the pressure will be infinite. This type of wormhole metric can never be regular. Sample plots displaying this behavior were obtained by numerical integration, and are shown in Fig. 3.

VII. NEGATIVE MATTER DENSITY WORMHOLES

Wormhole solutions have long been known to require the existence of exotic matter (matter which violates the weak energy condition) [10], so it is not surprising that this case also requires such. A study of wormhole solutions in topologically trivial spacetimes with nonzero cosmological constant also led to this conclusion, although the actual conditions on the exotic matter are modified [11]. The necessity of exotic matter is an unpleasant, but not prohibitive situation, the Casimir effect being perhaps the best known example of a manifestation of the violation of the energy conditions.

The most likely situation in which topological black holes have physical relevance is in the early universe [6] and is also one in which quantum fluctuations may produce (temporarily at least) regions in which the weak energy condition is violated. It is therefore natural to consider in more detail topological ''stars'' in which the energy conditions are violated. Indeed, a study of a dust cloud of negative energy density indicated that exotic matter may behave in a counterintuitive manner, collapsing to form black holes [5].

The simplest case which requires ρ_m to be negative is that for which the parameter α vanishes, as referred to earlier. The behavior of the pressure in this situation is representative of the more complicated $\alpha \neq 0$ cases. Here the matter density is forced to be negative in order that the metric be real. The pressure takes on the simple form

$$P_m = -|\rho_m| \times \left(\frac{\sqrt{\beta r^2+b} - \sqrt{\beta R^2+b}}{3|\rho_m|/|\rho_m| - |\Lambda|/(4\pi G)\sqrt{\beta R^2+b} - \sqrt{\beta r^2+b}} \right). \tag{30}$$

In this case the pressure is well behaved. It vanishes at the star's edge and decreases to a finite central value at $r = r_m$. The pressure here is always negative, so in fact it is a ten-

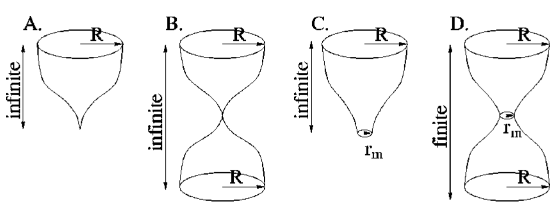

FIG. 2. (A) For a cusp, the star reaches a vanishing minimum radius, but this occurs an infinite proper distance from the exterior of the star. (B). The infinite hourglass is composed of a ''wormhole'' of infinite proper distance connecting two spaces where the radius may grow to infinity, e.g, two universes. The minimum radius is zero, so the infinite hourglass may also be considered two cusps joined at $r=0$. (C). The infinite trumpet is similar to the infinite hourglass, but has a nonzero, finite minimum radius. It still has an infinite proper length. (D). The finite throat is a proper wormhole analogue. Two spaces where the universe may grow to infinity are connected by a throat of finite proper length and nonzero, finite minimum radius.

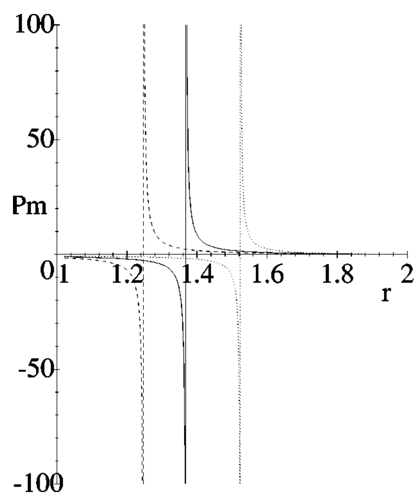

FIG. 3. The pressure, as found in Eq. (28), is evaluated for the special case where the largest root of $H(r,b)$ is a single root, $r_m=1$. The equation used is $H(r,b) = -(1/x)(\frac{8}{3}\pi\rho_m - |\Lambda|/3)r_m^2(x-1)\{x^2+x+1-b[r_m^2(\frac{8}{3}\pi\rho_m - |\Lambda|/3)]^{-1}\}$. All the curves have $|\Lambda|=1$, $\rho_m=0.5$, $R=2$ and $r_m=1$. The dashed curve has $b=-1$, the solid curve has $b=0$, and the dotted curve has $b=+1$.

