

How unique is the expected stress-energy tensor of a massive scalar field?

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We show that the set of ambiguities in the renormalized expected stress-energy tensor allowed by the Wald axioms is much larger for a massive scalar field (an infinite number of free parameters) than for a massless scalar field (two free parameters). We also use the closed-time-path effective action formalism of Schwinger to calculate the expected value of the stress-energy tensor in the incoming vacuum state, for a massive scalar field, on any spacetime which is a linear perturbation off Minkowski spacetime. This result generalizes an earlier result of Horowitz and also Jordan in the massless case, and can be used as a testbed for comparing different calculational methods. [S0556-2821(98)05622-7]

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I. INTRODUCTION AND SUMMARY

A. Background and motivation

In semiclassical gravity, a classical metric is coupled to quantum fields according to the semiclassical Einstein equation

$$G_{ab} = 8\pi G \langle \hat{T}_{ab} \rangle, \quad (1.1)$$

where G_{ab} is the Einstein tensor and G is Newton's gravitational constant. This equation is usually postulated rather than derived as there is no complete theory of quantum gravity from which it could be derived, although several formal derivations have been given [1]. There are several well-known difficulties associated with the semiclassical theory. First, there are difficulties associated with the existence of unphysical, exponentially growing "runaway" solutions of Eq. (1.1), which have not yet been completely resolved [2]. The second difficulty, which is the subject of this paper, is the non-uniqueness of the expected stress-energy tensor on the right hand side of Eq. (1.1).

For a scalar field, several methods have been suggested for calculating the expected stress energy tensor. These include (i) the "point splitting" algorithm [3], (ii) the DeWitt-Schwinger expansion method [4], and (iii) the closed-time-path or in-in effective action method [5–7]. There is no general agreement as to which method is correct. For example, it is claimed in Ref. [3] that the DeWitt-Schwinger method is invalid for a massive scalar field since it does not have a regular limit as $m \rightarrow 0$, where m is the mass.

As is well known, a theorem of Wald [8,3] plays a crucial role in this field. The theorem states that if one has two different prescriptions for obtaining stress tensors from metrics and from quantum states, and if these prescriptions obey a certain set of physically-motivated axioms, then the two prescriptions must agree up to a local conserved tensor [9]. Thus, if $\langle \hat{T}_{ab} \rangle$ and $\langle \tilde{T}_{ab} \rangle$ are two different such prescriptions for computing the expected stress-energy tensor, then the difference

$$t_{ab} \equiv \langle \hat{T}_{ab} \rangle - \langle \tilde{T}_{ab} \rangle \quad (1.2)$$

must be a conserved local curvature tensor.

In the case of a massless scalar field, there is no natural mass scale in the theory, and the following well-known argument based on dimensional analysis shows that $\langle \hat{T}_{ab} \rangle$ is unique up to a two parameter ambiguity. Let us use units in which $\hbar = c = 1$, but in which $G \neq 1$. Then, there are only two independent conserved local curvature tensors with the appropriate dimensions of (mass)⁴: namely

$$H_{ab}^{(1)}(x) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}(x)} \int d^4x' \sqrt{-g} R(x')^2 \quad (1.3)$$

and

$$H_{ab}^{(2)}(x) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}(x)} \int d^4x' \sqrt{-g} R_{cd}(x') R^{cd}(x'). \quad (1.4)$$

Thus we must have

$$t_{ab} = \alpha H_{ab}^{(1)} + \beta H_{ab}^{(2)}, \quad (1.5)$$

where α and β are two unknown dimensionless parameters. Hence, the expected stress tensor $\langle \hat{T}_{ab} \rangle$ is unique up to a two parameter ambiguity.

Consider now a massive scalar field. In this case the above argument fails, since there is a preferred mass scale present, namely the mass m of the field. Using this mass scale one can construct local conserved tensors of the form

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}(x)} \int d^4x' \sqrt{-g} m^4 \left[\frac{R(x')}{m^2} \right]^n, \quad (1.6)$$

which have dimension (mass)⁴ and are thus possible candidates for t_{ab} . Here n can be any integer greater than 2. Similar terms can be constructed from the Ricci and Riemann tensors. The conventional view has been to exclude such terms as being unphysical, since they diverge as $m \rightarrow 0$ (see, e.g., p. 90 of Ref. [3]). Thus, the conventional view has been that the allowed ambiguity for a massive field is no worse than that for a massless field, namely the two parameter ambiguity (1.5).

B. Ambiguity in expected stress tensor for a massive scalar field

The first main point of this paper is that the above conventional view is unfounded. This can be seen as follows. Consider the local conserved tensor

$$t_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}(x)} \int d^4x' \sqrt{-g} m^4 F \left[\frac{R(x')}{m^2} \right], \quad (1.7)$$

where $F(x)$ is any dimensionless function of a dimensionless argument x . In order for t_{ab} to be acceptable on physical grounds as a contribution to the expected stress tensor, it must satisfy the requirements that

$$t_{ab} \rightarrow 0 \quad \text{as} \quad m^2 \rightarrow 0 \quad (1.8)$$

and

$$t_{ab} \rightarrow 0 \quad \text{as} \quad R \rightarrow 0. \quad (1.9)$$

Now, the tensor (1.7) can be written as

$$t_{ab} = -m^2 R_{ab} F'(R/m^2) + m^4 F(R/m^2) g_{ab}/2 + F''(R/m^2) Y_{ab} + F^{(3)}(R/m^2) Z_{ab}/m^2, \quad (1.10)$$

where Y_{ab} and Z_{ab} are tensors constructed out of derivatives of R , of dimension (mass)⁴ and (mass)⁶ respectively. Suppose that we choose the function F to be smooth, to satisfy $F(0)=0$, and to satisfy $F^{(j)}(x)x^{j-2} \rightarrow 0$ as $x \rightarrow \infty$ for $j=0,1,2,3$, where $F^{(j)}(x)$ is the j th derivative of F . Examples of functions satisfying these requirements are $F(x) = x^2 \exp(-x^2)$ and $F(x) = x^2/(1+x^4)$. Then, the tensor (1.7) will satisfy the required properties (1.8) and (1.9) [10].

Note that when one expands the function $F(x)$ as a power series to obtain

$$F \left[\frac{R(x')}{m^2} \right] = \sum_n a_n \left[\frac{R(x')}{m^2} \right]^n, \quad (1.11)$$

it can be seen that the tensor (1.7) contains terms of the form (1.6), each of which individually has unacceptable behavior as $m \rightarrow 0$. However the sum (1.7) of all these terms does have acceptable behavior as $m \rightarrow 0$. Note also that it is not possible to exclude terms of the form (1.7) by postulating, as an additional axiom, that the stress tensor be an analytic function of m^2 , since for example the choice $F(x) = x^2/(1+x^4)$ yields a local conserved tensor t_{ab} which is an analytic function of m^2 in an open neighborhood of the real axis in the complex m^2 plane [11].

The ambiguity t_{ab} in the stress-energy tensor allowed by the Wald axioms is therefore much worse in the massive case than in the massless case. It is an infinite parameter ambiguity—one can specify a free function $F(x)$ —rather than a two parameter ambiguity [12]. Of course, it is still possible that the various conventional calculational methods still agree to within the two parameter ambiguity (1.5) [13]. However, there is no guarantee that this should be the case. Therefore it would be worthwhile to find some additional

axiom or physical principle, to augment the Wald axioms, that would further pin down the stress tensor in the massive case [14].

C. Nearly flat spacetimes

Spacetimes which are linear perturbations off Minkowski spacetime form a useful testbed in which to probe these issues [15]. The second principle purpose of this paper is to explicitly calculate the renormalized stress tensor of a massive scalar field in such spacetimes, using the closed-time-path or in-in effective action formalism [5–7]. If our calculation is repeated using the point-splitting or DeWitt-Schwinger methods, then it will be possible to compare the predictions of the different methods.

Now the Wald axioms imply that any prescription for calculating the stress tensor is determined by specifying the expected value of the stress tensor in the incoming vacuum state $|0, \text{in}\rangle$. The expected value in any other state is then uniquely determined [2]. Therefore, it suffices to consider the expected value of the stress tensor in the incoming vacuum state. In the massless case, calculations of the in-in expected stress tensor have already been performed using several different methods (see Horowitz [16] and Jordan [17]). The results of these different calculations agree up to the two parameter ambiguity (1.5), as they must according to Wald's theorem.

The result we obtain from the in-in effective action formalism [5–7] is [cf. Eq. (4.13) below]

$$\begin{aligned} \langle 0, \text{in} | \hat{T}_{ab}(x) | 0, \text{in} \rangle \\ = \alpha \dot{H}_{ab}^{(1)}(x) + \beta \dot{H}_{ab}^{(2)}(x) - \frac{1}{256\pi^2} \int d^4x' \\ \times [\dot{H}_{ab}^{(1)}(x') T_1(x-x') + \dot{H}_{ab}^{(2)}(x') T_2(x-x')]. \end{aligned} \quad (1.12)$$

Here it is assumed that the metric tensor is of the form

$$g_{ab} = \eta_{ab} + h_{ab}, \quad (1.13)$$

where η_{ab} is a flat, Minkowski metric, and that the coordinates x^a and x'^a are Lorentzian coordinates with respect to η_{ab} . Also α and β are arbitrary dimensionless constants, the distributions $T_1(x-x')$ and $T_2(x-x')$ are defined by Eqs. (4.14) and (4.15) below, and $\dot{H}_{ab}^{(1)}$ and $\dot{H}_{ab}^{(2)}$ are linearized versions of the local conserved curvature tensors (1.3) and (1.4).

The stress-energy tensor (1.12) is causal, as it must be, and reduces to the known result of the massless case [16,17] in the limit $m \rightarrow 0$. Furthermore it is not a smooth function of m^2 at $m^2=0$. The calculational method we use also automatically yields two undetermined parameters α and β , so the result (1.12) explicitly exhibits the two parameter ambiguity, just as in the massless case [16].

D. Organization of this paper

Section II reviews the in-out and in-in effective action formalisms. In Sec. III we calculate the in-in and in-out effective actions of a massive scalar field propagating on a spacetime which is a linear perturbation off Minkowski spacetime. In Sec. IV we find the expected stress-energy in the incoming vacuum state, and then in Sec. V we discuss its properties. Section VI summarizes our results.

Throughout we use units in which $\hbar = c = 1$, and use the metric signature and sign conventions of Misner, Thorne and Wheeler [18]. Further notational conventions are given in Appendix A.

II. THE IN-OUT AND IN-IN PATH INTEGRAL FORMALISMS

In this section we review both the standard in-out path integral formalism of quantum field theory (see, e.g., Refs. [19,4]), as applied to curved spacetimes, and the modified in-in formalism due to Schwinger [5]. This in-in or closed time path method was later adapted to curved spacetimes by Jordan [6,7], and has been extensively explored by Hu [22]. Our presentation of the in-in method in Sec. II C below differs from that of Refs. [6,7] in that all the fundamental definitions are explicitly coordinate independent.

