Covariant velocity and density perturbations in quasi-Newtonian cosmologies

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Recently a covariant approach to cold matter universes in the zero-shear hypersurfaces (or longitudinal) gauge has been developed. This approach reveals the existence of an integrability condition, which does not appear in standard non-covariant treatments. A simple derivation and generalization of the integrability condition is given, based on showing that the quasi-Newtonian models are a sub-class of the linearized ''silent'' models. The solution of the integrability condition implies a propagation equation for the acceleration. It is shown how the velocity and density perturbations are then obtained via this propagation equation. The density perturbations acquire a small relative-velocity correction on all scales, arising from the fully covariant general relativistic analysis. [S0556-2821(98)04622-0]

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I. INTRODUCTION

Density and velocity perturbations of cold matter universes are crucial to the understanding of structure formation in cosmology [1]. On scales well below the Hubble length, Newtonian theory may be used to analyze gravitational instability. However, current and upcoming observations and simulations are probing scales which are a significant fraction of the Hubble length, thus requiring a relativistic treatment [2]. Bardeen's gauge-invariant theory [3] is most often used for relativistic perturbations. Various choices of gauge are possible in the theory. The zero-shear hypersufaces (or longitudinal) gauge [3,4] sets up a frame of reference which emulates that of Eulerian observers in Newtonian theory, thus motivating the term ''quasi-Newtonian.'' Since the frame is non-comoving, there are relative-velocity effects on the density perturbations, and subtle issues arise in dealing with these effects, as pointed out recently by van Elst and Ellis [5]. In order to resolve these problems in a fully gauge-invariant way, a covariant approach may be adopted.

The covariant and gauge-invariant approach to perturbations was developed by Ellis and Bruni [6] on the basis of Hawking's paper [7]. It has various advantages over the non-covariant gauge-invariant theory (see [8] for further discussion). One advantage is that all quantities have a direct and immediate physical or geometric meaning, and no nonlocal decomposition into scalar, vector or tensor modes is required. A second advantage is that the covariant approach provides a natural and transparent setting to search for integrability conditions which may arise from constraints. These general relativistic constraints and the consequences that follow from their evolution, are often not made explicit. As a result, crucial general relativistic effects can sometimes be obscured or missed.

Covariant consistency analysis of constraints was developed by Maartens [9], building on the methods of Lesame *et al.* [10], in a form applicable to both the nonlinear exact theory and the case of linearized perturbations. (Further developments of the approach are given in [11–13].) Applications of a covariant approach to ''silent'' universes [14–16], to nonlinear gravitational radiation [17,18], and to non-accelerating fluid models [19,20] reveal the existence of crucial integrability conditions. The covariant characterization of scalar, vector and tensor perturbations also relies on such an approach [21–25].

Van Elst and Ellis [5] use a covariant consistency analysis first in nonlinear quasi-Newtonian cosmologies and then in the case of covariant linearization about a Friedmann-Lemaitre-Robertson-Walker (FLRW) background. They show that the nonlinear models are likely in general to be inconsistent, except for special cases such as FLRW solutions. Inconsistency of the nonlinear models is not surprising, since one is demanding for all possible dynamical evolutions of matter that there exists a shearfree and irrotational congruence, which in particular forces the magnetic part of the Weyl curvature to vanish [26]. This rules out gravitational radiation [15], and thus leads to severe restrictions on the gravitational field [15,27,28]. In the linearized case, one might expect that all the problematic terms which arise in evolving the constraints would be removed. It is implicitly or effectively assumed in standard non-covariant perturbation theory that there are no integrability conditions arising from the zero-shear hypersurfaces gauge.

However, it turns out that the linearized models are not in general consistent. Van Elst and Ellis introduce an ansatz for the evolution of the gravitational potential, which they motivate by a discussion of the lapse in Arnowitt-Deser-Misner-(ADM-)type approaches. Using this ansatz, they find an integrability condition in the linearized models. The integrability condition is satisfied by a particular value of the constant parameter in the ansatz.[1] The reason that the integrability condition is not revealed in some non-covariant treatments is probably either an implicit assumption that only evolution equations need be considered, or a gauge-dependent approximation that effectively neglects the velocity and removes the constraints (see [5] for further discussion of this point).

In this paper, extensions of the results of [5] on linearized quasi-Newtonian models are given. The main result is the determination of the velocity and density perturbations. A crucial part of the analysis is the combined use of the comoving (''Lagrangian'') and quasi-Newtonian (''Eulerian'') frames.

[1]For other values of the parameter, only very special models appear to be consistent [5].

Section II summarizes the necessary covariant equations and methods, with some details given in appendices. Section III uses a transformation to the comoving frame in order to show that the quasi-Newtonian models are a sub-class of the linearized silent models. This is the basis for a simple and direct approach to deriving the integrability condition in general, i.e., without introducing any ansatz for the gravitational potential. The general integrability condition reduces to the special form given in [5] when one imposes their ansatz for the gravitational potential. Their ansatz is also generalized. Furthermore, another integrability condition is derived by considering spatial consistency of the constraints. The van Elst-Ellis solution of the first integrability condition is shown to reduce the second condition to an identity.

The main result however, follows from the fact that the van Elst-Ellis solution itself implies a crucial propagation equation for the 4-acceleration. This propagation equation then leads to an equation determining the velocity perturbations. The equation is scale-independent, so that velocity perturbations have an effect on all scales. The velocity perturbation equation is readily solved analytically for a flat background.

