

**Acoustic oscillations and viscosity**

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(Received 2 February 1998; published 18 November 1998)

Using a simple thermo-hydrodynamic model that respects relativistic causality, we reexamine the analysis of the qualitative features of acoustic oscillations in the photon-baryon fluid. The growing photon mean free path introduces transient effects that can be modelled by the causal generalization of relativistic Navier-Stokes-Fourier theory. Causal thermodynamics provides a more satisfactory hydrodynamic approximation to kinetic theory than the quasi-stationary (and non-causal) approximations arising from standard thermodynamics or from expanding the photon distribution to first order in Thomson scattering time. The causal approach introduces small corrections to the dispersion relation obtained in quasi-stationary treatments. A dissipative contribution to the speed of sound slightly increases the frequency of the oscillations. The diffusion damping scale is slightly increased by the causal corrections. Thus quasi-stationary approximations tend to over-estimate the spacing and under-estimate the damping of acoustic peaks. In our simple model, the fractional corrections at decoupling are  $\geq 10^{-2}$ . [S0556-2821(98)05224-2]

PACS number(s): 98.80.Hw, 04.25.Nx, 04.40.Nr, 05.70.Ln

**I. INTRODUCTION**

Acoustic oscillations in the photon-baryon fluid before decoupling leave a vital imprint on the cosmic microwave background [1] (and possibly also on large-scale structure [2]). The form of this imprint encodes information on fundamental cosmological parameters, and increasingly accurate and refined numerical integrations are being performed to produce predictions that can be tested against current and upcoming observations. As a complement to detailed numerical models, it is also useful to analyze qualitatively and analytically the key physical features such as acoustic oscillations. As pointed out by Hu and Sugiyama [1], numerical integrations are sufficient for direct comparison of specific models with observations, but do not readily produce a qualitative and analytic understanding of the physical processes at play. In [1], acoustic oscillations and their damping due to photon diffusion are analyzed analytically via expanding the integrated Boltzmann multipoles in the Thomson scattering time  $\tau_T$ . To zero order in  $\tau_T$ , i.e. in the tight-coupling approximation, which is valid on scales much larger than the photon mean free path, the oscillations are undamped. Damping arises from the first order approximation, which introduces a shear viscosity via the radiation quadrupole. This approximation is hydrodynamic, in the sense that all multipoles beyond the quadrupole are neglected, and it is effectively equivalent (see [3]) to the non-equilibrium thermodynamic approach of Weinberg [4], which is based on the relativistic Navier-Stokes-Fourier theory developed by Eckart.

As pointed out by Peebles and Yu [5], expansions in  $\tau_T$ ,

even beyond first order (see [3]), cannot take proper account of the change in the photon mean free time, especially near decoupling. In other words, the  $O(\tau_T)$  approximation and the equivalent Navier-Stokes-Fourier approximation are inherently quasi-stationary, and provide a limited approximation to the Boltzmann equation. An improved hydrodynamic approximation to kinetic theory is the causal thermodynamics developed by Israel and Stewart [6], which generalizes the non-causal Eckart theory by incorporating transient non-quasi-stationary effects. In this paper, we use causal thermodynamics to refine aspects of the work by Weinberg [4] and Hu and Sugiyama [1]. Of course, all hydrodynamic approximations, which assume that the photon-baryon fluid is close to equilibrium via interactions, will break down when the Thomson interaction rate becomes too low, i.e. as decoupling is approached.

In a previous paper [7], we developed a general formalism for incorporating causal thermodynamics into perturbation theory, thus providing a self-consistent approach to density perturbations in dissipative cosmological fluids. Our formalism is based on the covariant perturbation theory of Ellis and Bruni [8,9], and the covariant causal thermodynamics of Israel and Stewart [6]. Here we apply the general formalism to the case of viscous damping of density perturbations in the photon-baryon fluid. This effect has been comprehensively analyzed via detailed study of the Boltzmann equation (see e.g. [5,10–12]). Dissipative hydrodynamics provides a simplified alternative to a full kinetic theory analysis, which can illuminate some of the key physical effects without the same level of detail and complexity. The approaches of Weinberg [4] and Hu and Sugiyama [1] neglect gravitational effects, which is not unreasonable on subhorizon scales. We refine their approach by including metric perturbations. More importantly, their approach is inherently quasi-stationary, and the main refinement we introduce is to incorporate transient

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effects via the causal generalization of Navier-Stokes-Fourier thermodynamics.