A. The classical theory

We consider a massive, minimally coupled scalar field ϕ for which the Einstein-Klein-Gordon action is

$$S[g^{ab}, \phi] = S_g[g^{ab}] + S_m[g^{ab}, \phi], \quad (2.1)$$

where

$$S_g[g^{ab}] = 2\mu_p^2 \int \sqrt{-g} R d^n x \quad (2.2)$$

and

$$S_m[g^{ab}, \phi] = -\frac{1}{2} \int \sqrt{-g} (g^{ab} \nabla_a \phi \nabla_b \phi + m^2 \phi^2) d^n x. \quad (2.3)$$

Here m is the mass of the scalar field, $\mu_p^2 = (32\pi G)^{-1}$ is the square of the Planck mass, and n is the number of spacetime dimensions (we shall be using the dimensional regularization scheme below). The corresponding equations of motion are $4\mu_p^2 G_{ab} = T_{ab}$ and $(\square - m^2)\phi = 0$, where the stress-energy tensor is

$$\begin{aligned} T_{ab} &= \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} S_m[g^{ab}, \phi] \\ &= \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi - \frac{1}{2} g_{ab} m^2 \phi^2. \end{aligned} \quad (2.4)$$

B. In-out formalism

We assume that the metric g_{ab} is asymptotically static at early and late times so the incoming and outgoing vacuum states $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ are well defined. In the usual way we define the generating functional

$$e^{iW[g^{ab}, J]} \equiv \langle 0, \text{out} | 0, \text{in} \rangle_J, \quad (2.5)$$

where the subscript J on the right hand side indicates that a source term

$$\langle J, \phi \rangle \equiv \int d^n x \sqrt{-g(x)} J(x) \phi(x) \quad (2.6)$$

has been added to the action. It can be shown that $e^{iW[g^{ab}, J]}$ has the path integral representation

$$e^{iW[g^{ab}, J]} = \int D\phi e^{i(S_m[g^{ab}, \phi] + \langle J, \phi \rangle)}, \quad (2.7)$$

and that time-ordered matrix elements are given by

$$\begin{aligned} \langle 0, \text{out} | T \hat{\phi}(x) \hat{\phi}(y) | 0, \text{in} \rangle_J \\ = \int D\phi \phi(x) \phi(y) e^{i(S_m[g^{ab}, \phi] + \langle J, \phi \rangle)}. \end{aligned} \quad (2.8)$$

In Eqs. (2.7) and (2.8), the usual boundary conditions on the path integral are assumed, namely that ϕ is purely negative frequency ($\propto e^{i\omega t}$ with $\omega > 0$) at early times and positive frequency at late times, or equivalently, that the mass squared parameter is understood to have a small negative imaginary part, $m^2 \rightarrow m^2 - i\epsilon$. From Eqs. (2.8) and (2.4) it follows that

$$\begin{aligned} \frac{-2}{\sqrt{-g}} \frac{\delta W[g^{ab}, 0]}{\delta g^{ab}} \\ = -ie^{-iW[g^{ab}, 0]} \int D\phi i \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} S_m[g^{ab}, \phi] e^{iS_m[g^{ab}, \phi]} \\ = \frac{\langle 0, \text{out} | \hat{T}_{ab} | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle}. \end{aligned} \quad (2.9)$$

The effective action is defined in the usual way as a Legendre transform of the generating functional:

$$\Gamma_m[g^{ab}, \bar{\phi}] \equiv W[g^{ab}, J] - \langle J, \bar{\phi} \rangle, \quad (2.10)$$

where

$$\bar{\phi} = \frac{1}{\sqrt{-g}} \frac{\delta W[g^{ab}, J]}{\delta J} = \langle 0, \text{out} | \hat{\phi} | 0, \text{in} \rangle_J. \quad (2.11)$$

Here and henceforth the subscript m in Γ_m indicates that the classical action from which Γ_m is computed [Eq. (2.7) above] includes only the matter part S_m and not the gravitational part S_g . Combining Eqs. (2.9)–(2.11) we now obtain

$$\frac{-2}{\sqrt{-g}} \frac{\delta \Gamma_m[g^{ab}, \bar{\phi}]}{\delta g^{ab}} \Big|_{\bar{\phi} = \bar{\phi}[g^{ab}]} = \frac{\langle 0, \text{out} | \hat{T}_{ab} | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle}. \quad (2.12)$$

Here the right hand side is a functional only of the metric g^{ab} , while on the left hand side $\bar{\phi}[g^{ab}]$ is the solution to Eq. (2.11) at $J=0$.

Since the action (2.3) is quadratic in ϕ , it is straightforward to compute the effective action exactly. The result is the standard, formal, expression [20]

$$\Gamma_m[g^{ab}, \bar{\phi}] = S_m[g^{ab}, \bar{\phi}] + \frac{i}{2} \text{tr} \ln \bar{A}, \quad (2.13)$$

where \bar{A} is the operator given by $\bar{A}\phi = (\square - m^2 + i\epsilon)\phi$. The operator \bar{A} is the natural operator associated with the quadratic form S_m and with the covariant inner product on functions on spacetime

$$\langle f, h \rangle \equiv \int d^n x \sqrt{-g(x)} f(x) * h(x). \quad (2.14)$$

The reason the inner product (2.14) is the appropriate inner product is that the measure $D\phi$ in Eq. (2.7) is determined by the metric g^{ab} [21].

Finally, we define the quantity $\Gamma[g^{ab}, \bar{\phi}]$ (as opposed to $\Gamma_m[g^{ab}, \bar{\phi}]$) to be the effective action obtained when one starts from the full action (2.1) rather than just the matter part (2.3). It is clear that

$$\Gamma[g^{ab}, \bar{\phi}] = \Gamma_m[g^{ab}, \bar{\phi}] + S_g[g^{ab}]. \quad (2.15)$$

C. In-in formalism

We introduce the generating functional

$$e^{iW[g_+^{ab}, g_-^{ab}, J_+, J_-]} \equiv \int D\alpha \langle 0, \text{in} | \alpha, T \rangle_{J_-} \langle \alpha, T | 0, \text{in} \rangle_{J_+}, \quad (2.16)$$

which depends on two independent sources $J_+(x)$ and $J_-(x)$, as well as two metrics g_+^{ab} and g_-^{ab} . Equation (2.16) includes an integration over a complete set of states $|\alpha, T\rangle$ on the hypersurface $x^0 = T$ at some future time T . We assume that $g_+^{ab} = g_-^{ab}$ for $x^0 \geq T$. Each of the matrix elements in Eq. (2.16) can be expressed as path integrals in the usual way:

$$\langle \alpha, T | 0, \text{in} \rangle_{J_\pm} = \int D\phi_\pm e^{i(S_m[g_\pm^{ab}, \phi_\pm] + \langle J_\pm, \phi_\pm \rangle_\pm)}, \quad (2.17)$$

where the measure $D\phi_+$ and inner product $\langle \dots \rangle_+$ are determined by the metric g_+^{ab} , the measure $D\phi_-$ and the inner product $\langle \dots \rangle_-$ by the metric g_-^{ab} . Combining Eqs. (2.16) and (2.17) yields

$$e^{iW[g_+^{ab}, g_-^{ab}, J_+, J_-]} = \int D\alpha \int D\phi_- e^{-i(S_m^*[g_-^{ab}, \phi_-] + \langle J_-, \phi_- \rangle_-)} \\ \times \int D\phi_+ e^{i(S_m[g_+^{ab}, \phi_+] + \langle J_+, \phi_+ \rangle_+)}, \quad (2.18)$$

with the boundary condition that $\phi_+ = \phi_- = \alpha$ on the hypersurface given by $x^0 = T$. Another boundary condition, needed to assure convergence of the path integrals, is that ϕ_+ be purely negative frequency and ϕ_- be purely negative frequency at early times, or equivalently that m^2 be interpreted as $m^2 - i\epsilon$ in the action (2.3). From now on we assume the second of these. We can rewrite the generating functional (2.18) as

$$e^{iW[g_+^{ab}, g_-^{ab}, J_+, J_-]} = \int D\phi_+ D\phi_- \exp i\{S_m[g_+^{ab}, \phi_+] \\ + \langle J_+, \phi_+ \rangle_+ - S_m^*[g_-^{ab}, \phi_-] \\ - \langle J_-, \phi_- \rangle_-\}, \quad (2.19)$$

where the integral $\int D\alpha$ is now included in the integration over ϕ_+ and ϕ_- , and the boundary condition is that $\phi_+ = \phi_-$ on the hypersurface given by $x^0 = T$. Below we will be taking the limit $T \rightarrow \infty$. From Eq. (2.19) it follows that

$$\frac{-2}{\sqrt{-g_+}} \frac{\delta W[g_+^{ab}, g_-^{ab}, 0, 0]}{\delta g_+^{ab}} \Big|_{g_+^{ab} = g_-^{ab} = g^{ab}} = \langle 0, \text{in} | \hat{T}_{ab} | 0, \text{in} \rangle. \quad (2.20)$$

The effective action is defined to be a Legendre transform of the generating functional as before:

$$\Gamma_m[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-] \equiv W[g_+^{ab}, g_-^{ab}, J_+, J_-] \\ - \langle J_+, \bar{\phi}_+ \rangle_+ + \langle J_-, \bar{\phi}_- \rangle_-, \quad (2.21)$$

where

$$\bar{\phi}_\pm[g_+^{ab}, g_-^{ab}, J_+, J_-] = \pm \frac{1}{\sqrt{g_\pm}} \frac{\delta W[g_+^{ab}, g_-^{ab}, J_+, J_-]}{\delta J_\pm}, \quad (2.22)$$

and where we use the shorthand notation (2.6). Note that when $g_+^{ab} = g_-^{ab}$ and $J_+ = J_-$, we find from Eqs. (2.19) and (2.21) that

$$\bar{\phi}_\pm[g^{ab}, g^{ab}, J, J] = \langle 0, \text{in} | \hat{\phi} | 0, \text{in} \rangle_J. \quad (2.23)$$

From Eqs. (2.20) and (2.21) we obtain the expected stress-energy tensor in the in-coming vacuum state:

$$T_{ab \text{ in-in}} \equiv \langle 0, \text{in} | \hat{T}_{ab} | 0, \text{in} \rangle = \frac{-2}{\sqrt{-g_+}} \frac{\delta \Gamma_m}{\delta g_+^{ab}} \Big|_{\bar{\phi}_\pm = \bar{\phi}[g^{ab}], g_\pm^{ab} = g^{ab}}. \quad (2.24)$$

Here $\bar{\phi}[g^{ab}]$ is given by Eq. (2.22) at $J_+ = J_- = 0$ and $g_+^{ab} = g_-^{ab} = g^{ab}$:

$$\bar{\phi}[g^{ab}] = \bar{\phi}_+[g^{ab}, g^{ab}, 0, 0]. \quad (2.25)$$

From Eq. (2.23) it can be seen that $\bar{\phi}[g^{ab}]$ is the expected value of the field in the incoming vacuum state. Equation (2.24) is the formula we will use below to compute the stress tensor.

We introduce the shorthand notation $\phi_s = (\phi_+, \phi_-)$,

$$\hat{J}_s = \begin{pmatrix} +J_+ \\ -J_- \end{pmatrix}, \quad (2.26)$$

and

$$S_m[g_s^{ab}, \phi_s] = S_m[g_+^{ab}, \phi_+] - S_m^*[g_-^{ab}, \phi_-], \quad (2.27)$$

where the index s takes the values $+$ and $-$. The generating functional (2.19) can be rewritten using this notation as

$$e^{iW[g_s^{ab}, \hat{J}_s]} = \int D\phi_r e^{i(S_m[g_s^{ab}, \phi_s] + \langle \hat{J}_t, \phi_t \rangle_t)}, \quad (2.28)$$

where a sum over the repeated index t is understood.