In Sec. IV, the density perturbations are found by a direct and simple approach. The complicated calculations in [5] of the energy flux (momentum density) source term in the density perturbation equation are by-passed by considering the density perturbations in the comoving frame. Furthermore, the equation is solved for a flat background. The covariant correction to density perturbations that arises from relative-velocity effects is a simple comoving divergence term, which can be found from the velocity perturbations. This correction affects all scales, although it is rapidly dominated by the usual solution. The correction to the growing mode is consistent with the result of Takada and Futamase [2], who use non-covariant theory, but not in the zero-shear hypersurfaces gauge.

Concluding remarks are made in Sec. V.

II. COVARIANT EQUATIONS

Given a choice of 4-velocity field u^a (with $u^a u_a = -1$), the Ehlers-Ellis approach [29,30] employs only fully covariant quantities and equations with transparent physical and geometric meaning. The quantities are split into spacetime scalars and spatially projected tensors, while the equations split into evolution equations along u^a and constraint equations involving only spatial covariant derivatives. These equations arise from the Ricci identity for u^a and the Bianchi identities, with Einstein's field equations incorporated via algebraic replacement of the Einstein tensor by the energy-momentum tensor T_{ab}. The covariant linearization of the equations is the basis for the Ellis-Bruni perturbation theory [6]. Integrability conditions, at both the nonlinear and linear levels, arise from investigating the derivatives of constraint equations (including additional conditions that may be imposed by physical or geometric assumptions), using covariant differential identities and the evolution equations. The streamlined and developed version of the Ehlers-Ellis formalism given by Maartens [9] (see also [23,31,32]) greatly facilitates such investigations, especially by making explicit the irreducible quantities and derivatives, which significantly simplifies the equations, and by developing the covariant identities which these quantities and derivatives obey.

The projection tensor $h_{ab} = g_{ab} + u_a u_b$, where g_{ab} is the spacetime metric, and the projected alternating tensor $\varepsilon_{abc} = \eta_{abcd} u^d$, where $\eta_{abcd} = -\sqrt{|g|}\,\delta^0{}_{[a}\delta^1{}_b\delta^2{}_c\delta^3{}_{d]}$ is the spacetime alternating tensor, are the basis for covariant irreducible splitting of tensors and derivatives. Projected rank-2 tensors S_{ab} are split into a scalar trace, a projected vector spatially dual to the skew part, and a projected symmetric tracefree part:

$$S_{ab} = \tfrac{1}{3} S_{cd} h^{cd} h_{ab} + \varepsilon_{abc} S^c + S_{\langle ab \rangle},$$

where $S_a = \frac{1}{2}\varepsilon_{abc} S^{[bc]} = S_{\langle a \rangle} \equiv h_{ab} S^b$ and $S_{\langle ab \rangle} \equiv [h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h^{cd} h_{ab}] S_{cd}$. Covariant time and spatial derivatives are defined by

$$\dot{S}^{a\cdots}{}_{b\cdots} = u^c \nabla_c S^{a\cdots}{}_{b\cdots},$$

$$\mathrm{D}_c S^{a\cdots}{}_{b\cdots} = h_c{}^f h^a{}_d \cdots h_b{}^e \cdots \nabla_f S^{d\cdots}{}_{e\cdots},$$

and then the covariant spatial divergence and curl are [9,23]

$$\mathrm{div}\, V = \mathrm{D}^a V_a, \quad (\mathrm{div}\, S)_a = \mathrm{D}^b S_{ab}, \tag{1}$$

$$\mathrm{curl}\, V_a = \varepsilon_{abc} \mathrm{D}^b V^c, \quad \mathrm{curl}\, S_{ab} = \varepsilon_{cd(a} \mathrm{D}^c S_{b)}{}^d. \tag{2}$$

The dynamic quantities are the energy density ρ, the pressure p, the energy flux $q_a = q_{\langle a \rangle}$, and the anisotropic stress $\pi_{ab} = \pi_{\langle ab \rangle}$, so that

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab}.$$

The kinematic quantities are given by

$$\nabla_b u_a = \mathrm{D}_b u_a - A_a u_b, \quad \mathrm{D}_b u_a = \tfrac{1}{3}\Theta h_{ab} + \sigma_{ab} + \varepsilon_{abc}\omega^c,$$

where $\Theta = \mathrm{D}^a u_a$ is the expansion, $A_a = \dot{u}_a = A_{\langle a \rangle}$ is the 4-acceleration, $\omega_a = -\frac{1}{2}\mathrm{curl}\, u_a = \omega_{\langle a \rangle}$ is the vorticity, and $\sigma_{ab} = \mathrm{D}_{\langle a} u_{b \rangle}$ is the shear. Finally, the gravito-electromagnetic fields are

$$E_{ab} = C_{acbd} u^c u^d = E_{\langle ab \rangle}, \quad H_{ab} = \tfrac{1}{2}\varepsilon_{acd} C^{cd}{}_{be} u^e = H_{\langle ab \rangle},$$

where C_{abcd} is the Weyl tensor, which represents the locally free gravitational field [30,23]. The FLRW background is then covariantly and gauge-invariantly characterized by: dynamics—$\mathrm{D}_a\rho = 0 = \mathrm{D}_a p, q_a = 0, \pi_{ab} = 0$; kinematics—$\mathrm{D}_a\Theta = 0, A_a = 0 = \omega_a, \sigma_{ab} = 0$; gravito-electromagnetic field—$E_{ab} = 0 = H_{ab}$.