It turns out that the relaxational effects which are incorporated into the causal transport equation, but which are neglected in the non-causal theory, have a small but interesting impact. The sound speed, which determines the frequency of acoustic oscillations for each mode, acquires a positive dissipative correction. This produces a small increase in the frequency of acoustic oscillations on each scale. The viscous damping rate found in quasi-stationary approximations [1,4] also acquires a positive correction, leading to a small increase in the diffusion cutoff scale. A rough estimate based on our simplified model is that the fractional corrections are  $\gtrsim 10^{-2}$  by decoupling.

For simplicity, since we are interested in qualitative features rather than specific numerical predictions, and since we wish to focus on relaxational corrections to quasi-stationary models, we assume a flat background universe with negligible non-baryonic content. We treat the mixture of baryonic matter and radiation as a single dissipative fluid, with dissipation arising from the growing mean free path of photons in their Thomson scattering off electrons. We consider the era between matter-radiation equality and decoupling, neglecting the matter pressure. Dissipative effects are expected to be greatest near decoupling. We will also neglect thermal conduction for simplicity, while the bulk viscous stress is necessarily negligible since the fluid is in the non-relativistic regime. Thus the dissipative effect of photon diffusion is reduced (within a radiative hydrodynamic model) to a shear viscous stress  $\pi_{ab}$ .

In non-causal thermodynamics [and in an  $O(\tau_T)$  expansion of the Boltzmann multipoles], this shear viscous stress is algebraically determined by the shear  $\sigma_{ab}$  via [4]

$$\pi_{ab} = -2\eta\sigma_{ab}, \quad (1)$$

where  $\eta$  is the viscosity. The implicit instantaneous relation between the cause (shear, i.e. anisotropic expansion rate) and the effect (anisotropic stress) is what underlies the pathologies of the theory, i.e. that it admits dissipative signals with infinite wavefront speeds and that its equilibrium states are generically unstable [13]. In the causal theory, the stress is no longer algebraically and ‘‘instantaneously’’ determined by the shear, but satisfies an evolution equation in which there is a time-lag between cause and effect [6]:

$$\tau\dot{\pi}_{ab} + \pi_{ab} = -2\eta\sigma_{ab}, \quad (2)$$

where  $\tau$  is the relaxation time-scale. This causal transport leads to a subluminal speed of viscous pulses, and introduces small but interesting corrections to the quasi-stationary dispersion relation.

Causal thermodynamics has a solid kinetic theory motivation via the relativistic Grad 14-moment approximation to the Boltzmann equation [6]. This approximation effectively restores second-order non-equilibrium terms in the entropy that are neglected in the relativistic Chapman-Enskog approximation [14], which leads to the non-causal theory. Thus causal thermodynamics provides a more satisfactory and less incomplete hydrodynamic approximation to kinetic theory

than the standard non-causal theory. The differences are greatest for high frequency phenomena, on scales comparable to the mean free time or path, when the Chapman-Enskog approximation breaks down, while the Grad approximation remains valid.

We follow the notation of [7]. Units are such that  $c = 1 = 8\pi G$ , except where numerical values are given in specified units. The present Hubble rate is  $H_0 = 100h$  km/s/Mpc.

## II. CAUSAL VISCOUS PERTURBATIONS

Reference [7] contains a general discussion and general covariant perturbation equations for causal dissipative hydrodynamics in cosmology. In the interest of brevity, we will not repeat here the derivations and motivations of [7], but simply quote the results as they are needed.

In the era between matter-radiation equality and decoupling, the mixture of photons, baryons and electrons is treated as a single dissipative fluid, with dissipation due to photon Silk diffusion [15]. The baryons and electrons are tightly coupled, culminating in their recombination, while the photon coupling to baryonic matter through Thomson scattering weakens as the mean free path grows. We assume for simplicity that there is no non-baryonic cold dark matter. We also assume negligible thermal conduction and particle flux, so that the particle and energy frames coincide [6], and we can choose the fluid four-velocity  $u^a$  (where  $u^a u_a = -1$ ) as the four-velocity of this frame. The bulk viscous stress is negligible, since the fluid is non-relativistic.