Next, we derive the analog of the formal expression (2.13) for the in-in effective action. We define the operator \bar{A} on pairs of functions $\phi_s = (\phi_+, \phi_-)$ by

$$(\bar{A}\phi)_s(x) = \int d^n y \sqrt{-g_t(y)} \bar{A}_{st}(x, y) \phi_t(y), \quad (2.29)$$

where

$$\bar{A}_{st}(x, y) \equiv \frac{1}{\sqrt{g_s(x)g_t(y)}} \frac{\delta^2 S_m[g_r^{ab}, \phi_r]}{\delta\phi_s(x)\delta\phi_t(y)}, \quad (2.30)$$

which can be written using Eqs. (2.3) and (2.27) as

$$\bar{A}_{st}(x, y) = \delta^n(x-y) \begin{bmatrix} \frac{\square_{+,x} - m^2 + i\epsilon}{\sqrt{-g_+(x)}} & 0 \\ 0 & \frac{-\square_{-,x} + m^2 + i\epsilon}{\sqrt{-g_-(x)}} \end{bmatrix}. \quad (2.31)$$

Here $\square_{+,x}$ denotes the wave operator associated with the metric g_+^{ab} acting on the coordinates $x = x^a$, and similarly for $\square_{-,x}$. Using Eqs. (2.21), (2.28) and (2.30), we then find the following analog of Eq. (2.13):

$$\Gamma_m[g_r^{ab}, \bar{\phi}_r] = S_m[g_r^{ab}, \bar{\phi}_r] + \frac{i}{2} \text{Tr} \ln(\bar{A}_{st}), \quad (2.32)$$

where Tr denotes the appropriate trace over both spacetime variables and over the indices s, t . The operator \bar{A} is the

natural operator associated with the quadratic form (2.27) and with the covariant inner product on pairs of functions (f_+, f_-) given by

$$\langle (f_+, f_-), (h_+, h_-) \rangle \equiv \sum_{s=+,-} \int d^n x \sqrt{-g_s(x)} f_s(x) * h_s(x). \quad (2.33)$$

The reason the inner product (2.33) is the appropriate inner product is that the measures $D\phi_+$ and $D\phi_-$ in Eq. (2.19) are determined by the metrics g_+^{ab} and g_-^{ab} respectively.

In our perturbative computations below, we shall derive an expression for the effective action in terms of a series of products of operators. For that purpose, it will be convenient to use the Hilbert space structure associated with the coordinate dependent inner product

$$\langle (f_+, f_-), (h_+, h_-) \rangle^c \equiv \sum_{s=+,-} \int d^n x f_s(x) * h_s(x), \quad (2.34)$$

instead of that associated with the covariant inner product (2.33). We will always choose the coordinate system x^a appearing in Eq. (2.34) to be a Lorentzian coordinate system associated with the flat metric η_{ab} .

Finally, we define the quantity $\Gamma[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-]$ (as opposed to Γ_m) to be the effective action obtained when one starts from the full action (2.1) rather than just the matter part (2.3). It is clear that

$$\begin{aligned} \Gamma[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-] \\ = \Gamma_m[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-] + S_g[g_+^{ab}] - S_g[g_-^{ab}]. \end{aligned} \quad (2.35)$$

The semiclassical equations of motion are given by

$$\left. \frac{\delta\Gamma}{\delta g_+^{ab}} \right|_{\bar{\phi}_\pm = \bar{\phi}, g_\pm^{ab} = g^{ab}} = \left. \frac{\delta\Gamma}{\delta \bar{\phi}_+} \right|_{\bar{\phi}_\pm = \bar{\phi}, g_\pm^{ab} = g^{ab}} = 0. \quad (2.36)$$

The second of these equations is automatically solved when we choose $\bar{\phi} = \bar{\phi}[g^{ab}]$, cf. Eq. (2.25) above, corresponding to the incoming vacuum state. Thus the equation of motion reduces to

$$\left. \frac{\delta\Gamma}{\delta g_+^{ab}} \right|_{\bar{\phi}_\pm = \bar{\phi}[g^{ab}], g_\pm^{ab} = g^{ab}} = 0. \quad (2.37)$$

III. THE EFFECTIVE ACTION FOR NEARLY FLAT SPACETIMES

In this section we specialize to almost flat spacetimes of the form (1.13), and calculate the in-out effective action (2.15) and the in-in effective action (2.35) as series expansions in powers of the metric perturbation h_{ab} . We use the methods of Hartle and Horowitz [23] and of Jordan [6,17], who performed similar calculations in the massless case.

We start by further simplifying Eq. (2.32). Note that

$$\frac{\delta}{\delta g_+^{ab}} \text{Tr} \ln[\bar{A}_{st}] = \text{Tr} \left(\bar{A}_{sr}^{-1} \frac{\delta \bar{A}_{rt}}{\delta g_+^{ab}} \right) \quad (3.1)$$

$$= \text{tr} \left(\bar{A}_{++}^{-1} \frac{\delta \bar{A}_{++}}{\delta g_+^{ab}} \right), \quad (3.2)$$

since in the sum over r in Eq. (3.1), only \bar{A}_{++} depends on g_+^{ab} . Here tr denotes a trace over the spacetime variables x, y only, not including a sum over the s, t indices. Combining Eqs. (2.32) and (3.2) we now obtain

$$\Gamma_m[g_r^{ab}, \bar{\phi}_r] = S_m[g_r^{ab}, \bar{\phi}_r] + \frac{i}{2} \text{tr} \ln \bar{A}_{++} + F_1[g_+^{ab}], \quad (3.3)$$

where $F_1[g_+^{ab}]$ is some functional of g_+^{ab} . This term $F_1[g_+^{ab}]$ will not contribute to the functional derivative in Eq. (2.24) and hence will not contribute to the in-in expected stress tensor.

Next, we define the coordinate dependent operator $A = A_{rs}$ acting on pairs of functions (ϕ_+, ϕ_-) by

$$(A\phi)_r(x) = \sqrt{-g_r(x)} (\bar{A}\phi)_r(x). \quad (3.4)$$

Then from Eqs. (2.29) and (2.31) the kernel $A_{rs}(x, y)$ of A with respect to the coordinate dependent inner product (2.34) is given by

$$A_{rs}(x, y) = \sqrt{-g_r(x)} \sqrt{-g_s(y)} \bar{A}_{rs}(x, y). \quad (3.5)$$

[Here $\bar{A}_{rs}(x, y)$ is the kernel of the operator \bar{A} with respect to the inner product (2.33), given by Eq. (2.31)]. Combining Eqs. (2.31) and (3.3)–(3.5) now yields

$$\begin{aligned} \Gamma_m[g_r^{ab}, \bar{\phi}_r] &= S_m[g_r^{ab}, \bar{\phi}_r] + F_1[g_+^{ab}] + \frac{i}{2} \text{tr} \ln A_{++} + F_2[\det(g_+)], \end{aligned} \quad (3.6)$$

where $F_2[\det(g_+)]$ is some functional of the determinant $\det[g_+^{ab}(x)]$. When we take the variational derivative of Γ_m in Eq. (2.24) to calculate the stress tensor, the term F_2 will contribute a term proportional to the metric g_{ab} and hence will contribute only to the renormalization of the cosmological constant.

We define the propagator \bar{G}_{st} to be the inverse of the operator \bar{A}_{st} . Its kernel $\bar{G}_{st}(x, y)$ with respect to the inner product (2.33) is given by

$$\int d^n x' \sqrt{-g_s(x')} \bar{A}_{rs}(x, x') \bar{G}_{st}(x', y) = - \frac{\delta^n(x-y)}{\sqrt{-g_t(y)}} \delta_{rt}. \quad (3.7)$$

Note that the operation of taking the inverse is unique by virtue of the boundary conditions imposed on the path inte-

grals and the fact that the mass squared parameter is assumed to have a small negative imaginary part (see Appendix C). We also define a coordinate dependent operator G_{st} to be the inverse of the operator A_{st} ; its kernel $G_{st}(x, y)$ with respect to the inner product (2.34) is given by

$$\int d^n x' A_{rs}(x, x') G_{st}(x', y) = - \delta^n(x-y) \delta_{rt}. \quad (3.8)$$

From Eqs. (3.5), (3.7) and (3.8) it follows that $G_{rs}(x, y) = \bar{G}_{rs}(x, y)$, that is, the above two kernels coincide.

A. Perturbation expansion for the in-in effective action

We now expand the operator A_{st} as

$$A_{rs}(x, x') = A_{rs}^0(x, x') + V_{rs}(x, x'), \quad (3.9)$$

where

$$V_{rs}(x, x') = V_{rs}^{(1)}(x, x') + V_{rs}^{(2)}(x, x') + \dots \quad (3.10)$$

Here A_{rs}^0 is the Minkowski spacetime operator, and the terms $V^{(1)}$ and $V^{(2)}$ are the pieces of A_{rs} that are linear and quadratic in the metric perturbation h_{ab} , respectively [see Eqs. (3.27) and (3.28) below]. Note that from Eqs. (2.31), (3.5) and (3.9) it follows that the operator V_{rs} is diagonal in the indices r and s and is of the form

$$\begin{aligned} V_{rs}[g_+^{ab}, g_-^{ab}] &= \begin{bmatrix} V_{++}[g_+^{ab}, g_-^{ab}] & V_{+-}[g_+^{ab}, g_-^{ab}] \\ V_{-+}[g_+^{ab}, g_-^{ab}] & V_{--}[g_+^{ab}, g_-^{ab}] \end{bmatrix} \\ &= \begin{bmatrix} V[g_+^{ab}] & 0 \\ 0 & -V[g_-^{ab}]^* \end{bmatrix}, \end{aligned} \quad (3.11)$$

for some functional $V = V[g^{ab}]$.

We similarly expand the propagator G_{st} as

$$G_{st} = G_{st}^0 + G_{st}^{(1)} + G_{st}^{(2)} + \dots \quad (3.12)$$

The Minkowski spacetime propagator G_{st}^0 can be obtained by combining Eqs. (2.31), (3.5) and (3.8) and using $g_{ab} = \eta_{ab}$. In Appendix C we show that this yields

$$\begin{aligned} G_{st}^0(x, y) &= \begin{bmatrix} G_{++}^0(x, y) & G_{+-}^0(x, y) \\ G_{-+}^0(x, y) & G_{--}^0(x, y) \end{bmatrix} \\ &= \begin{bmatrix} G(x-y) & \Delta^+(x-y) \\ -\Delta^+(x-y)^* & -G(x-y)^* \end{bmatrix}, \end{aligned} \quad (3.13)$$

where

$$G(x) = \int \frac{d^n p}{(2\pi)^n} \frac{e^{ipx}}{p^2 + m^2 - i\epsilon} \quad (3.14)$$

is the free Feynman propagator and

$$\Delta^+(x) = 2\pi i \int \frac{d^n p}{(2\pi)^n} e^{ipx} \delta(p^2 + m^2) \Theta(-p^0) \quad (3.15)$$

is the positive Wightman function.