In this paper only the linearized quasi-Newtonian cosmologies are considered, since the main focus here is on perturbations and structure formation. The covariant linearized evolution equations in the general case are [32]

$$\dot{\rho} = -(\rho + p)\Theta - \mathrm{div}\, q, \tag{3}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - \tfrac{1}{2}(\rho + 3p) + \mathrm{div}\, A, \tag{4}$$

$$\dot{q}_a = -4Hq_a - (\rho + p)A_a - \mathrm{D}_a p - (\mathrm{div}\ \pi)_a, \tag{5}$$

$$\dot{\omega}_a = -2H\omega_a - \tfrac{1}{2}\mathrm{curl}\ A_a, \tag{6}$$

$$\dot{\sigma}_{ab} = -2H\sigma_{ab} - E_{ab} + \tfrac{1}{2}\pi_{ab} + \mathrm{D}_{\langle a}A_{b\rangle}, \tag{7}$$

$$\dot{E}_{ab} = -3HE_{ab} + \mathrm{curl}\ H_{ab} - \tfrac{1}{2}\dot{\pi}_{ab} - \tfrac{1}{2}(\rho + p)\sigma_{ab} - \tfrac{1}{2}\mathrm{D}_{\langle a}q_{b\rangle} - \tfrac{1}{2}H\pi_{ab}, \tag{8}$$

$$\dot{H}_{ab} = -3HH_{ab} - \mathrm{curl}\ E_{ab} + \tfrac{1}{2}\mathrm{curl}\ \pi_{ab}, \tag{9}$$

and the linearized constraint equations are

$$\mathcal{C}^1 \equiv \mathrm{div}\ \omega = 0, \tag{10}$$

$$\mathcal{C}^2{}_a \equiv (\mathrm{div}\ \sigma)_a - \mathrm{curl}\ \omega_a - \tfrac{2}{3}\mathrm{D}_a\Theta + q_a = 0, \tag{11}$$

$$\mathcal{C}^3{}_{ab} \equiv \mathrm{curl}\ \sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} - H_{ab} = 0, \tag{12}$$

$$\mathcal{C}^4{}_a \equiv (\mathrm{div}\ E)_a + \tfrac{1}{2}(\mathrm{div}\ \pi)_a - \tfrac{1}{3}\mathrm{D}_a\rho + Hq_a = 0, \tag{13}$$

$$\mathcal{C}^5{}_a \equiv (\mathrm{div}\ H)_a + \tfrac{1}{2}\mathrm{curl}\ q_a - (\rho + p)\omega_a = 0, \tag{14}$$

where $H = \dot{a}/a$ is the background Hubble rate, related to the background values of ρ and p by

$$\rho = 3\left(H^2 + \frac{\mathcal{K}}{a^2}\right), \quad \dot{H} + H^2 = -\tfrac{1}{6}(\rho + 3p), \tag{15}$$

with $\mathcal{K} = 0, \pm 1$ the curvature index. The differential identities that are needed for investigating consistency and deriving perturbation equations are collected in Appendix A.

If another 4-velocity $\tilde{u}^a$ is chosen, the corresponding kinematic, dynamic and gravito-electromagnetic quantities undergo transformations. For completeness, and since they do not appear elsewhere, the exact nonlinear form of these transformations is given in Appendix B. Only the linearized form of the expressions is required below.

The covariant characterization of quasi-Newtonian cosmologies is as follows [5]: they are almost-FLRW dust universes, with a congruence of observers whose 4-velocity field u^a is irrotational, shearfree and nonrelativistic relative to comoving observers. The comoving 4-velocity $\tilde{u}^a$ is given by the linearized form of Eq. (B1):

$$\tilde{u}^a = u^a + v^a, \tag{16}$$

where v^a is the nonrelativistic relative velocity, which vanishes in the background. The models are thus defined in the quasi-Newtonian frame by

$$(\text{dynamics})\quad p = 0, \quad q_a = \rho v_a, \quad \pi_{ab} = 0, \tag{17}$$

$$(\text{kinematics})\quad \omega_a = 0, \quad \sigma_{ab} = 0, \tag{18}$$

as shown by the linearized form of equations (B3)–(B10). Thus quasi-Newtonian cosmologies are irrotational, shearfree dust spacetimes with energy flux (momentum density) arising purely from a particle flux in the quasi-Newtonian frame relative to the comoving frame. Note that the isotropic and anisotropic stresses that arise from relative motion are second order in v_a, as given by Eqs. (B8) and (B10).

The gravito-magnetic constraint equation (12), together with Eq. (18), shows that

$$H_{ab} = 0. \tag{19}$$

Thus there is no gravitational radiation [15], which further justifies the term ''quasi-Newtonian.'' In addition, the div H constraint (14), together with Eq. (18), shows that q_a is irrotational, and thus so is v_a:

$$\mathrm{curl}\ v_a = 0 = \mathrm{curl}\ q_a. \tag{20}$$

Since the vorticity vanishes, it follows (see [4]) that $v_a = \mathrm{D}_a\psi$, where the velocity potential ψ is determined below.

III. INTEGRABILITY CONDITIONS

In irrotational dust models with vanishing energy flux and anisotropic stress, the constraint equations $\mathcal{C}^A = 0$ evolve consistently with the evolution equations, even at the nonlinear level, in the sense that [9] (see also [11–13])

$$\dot{\mathcal{C}}^A = F^A{}_B\mathcal{C}^B + G^A{}_{Ba}\mathrm{D}^a\mathcal{C}^B,$$

where F and G depend only on the kinematic, dynamic and gravito-electromagnetic quantities (and not their derivatives). If one imposes the ''silent'' constraint (19), then the nonlinear models are generically inconsistent, but the linearized models are consistent [14,15]. A very simple approach to the integrability conditions for quasi-Newtonian cosmologies follows from showing that these models are in fact a subclass of the linearized silent models. This can be seen by transforming to the comoving frame.