The fundamental quantity in the covariant approach to density perturbations [8] is

$$\delta = a D^a \delta_a = a^2 \frac{D^2 \rho}{\rho}, \quad (3)$$

where  $D_a$  is the covariant spatial derivative,  $a$  is the background scale factor, and

$$\delta_a = a \frac{D_a \rho}{\rho}$$

is the comoving fractional energy density gradient, which is a covariant measure of density inhomogeneity. The total energy density  $\rho = \rho_B + \rho_R$  is made up of baryonic and radiation parts, and  $\delta$  is the density fluctuation scalar for the photon-baryon system, considered as a combined single fluid.

Covariant entropy perturbations in a single dissipative fluid model (as opposed to a 2-component model) are sourced by heat flux and bulk viscous stress, and shear viscous stress has no effect to linear order [Eq. (18) of [7]]. If, as we assume, the initial entropy perturbation is zero, there are therefore no entropy perturbations.<sup>1</sup> Then  $\delta$  satisfies the evolution equation [Eq. (28) of [7]]

<sup>1</sup>In any case, the entropy perturbations are decoupled from the density perturbations, because the non-barotropic index  $(\partial p / \partial s)_\rho$  vanishes in the background, so that the entropy source term in the density perturbation equation is zero [Eq. (28) of [7]].

$$\begin{aligned} \delta + 2H \left( 1 - 3w + \frac{3}{2} c_s^2 \right) \dot{\delta} \\ - \frac{3}{2} H^2 (1 + 8w - 3w^2 - 6c_s^2) \delta - c_s^2 D^2 \delta = \mathcal{S}[\pi], \end{aligned} \quad (4)$$

where an overdot denotes the covariant derivative along  $u^a$ ,  $H = \dot{a}/a$ ,  $w = p/\rho$ , with  $p = p_B + p_R$  the total isotropic pressure, and  $c_s^2 = \dot{p}/\dot{\rho}$  is the adiabatic sound speed. The shear viscous source term is given by [Eq. (32) of [7]]

$$\mathcal{S}[\pi] = 3H\dot{\mathcal{S}} - 3H^2(1 + 6w - 3c_s^2)\mathcal{S} + D^2\mathcal{S}, \quad (5)$$

where the covariant shear viscous stress scalar is

$$\mathcal{S} = a^2 \frac{D^a D^b \pi_{ab}}{\rho}.$$

The viscous transport equation (2) leads to the following evolution equation for  $\mathcal{S}$  [Eq. (46) of [7]]:<sup>2</sup>

$$\begin{aligned} \tau \dot{\mathcal{S}} + \left[ 1 - H \left\{ 3(1+w)\tau - \frac{4\eta}{\rho(1+w)} \right\} \right] \mathcal{S} \\ = \frac{4\eta}{3\rho(1+w)} [\dot{\delta} - 3wH\delta]. \end{aligned} \quad (6)$$

Equations (4)–(6) form a coupled system that governs the evolution of density perturbations with causal viscosity.

The thermal energy of matter is much less than the rest mass energy after the epoch  $t_e$  of matter-radiation equality, so that  $p_B \approx 0$ , and then  $p \approx \frac{1}{3}\rho_R$ . Taking  $a = 1$  at the present epoch, we have, in the background (where matter and radiation are non-interacting),

$$w = \frac{1}{3} \left[ \left( \frac{a}{a_e} \right) + 1 \right]^{-1} < \frac{1}{6}, c_s^2 = \frac{4}{3} \left[ 3 \left( \frac{a}{a_e} \right) + 4 \right]^{-1} < \frac{4}{21}, \quad (7)$$

for  $a > a_e$ . To a reasonable approximation (sufficient for the purposes of our simple model), we can neglect these quantities relative to 1 in the parentheses in Eqs. (4)–(6). Furthermore, since  $w \ll 1$  and  $\eta H/\rho \ll 1$ , we can neglect the last term on the right of Eq. (6).