Next, by combining the expansions (3.9) and (3.12) together with the definition (3.8), we find the following expression for the logarithmic term appearing in the effective action (3.3) (see Appendix D):

$$\begin{aligned} \ln[G_{++}] &= \ln[G_{++}^0] + V_{++}^{(1)} G_{++}^0 \\ &+ V_{++}^{(2)} G_{++}^0 + \frac{1}{2} V_{++}^{(1)} G_{++}^0 V_{++}^{(1)} G_{++}^0 \\ &+ V_{++}^{(1)} G_{+-}^0 - V_{--}^{(1)} G_{-+}^0 + O(h^3). \end{aligned} \quad (3.16)$$

The products on the right hand side of Eq. (3.16) are operator products, where the kernels are understood to refer to the inner product (2.34), so that, for example, $[V_{++}^{(1)} G_{++}^0](x, y) \equiv \int d^n y' V_{++}^{(1)}(x, y') G_{++}^0(y', y)$. Combining Eqs. (3.16) and (3.3) we finally obtain the perturbation expansion of the in-in effective action

$$\begin{aligned} \Gamma_m[g_r^{ab}, \bar{\phi}_r] &= S_m[g_r^{ab}, \bar{\phi}_r] - \frac{i}{2} \text{tr} \ln G_{++}^0 \\ &- \frac{i}{2} \text{tr} \left[V_{++}^{(1)} G_{++}^0 + V_{++}^{(2)} G_{++}^0 \right. \\ &+ \frac{1}{2} V_{++}^{(1)} G_{++}^0 V_{++}^{(1)} G_{++}^0 \\ &\left. + V_{++}^{(1)} G_{+-}^0 - V_{--}^{(1)} G_{-+}^0 \right] + F_1[g_-^{ab}] \\ &+ F_2[\det(g_+)] + O(h^3). \end{aligned} \quad (3.17)$$

B. Perturbation expansion for the in-out effective action

Consider now the corresponding calculation in the in-out formalism. If we compare the generating functionals (2.7) of the in-out formalism and (2.19) of the in-in formalism, we see that the terms in ϕ and in ϕ_+ in these equations coincide. Hence, from the definitions (2.10) and (2.21) of the effective actions in terms of the generating functionals, it follows that the in-out propagator and effective action can be obtained from the corresponding in-in quantities simply by replacing ϕ_+ with ϕ , J_+ with J , and dropping all terms generated by ϕ_- and J_- . The resulting in-out action is

$$\begin{aligned} \Gamma_m[g^{ab}, \bar{\phi}] &= S_m[g^{ab}, \bar{\phi}] - \frac{i}{2} \text{tr} \ln G^0 \\ &- \frac{i}{2} \text{tr} \left[V^{(1)} G^0 + V^{(2)} G^0 + \frac{1}{2} V^{(1)} G^0 V^{(1)} G^0 \right] \\ &+ F_2[\det(g)] + O(h^3), \end{aligned} \quad (3.18)$$

where $G^0(x, y) = G(x - y)$ is the usual free Feynman propagator (3.14), and $V^{(1)}$ and $V^{(2)}$ are the pieces of the operator V defined in Eq. (3.11) that are linear and quadratic in h_{ab} . Comparing Eq. (3.18) with the in-in effective action (3.17) and using Eqs. (3.11) and (3.13), we see that the only difference is the term involving $G_{-+}^0(x, y)$, since $G_{++}^0 = G^0$ and $V_{++} = V$. Hence we can write

$$\begin{aligned} \Gamma_m[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-]_{\text{in-in}} \\ = \Gamma_m[g_+^{ab}, \bar{\phi}_+]_{\text{in-out}} + F_3[g_-^{ab}] + U[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-], \end{aligned} \quad (3.19)$$

where

$$U[g_+^{ab}, g_-^{ab}, \bar{\phi}_+, \bar{\phi}_-] = -\frac{i}{2} \text{tr} [V_{++}^{(1)} G_{+-}^0 - V_{--}^{(1)} G_{-+}^0] \quad (3.20)$$

and $F_3[g_-^{ab}]$ is a term which does not depend on g_+^{ab} or $\bar{\phi}_+$.

C. Explicit calculations

We write the spacetime metric as

$$g_{ab} = \eta_{ab} + h_{ab}, \quad (3.21)$$

where η_{ab} is a flat Minkowski metric. From now on indices are raised and lowered with η_{ab} , and derivatives denoted by a comma are coordinate derivatives in a Lorentzian coordinate system associated with the metric η_{ab} . Expanding the action (2.1) to second order in h_{ab} yields (see Appendix A)

$$S[g^{ab}, \phi] = \dot{S}_g[h_{ab}] + \dot{S}_m[h_{ab}, \phi] + O(h^3), \quad (3.22)$$

where the quadratic actions \dot{S}_g and \dot{S}_m are given by

$$\dot{S}_g[h_{ab}] = 2\mu_p^2 \int (h_{ab,ab} - h_{,a}{}^a) d^n x \quad (3.23)$$

and

$$\begin{aligned} \dot{S}_m[h_{ab}, \phi] \\ = -\frac{1}{2} \int (\eta^{ab} \phi_{,a} \phi_{,b} + m^2 \phi^2) d^n x - \frac{1}{2} \int \left[\frac{1}{2} h \eta^{ab} \phi_{,a} \phi_{,b} \right. \\ \left. - h^{ab} \phi_{,a} \phi_{,b} + \frac{1}{2} h m^2 \phi^2 \right] d^n x \\ - \frac{1}{2} \int \left[\left(\frac{1}{8} h^2 - \frac{1}{4} h^{ab} h_{ab} \right) \eta^{ab} \phi_{,a} \phi_{,b} \right. \\ \left. + \left(\frac{1}{8} h^2 - \frac{1}{4} h^{ab} h_{ab} \right) m^2 \phi^2 \right. \\ \left. - \frac{1}{2} h h^{ab} \phi_{,a} \phi_{,b} + h^{ac} h^b{}_c \phi_{,a} \phi_{,b} \right] d^n x. \end{aligned} \quad (3.24)$$

Here h is the trace $h \equiv \eta^{ab} h_{ab}$.

Next, we find from Eqs. (2.31) and (3.5) the formula for the operator A_{++}

$$A_{++}(x,y) = \delta^n(x-y)(\eta^{ab}\partial_a\partial_b - m^2 + V), \quad (3.25)$$

where [cf. Eq. (3.11) above]

$$V = V^{(1)} + V^{(2)} + O(h^3) \quad (3.26)$$

with

$$\begin{aligned} V^{(1)} &= \frac{1}{2}\partial_a h \partial^a - \partial_a h^{ab}\partial_b - \frac{1}{2}m^2 h \\ &= -\partial_a \bar{h}^{ab}\partial_b + \frac{1}{n-2}m^2 \bar{h} \end{aligned} \quad (3.27)$$

and

$$\begin{aligned} V^{(2)} &= \partial_a \bar{h}^{ac} \bar{h}^b{}_c \partial_b - \frac{1}{4}\partial_a \bar{h}^{cd} \bar{h}_{cd} \eta^{ab} \partial_b \\ &\quad - \frac{1}{n-2}\partial_a \bar{h} \bar{h}^{ab} \partial_b + \frac{1}{4(n-2)}\partial_a \bar{h}^2 \eta^{ab} \partial_b \\ &\quad + \frac{m^2}{4}\left[\bar{h}^{cd} \bar{h}_{cd} - \frac{1}{n-2}\bar{h}^2\right]. \end{aligned} \quad (3.28)$$

Here for simplicity we have written h_{ab+} simply as h_{ab} . Also we are using a notational convention where, for example, $(\partial_a h \partial^a)\varphi \equiv \partial_a(h \partial^a \varphi)$. We have also introduced the quantity

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}h \eta_{ab} \quad (3.29)$$

and its trace $\bar{h} = \eta^{ab}\bar{h}_{ab}$. Note that \bar{h}_{ab} is the trace-reversal of the metric perturbation h_{ab} when the dimension n of spacetime is 4, but not otherwise.

Using Eqs. (2.35), (3.11), (3.17), (3.27), and (3.28), we now find that the in-in effective action is

$$\begin{aligned} \Gamma[h_{ab+}, h_{ab-}, \bar{\phi}_+, \bar{\phi}_-] \\ &= \dot{S}[h_{ab+}, h_{ab-}, \bar{\phi}_+, \bar{\phi}_-] + U[h_{ab+}, h_{ab-}] + K_1[h_{ab+}] \\ &\quad + K_2[h_{ab+}] + L[h_{ab+}] + F_4[g_-^{ab}, \det(g_+)] \\ &\quad + O(h^3), \end{aligned} \quad (3.30)$$

where

$$\begin{aligned} \dot{S}[h_{ab+}, h_{ab-}, \bar{\phi}_+, \bar{\phi}_-] \\ &= \dot{S}_g[h_{ab+}] + \dot{S}_m[h_{ab+}, \phi_+] \\ &\quad - \dot{S}_g[h_{ab-}] - \dot{S}_m^*[h_{ab-}, \phi_-]. \end{aligned} \quad (3.31)$$

In Eq. (3.30), we have absorbed the constant $\text{tr} \ln G_{++}^0$ and the terms F_1 and F_2 into the functional $F_4[g_-^{ab}, \det(g_+)]$. As before this term is a quantity which depends on the metric g_+^{ab} only through its determinant, and which thus affects the in-in equation of motion only via a renormalization of the cosmological constant. The terms $U[h_{ab+}, h_{ab-}]$, $K_1[h_{ab+}]$, $K_2[h_{ab+}]$ and $L[h_{ab+}]$ in Eq. (3.30) are given by

$$U[h_{ab+}, h_{ab-}] = -\frac{i}{2} \int d^n x \int d^n x' \delta^n(x-x') V^{(1)}[h_{ab+}(x)] G_{+-}^0(x-x') \{-V^{(1)*}[h_{ab-}(x')]\} G_{-+}^0(x'-x), \quad (3.32)$$

$$\begin{aligned} K_1[h_{ab}] &= -\frac{i}{2} \int d^n x \int d^n x' \delta^n(x-x') V^{(1)}(x) G(x-x') \\ &= -\frac{i}{2} \int d^n x \int d^n x' \delta^n(x-x') \left[-\partial_a \bar{h}^{ab} \partial_b + \frac{1}{n-2} m^2 \bar{h} \right] G(x-x'), \end{aligned} \quad (3.33)$$

$$\begin{aligned} K_2[h_{ab}] &= -\frac{i}{2} \int d^n x \int d^n x' \delta^n(x-x') V^{(2)}(x) G(x-x') \\ &= -\frac{i}{2} \int d^n x \int d^n x' \delta^n(x-x') \left[\partial_a \bar{h}^{ac} \bar{h}^b{}_c \partial_b - \frac{1}{4} \partial_a \bar{h}^{cd} \bar{h}_{cd} \eta^{ab} \partial_b - \frac{1}{n-2} \partial_a \bar{h} \bar{h}^{ab} \partial_b + \frac{1}{4(n-2)} \partial_a \bar{h} \bar{h} \eta^{ab} \partial_b \right. \\ &\quad \left. + \frac{m^2}{4} \left(\bar{h}^{cd} \bar{h}_{cd} - \frac{1}{n-2} \bar{h} \bar{h} \right) \right] G(x-x'), \end{aligned} \quad (3.34)$$

and

$$\begin{aligned} L[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' V^{(1)}(x) G(x-x') V^{(1)}(x') \\ &\quad \times G(x'-x) \\ &= L_1[h_{ab}] + L_2[h_{ab}] + L_3[h_{ab}] + L_4[h_{ab}]. \end{aligned} \quad (3.35)$$

We have also defined

$$\begin{aligned} L_1[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' \bar{h}(x)^{ab} G(x-x')_{,bc'} \\ &\quad \times \bar{h}(x')^{c'd'} G(x'-x)_{,d'a}, \end{aligned} \quad (3.36)$$

$$\begin{aligned} L_2[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' \frac{m^2}{n-2} \bar{h}(x) G(x-x')_{,c'} \\ &\quad \times \bar{h}(x')^{c'd'} G(x'-x)_{,d'}, \end{aligned} \quad (3.37)$$

$$\begin{aligned} L_3[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' \bar{h}(x)^{ab} G(x-x')_{,b} \\ &\quad \times \frac{m^2}{n-2} \bar{h}(x') G(x'-x)_{,a}, \end{aligned} \quad (3.38)$$

and

$$\begin{aligned} L_4[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' \frac{m^2}{n-2} \bar{h}(x) G(x-x') \\ &\quad \times \frac{m^2}{n-2} \bar{h}(x') G(x'-x). \end{aligned} \quad (3.39)$$

In Eqs. (3.32) and (3.35), the differential operators in each factor of $V^{(1)}$ act only on the propagator immediately to the right of such factors.