Linearizing the expressions in appendix B for the case where u^a and $\tilde{u}^a$ are any frames in nonrelativistic relative motion, one finds for the kinematic quantities

$$\tilde{\Theta} = \Theta + \mathrm{div}\ v, \tag{21}$$

$$\tilde{A}_a = A_a + \dot{v}_a + Hv_a, \tag{22}$$

$$\tilde{\omega}_a = \omega_a - \tfrac{1}{2}\mathrm{curl}\ v_a, \tag{23}$$

$$\tilde{\sigma}_{ab}=\sigma_{ab}+\mathrm{D}_{\langle a}v_{b\rangle}, \tag{24}$$

for the dynamic quantities

$$\tilde{\rho}=\rho, \quad \tilde{p}=p, \quad \tilde{q}_a=q_a-(\rho+p)v_a, \quad \tilde{\pi}_{ab}=\pi_{ab}, \tag{25}$$

and for the gravito-electromagnetic field

$$\tilde{E}_{ab}=E_{ab}, \quad \tilde{H}_{ab}=H_{ab}. \tag{26}$$

For u^a the quasi-Newtonian frame and $\tilde{u}^a$ the comoving frame, it follows from the Eqs. (17)–(26) that

$$\tilde{p}=0, \quad \tilde{q}_a=0, \quad \tilde{\pi}_{ab}=0, \tag{27}$$

$$\tilde{A}_a=0, \quad \tilde{\omega}_a=0, \quad \tilde{\sigma}_{ab}=\mathrm{D}_{\langle a}v_{b\rangle}, \tag{28}$$

$$\tilde{E}_{ab}=E_{ab}, \quad \tilde{H}_{ab}=0. \tag{29}$$

Equations (27)–(29) constitute a covariant characterization of linearized silent universes, except that the shear takes a special form. Thus quasi-Newtonian models are linearized silent models with a special form of shear, and integrability conditions arise only from the restriction on the shear.

It is now convenient to return to the quasi-Newtonian frame, where the restriction on the shear is that it vanishes. Integrability conditions arise directly from the fact that the shear propagation equation (7) is turned into a constraint, i.e. $E_{ab}=\mathrm{D}_{\langle a}A_{b\rangle}$. This can be simplified, using curl $A_a=0$, which follows from the vorticity propagation equation (6), and identity (A1). Thus

$$A_a=\mathrm{D}_a\varphi, \tag{30}$$

where φ is the covariant relativistic generalization of the Newtonian potential. Then the shear constraint becomes

$$\mathcal{E}_{ab}\equiv E_{ab}-\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi=0. \tag{31}$$

In summary, the only independent new constraint is Eq. (31), and any conditions that follow from its derivatives. What is happening here is that the consistent evolution of the basic constraints $\mathcal{C}^A$ $(A=1,2,\ldots,5)$ is not affected by introducing a new constraint. It is the new constraint $\mathcal{E}$ which leads to integrability conditions. The freedom in the gravito-electric field is clearly central to the consistency of the silent models, and conversely, it is the longitudinal condition (31) on that field which produces integrability conditions in the quasi-Newtonian subcase.

A. Time evolution

The time derivative of Eq. (31) follows from the gravito-electric propagation equation (8) and the identity

$$\{\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\sigma\}^{\bullet}=\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\dot{\varphi}+(\dot{\varphi}-2H)\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi,$$

which is proved using identities (A3) and (A4) and Eq. (30). It follows that

$$\dot{\mathcal{E}}_{ab}=-3H\mathcal{E}_{ab}-\tfrac{1}{2}\mathrm{D}_{\langle a}\mathcal{C}^2{}_{b\rangle}-(\dot{\varphi}+H)\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi -\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}(\dot{\varphi}+\tfrac{1}{3}\Theta), \tag{32}$$

on using Eq. (11). Thus $\mathcal{E}_{ab}$ evolves consistently if

$$\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}(\dot{\varphi}+\tfrac{1}{3}\Theta)+(\dot{\varphi}+\tfrac{1}{3}\Theta)\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi=0. \tag{33}$$

This is the first integrability condition in quasi-Newtonian cosmologies. It represents an extension of the condition derived in [5], since there a particular ansatz is assumed *a priori* for $\dot{\varphi}$, i.e.,

$$\dot{\varphi}=\alpha\Theta,$$

where α is a constant parameter. Choosing $\alpha=-\frac{1}{3}$, i.e.

$$\dot{\varphi}+\tfrac{1}{3}\Theta=0, \tag{34}$$

it is clear that the integrability condition (33) is reduced to an identity.

Equation (34) will be adopted here, since it is covariant, has a clear geometric motivation (as given in [5]), and guarantees consistent evolution of the gravito-electric constraint. Before proceeding with the van Elst-Ellis solution, it is interesting to ask whether it may be generalized. The integrability condition (33) may be rewritten as

$$\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\{\mathrm{e}^{\varphi}(\dot{\varphi}+\tfrac{1}{3}\Theta)\}=0. \tag{35}$$

This shows clearly how the van Elst–Ellis solution may be generalized to

$$\dot{\varphi}+\tfrac{1}{3}\Theta=\beta\mathrm{e}^{-\varphi}, \tag{36}$$

where $\mathrm{D}_a\beta=0$, i.e. β is an arbitrary background scalar. There does not appear to be any advantage in adopting the generalized solution (36). It is not clear whether more general solutions of the condition may be found.

What are the immediate consequences of the van Elst-Ellis solution to the integrability condition? First, the time evolution of Eq. (34) itself leads to the covariant modified Poisson equation

$$\mathrm{D}^2\varphi=\tfrac{1}{2}\rho-(3\ddot{\varphi}+\Theta\dot{\varphi}), \tag{37}$$

after using the Raychaudhuri equation (4). This equation governs the relativistic gravitational potential for a given energy density.