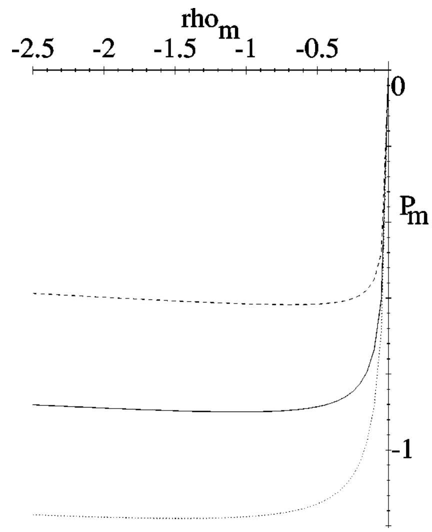

FIG. 4. The matter pressure as a function of matter density for $b=0,+1,-1$, represented by the solid, dotted and dashed lines. All plots use $|\Lambda|/G=1$, $r=1.001$, $R=2$ and $r_m=1$ in the equation shown for Fig. 3. The cutoff of the allowed region is at $\rho_m=-1.06$, -1.45, and -0.69 for $b=0$, $+1$, and, -1, respectively.

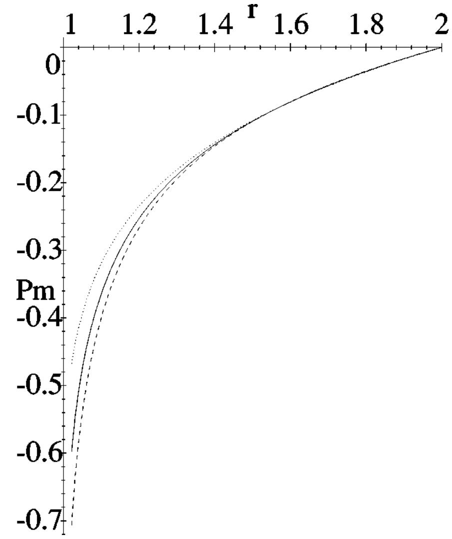

FIG. 5. The matter pressure, as found in Eq. (28), is evaluated for the same equation for $H(r,b)$ as before, but now $\rho_m<0$ is permitted. The parameters used to find the solid, dotted and dashed lines are $b=0$, $+1$, -1. All plots use $|\Lambda|/G=1$, $\rho_m=-0.5$, $R=2$ and $r_m=1$. The lines here approach finite values at the throat, $r=r_m$.

sion. Generalizing the constraint from Ref. [9] to include negative matter pressure and density yields

$$\frac{dP_m}{d\rho_m}\geqslant 0\geqslant \frac{dP_m}{d|\rho_m|}. \tag{31}$$

This will guarantee that the matter pressure P_m must become more negative as the matter density becomes more negative, as is physically reasonable (see Fig. 4). This constraint may also be interpreted as a limit on the matter density, since when it becomes too negative, the inequality will no longer hold throughout the star. This constraint is most easily evaluated numerically.

The behavior of the matter pressure when α is non-zero is qualitatively similar. The function is well-behaved, zero at the outer edge and finite at the throat. The same constraint (31) as before is employed to produce physically reasonable results (see Fig. 4). The strongest constraint arises at a radius near the throat radius, as was done in Fig. 5.

VIII. CONCLUSIONS

Although the collapse of a pressureless dust cloud of positive energy to a topological black hole proceeds in a manner somewhat analogous to that in the usual spherical case (with genus $g=0$) [3], the formation of regular central structures in topologically non-trivial anti–de Sitter spacetimes differs considerably from the topologically trivial case. Indeed geometric requirements are in conflict with energy positivity requirements, implying that there are no regular central structures formed from a perfect fluid respecting the energy conditions and whose exterior metric is given by Eq. (1).

The only ''regular'' solutions are those in which the matter density within the star is negative, with a magnitude smaller than a critical value determined by Eq. (31). In this case the topological star consists of a fluid of gravitationally repulsive negative energy, held together by a sufficiently large tension (i.e., negative pressure). Both pressure and density are finite everywhere throughout the star. This is the reverse of a normal star, whose gravitationally self-attractive density is prevented from collapsing by its pressure. Should the pressure of the negative-mass star decrease below a certain threshold during its evolution, it will either explode due to gravitational self-repulsion or collapse to a black hole of negative mass. The former situation will occur if the magnitude of the density is sufficiently large relative to $|\Lambda|/G$. Otherwise, the evolution of the star should proceed along the lines described in Ref. [5], ultimately reaching a black hole of negative mass as its final state.

The most likely physical situation in which any of these scenarios is relevant is in the early universe. Topological black holes can be formed via pair-production in the presence of domain walls [6] or from the collapse of a dust cloud (of either positive or negative density) in a topologically suitable setting [3,5]. However, if it is possible to produce exotic matter in such settings, the results of this paper indicate that regular (wormhole-type) central structures can form. The stability of such solutions, as well as the existence of regular stable solutions with variable density remain interesting open questions.

ACKNOWLEDGMENTS

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