We decompose into Fourier modes  $\delta(t, \vec{x}) \rightarrow \Delta(t, \vec{k})$  and  $\mathcal{S}(t, \vec{x}) \rightarrow \Sigma(t, \vec{k})$ , where  $k$  is the comoving wave number, so that the proper wave number is  $K = k/a$ . The proper wavelength  $\lambda = 2\pi/K$  is constrained by  $\lambda_T < \lambda < \lambda_H$ , where  $\lambda_H = H^{-1}$  is the Hubble scale (above which thermohydrodynamic effects do not operate), and the minimum scale  $\lambda_T$  is the photon mean free path for Thomson scattering:

$$\lambda_T = \frac{1}{n\sigma_T}. \quad (8)$$

Here  $\sigma_T$  is the Thomson cross section and  $n$  is the free electron number density. Perturbations on scales below  $\lambda_T$  will be wiped out by photon streaming (and the hydrodynamic model breaks down on these scales). Causal thermodynamics applies down to the Thomson scale, but the non-causal theory requires  $\lambda \gg \lambda_T$ . The damping effect of photon streaming operates beyond the Thomson mean free scale, since photon diffusion out of over-dense regions rapidly destroys perturbations below a critical scale, as shown by Silk [15]. The critical cutoff scale is the diffusion scale  $\lambda_D$ , which is considerably greater than the Thomson mean free scale, as we will confirm below.

Collecting the above points, and using the background Friedmann equation  $\rho = 3H^2$ , the coupled system becomes

$$\begin{aligned} \ddot{\Delta} + 2H\dot{\Delta} - \frac{3}{2} H^2 \left( 1 - \frac{2}{3} c_s^2 K_*^2 \right) \Delta \\ = 3H\dot{\Sigma} - 3H^2 \left( 1 + \frac{1}{3} K_*^2 \right) \Sigma, \end{aligned} \quad (9)$$

$$3H\tau\dot{\Sigma} + 3H(1 + \eta_* - 3H\tau)\Sigma = \eta_*\dot{\Delta}, \quad (10)$$

where we have defined the dimensionless expansion-normalized shear viscosity

$$\eta_* = \frac{4\eta}{3H} \quad (11)$$

and proper wave number

$$K_* = \frac{K}{H} = 2\pi \frac{\lambda_H}{\lambda}. \quad (12)$$

The relevant scales constrain  $K_*$  by

$$2\pi < K_* < 2\pi \frac{\lambda_H}{\lambda_T}. \quad (13)$$

Radiative shear viscosity due to Thomson scattering is given by (see [4], and [16–18] for refinements using kinetic theory<sup>3</sup>)

$$\eta = \frac{4}{15} r_0 T^4 \tau_T, \quad (14)$$

where  $r_0$  is the blackbody radiation constant,  $T$  is the photon temperature and  $\tau_T = \lambda_T$  is the photon mean free time for Thomson scattering. The causal relaxation time-scale  $\tau$  is by Eq. (2) the characteristic time taken for the fluid to relax toward equilibrium if the viscous “driving force” were to be

<sup>3</sup>Note that [16] claims a correction factor of  $\frac{10}{9}$  in Weinberg’s expression for  $\eta$ . This arises from anisotropic scattering effects, and a further small correction is also induced by polarization effects [1]. These corrections are not important for our simple model.

<sup>2</sup>The  $\rho$  factors are mistakenly omitted in Eqs. (46) and (54) of [7].

“switched off.” Thus we expect  $\tau$  is of the order of a few Thomson interaction times, and below the characteristic diffusion time:

$$\tau_T \lesssim \tau < \tau_D, \quad (15)$$

where  $\tau_D = \lambda_D$ .

In the era we are considering,  $H$  decays approximately like  $a^{-3/2}$ ,  $n$  decays faster than  $a^{-3}$  because of recombination effects, and  $T$  decays as  $a^{-1}$ . Thus, to lowest order,

$$K_* = (K_*)_e \left( \frac{a}{a_e} \right)^{1/2}, \quad (16)$$

$$\lambda_H = (\lambda_H)_e \left( \frac{a}{a_e} \right)^{3/2}, \quad (17)$$

$$\lambda_T = (\lambda_T)_e \left( \frac{a}{a_e} \right)^{3+I}, \quad (18)$$

$$\tau_T H = \left( \frac{\lambda_T}{\lambda_H} \right)_e \left( \frac{a}{a_e} \right)^{3/2+I}, \quad (19)$$

where  $I > 0$  encodes the growing effects of recombination on the number density of free electrons, the details of which are not important for our simple model. Using the standard numerical values [19] of  $t_e \approx 3 \times 10^{10} h^{-4}$  sec, redshift  $z_e \approx 2 \times 10^4 h^2$ , and present baryon number density  $n_B \approx 10^8 h^8 \text{ cm}^{-3}$ , with the present baryonic (dark and luminous) density fraction  $\Omega_B \approx 1$ , we find that