Secondly, one can get a crucial evolution equation for the 4-acceleration [5]. Such an evolution equation is not present in the set of general evolution equations. It arises via the shearfree condition, as a consequence of Eq. (34). Taking the gradient of Eq. (34), and using identity (A3) and the div σ constraint (11), one gets[2]

$$\dot{A}_a+2HA_a=-\tfrac{1}{2}\rho v_a. \tag{38}$$

[2]Note that the same evolution equation (38) is obtained if the generalized solution (36) is used, with $\beta\neq 0=\mathrm{D}_a\beta$.

Now there is also an evolution equation for v_a [5]:

$$\dot{v}_a + H v_a = -A_a, \tag{39}$$

as follows from the conservation equations (3) and (5), or from the comoving frame equations (22) and (28). This is just the relativistic generalization of the Newtonian equation for relative acceleration:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\varphi.$$

The coupled evolution equations (38) and (39) may be decoupled to produce second order equations in either quantity. For v_a:

$$\ddot{v}_a + 3H\dot{v}_a - \left(H^2 + \frac{2\mathcal{K}}{a^2}\right)v_a = 0, \tag{40}$$

on using Eq. (15), while for A_a:

$$\ddot{A}_a + 6H\dot{A}_a + \frac{1}{2}\left(7H^2 - \frac{5\mathcal{K}}{a^2}\right)A_a = 0. \tag{41}$$

These equations may be solved to find the velocity perturbations v_a and the 4-acceleration A_a. Since there are no spatial derivatives in Eq. (40), the velocity perturbations are independent of scale.

For a flat background, with $\mathcal{K}=0$ and $H \propto a^{-3/2}$, Eq. (40) is readily solved analytically:

$$v_a = \Lambda_a^{(+)} a^{1/2} + \Lambda_a^{(-)} a^{-2}, \tag{42}$$

where $(+)$ and $(-)$ denote the growing and decaying modes, and $\dot{\Lambda}_a^{(\pm)} = 0$. Using Eq. (39), the solution for the acceleration follows as

$$A_a = -\Lambda_a^{(+)} a^{-1} + \tfrac{2}{3}\Lambda_a^{(-)} a^{-7/2}. \tag{43}$$

Finally, one can derive another equation for φ using Eq. (34), which implies, together with Eq. (11), that

$$v_a = -\frac{2}{\rho}\,\mathrm{D}_a\dot{\varphi}. \tag{44}$$

Then using Eq. (44) in Eq. (40), it follows that

$$\mathrm{D}_a\ddot{\varphi} + 3H\mathrm{D}_a\dot{\varphi} - \tfrac{1}{3}(\rho - 3H^2)\mathrm{D}_a\varphi = 0. \tag{45}$$

Note that the background energy conservation equation and Eq. (44) determine the velocity potential in terms of the gravitational potential:

$$v_a = \mathrm{D}_a\psi \quad \text{where} \quad \psi = -\left(\frac{2}{\rho_0 a_0^3}\right)a^3\dot{\varphi}.$$

B. Spatial consistency

Spatial derivatives of the new constraint $\mathcal{E}$ determine whether it is consistent with $\mathcal{C}^A$ on an initial spatial surface. Taking the curl of Eq. (31) produces no restrictions, since curl $E_{ab}=0$ by the gravito-magnetic propagation equation (9) and constraint equation (12), and since curl $\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi$ vanishes identically. This follows from using identities (A1) and (A6):

$$\begin{aligned}\text{curl }\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi &= \text{curl }\mathrm{D}_{(a}\mathrm{D}_{b)}\varphi = \text{curl }\mathrm{D}_a\mathrm{D}_b\varphi \\ &= \varepsilon_{cd(a}\mathrm{D}^{[c}\mathrm{D}^{d]}\mathrm{D}_{b)}\varphi \\ &= (H^2 - \tfrac{1}{3}\rho)\varepsilon_{cd(a}h^d{}_{b)}\mathrm{D}^c\varphi = 0.\end{aligned}$$

The divergence however gives a nontrivial condition. First, one needs an identity for the divergence of the distortion:

$$\mathrm{D}^b\mathrm{D}_{\langle a}A_{b\rangle} = \tfrac{1}{2}\mathrm{D}^2A_a + \tfrac{1}{6}\mathrm{D}_a(\text{div } A) + \tfrac{1}{3}(\rho - 3H^2)A_a, \tag{46}$$

which holds for any projected vector A_a, and follows from identity (A6). Using Eqs. (30) and (A2), it follows that

$$\mathrm{D}^b\mathrm{D}_{\langle a}\mathrm{D}_{b\rangle}\varphi = \tfrac{2}{3}\mathrm{D}_a(\mathrm{D}^2\varphi) + \tfrac{2}{3}(\rho - 3H^2)\mathrm{D}_a\varphi. \tag{47}$$

Now use Eqs. (47), (13) and (11) to get

$$\begin{aligned}(\text{div }\mathcal{E})_a = {}& \mathcal{C}^4{}_a - H\mathcal{C}^2{}_a + \tfrac{1}{3}\mathrm{D}_a\rho - \tfrac{2}{3}H\mathrm{D}_a\Theta \\ & - \tfrac{2}{3}\mathrm{D}_a(\mathrm{D}^2\varphi) - \tfrac{2}{3}(\rho - 3H^2)\mathrm{D}_a\varphi.\end{aligned}$$

Thus there is a second integrability condition arising from Eq. (31), i.e.

$$\mathrm{D}_a\rho - 2H\mathrm{D}_a\Theta - 2\mathrm{D}_a(\mathrm{D}^2\varphi) - 2(\rho - 3H^2)\mathrm{D}_a\varphi = 0. \tag{48}$$

In general, this appears to be independent of the first integrability condition (33). However, if one uses the van Elst–Ellis solution (34) of the first condition, then the second condition is identically satisfied. This can be seen as follows. Taking the gradient of Eq. (37) and using Eq. (34), one finds that the second integrability condition (48) becomes

$$2[\mathrm{D}_a\ddot{\varphi} + 3H\mathrm{D}_a\dot{\varphi} - \tfrac{1}{3}(\rho - 3H^2)\mathrm{D}_a\varphi] = 0, \tag{49}$$

which is identically satisfied by virtue of Eq. (45).