It is straightforward to obtain the in-out effective action from the in-in effective action (3.30) using the method of Sec. III B above. Using Eqs. (3.19) and (3.30) we obtain

$$\begin{aligned} \Gamma[h_{ab}, \bar{\phi}] &= S[h_{ab}, \bar{\phi}] + K_1[h_{ab}] \\ &\quad + K_2[h_{ab}] + L[h_{ab}] + F_2[\det(g)] + O(\hbar^3). \end{aligned} \quad (3.40)$$

As already mentioned [cf. Eq. (3.19) above] the term $U[h_{ab+}, h_{ab-}]$ in Eq. (3.32) contains the differences between the in-in and in-out formalisms, and has to be added to the in-out effective action $\Gamma[h_{ab}, \bar{\phi}]$ to obtain the in-in effective action $\Gamma[h_{ab+}, \bar{\phi}_+, h_{ab-}, \bar{\phi}_-]$.

Next, we insert the expression (3.14) for the Feynman propagator $G(x-x')$ into Eqs. (3.33)–(3.39) to evaluate the quantities U , K_1 , K_2 and L . To simplify the calculations, we work in the Lorentz gauge where $\bar{h}^{ab}_{,b} = 0$, and we regularize the results using dimensional regularization. The results are written in terms of curvature invariants using Appendix B, and are listed in Appendix E.

As an example, we show how to compute the term (3.38):

$$\begin{aligned} L_3[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' \bar{h}^{ab}(x) \frac{m^2}{n-2} \bar{h}(x') \\ &\quad \times \int \frac{d^n p}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} \frac{p_b q_a e^{i(p-q)(x-x')}}{(p^2 + m^2 - i\epsilon)(q^2 + m^2 - i\epsilon)} \\ &= -\frac{i}{4} \int d^n x \int d^n x' \bar{h}(x)^{ab} \frac{m^2}{n-2} \bar{h}(x') \\ &\quad \times \int \frac{d^n k}{(2\pi)^n} e^{ik(x-x')} \int \frac{d^n q}{(2\pi)^n} \\ &\quad \times \frac{q_a(k_b + q_b)}{((k+q)^2 + m^2 - i\epsilon)(q^2 + m^2 - i\epsilon)}. \end{aligned} \quad (3.41)$$

We can drop the k_b in the term $(k_b + q_b)$, since we are working in the Lorentz gauge. This yields

$$\begin{aligned} L_3[h_{ab}] &= -\frac{i}{4} \int d^n x \int d^n x' \bar{h}(x)^{ab} \frac{m^2}{n-2} \bar{h}(x') \\ &\quad \times \int \frac{d^n k}{(2\pi)^n} e^{ik(x-x')} I_{ab}(k) \end{aligned} \quad (3.42)$$

where

$$I_{ab}(k) = \int \frac{d^n q}{(2\pi)^n} \frac{q_a q_b}{((k+q)^2 + m^2 - i\epsilon)(q^2 + m^2 - i\epsilon)}. \quad (3.43)$$

In order to perform this integral, we analytically continue to Euclidean signature. Defining $I_{ab}^E(k^0, k^j) \equiv I_{ab}(-ik^0, k^j)$ and using the transformations

$$q^0 \rightarrow -iq^0 \quad \text{and} \quad k^0 \rightarrow -ik^0 \quad (3.44)$$

in Eq. (3.43), we obtain

$$\begin{aligned} I_{ab}^E(k) &= \int_0^1 dx \int \frac{id^n q}{(2\pi)^n} \frac{(i)^{\delta_a^0 + \delta_b^0} q_a q_b}{[q^2 + m^2 + 2qk(1-x) + k^2(1-x)]^2}. \end{aligned} \quad (3.45)$$

In Eq. (3.45) we have also introduced the Feynman parameter x , and dropped the $i\epsilon$. The integral over q can now be evaluated (see, e.g., Ramond [24]). The result is

$$\begin{aligned} I_{ab}^E(k) &= \int_0^1 dx \frac{i}{(4\pi)^{n/2} \Gamma(2)} \left\{ k_a k_b f(k, x) \right. \\ &\quad \left. + \frac{(i)^{\delta_a^0 + \delta_b^0} \delta_{ab} \Gamma(1 - n/2)/2}{[m^2 + k^2(1-x) - k^2(1-x)^2]^{1-n/2}} \right\}. \end{aligned} \quad (3.46)$$

Next, the term $k_a k_b f(k, x)$ can be dropped, since it will not contribute to $L_3[h_{ab}]$ as we are working in the Lorentz

gauge. Also, when we analytically continue back to Lorentzian signature using $k^0 \rightarrow ik^0$, we find $(i)^{\delta_a^0 + \delta_b^0} \delta_{ab} \rightarrow \eta_{ab}$. Thus we obtain

$$I_{ab}(k) = \frac{i}{(4\pi)^2} \frac{\eta_{ab}}{2-n} \left[\frac{2}{4-n} - \gamma + \ln 4\pi \right] \\ \times \int_0^1 dx [m^2 + k^2(1-x)x] \\ \times \{1 - (2-n/2) \ln[m^2 + k^2(1-x)x]\} \\ + O[(4-n)], \quad (3.47)$$

where γ is Euler's constant. The integration over x can now be performed with the result

$$I_{ab}(k) = \frac{i}{(4\pi)^2} \frac{\eta_{ab}}{2-n} \left[\left(\frac{2}{4-n} - \gamma \right. \right. \\ \left. \left. - \ln \frac{m^2}{4\pi} + \frac{5}{3} \right) \left(m^2 + \frac{k^2}{6} \right) - \frac{m^2}{3} - \frac{1}{3} (k^2 + 4m^2) \right] \\ \times \sqrt{\frac{k^2 + 4m^2}{k^2}} \operatorname{arctanh} \sqrt{\frac{k^2}{k^2 + 4m^2}} \\ + O[(4-n)]. \quad (3.48)$$

Below we will write the function $\operatorname{arctanh}(K) = \ln[(1+K)/(1-K)]/2$ appearing in Eq. (3.48) in terms of a logarithm. When the $i\epsilon$ from the mass-term is included this will lead to logarithms of the form

$$\ln(K - i\epsilon) \equiv \ln|K| - i\pi\Theta(-K), \quad (3.49)$$

where $\Theta(K)$ is the Heaviside step function. Henceforth when we write \ln we shall mean the logarithm defined in Eq. (3.49), which has a branchcut along the negative real axis.

When the remaining terms in Eq. (3.30) are evaluated in the same fashion, and written in terms of curvature invariants (see Appendix B), we find the results listed in Appendix E. The effective action (3.40) then becomes

$$\Gamma[h_{ab}, \bar{\phi}] = S[h_{ab}, \bar{\phi}] + W, \quad (3.50)$$

with

$$W = \frac{1}{512\pi^2} \int d^4x \left[8m^4 A_\infty \sqrt{-g} - 8 \frac{m^2}{3} B_\infty \sqrt{-g} R \right. \\ \left. + \frac{C_\infty}{30} \sqrt{-g} 8R_{ab} R^{ab} + \frac{D_\infty}{30} \sqrt{-g} 4R^2 \right] + W_{\text{nl}} \\ + F_2[\det(g)]. \quad (3.51)$$

Here

$$W_{\text{nl}} = \frac{1}{512\pi^2} \int d^4x \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \\ \times [R(x)R(x') \tilde{Q}_1(k) \\ + R_{ab}(x)R^{ab}(x') \tilde{Q}_2(k)] \quad (3.52)$$

is the non-local part of the effective action. Furthermore we have defined the functions

$$\tilde{Q}_1(k) = 4 \left[-\frac{4}{15} \frac{m^4}{k^4} - \frac{37}{45} \frac{m^2}{k^2} - \frac{1}{30} \ln \frac{m^2 - i\epsilon}{\mu^2} \right. \\ \left. - \left(\frac{(k^2 + 4m^2)^2}{30k^4} - \frac{2m^2(k^2 + 4m^2)}{3k^4} + \frac{2m^4}{k^4} \right) \right. \\ \left. \times \sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} \right. \\ \left. \times \ln \left(\frac{\sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} + 1}{\sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} - 1} \right) \right] \quad (3.53)$$

and

$$\tilde{Q}_2(k) = 8 \left[\frac{16}{15} \frac{m^4}{k^4} + \frac{28}{45} \frac{m^2}{k^2} - \frac{1}{30} \ln \frac{m^2 - i\epsilon}{\mu^2} - \frac{(k^2 + 4m^2)^2}{30k^4} \right. \\ \left. \times \sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} \right. \\ \left. \times \ln \left(\frac{\sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} + 1}{\sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} - 1} \right) \right]. \quad (3.54)$$

The logarithm used here is the one defined in Eq. (3.49). The constants appearing in Eq. (3.51) are

$$A_\infty = \frac{2}{4-n} + \ln 4\pi - \gamma - \ln \mu^2 + \frac{3}{2} - \ln \frac{m^2}{\mu^2}, \quad (3.55)$$

$$B_\infty = \frac{2}{4-n} + \ln 4\pi - \gamma - \ln \mu^2 + 1 - \ln \frac{m^2}{\mu^2}, \quad (3.56)$$

$$C_\infty = \frac{2}{4-n} + \ln 4\pi - \gamma - \ln \mu^2 + \frac{46}{15} \quad (3.57)$$

and

$$D_\infty = \frac{2}{4-n} + \ln 4\pi - \gamma - \ln \mu^2 + \frac{1}{15}. \quad (3.58)$$

Note that both $\tilde{Q}_1(k)$ and $\tilde{Q}_2(k)$ are finite at $k^2=0$ and that they reduce to

$$\begin{aligned} \tilde{Q}_2(k) &= 2\tilde{Q}_1(k) = -\frac{4}{15} \ln \left(\frac{k^2 - i\epsilon}{\mu^2} \right) \\ &= -\frac{4}{15} \left[\ln \left| \frac{k^2}{\mu^2} \right| - i\pi \Theta(-k^2) \right] \end{aligned} \quad (3.59)$$

if $m=0$. Note also that the constant μ^2 appearing in Eqs. (3.53)–(3.58) drops out when these equations are inserted in Eqs. (3.51) and (3.52) in the $m \neq 0$ case. The constant μ^2 has dimension (mass)² and has been inserted to yield the correct dimensions in the logarithms of Eqs. (3.53) and (3.54).

Now the functional $F_2[\det(g)]$ in Eq. (3.51) must be a coordinate invariant, since Γ_m and the rest of the terms in that equation are. It follows that

$$F_2[\det(g)] \propto \int d^n x \sqrt{-g(x)}. \quad (3.60)$$

D. Renormalization of the in-out effective action

We now rewrite the classical action (2.1) in terms of some bare coupling constants $\mu_{p_b}^2$, Λ_b , α_b and β_b :

$$\begin{aligned} S[g_{ab}, \bar{\phi}] &= -\frac{1}{2} \int d^n x \sqrt{-g} (\nabla_a \bar{\phi} \nabla^a \bar{\phi} + m^2 \bar{\phi}^2) \\ &+ \int d^n x \sqrt{-g} \left[2\mu_{p_b}^2 (R - 2\Lambda_b) \right. \\ &\left. - \frac{1}{2} \beta_b R_{ab} R^{ab} - \frac{1}{2} \alpha_b R^2 \right]. \end{aligned} \quad (3.61)$$

From Eqs. (3.50), (3.51) and (3.61) we then find

$$\begin{aligned} \Gamma[h_{ab}, \bar{\phi}] &= -\frac{1}{2} \int d^4 x \sqrt{-g} [\nabla_a \bar{\phi} \nabla^a \bar{\phi} + m^2 \bar{\phi}^2] \\ &+ \int d^4 x \sqrt{-g} \left[2\mu_p^2 (R - 2\Lambda) \right. \\ &\left. - \frac{1}{2} \beta R_{ab} R^{ab} - \frac{1}{2} \alpha R^2 \right] + W_{\text{nl}}. \end{aligned} \quad (3.62)$$

Here μ_p^2 , Λ , α and β are the renormalized values of the parameters, given by

$$\mu_p^2 = \mu_{p_b}^2 - \frac{1}{384\pi^2} m^2 B_\infty, \quad (3.63)$$

$$\mu_p^2 \Lambda = \mu_{p_b}^2 \Lambda_b - \frac{1}{256\pi^2} m^4 A_\infty + \Delta, \quad (3.64)$$

$$\alpha = \alpha_b - \frac{1}{1920\pi^2} D_\infty, \quad (3.65)$$

and

$$\beta = \beta_b - \frac{1}{960\pi^2} C_\infty. \quad (3.66)$$

In the usual way, the renormalized values of the parameters are finite when we choose the bare parameters suitably. In Eq. (3.64), the quantity Δ is the (uncalculated) contribution to the renormalization of the cosmological constant due to the term (3.60). Note also that the parameters α and β which appear in the local part of the renormalized effective action and the parameter μ which appears in the non-local part W_{nl} are not all independent: from Eqs. (3.52)–(3.54) and (3.62) it can be seen that a change in μ can be compensated for by changes in α and β .