$$(\lambda_H)_e \approx 1.5 \times 10^{21} h^{-4} \text{ cm}, \quad (\lambda_T)_e \approx 10^{16} h^{-8} \text{ cm},$$

$$(\eta_*)_e \approx 1.5 \times 10^{-6} h^{-4}. \quad (21)$$

Equations (9) and (10) govern the evolution of density perturbation modes within a simplified causal viscous fluid model. The system may be decoupled to produce a third-order equation in  $\Delta$ . In practice we analyze the coupled system, but for completeness, the decoupled equation is

$$\begin{aligned} \tau \ddot{\Delta} + \left[ 1 + \frac{1}{2} \tau H + \dot{\tau} + F \right] \dot{\Delta} + H \left[ 2 + \left( \frac{3}{2} + \frac{1}{3} K_*^2 \right) \eta_* - \frac{15}{2} \tau H + 2 \dot{\tau} - (\tau H)^{-1} \{ \tau \dot{\eta}_* + F(\eta_* - 2 \tau H) \} \right] \Delta \\ - \frac{3}{2} H^2 \left[ \left( 1 + \eta_* - \frac{7}{2} \tau H + \dot{\tau} + F \right) \left( 1 - \frac{2}{3} c_s^2 K_*^2 \right) - \tau H - \frac{2}{3} \tau K_*^2 (c_s^2) \right] \Delta = 0, \end{aligned} \quad (22)$$

where

$$F \equiv -\tau \left[ \ln |\tau H (6 - K_*^2) - 3(1 + \eta_*)| \right] \dot{\phantom{x}},$$

and we used the background field equation  $\dot{H} = -\frac{3}{2} H^2$ . In the non-causal case  $\tau = 0$ , Eq. (22) reduces to the second order equation

$$\dot{\Delta} + H \left[ 2 + \left( \frac{3}{2} + \frac{1}{3} K_*^2 \right) \eta_* - \frac{\dot{\eta}_*}{H(1 + \eta_*)} \right] \Delta - \frac{3}{2} H^2 \left[ (1 + \eta_*) \left( 1 - \frac{2}{3} c_s^2 K_*^2 \right) \right] \Delta = 0. \quad (23)$$

### III. VISCOUS ACOUSTIC OSCILLATIONS

The key features of the solutions are readily analyzed qualitatively, given the simplicity of our model. We assume that the coefficients in Eqs. (9) and (10) are slowly varying, which is reasonable for small-scale modes. Then we can try a WKB solution of the form  $\Delta(t, k) = a^{-1} A(k) \exp i f \omega dt$  and  $\Sigma(t, k) = a^{-1} B(k) \exp i f \omega dt$ , where the scale factor terms remove the non-dissipative effects of expansion on small-scale modes, facilitating comparison with Weinberg's results, which neglect expansion. This gives

$$\begin{bmatrix} -\omega^2 - \frac{1}{2} H^2 (5 - 2 c_s^2 K_*^2) & -3 i H \omega + (6 + K_*^2) H^2 \\ -i \eta_* \omega + \eta_* H & 3 i \tau H \omega + 3(1 + \eta_* - 4 \tau H) H \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0.$$

Non-trivial solutions require a vanishing determinant of the coefficient matrix, leading to the dispersion equation

$$\begin{aligned}
& 2i\tau\omega^3 + 2(1-4\tau H)\omega^2 \\
& - i \left[ 2\eta_* \left( 2 + \frac{1}{3}K_*^2 \right) + \tau H(2c_s^2 K_*^2 - 5) \right] H\omega \\
& + \left[ (1 + \eta_* - 4\tau H)(5 - 2c_s^2 K_*^2) + 2\eta_* \left( 1 + \frac{1}{3}K_*^2 \right) \right] H^2 \\
& = 0.
\end{aligned} \tag{24}$$

In general, the solutions will have the form

$$\omega = \pm Kv + i\Gamma, \tag{25}$$

where  $v$  is the (real) phase speed of oscillations and  $\Gamma$  is the viscous damping rate. Damping will be severe for scales  $\lambda < \lambda_D$ , where the diffusion cutoff scale is defined by the equality of the damping rate and the expansion rate, i.e.  $\Gamma(\lambda_D) = H$ .