IV. DENSITY PERTURBATIONS

In [5] the density perturbation equation is found in the quasi-Newtonian frame, which requires incorporating a complicated energy flux source term. A simple alternative is to work in the comoving frame, which leads directly to solutions of the density perturbation equation in terms of the velocity perturbations.

The covariant density perturbation scalar [6] in the comoving frame is

$$\tilde{\delta} = a\tilde{\mathrm{D}}^a\tilde{\delta}_a \quad \text{where} \quad \tilde{\delta}_a = \frac{a\tilde{\mathrm{D}}_a\rho}{\rho}. \tag{50}$$

Using Eq. (50) and the identity

$$\tilde{\mathrm{D}}_a f = \mathrm{D}_a f + \dot{f} v_a,$$

it follows that to linear order the density perturbation scalars in the comoving and quasi-Newtonian frames are related by

$$\delta = \tilde{\delta} + 3a^2 H \mathrm{D}^a v_a . \tag{51}$$

Covariant time derivatives in the comoving and quasi-Newtonian frames agree to linear order: $\tilde{u}^a \nabla_a S = \dot{S}$. Thus $\tilde{\delta}$ satisfies the standard equation (with $\mathcal{K}=0$) for dust [6]

$$(\tilde{\delta})^{\cdot\cdot} + 2H(\tilde{\delta})^{\cdot} - \tfrac{3}{2}H^2\tilde{\delta} = 0. \tag{52}$$

Since the solution $\tilde{\delta}$ of Eq. (52) is well known, one can use Eq. (51) to write down the density perturbations in the quasi-Newtonian frame:

$$\delta = \Delta^{(+)} a + \Delta^{(-)} a^{-3/2} + \gamma a^{-1/2}\mathcal{V}, \tag{53}$$

where $\dot{\Delta}^{(\pm)} = 0$, and $\gamma = 2a_1^{3/2}$ is a constant, with $a_1 = a/t^{2/3}$, and

$$\mathcal{V} = a\mathrm{D}^a v_a$$

is the comoving divergence of the peculiar velocity field. This quantity encodes the scalar contribution of velocity perturbations to density perturbations. Using the solution (42) for v_a, one finds that

$$\mathcal{V} = \Gamma^{(+)} a^{1/2} + \Gamma^{(-)} a^{-2} \quad \text{where} \quad \Gamma^{(\pm)} = a\mathrm{D}^a \Lambda_a^{(\pm)}. \tag{54}$$

It follows from the identity (A4) that $\dot{\Gamma}^{\pm} = 0$. Then using Eq. (54) in Eq. (53) gives

$$\delta = \Delta^{(+)} a + \Delta^{(-)} a^{-3/2} + \gamma\Gamma^{(+)} + \gamma\Gamma^{(-)} a^{-5/2}. \tag{55}$$

The correction to the standard solution affects all scales. The growing mode is shifted by an amount that is a comoving constant, which will be dominated by the term that grows as a. A similar feature arises in a non-covariant Lagrangian perturbation analysis (see [2]). The decaying mode (not considered in [2]) is corrected by an amount which dies away more rapidly, and so becomes negligible.

V. CONCLUDING REMARKS

The covariant approach to quasi-Newtonian models developed by van Elst and Ellis [5] has been applied and extended here, in order to derive and solve the equations governing density and velocity perturbations. The quasi-Newtonian zero-shear hypersurfaces gauge (or longitudinal gauge) implicitly involves integrability conditions—i.e. it incorporates a dynamical condition, and is not pure gauge. This has been overlooked in some non-covariant treatments, effectively amounting to gauge-dependent assumptions about relative-velocity effects. A fully covariant general relativistic treatment has uncovered these integrability conditions and their implication for the perturbations. The perturbations have been determined, making explicit the relative-velocity effects.

The integrability condition found in [5] was generalized, via a simple approach that showed how the quasi-Newtonian models are in fact a sub-class of the linearized silent models. The generalized first integrability condition (35) naturally leads to the generalization (36) of the van Elst-Ellis solution (34). Furthermore, a second integrability condition (48) was found by investigating spatial consistency. The integrability conditions were shown to be crucial for determining the perturbations, since the solution (34) of the integrability conditions implies a propagation equation (38) for the 4-acceleration. In turn, this propagation equation leads to the velocity perturbation equation (40), and the solution (42) was given for a flat background. By transforming to the comoving frame, a simple relation (51) was derived for the density perturbations, which were given by Eq. (53) in terms of the velocity perturbations and the standard dust solution for a flat background. The density perturbations were given analytically by Eq. (55), which showed how relative-velocity effects produce small corrections to both the growing and decaying modes on all scales.

These results underline the importance of general relativistic constraint equations and the integrability conditions that can arise from imposing various physical and geometric assumptions in cosmology. A covariant approach [30,6] avoids many of the intricate problems arising from gauge-dependent approaches. The improved covariant formalism of [9], summarized in Sec. II and supplemented by the identities of Appendix A and the new results in Appendix B on transformations of the covariant quantities, has been central to deriving these results. Similar methods can in principle be applied to investigate the consistency and underlying implications of other gauge choices or of various approximations (such as the Zel'dovich approximation) in general relativistic cosmological perturbations. This is a subject of further research.

