

Matter scalar field in a closed universe

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We investigate the possibility that the matter of the Universe has a significant component (the quintessence component) determined by the equation of state $p = w\rho$, with $w < 0$. Here, we find conditions under which a closed model may look similar to a flat Friedmann-Robertson-Walker universe at low redshift. We study this problem in Einstein's general relativity and Brans-Dicke theories. In both cases we obtain explicit expressions for the quintessence scalar potential $V(Q)$, and the angular size as a function of the redshift.
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I. INTRODUCTION

The Friedmann-Roberson-Walker (FRW) model may describe a nonflat universe motivated by the observational evidences that the total matter density of the universe is different from its critical amount. In fact, the measured matter density of baryonic and nonbaryonic components is less than its critical value. However, theoretical arguments derived from inflationary models [1] and from some current microwave anisotropy measurements favor a flat universe where the total energy density equals the critical density.

In this respect, other forms of matter (components) are added which contribute to the total energy density, so it becomes possible to fulfill the prediction of inflation. Examples of these kinds are the cosmological constant Λ (or vacuum energy density), together with the cold dark matter (CDM) component, which forms the famous Λ CDM model, that, among others, seems to be the model which best fits existing observational data [2].

Recent cosmological observations, including those related to the relation between the magnitude and the redshift [3], constrain the cosmological parameters. The test of the standard model, which includes spacetime geometry, galaxy peculiar velocities, structure formation, and early universe physics, favors in many of these cases a flat universe model with the presence of a cosmological constant [4]. In fact, the luminosity distance-redshift relation (the Hubble diagram) for the IA supernova seems to indicate that the ratio of the matter content to its critical value Ω_0 and the cosmological constant fits best the values $\Omega_0 = 0.25$ and $\Lambda = 0.75$.

From a theoretical point of view, another possibility has risen, which is to consider a closed universe. It seems that quantum field theory is more consistent on compact spatial surfaces than in hyperbolic spaces [5]. Also, in quantum cosmology, the "birth" of a closed Universe from nothing is considered, which is characterized by having a vanishing total energy, momentum, and charge [6].

Motivated mainly by inflationary universe models, on the one hand, and by quantum cosmology, on the other hand, we describe in this paper the conditions under which a closed universe model may look flat at low redshift. This kind of situation has been considered in the literature [7]. There, a closed model was considered together with a nonrelativistic-matter density with $\Omega_0 < 1$, and the openness is obtained by adding a matter density whose equation of state is $p = -\rho/3$. Texture or tangled strings represent this kind of equation of state [8]. In a universe with texture, the additional energy density is redshifted as a^{-2} , where a is the scale factor. Thus it mimics a negative-curvature term in Einstein's equations. As a result, the kinematics of the model is the same as in an open universe in which $\Omega_0 < 1$. The first person who studied a universe filled with a matter content with an equation of state given by $p = -\rho/3$ seems to be Kolb [9]. He found that a closed universe may expand eternally at constant velocity (coasting cosmology). Also, he distinguishes a model universe with multiple images at different redshifts of the same object and a closed universe with a radius smaller than H_0^{-1} , among other interesting consequences.

Very recently, there has been quite a lot of work including in the CDM model a new component called the "quintessence" component, with the effective equation of state given by $p = w\rho$ with $-1 < w < 0$. This is the so-called QCDM model [10]. The differences between this model and the Λ model are, first, the Λ model has an equation of state with $w = -1$, whereas the Q model has a greater value and second, the energy density associated to the Q field in general varies with time, at difference of the Λ model. The final, and perhaps the most important, difference is that the Q model is spatially inhomogeneous and can cluster gravitationally, whereas the Λ model is totally spatially uniform. This latter difference is relevant in the sense that the fluctuations of the Q field could have an important effect on the observed cosmic microwave background (CMB) radiation and large scale structure [11]. However, it has been noticed that a degeneracy problem for the CMB anisotropy arises [12], since any given Λ model seems to be indistinguishable from the subset of quintessence models when CMB anisotropy power spectra

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are compared. However, they become different when the w parameter varies rapidly or becomes a constant restricted to $w > -\Omega_Q/2$. From the observational point of view, there have been attempts to restrict the value of this parameter. Astronomical observations of a type IA supernova have indicated that for a flat Universe, the ratio of the pressure of the Q component to its density is restricted to $w < -0.6$, and if the model considered is open, then $w < -0.5$ [13]. Certainly, improvement either in the study of the CMB anisotropy or the type IA supernova will help us to elucidate the exact amount of the Q component in the matter content of the universe.

In this paper we discuss cosmological FRW models with a Q field in both Einstein's theory of general relativity and Brans-Dicke (BD) theories [14]. We shall restrict ourselves to the case in which the w parameter remains constant. We obtain the potential $V(Q)$ associated to the Q field, and also determine the angular size as a function of the redshift.

II. EINSTEIN THEORY

In this section we review the situation in which the quintessence component of the matter density, whose equation of state $p = w\rho$, with w a constant less than zero, contributes to the effective Einstein action which is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} R + \frac{1}{2} (\partial_\mu Q)^2 - V(Q) + L_M \right]. \quad (1)$$

Here, G is Newton's gravitational constant, R the scalar curvature, Q the quintessence scalar field with associated potential $V(Q)$, and L_M represents the matter contributions other than the Q component.