In the dispersion relation (24), we can neglect  $\eta_*$  relative to 1 by Eqs. (20) and (21). Equations (7), (12), (17) and (21) show that for the relevant small scales ( $\lambda \ll \lambda_H$ ), we have<sup>4</sup>  $c_s^2 K_*^2 \gg 1$ . Then the dispersion cubic may be written

$$\begin{aligned}
& i\tau_* \omega_*^3 + (1-4\tau_*)\omega_*^2 \\
& - i \left( \frac{1}{3}\eta_* K_*^2 + \tau_* c_s^2 K_*^2 \right) \omega_* - (1-4\tau_*)c_s^2 K_*^2 = 0,
\end{aligned} \tag{26}$$

where  $\omega_* = \omega/H$  and  $\tau_* = \tau/H^{-1}$  are dimensionless expansion-normalized variables. For the non-causal theory, with  $\tau=0$ , Eq. (26) becomes

$$3\bar{\omega}_*^2 - (i\eta_* K_*^2)\bar{\omega}_* - 3c_s^2 K_*^2 = 0, \tag{27}$$

where an overbar indicates a quantity in the non-causal theory. Equation (27) has the solution

$$\bar{v}^2 = c_s^2 \left[ 1 - \frac{\Gamma^2}{c_s^2 K^2} \right], \quad \bar{\Gamma} = \frac{1}{6}\eta_* K_*^2 H.$$

Since  $\bar{\Gamma}/H < 1$  for the scales that survive viscous damping, and  $c_s^2 K_*^2 \gg 1$  for the relevant sub-horizon scales, we have  $\bar{v}^2 = c_s^2$  to a good approximation. Thus

$$\bar{v} = c_s, \quad \bar{\Gamma} = \left( \frac{2\eta}{3\rho} \right) K^2, \tag{28}$$

which agrees (when the isotropic pressure, heat flux and bulk stress are negligible) with Weinberg's special relativistic generalization of the non-relativistic Landau-Lifshitz result [4].<sup>5</sup> The diffusion cutoff scale follows from Eq. (28) as

$$\bar{\lambda}_D = \frac{2\pi}{3} \sqrt{\frac{2\eta}{H}} \lambda_H. \tag{29}$$

By Eqs. (17), (18) and (20), the diffusion scale is well above the Thomson scale, although the ratio decreases with expansion:

$$\frac{\bar{\lambda}_D}{\lambda_T} \approx 500h^2 \left( \frac{a_e}{a} \right)^{5/4+I}. \tag{30}$$

Now we consider the causal corrections to Eq. (28). From Eqs. (15), (19), (21) and (29), it follows that  $\tau_* \equiv \tau H \ll 1$ . We write  $\omega_* = \bar{\omega}_*(1 + y\tau_*)$ , where  $\bar{\omega}_*$  satisfies Eq. (27), and solve the dispersion cubic equation (26) to first order in  $\tau_*$ . This gives

$$y = \frac{\bar{\Gamma}}{H} \left[ 1 + i \frac{(\bar{\Gamma} + 4H)}{\pm c_s K} \right],$$

on using  $c_s^2 K_*^2 \gg 1$ . It follows that

$$v = c_s(1 + \tau\bar{\Gamma}), \tag{31}$$

$$\Gamma = \bar{\Gamma}[1 + 2\tau(\bar{\Gamma} + 2H)]. \tag{32}$$

Thus the causal generalization of standard thermodynamics leads to corrections that increase the effective sound speed and damping rate. The latter increase leads to an increase in the diffusion cutoff scale, which is determined by setting  $\bar{\Gamma} \approx H$  in Eq. (32):

$$\lambda_D = \bar{\lambda}_D(1 + 3\tau H). \tag{33}$$

In order to estimate the size of the corrections, we need an expression for the causal relaxation time-scale  $\tau$ . In the case of a simple (i.e. single-species) relativistic gas (obeying classical or quantum statistics), the thermodynamic coefficients such as  $\tau$  and  $\eta$  are in principle known via complicated kinetic theory formulas involving integrals over energy and collision cross-sections [6]. In the radiative transfer context, where 2 particle species are involved, one treated hydrodynamically, the shear viscosity is given by Eq. (14), while  $\tau$  is implicitly assumed to be zero in quasi-stationary treatments [4,16,17]. If Grad-type moment approximations are used, then  $\tau$  is equal to the mean collision time as a consequence of the hydrodynamic assumptions [18,23]. However, it seems physically reasonable that  $\tau$  could be greater than  $\tau_T$ , if the hydrodynamic assumptions in radiative transfer are somewhat relaxed. Thus a simple generalization

$$\tau = \alpha\tau_T, \tag{34}$$

where  $\alpha \geq 1$  is a constant, seems reasonable within the phenomenology of thermodynamics.