APPENDIX A: LINEARIZED IDENTITIES

Useful differential identities [32]:

$$\mathrm{curl}\ \mathrm{D}_a f = -2\dot{f}\omega_a, \tag{A1}$$

$$\mathrm{D}^2(\mathrm{D}_a f) = \mathrm{D}_a(\mathrm{D}^2 f) + \tfrac{2}{3}(\rho - 3H^2)\mathrm{D}_a f + 2\dot{f}\,\mathrm{curl}\ \omega_a, \tag{A2}$$

$$(\mathrm{D}_a f)^{\cdot} = \mathrm{D}_a \dot{f} - H\mathrm{D}_a f + \dot{f} A_a, \tag{A3}$$

$$(\mathrm{D}_a S_{b\cdots})^{\cdot} = \mathrm{D}_a \dot{S}_{b\cdots} - H\mathrm{D}_a S_{b\cdots}, \tag{A4}$$

$$(\mathrm{D}^2 f)^{\cdot} = \mathrm{D}^2\dot{f} - 2H\mathrm{D}^2 f + \dot{f}\ \mathrm{div}\ A, \tag{A5}$$

$$\mathrm{D}_{[a}\mathrm{D}_{b]}V_c = \tfrac{1}{3}(3H^2 - \rho)V_{[a}h_{b]c}, \tag{A6}$$

$$\mathrm{D}_{[a}\mathrm{D}_{b]}S^{cd} = \tfrac{2}{3}(3H^2 - \rho)S_{[a}{}^{(c}h_{b]}{}^{d)}, \tag{A7}$$

$$\mathrm{div}\ \mathrm{curl}\ V = 0 \tag{A8}$$

$$(\mathrm{div}\ \mathrm{curl}\ S)_a = \tfrac{1}{2}\mathrm{curl}(\mathrm{div}\ S)_a, \tag{A9}$$

$$\mathrm{curl}\ \mathrm{curl}\ V_a = \mathrm{D}_a(\mathrm{div}\ V) - \mathrm{D}^2 V_a + \tfrac{2}{3}(\rho - 3H^2)V_a, \tag{A10}$$

$$\mathrm{curl\ curl}\ S_{ab}=\tfrac{3}{2}\mathrm{D}_{\langle a}(\mathrm{div}\ S)_{b\rangle}-\mathrm{D}^2S_{ab}+(\rho-3H^2)S_{ab}\,, \tag{A11}$$

where the vectors and tensors vanish in the background, $S_{ab}=S_{\langle ab\rangle}$, and all identities except Eq. (A1) are linearized. (Nonlinear identities are given in [9,11,23].)

APPENDIX B: TRANSFORMATIONS UNDER CHANGE OF FRAME

Change in 4-velocity:

$$\tilde{u}_a=\gamma(u_a+v_a)\mathrm{where}\ \ v_au^a=0,\quad \gamma=(1-v^2)^{-1/2}. \tag{B1}$$

The following algebraic relations are needed:

$$g_{ab}=h_{ab}-u_au_b=\tilde{h}_{ab}-\tilde{u}_a\tilde{u}_b\,,$$

$$\tilde{h}_{ab}=h_{ab}+\gamma^2[v^2u_au_b+2u_{(a}v_{b)}+v_av_b],$$

$$\eta_{abcd}=2u_{[a}\varepsilon_{b]cd}-2\varepsilon_{ab[c}u_{d]}=2\tilde{u}_{[a}\tilde{\varepsilon}_{b]cd}-2\tilde{\varepsilon}_{ab[c}\tilde{u}_{d]}\,,$$

$$\tilde{\varepsilon}_{abc}=\gamma\varepsilon_{abc}+\gamma\{2u_{[a}\varepsilon_{b]cd}+u_c\varepsilon_{abd}\}v^d,$$

$$C_{ab}{}^{cd}=4\{u_{[a}u^{[c}+h_{[a}{}^{[c}\}E_{b]}{}^{d]}+2\varepsilon_{abe}u^{[c}H^{d]e}+2u_{[a}H_{b]e}\varepsilon^{cde}$$
$$=4\{\tilde{u}_{[a}\tilde{u}^{[c}+\tilde{h}_{[a}{}^{[c}\}\tilde{E}_{b]}{}^{d]}+2\tilde{\varepsilon}_{abe}\tilde{u}^{[c}\tilde{H}^{d]e}+2\tilde{u}_{[a}\tilde{H}_{b]e}\tilde{\varepsilon}^{cde},$$

together with the decomposition [23]

$$\nabla_bv_a=-u_b\{\dot{v}_{\langle a\rangle}+A_cv^cu_a\}+u_a\{\tfrac{1}{3}\Theta v_b+\sigma_{bc}v^c+[\omega,v]_b\}+\tfrac{1}{3}(\mathrm{div}\ v)h_{ab}-\tfrac{1}{2}\varepsilon_{abc}\mathrm{curl}\ v^c+\mathrm{D}_{\langle a}v_{b\rangle}\,, \tag{B2}$$

where $[W,V]_a\equiv\varepsilon_{abc}W^bV^c$. Then the following exact nonlinear transformations may be derived.