Finally, the renormalized in-in effective action is given by combining Eqs. (3.19), (3.62), and Eq. (E8) from Appendix E below.

IV. THE STRESS ENERGY TENSOR

A. Equations of motion in the in-out formalism

The semiclassical equations of motion are obtained from

$$\frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \Gamma[h_{cd}, \bar{\phi}] = 0, \quad (4.1)$$

and

$$\frac{\delta}{\delta \bar{\phi}} \Gamma[h_{cd}, \bar{\phi}] = 0. \quad (4.2)$$

Equation (4.2) is automatically solved when we choose

$$\bar{\phi} = \bar{\phi}[g^{ab}], \quad (4.3)$$

the functional given by Eq. (2.11) above at $J=0$. When we insert Eqs. (3.62) and (4.3) into Eq. (4.1), the functional derivative of the first term on the right hand side of Eq. (3.62) can be dropped since it is of order $O(\hbar^2)$ [as $\bar{\phi}$ from Eq. (2.11) for $J=0$ is of order $O(\hbar)$]. The resulting equation of motion is

$$G_{ab}(x) + \Lambda g_{ab}(x) = \frac{1}{4\mu_p^2} T_{ab}(x)_{\text{in-out}}, \quad (4.4)$$

where we have defined the in-out expected stress-energy tensor

$$\begin{aligned} T_{ab}(x)_{\text{in-out}} &\equiv \frac{\langle 0, \text{out} | \hat{T}_{ab} | 0, \text{in} \rangle}{\langle 0, \text{out} | 0, \text{in} \rangle} \\ &= \frac{-1}{256\pi^2} \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \\ &\quad \times [\dot{H}_{ab}^{(1)}(x') \tilde{Q}_1(k) + \dot{H}_{ab}^{(2)}(x') \tilde{Q}_2(k)] \\ &\quad + \alpha \dot{H}_{ab}^{(1)}(x) + \beta \dot{H}_{ab}^{(2)}(x). \end{aligned} \quad (4.5)$$

In writing this tensor we have also introduced the linearized versions $\dot{H}_{ab}^{(1)}(x)$ and $\dot{H}_{ab}^{(2)}(x)$ [see Eqs. (B14) and (B15) below] of the conserved local curvature tensors

$$\begin{aligned} H_{ab}^{(1)}(x) &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}(x)} \int d^4x' \sqrt{-g} R(x') R(x') \\ &= 2g_{ab} \square R - 2\nabla_a \nabla_b R - \frac{1}{2} g_{ab} R^2 + 2RR_{ab} \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} H_{ab}^{(2)}(x) &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}(x)} \int d^4x' \sqrt{-g} R_{cd}(x') R^{cd}(x') \\ &= \frac{1}{2} g_{ab} \square R + \square R_{ab} - 2\nabla_c \nabla_a R^c_b + 2R^c_a R_{cb} \\ &\quad - \frac{1}{2} g_{ab} R_{cd} R^{cd}. \end{aligned} \quad (4.7)$$

As is well known, Eq. (4.4) is not a physically realistic equation for semiclassical gravity since the right hand side is complex and not real [6,17].

B. The in-in expected stress-energy tensor

By combining Eqs. (2.25), (2.37), (3.30), (3.40), (4.1) and (4.4) we obtain the equations of motion in the in-in formalism

$$G_{ab}(x) + \Lambda g_{ab}(x) = \frac{1}{4\mu_p^2} [T_{ab}(x)_{\text{in-out}} + T'_{ab}(x)], \quad (4.8)$$

where $T_{ab}(x)_{\text{in-out}}$ is given by Eq. (4.5), and where the additional term $T'_{ab}(x)$ due to the term U in Eq. (3.30) is given by

$$\begin{aligned} T'_{ab}(x) &= \frac{-2}{\sqrt{-g}} \frac{\delta U[h_{ab+}, h_{ab-}]}{\delta g_+^{ab}} \Big|_{g_+^{ab} = g_-^{ab} = g_{ab}} \\ &= \frac{-1}{256\pi^2} \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} [\dot{H}_{ab}^{(1)}(x') \tilde{Q}'_1(k) \\ &\quad + \dot{H}_{ab}^{(2)}(x') \tilde{Q}'_2(k)]. \end{aligned} \quad (4.9)$$

Here we used Eq. (E8) from Appendix E below, and have defined

$$\begin{aligned} \tilde{Q}'_1(k) &= -4 \left[\frac{(k^2 + 4m^2)^2}{30k^4} - \frac{2m^2(k^2 + 4m^2)}{3k^4} + \frac{2m^4}{k^4} \right] \\ &\quad \times \sqrt{1 + \frac{4m^2}{k^2}} 2\pi i \Theta(-k^2 - 4m^2) \Theta(-k^0) \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \tilde{Q}'_2(k) &= -8 \left[\frac{(k^2 + 4m^2)^2}{30k^4} \right] \sqrt{1 + \frac{4m^2}{k^2}} \\ &\quad \times 2\pi i \Theta(-k^2 - 4m^2) \Theta(-k^0). \end{aligned} \quad (4.11)$$

The in-in expected stress-energy tensor

$$T_{ab}(x)_{\text{in-in}} \equiv \frac{\langle 0, \text{in} | \hat{T}_{ab} | 0, \text{in} \rangle}{\langle 0, \text{in} | 0, \text{in} \rangle} \quad (4.12)$$

is therefore given by

$$\begin{aligned} T_{ab}(x)_{\text{in-in}} &= T_{ab}(x)_{\text{in-out}} + T'_{ab}(x) \\ &= \frac{-1}{256\pi^2} \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} [\dot{H}_{ab}^{(1)}(x') \tilde{T}_1(k) \\ &\quad + \dot{H}_{ab}^{(2)}(x') \tilde{T}_2(k)] + \alpha \dot{H}_{ab}^{(1)}(x) + \beta \dot{H}_{ab}^{(2)}(x). \end{aligned} \quad (4.13)$$

Here we have defined

$$\tilde{T}_1(k) = \tilde{Q}_1(k) + \tilde{Q}'_1(k) = 4 \left\{ \begin{aligned} & -\frac{4}{15} \frac{m^4}{k^4} - \frac{37}{45} \frac{m^2}{k^2} \\ & -\frac{1}{30} \ln \frac{m^2 - i\epsilon}{\mu^2} - \left[\frac{(k^2 + 4m^2)^2}{30k^4} \right. \\ & \left. - \frac{2m^2(k^2 + 4m^2)}{3k^4} + \frac{2m^4}{k^4} \right] \sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} \\ & \times \ln \left(\frac{\sqrt{1 + \frac{4(m^2 - i\epsilon \text{Sgn}k^0)}{k^2}} + 1}{\sqrt{1 + \frac{4(m^2 - i\epsilon \text{Sgn}k^0)}{k^2}} - 1} \right) \end{aligned} \right\} \quad (4.14)$$

and

$$\begin{aligned} \tilde{T}_2(k) &= \tilde{Q}_2(k) + \tilde{Q}'_2(k) \\ &= 8 \left[\begin{aligned} & \frac{16}{15} \frac{m^4}{k^4} + \frac{28}{45} \frac{m^2}{k^2} - \frac{1}{30} \ln \frac{m^2 - i\epsilon}{\mu^2} - \frac{(k^2 + 4m^2)^2}{30k^4} \\ & \times \sqrt{1 + \frac{4(m^2 - i\epsilon)}{k^2}} \\ & \times \ln \left(\frac{\sqrt{1 + \frac{4(m^2 - i\epsilon \text{Sgn}k^0)}{k^2}} + 1}{\sqrt{1 + \frac{4(m^2 - i\epsilon \text{Sgn}k^0)}{k^2}} - 1} \right) \end{aligned} \right]. \quad (4.15) \end{aligned}$$

It is easy to see that $\tilde{T}_1(k)$ and $\tilde{T}_2(k)$ are finite at $k^2=0$ (for $m \neq 0$), and that they are sufficiently regular that their Fourier transforms $T_1(x)$ and $T_2(x)$ exist as distributions.

V. PROPERTIES OF THE IN-IN EXPECTED STRESS-ENERGY TENSOR

The stress-energy tensor given in Eq. (4.13) is determined by the Green functions $\tilde{T}_1(k)$ and $\tilde{T}_2(k)$ in Eqs. (4.14) and (4.15). In this section we show that in the limit $m \rightarrow 0$, these Green functions reduce to the previously obtained Green functions for a massless field. We also show that they are causal, i.e., that their Fourier transforms $T_1(x)$ and $T_2(x)$ have support only inside the past light cone. These properties

serve as a check of our calculation.

A. The massless limit

The Green functions $\tilde{T}_1(k)$ and $\tilde{T}_2(k)$ in Eqs. (4.14) and (4.15) reduce to

$$\begin{aligned} \tilde{T}_2(k) &= 2\tilde{T}_1(k) = -\frac{4}{15} \ln \left(\frac{k^2 - i\epsilon \text{Sgn}(k^0)}{\mu^2} \right) \\ &= -\frac{4}{15} \left[\ln \left(\frac{|k^2|}{\mu^2} \right) - i\pi \Theta(-k^2) \text{Sgn}(k^0) \right] \quad (5.1) \end{aligned}$$

if $m=0$. This Green function together with Eq. (4.13) yields exactly the same the stress-energy tensor as found by Horowitz [16] and Jordan [17].

Note that the Green functions are not smooth in m^2 near $m=0$. For $k^0 > 0$ we find

$$\begin{aligned} \left. \frac{\partial \tilde{T}_2}{\partial m^2} \right|_{m^2=0} &= 8 \left[\begin{aligned} & \frac{28}{45k^2} - \left(\frac{1}{15k^2 \sqrt{1 - \frac{4i\epsilon}{k^2}}} + \frac{4 \sqrt{1 - \frac{4i\epsilon}{k^2}}}{15k^2} \right) \\ & \times \ln \left(\frac{\sqrt{1 - \frac{4i\epsilon}{k^2}} + 1}{\sqrt{1 - \frac{4i\epsilon}{k^2}} - 1} \right) \end{aligned} \right], \quad (5.2) \end{aligned}$$

which diverges in the limit $\epsilon \rightarrow 0$. Hence the first derivative of the stress tensor with respect to m^2 does not exist at $m=0$.