The correction to the diffusion scale and the maximal correction to the sound speed (which occurs at  $\bar{\Gamma} = H$ ) are determined by  $\tau H$ . By Eqs. (34), (19) and (21), we find that, in our simple model,

<sup>4</sup>Numerical integrations in the special case  $c_s^2 K_*^2 \ll 1$  are given in [20].

<sup>5</sup>See also [21] for a kinetic theory approach, and [22] for the inclusion of magnetic fields.

$$\tau H = \alpha \left( \frac{\lambda_T}{\lambda_H} \right) \left( \frac{a}{a_e} \right)^{3/2+I} \approx 7\alpha \times 10^{-6} h^{-4} \left( \frac{a}{a_e} \right)^{3/2+I}. \quad (35)$$

The correction factor grows with expansion, as expected, since the mean free photon path is growing. We take  $h = 0.6$ . At matter-radiation equality, the fractional correction is then  $\approx 5\alpha \times 10^{-5} \geq 5 \times 10^{-5}$ . As decoupling is approached, the ionization parameter  $I$  becomes more important. However, for a rough under-estimate, we can set  $I = 0$ . (In any case, all hydrodynamically based models will break down near decoupling.) With  $z_{\text{dec}} \approx 1100$ , we find that the maximal fractional correction introduced by relaxational effects is

$$(\tau H)_{\text{max}} > \alpha \times 10^{-3} > 10^{-3}, \quad (36)$$

in our simple model. This is the relevant estimate, since the imprint of acoustic oscillations is frozen in at decoupling.

#### IV. CONCLUSIONS

Our principal result is the causal generalization and upward correction of the quasi-stationary sound speed, given in Eq. (31), and damping rate, given in Eq. (32). These corrections are small, at roughly the 1% level, as given in Eq. (36), but they show how quasi-stationary approximations tend to under-estimate the frequency and damping of acoustic oscillations, and they indicate the refinements that would arise from a more complete kinetic theory approach, which involves far greater computational complexity. The qualitative features of the refinements are (a) a higher threshold for the survival of density perturbations, i.e. an increased diffusion cutoff scale, given by Eq. (33), (b) the appearance of a dissipative contribution to the sound speed of the photon-baryon fluid (consistent with the general theoretical results for perturbations of stationary fluids in flat spacetime [6,13])

which increases the frequency of oscillations, (c) the scale-dependence of the corrections to the damping rate and sound speed, and (d) the central role played by the relaxation time-scale  $\tau$ , which provides a simple scalar encoding for the non-quasi-stationary effects of photon diffusion.

This paper shows that in principle dissipative hydrodynamics with causal transport equations can provide an economical alternative to kinetic theory models, with some additional insights into the nature and source of acoustic oscillations. The simplified model used here could be refined, principally by incorporating thermal conduction, to produce an improved model that nevertheless will still allow a qualitative and analytic treatment, unlike the full kinetic theory approach.

Our model can also be seen as part of a growing body of examples in cosmology (e.g. [24]), relativistic astrophysics (e.g. [25]), and other areas of physics (e.g. [26]),<sup>6</sup> which show that interesting and sometimes significant physical differences can arise from the causal approach to thermodynamics. (A general overview is given in [27].) Even if the causal corrections are small, as is the case here, they provide further insight into the physics, with the added advantage that one avoids the essentially unsatisfactory features of non-causal thermodynamics. The predictions of the non-causal theory are readily recovered in the appropriate limit.

#### ACKNOWLEDGMENTS

J.T. was supported by the DGICYT under Grant No. PB94-0718, and would also like to thank the Relativity and Cosmology Group at Portsmouth University, where part of this work was done, for hospitality.

<sup>6</sup>In non-relativistic physics, causal theories are usually called ‘‘extended’’ or ‘‘hyperbolic.’’

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