Kinematic quantities (using $\nabla_a\gamma=\gamma^3v^b\nabla_av_b$):

$$\tilde{\Theta}=\gamma\Theta+\gamma(\mathrm{div}\ v+A^av_a)+\gamma^3W, \tag{B3}$$

$$\tilde{A}_a=\gamma^2A_a+\gamma^2\{\dot{v}_{\langle a\rangle}+\tfrac{1}{3}\Theta v_a+\sigma_{ab}v^b-[\omega,v]_a+(\tfrac{1}{3}\Theta v^2+A^bv_b+\sigma_{bc}v^bv^c)u_a+\tfrac{1}{3}(\mathrm{div}\ v)v_a+\tfrac{1}{2}[v,\mathrm{curl}\ v]_a+v^b\mathrm{D}_{\langle b}v_{a\rangle}\}+\gamma^4W(u_a+v_a), \tag{B4}$$

$$\tilde{\omega}_a=\gamma^2\{(1-\tfrac{1}{2}v^2)\omega_a-\tfrac{1}{2}\mathrm{curl}\ v_a+\tfrac{1}{2}v_b(2\omega^b-\mathrm{curl}\ v^b)u_a+\tfrac{1}{2}v_b\omega^bv_a+\tfrac{1}{2}[A,v]_a+\tfrac{1}{2}[\dot{v},v]_a+\tfrac{1}{2}\varepsilon_{abc}\sigma^b{}_dv^cv^d\}, \tag{B5}$$

$$\begin{aligned}\tilde{\sigma}_{ab}=&\gamma\sigma_{ab}+\gamma(1+\gamma^2)u_{(a}\sigma_{b)c}v^c+\gamma^2A_{(a}[v_{b)}+v^2u_{b)}]+\gamma\mathrm{D}_{\langle a}v_{b\rangle}-\tfrac{1}{3}h_{ab}[A_cv^c+\gamma^2(W-\dot{v}_cv^c)]+\gamma^3u_au_b[\sigma_{cd}v^cv^d\\&+\tfrac{2}{3}v^2A_cv^c-v^cv^d\mathrm{D}_{\langle c}v_{d\rangle}+(\gamma^4-\tfrac{1}{3}v^2\gamma^2-1)W]+\gamma^3u_{(a}v_{b)}[A_cv^c+\sigma_{cd}v^cv^d-\dot{v}_cv^c+2\gamma^2(\gamma^2-\tfrac{1}{3})W]\\&+\tfrac{1}{3}\gamma^3v_av_b[\mathrm{div}\ v-A_cv^c+\gamma^2(3\gamma^2-1)W]+\gamma^3v_{\langle a}\dot{v}_{b\rangle}+v^2\gamma^3u_{(a}\dot{v}_{\langle b\rangle)}+\gamma^3v_{(a}\sigma_{b)c}v^c-\gamma^3[\omega,v]_{(a}\{v_{b)}+v^2u_{b)}\}\\&+2\gamma^3v^c\mathrm{D}_{\langle c}v_{(a\rangle}\{v_{b)}+u_{b)}\},\end{aligned} \tag{B6}$$

where $W\equiv\dot{v}_cv^c+\frac{1}{3}v^2\ \mathrm{div}\ v+v^cv^d\mathrm{D}_{\langle c}v_{d\rangle}$.

Dynamic quantities (compare [33]):

$$\tilde{\rho}=\rho+\gamma^2[v^2(\rho+p)-2q_av^a+\pi_{ab}v^av^b], \tag{B7}$$

$$\tilde{p}=p+\tfrac{1}{3}\gamma^2[v^2(\rho+p)-2q_av^a+\pi_{ab}v^av^b], \tag{B8}$$

$$\tilde{q}_a=\gamma q_a-\gamma\pi_{ab}v^b-\gamma^3[(\rho+p)-2q_bv^b+\pi_{bc}v^bv^c]v_a-\gamma^3[v^2(\rho+p)-(1+v^2)q_bv^b+\pi_{bc}v^bv^c]u_a\,, \tag{B9}$$

$$\begin{aligned}\tilde{\pi}_{ab}=&\pi_{ab}+2\gamma^2v^c\pi_{c(a}\{u_{b)}+v_{b)}\}-2v^2\gamma^2q_{(a}u_{b)}-2\gamma^2q_{(a}v_{b)}-\tfrac{1}{3}\gamma^2[v^2(\rho+p)-2q_cv^c+\pi_{cd}v^cv^d]h_{ab}\\&+\tfrac{1}{3}\gamma^4[2v^4(\rho+p)-4v^2q_cv^c+(3-v^2)\pi_{cd}v^cv^d]u_au_b+\tfrac{2}{3}\gamma^4[2v^2(\rho+p)-(1+3v^2)q_cv^c+2\pi_{cd}v^cv^d]u_{(a}v_{b)}\\&+\tfrac{1}{3}\gamma^4[(3-v^2)(\rho+p)-4q_cv^c+2\pi_{cd}v^cv^d]v_av_b\,.\end{aligned} \tag{B10}$$

Gravito-electromagnetic field:

$$\tilde{E}_{ab}=\gamma^2\{(1+v^2)E_{ab}+v^c[2\varepsilon_{cd(a}H_{b)}{}^d+2E_{c(a}u_{b)}+(u_au_b+h_{ab})E_{cd}v^d-2E_{c(a}v_{b)}+2u_{(a}\varepsilon_{b)cd}H^{de}v_e]\}, \tag{B11}$$

$$\tilde{H}_{ab}=\gamma^2\{(1+v^2)H_{ab}+v^c[-2\varepsilon_{cd(a}E_{b)}{}^d+2H_{c(a}u_{b)}+(u_au_b+h_{ab})H_{cd}v^d-2H_{c(a}v_{b)}-2u_{(a}\varepsilon_{b)cd}E^{de}v_e]\}. \tag{B12}$$

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