B. Causality

It is difficult to find the Fourier transforms $T_j(x)$, $j=1,2$, of the Green functions (4.14) and (4.15). However, it is not necessary to explicitly perform these Fourier transforms in order to demonstrate causality. By Lorentz invariance it is sufficient to show that

$$T_j(t,0,0,0) = 0 \quad \text{for } t < 0 \quad (5.3)$$

and

$$T_j(0,r,0,0) = 0 \quad \text{for } r \neq 0, \quad (5.4)$$

for $j=1,2$. In other words, the Green functions $T_j(x)$ must be zero inside the past light cone and outside the light cone. To check the condition (5.3) we write

$$T_j(t,0,0,0) = \frac{1}{(2\pi)^4} \int d^3k \int dk^0 e^{-ik^0 t} \tilde{T}_j(k). \quad (5.5)$$

Now from Eqs. (4.14) and (4.15) we see that the logarithmic terms in both $\tilde{T}_1(k)$ and $\tilde{T}_2(k)$ have branchcuts in the lower complex k^0 plane, but no poles elsewhere. It is therefore possible to deform the contour of the k^0 integration into the usual semi-circle with infinite radius in the upper complex k^0 plane. Since $t < 0$ the integral vanishes. This immediately shows that $T_j(t, 0, 0, 0) = 0$ for $t < 0$. A similar argument can be used to show that $T_j(0, r, 0, 0) = 0$.

VI. CONCLUSIONS

We have shown that the Wald axioms determine the stress-energy tensor (up to two parameters) only in the case of a massless field. In the case of a massive scalar field, the Wald axioms allow for a much larger ambiguity. We have calculated the expectation value of the stress-energy tensor in the incoming vacuum state for a massive scalar field on any spacetime which is a linear perturbation off Minkowski spacetime, generalizing an earlier formula of Horowitz [16] and Jordan [17] in the massless case. In our calculation we used the in-in effective action formalism [5–7]. As expected, the resulting stress-energy tensor is causal and reduces to the known result in the massless case in the limit $m \rightarrow 0$. As in the massless case, we find a two parameter ambiguity in the stress-energy tensor, even though this is not guaranteed by the Wald axioms.

After this paper was submitted for publication, we became aware of related work by Dalvit and Mazzitelli [25]. Dalvit and Mazzitelli calculate an in-in effective action using the method of expanding in powers of curvature pioneered by Vilkovisky and collaborators [26]. Our result (1.12) can be derived from Eqs. (38) and (41) of Ref. [25] by evaluating the integrals over t , by specializing to linear perturbations about flat spacetime, and by representing the d'Alembertian operators in Fourier space. Our derivation of the result (1.12) has the advantage that it is more direct and simple than that of Ref. [25], since it does not rely on the Vilkovisky expansion formalism.

We conclude by listing some open questions. First, are there additional axioms which would reduce the ambiguity in the stress-energy tensor? Second, will the same stress-energy tensor (1.12) be predicted (up to the two parameter ambiguity) by the point splitting method [3] or by the DeWitt-Schwinger method [4]? There is no a-priori guarantee that this will be the case.

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APPENDIX A: NOTATION AND CONVENTIONS

We use units in which $\hbar = c = 1$, but in which $G \neq 1$, so that the Planck mass is given by

$$\mu_p = \sqrt{\frac{1}{32\pi G}}. \quad (\text{A1})$$

Throughout we use the same sign conventions for metric and curvature tensors as in the book of Misner, Thorne, and Wheeler [18]. Specifically the metric g_{ab} has signature $(-, +, +, +)$. Indices i, j, k, \dots run over the spatial indices 1, 2, 3 while indices a, b, c, \dots run over 0, 1, 2, 3.

We introduce the metric perturbation

$$h_{ab} = g_{ab} - \eta_{ab} \quad (\text{A2})$$

and its trace $h = h_{ab}\eta^{ab}$, where η_{ab} is a flat metric. In expressions involving h_{ab} , indices are raised and lowered with the flat spacetime metric η^{ab} . The coordinate derivative of a tensor T^a_b in a Lorentzian coordinate system with respect to the metric η_{ab} is denoted in the usual way:

$$T^a_{b,c} = \partial_c T^a_b. \quad (\text{A3})$$

The Fourier transform of any function $F(x)$ on Minkowski spacetime is defined as

$$\tilde{F}(k) = \int d^4x e^{-ikx} F(x). \quad (\text{A4})$$

Throughout we use

$$kx = \eta_{ab} k^a x^b \quad (\text{A5})$$

to denote the dot product of two 4-vectors k^a and x^a in Minkowski spacetime. We use $\Theta(x)$ to denote the step function, $\Theta(x) = 1$ for $x > 0$ and 0 otherwise.

Products of operators A, B are defined as

$$(AB)(x, z) = \int d^4y A(x, y) B(y, z). \quad (\text{A6})$$

Thus, factors of $\sqrt{-g}$ are not implicit in expressions such as GV in our calculations. We show such factors explicitly when they are required, with the exception of the notation for products of functions

$$\langle f_1, f_2 \rangle \equiv \int d^n x \sqrt{-g(x)} f_1(x) * f_2(x). \quad (\text{A7})$$

APPENDIX B: EXPRESSIONS FOR CURVATURE INVARIANTS

In this appendix we expand the various possible local counterterms in the effective action to second order in the metric perturbation, and write them in terms of the quantity \bar{h}_{ab} . We define

$$\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2} h \eta_{ab}, \quad (\text{B1})$$

where $h = \eta^{ab} h_{ab}$. In n dimensions we then find

$$h = -\frac{2}{n-2} \bar{h} \quad (\text{B2})$$

and thus

$$h_{ab} = \bar{h}_{ab} - \frac{1}{n-2} \bar{h} \eta_{ab}. \quad (\text{B3})$$

In our calculation of the effective action, when we expand quantities such as $\sqrt{-g}$, R or R_{ab} in terms of \bar{h}_{ab} and \bar{h} , we find terms of order $O(n-4)$. Such terms must sometimes be kept and not discarded, since they can give rise to finite contributions when multiplied by infinite terms of the form $1/(n-4)$.

We find for the determinant of the metric tensor

$$\sqrt{-g} = 1 - \frac{1}{n-2} \bar{h} - \frac{1}{4} \bar{h}^{cd} \bar{h}_{cd} + \frac{1}{4(n-2)} \bar{h} \bar{h} + O(h^3). \quad (\text{B4})$$

The curvature scalar becomes

$$R = R^{(1)} + R^{(2)} + O(h^3), \quad (\text{B5})$$

where

$$R^{(1)} = \frac{1}{n-2} \bar{h}_{,a}{}^a + \bar{h}_{ab}{}^{,ab} \quad (\text{B6})$$

and

$$\begin{aligned} R^{(2)} = & \frac{3}{4} \bar{h}_{ab,c} \bar{h}^{ab,c} + \bar{h}_{ab} \bar{h}^{ab}{}_{,c}{}^c \\ & - \frac{3n-10}{4(n-2)^2} \bar{h}_{,c} \bar{h}^{,c} - \frac{(n-4)}{(n-2)^2} \bar{h}_{,c}{}^c \bar{h} - \frac{1}{2} \bar{h}^{ca,b} \bar{h}_{cb,a} \\ & + \frac{1}{n-2} (2\bar{h} \bar{h}_{ab}{}^{,ab} + \bar{h}_{,a} \bar{h}^{ab}{}_{,b}) - 2\bar{h}^{ab} \bar{h}_{ac,b}{}^c \\ & - \bar{h}^{ab}{}_{,b} \bar{h}_{ac}{}^{,c}. \end{aligned} \quad (\text{B7})$$

Other useful scalars are

$$R^2 = \frac{1}{(n-2)^2} \bar{h}_{,a}{}^a \bar{h}_{,b}{}^b + \left(\frac{2}{n-2} \bar{h}_{,c}{}^c + \bar{h}_{cd}{}^{,cd} \right) \bar{h}_{ab}{}^{,ab} + O(h^3) \quad (\text{B8})$$

and

$$\begin{aligned} R_{ab} R^{ab} = & \frac{1}{4} \left[\bar{h}_{ab}{}_{,c} \bar{h}^{ab}{}_{,d} - \frac{(n-4)}{(n-2)^2} \bar{h}_{,a}{}^a \bar{h}_{,b}{}^b \right] \\ & + \frac{1}{4} \left(-2\bar{h}_{ab,c}{}^c + \frac{2}{n-2} \bar{h}_{,c}{}^c \eta_{ab} \right. \\ & \left. + \bar{h}_{ca,b}{}^c + \bar{h}_{cb,a}{}^c \right) (\bar{h}_c{}^{a,bc} + \bar{h}_c{}^{b,ac}) + O(h^3). \end{aligned} \quad (\text{B9})$$

Note that many of the terms vanish if the Lorentz gauge $\bar{h}^{ab}{}_{,b} = 0$ is used. In the Lorentz gauge we find that (discarding surface terms)

$$\begin{aligned} & \int d^n x 8 \sqrt{-g} \\ & = \int d^4 x (8 - 4\bar{h} - 2\bar{h}_{ab} \bar{h}^{ab} + \bar{h} \bar{h}) + \frac{1}{2} \delta \int d^4 x (4\bar{h} - \bar{h} \bar{h}) \\ & \quad + O(\delta^2), \end{aligned} \quad (\text{B10})$$

$$\begin{aligned} & \int d^n x 8 \sqrt{-g} R \\ & = \int d^4 x (2\bar{h}_{ab,c}{}^c \bar{h}^{ab} - \bar{h}_{,c}{}^c \bar{h}) + \frac{1}{2} \delta \int d^4 x \bar{h}_{,c}{}^c \bar{h} + O(\delta^2), \end{aligned} \quad (\text{B11})$$

$$\begin{aligned} & \int d^n x 4 \sqrt{-g} R^2 \\ & = \int d^4 x \bar{h}_{,a}{}^a \bar{h}_{,b}{}^b - \frac{1}{2} \delta \int d^4 x 2\bar{h}_{,a}{}^a \bar{h}_{,b}{}^b + O(\delta^2), \end{aligned} \quad (\text{B12})$$

and

$$\begin{aligned} & \int d^n x 8 \sqrt{-g} R^{ab} R_{ab} \\ & = \int d^4 x 2\bar{h}^{ab}{}_{,c}{}^c \bar{h}_{ab,d}{}^d - \frac{1}{2} \delta \int d^4 x \bar{h}_{,c}{}^c \bar{h}_{,d}{}^d + O(\delta^2), \end{aligned} \quad (\text{B13})$$

where $\delta = n - 4$.

Using the Lorentz gauge and specializing to four dimensions, we find for the linearized versions of the local curvature tensors (4.6) and (4.7)

$$\dot{H}_{ab}^{(1)}(x) = \eta_{ab} \bar{h}_{,c}{}^c d^d - \bar{h}_{,abc}{}^c + O(h^2) \quad (\text{B14})$$

and

$$\dot{H}_{ab}^{(2)}(x) = \frac{1}{2} (\eta_{ab} \bar{h}_{,c}{}^c d^d - \bar{h}_{,abc}{}^c - \bar{h}_{ab,c}{}^c d^d) + O(h^2). \quad (\text{B15})$$

APPENDIX C: THE IN-IN PROPAGATOR IN MINKOWSKI SPACETIME

In this appendix we combine Eqs. (2.31), (3.5) and (3.8) specialized to Minkowski spacetime to obtain Eqs. (3.13)–(3.15). Using Eq. (3.8) and the expansions (3.9) and (3.12), we obtain

$$\int d^n x' A_{rs}^0(x, x') G_{st}^0(x', y) = -\delta^n(x-y) \delta_{rt}. \quad (\text{C1})$$

Note that it follows from the form of $A_{rs}(x, y)$ given by Eqs. (2.31) and (3.5) that $G_{--}^0(x, y) = -G_{++}^{0*}(x, y)$ and that $G_{-+}^0(x, y) = -G_{+-}^{0*}(x, y)$.

First, the relation

$$G_{++}^0(x,y) = G(x-y), \quad (\text{C2})$$

follows immediately from translational invariance and the equation

$$[\square_x - (m^2 - i\epsilon)]G_{++}^0(x,y) = -\delta^n(x-y) \quad (\text{C3})$$

which follows from Eq. (C1).

The equation determining $G_{-+}^0(x,y)$ is, from Eq. (C1),

$$[\square_x - (m^2 + i\epsilon)]G_{-+}^0(x,y) = 0, \quad (\text{C4})$$

which upon Fourier transforming and using translational invariance becomes

$$[-p^2 - (m^2 + i\epsilon)]\tilde{G}_{-+}^0(p) = 0. \quad (\text{C5})$$

Any function $\tilde{G}_{-+}^0(p)$ which has support only on the hypersurface $p^2 = -(m^2 + i\epsilon)$ will be a solution to this equation, and hence $\tilde{G}_{-+}^0(p)$ is not uniquely determined by Eq. (C5). What we have not used yet is the boundary condition that $\phi_+ = \phi_-$ on the hypersurface given by $x^0 = T$, where $T \rightarrow \infty$. To make use of this boundary condition note that the classical equations for ϕ_{\pm} are

$$\int d^n y A_{rs}^0(x,y) \phi_s(y) = -\hat{J}_r(x), \quad (\text{C6})$$

and that the solutions to these equations are

$$\phi_s(x) = \int d^n y G_{st}^0(x,y) \hat{J}_t(y). \quad (\text{C7})$$

Enforcing the above mentioned boundary condition yields

$$\int d^n y G_{+s}^0(x,y) \hat{J}_s(y) = \int d^n y G_{-s}^0(x,y) \hat{J}_s(y) \quad (\text{C8})$$

at $x = (T, x^1, x^2, x^3)$, which using Eq. (2.26) simplifies to

$$\int d^n y [G_{++}^0(x,y) - G_{-+}^0(x,y)] J_+(y) + \int d^n y [G_{+-}^0(x,y) - G_{--}^0(x,y)] J_-(y) = 0. \quad (\text{C9})$$

Since the sources $J_{\pm}(y)$ are completely arbitrary, Eqs. (C2) and (C9) imply that

$$G_{-+}^0(x,y) = G(x-y) \quad (\text{C10})$$

for $x = (T, x^1, x^2, x^3)$. Now the Feynman propagator $G(x)$ can be written as

$$G(x) = -\Theta(x^0) \Delta^-(x) + \Theta(-x^0) \Delta^+(x), \quad (\text{C11})$$

where

$$\Delta^{\pm}(x) = \pm 2\pi i \int \frac{d^n p}{(2\pi)^n} e^{ip(x-y)} \delta(p^2 + m^2) \Theta(\mp p^0) \quad (\text{C12})$$

are the positive and negative Wightman functions [19]. From Eqs. (C10) and (C11) it follows that

$$G_{-+}^0(x,y) = -\Delta^-(x-y) \quad (\text{C13})$$

for large x^0 . Equation (C13) now implies that the appropriate solution of Eq. (C5) which fulfills the boundary condition at $x^0 = T$ corresponds to $G_{-+}^0(x,y) = -\Delta^-(x-y)$, which yields Eq. (3.13) above.

APPENDIX D: EXPANSION OF THE PROPAGATOR

In this appendix we obtain the expansion (3.16) for the operator G_{++} . From Eqs. (3.8), (3.9), (3.12) and (C1) it follows that

$$G_{rt} = G_{rs}^0 [\delta_{st} + V_{ss'} G_{s't}^0 + V_{ss'} G_{s't's''}^0 V_{s''t'} G_{t't}^0] + O(V^3). \quad (\text{D1})$$

This implies that

$$\begin{aligned} G_{++} &= G_{++}^0 + G_{+s}^0 V_{ss'} G_{s't}^0 + G_{+s}^0 V_{ss'} G_{s't's''}^0 V_{s''t'} G_{t't}^0 + \dots \\ &= G_{++}^0 [1 + V_{+s} G_{s+}^0 \\ &\quad + V_{+s} G_{s's'}^0 V_{s't} G_{t+}^0 + O(V^3)], \end{aligned} \quad (\text{D2})$$

where we have used the fact that $G_{++}^{0-1} G_{+-}^0 = A_{++}^0 G_{+-}^0 = 0$. Hence we can write the logarithm of the propagator as

$$\begin{aligned} \ln(G_{++}) &= \ln(G_{++}^0) + V_{+s} G_{s+}^0 \\ &\quad + V_{+s} G_{s's'}^0 V_{s't} G_{t+}^0 - \frac{1}{2} V_{+s} G_{s+}^0 V_{+t} G_{t+}^0 \\ &\quad + O(V^3). \end{aligned} \quad (\text{D3})$$

Next we use the fact that V_{rs} is diagonal [cf. Eq. (3.11) above] to obtain

$$\begin{aligned} \ln(G_{++}) &= \ln(G_{++}^0) + V_{++} G_{++}^0 \\ &\quad + \frac{1}{2} V_{++} G_{++}^0 V_{++} G_{++}^0 + V_{++} G_{+-}^0 V_{--} G_{-+}^0 \\ &\quad + O(V^3) \\ &= \ln(G_{++}^0) + V_{++}^{(1)} G_{++}^0 + V_{++}^{(2)} G_{++}^0 \\ &\quad + \frac{1}{2} V_{++}^{(1)} G_{++}^0 V_{++}^{(1)} G_{++}^0 \\ &\quad + V_{++}^{(1)} G_{+-}^0 V_{--}^{(1)} G_{-+}^0 + O(V^3), \end{aligned} \quad (\text{D4})$$

which yields Eq. (3.16).

APPENDIX E: TERMS IN THE EFFECTIVE ACTION

In this appendix we list the terms contributing to the effective action (3.30) given by Eqs. (3.32)–(3.39). Throughout we use the Lorentz gauge. We find

$$K_1[h_{ab}] = \frac{1}{512\pi^2} \int d^4x \bar{h}(x) (-4m^2) \left(Y + \frac{5}{2} \right), \quad (\text{E1})$$

$$K_2[h_{ab}] = \frac{1}{512\pi^2} \int d^4x \left[\bar{h}(x) \bar{h}(x) (2m^4) \left(Y + \frac{5}{2} \right) + 2\bar{h}(x)^{ab} \bar{h}(x)_{ab} (-2m^4) \left(Y + \frac{3}{2} \right) \right], \quad (\text{E2})$$

$$L_1[h_{ab}] = \frac{1}{512\pi^2} \int d^4x \int d^4x' \left\{ [\bar{h}(x) \bar{h}(x')] + 2\bar{h}^{ab}(x) \bar{h}_{ab}(x')] \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \left[\left(Y + \frac{46}{15} \right) \times \left(m^4 + \frac{m^2}{3} k^2 + \frac{k^4}{30} \right) - \frac{m^4}{2} - \frac{m^2}{15} k^2 - \frac{1}{15} (k^2 + 4m^2)^2 F_1(k) \right] \right\}, \quad (\text{E3})$$

$$L_2[h_{ab}] = L_3[h_{ab}] = \frac{1/2}{512\pi^2} \int d^4x \int d^4x' \times \left\{ \bar{h}(x) \bar{h}(x') \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \times \left[\left(Y + \frac{11}{3} \right) \left(-4m^4 - \frac{2m^2}{3} k^2 \right) + \frac{4m^4}{3} + \frac{4}{3} m^2 (k^2 + 4m^2) F_1(k) \right] \right\} \quad (\text{E4})$$

and

$$L_4[h_{ab}] = \frac{1}{512\pi^2} \int d^4x \int d^4x' \times \left\{ \bar{h}(x) \bar{h}(x') \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} [(Y+4)(2m^4) - 4m^4 F_1(k)] \right\}. \quad (\text{E5})$$

Here we have defined the constant

$$Y = \frac{2}{4-n} + \ln \frac{4\pi}{\mu^2} - \gamma - \ln \frac{m^2}{\mu^2}, \quad (\text{E6})$$

and the function

$$F_1(k) = \sqrt{\frac{k^2 + 4m^2}{k^2}} \operatorname{arctanh} \left(\sqrt{\frac{k^2}{k^2 + 4m^2}} \right). \quad (\text{E7})$$

The term which contains the differences between the in-out and in-in effective action is

$$U[h_{ab+}, h_{ab-}] = \frac{1}{256\pi^2} \int d^4x \int d^4x' \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \times \left\{ \bar{h}_+(x) \bar{h}_-(x') \left[-4m^4 + \frac{4m^2}{3} (k^2 + 4m^2) - \frac{1}{15} (k^2 + 4m^2)^2 \right] G_1(k) + 2\bar{h}_+^{ab}(x) \times \bar{h}_{ab-}(x') \left[-\frac{1}{15} (k^2 + 4m^2)^2 \right] G_1(k) \right\}, \quad (\text{E8})$$

where

$$G_1(k) = i\pi \sqrt{\frac{k^2 + 4m^2}{k^2}} \Theta(-k^2 - 4m^2) \Theta(-k^0). \quad (\text{E9})$$

- [1] For a list of some of these formal derivations, see, for example, Sec. II B of Ref. [2].
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- [8] R. M. Wald, *Commun. Math. Phys.* **54**, 1 (1977); *Phys. Rev. D* **17**, 1477 (1978).
- [9] By “local tensor” is meant a tensor which is a functional of the metric, and whose value at any point p depends only on the metric in an arbitrarily small neighborhood of p .
- [10] Further arguments based on dimensional analysis cannot eliminate the ambiguous term (1.7): If one reverts to units in which $\hbar \neq 1$ and $G \neq 1$, one can argue that the expected stress tensor should be of the form

$$T_{ab} = \hbar T_{ab}^{(1)} + O(\hbar^2),$$

where $T_{ab}^{(1)}$ is independent of \hbar . Also it can be argued that $T_{ab}^{(1)}$ should be independent of G . These requirements are satisfied when one takes for $T^{(1)}$ the expression (1.7), which has dimensions $(\text{length})^{-4}$. [Note that in this context m is fundamentally an inverse length scale, not a mass scale, from Eq. (2.3).]

- [11] In addition, this postulate is incompatible with the result we find in Sec. V A that the expected stress tensor predicted by the in-in effective action formalism is not an analytic function of m^2 .
- [12] This ambiguity is presumably related to a freedom in choosing the quantity conventionally denoted $w_0(x,y)$ in the point splitting algorithm [3].
- [13] For a preliminary investigation into the differences between ζ -function techniques and point-splitting techniques, see V. Moretti, *Phys. Rev. D* **56**, 7797 (1997); also V. Moretti, “Local ζ -function techniques vs point-splitting procedure: a few rigorous results,” gr-qc/9805091.
- [14] Note that the axiom used in Ref. [16] in the context of linear perturbation theory—that the stress tensor does not depend on derivatives of the metric of differential order higher than four—will not work outside of linear perturbation theory, as the ambiguous tensor (1.7) satisfies this property.
- [15] Our example (1.7) of an infinite parameter ambiguity in fact is not present in linear perturbation theory. When one picks out the piece of the ambiguous contribution (1.10) to the stress tensor that is linear in the metric perturbation, one finds that the resulting tensor contains only the two parameter ambiguity. However, there are other ambiguous terms with infinitely many free parameters that are present at linear order, for example, terms generated by taking a variational derivative of
- $$\int d^4x \sqrt{-g(x)} m^2 F(\square/m^2) R,$$
- where F is a suitably chosen smooth function.
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- [20] See, for example, Eq. (11.63) of Ref. [19].
- [21] See, for example, p. 156 of Ref. [4].
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