Considering the FRW metric for a closed universe

$$ds^2 = dt^2 - a(t)^2 d\Omega_{k=1}^2, \quad (2)$$

with $d\Omega_{k=1}^2$ representing the spatial line element associated to the hypersurfaces of homogeneity, corresponding to a three sphere, and where $a(t)$ represents the scale factor, which together with the assumption that the Q scalar field is homogeneous, i.e., $Q = Q(t)$, we obtain the following Einstein field equations:

$$H^2 = \frac{8\pi G}{3} (\rho_M + \rho_Q) - \frac{1}{a^2} \quad (3)$$

and

$$\ddot{Q} + 3H\dot{Q} = -\frac{\partial V(Q)}{\partial Q}, \quad (4)$$

where the overdots specify derivatives respect to t , $H = \dot{a}/a$ defines the Hubble expansion rate, ρ_M is the average energy

density of nonrelativistic matter, and ρ_Q is the average energy density associated to the quintessence field defined by $\rho_Q = \frac{1}{2}\dot{Q}^2 + V(Q)$ and average pressure $p_Q = \frac{1}{2}\dot{Q}^2 - V(Q)$. As was mentioned in the Introduction we shall consider a model where the Q component has an equation of state defined by $p_Q = w\rho_Q$, where w is considered to lie in the range $-1 < w < 0$, in order to be in agreement with the current observational data [11].

In order to have a universe which is closed, but still have a nonrelativistic-matter density whose value corresponds to that of a flat universe, we should impose the following relation:

$$\rho_Q = \frac{3}{8\pi G a^2}. \quad (5)$$

This kind of situation has been recently considered in Ref. [7], where a matter density with $\Omega_0 < 1$ in a closed universe was described.

Under condition (5), Einstein's equations becomes analogous to that of a flat universe in which the matter density ρ_M corresponding to dust is equal to $\rho_M^0 [a_0/a(t)]^3$ and the scale factor $a(t)$ is given by $a_0(t/t_0)^{2/3}$. Using the expressions for ρ_Q and p_Q defined above, we obtain

$$Q(t) = Q_0 \left(\frac{t}{t_0} \right)^{1/3}. \quad (6)$$

with Q_0 defined by $Q_0 = 3\sqrt{3(1+w)/8\pi G}(t_0/a_0)$. The quantities denoted by the subscript 0 correspond to quantities of the current epoch.

From solution (6), together with the definitions of ρ_Q and p_Q we obtain an expression for the scalar potential $V(Q)$ given by

$$V(Q) = V_0 \left(\frac{Q_0}{Q} \right)^4, \quad (7)$$

where V_0 is the present value of the scalar quintessence potential given by $V_0 = 3(1-w)/16\pi G a_0^2$. When both solutions (6) and (7) are introduced into the field equation (4) the w parameter necessarily will be equal to $-\frac{1}{3}$, as is expected from the approach followed in Ref. [7].

To see that a closed model at low redshift is indistinguishable from a flat one, we could consider the angular size or the number-redshift relation as a function of the redshift z , as was done in Ref. [7]. Here, we shall restrict ourselves to consider the angular size only. The results will be compared with the corresponding analogous results obtained in BD theory.

The angular-diameter distance d_A between a source at a redshift z_2 and $z_1 < z_2$, is defined by

$$d_A(z_1, z_2) = \frac{a_0 \sin[\Delta\chi(z_1, z_2)]}{1 + z_2}, \quad (8)$$

where $\Delta\chi(z_1, z_2)$ is the polar-coordinate distance between a source at z_1 and another at z_2 , in the same line of sight (in a flat background) and is given by

$$\Delta\chi(z_1, z_2) = \frac{2}{a_0 H_0} \left[\frac{1}{\sqrt{1+z_1}} - \frac{1}{\sqrt{1+z_2}} \right]. \quad (9)$$

Here, H_0 corresponds to the present value of the Hubble constant, defined by $H_0 = \sqrt{8\pi G \rho_M^0/3}$. The corresponding angular size of an object of proper length l at a redshift z results in $\Theta \approx l/d_A(0, z)$, which becomes (in units of lH_0)

$$\Theta = \frac{1}{a_0 H_0} \frac{1+z}{\sin\{(2/a_0 H_0)[1-1/\sqrt{1+z}]\}}. \quad (10)$$

For a small redshift (or equivalently, for a small time interval) the angular size is given by

$$\Theta \approx \frac{1}{z} + \frac{7}{4} + \left[\frac{6}{(a_0 H_0)^2} + \frac{33}{48} \right] z + O(z^2). \quad (11)$$

Since for $\Omega_0 > 1$ it is found that

$$\Theta = \sqrt{\Omega - 1} \frac{1+z}{\sin\{2\sqrt{(\Omega-1)/(\Omega_0-1)}[\tan^{-1}(\sqrt{(\Omega_0 z+1)/(\Omega_0-1)}) - \tan^{-1}(-\sqrt{1/(\Omega_0-1)})]\}}, \quad (12)$$

where, Ω represents the sum of matter and the quintessence contribution to the total matter density, we obtain that, at low redshift, $\Theta(\Omega_0 > 1) \sim 1/z$, which coincides with the first term of the expansion (11). Therefore, it is expected that the models with $\Omega_0 = 1$ and $\Omega_0 > 1$ become indistinguishable at a low enough redshift. In Fig. 1 we have plotted Θ as a function of the redshift z in the range $0.01 \leq z \leq 10$, for $\Omega_0 = 1$ and $\Omega_0 = 3/2$. We have determined the value of $a_0 H_0$ by

fixing the polar-coordinate distance at the last scattering surface given by $\Delta\chi(z_{LS}) = \pi$, with $z_{LS} \approx 1100$, as was done in Ref. [7]

III. BD THEORY

In this section we discuss the quintessence matter model in a theory where the ‘‘gravitational constant’’ is considered to be a time-dependent quantity. The effective action associated to the generalized BD theory [15] is given by

$$S = \int d^4x \sqrt{-g} \left[\Phi R - \frac{\omega_0}{\Phi} (\partial_\mu \Phi)^2 - V(\Phi) + \frac{1}{2} (\partial_\mu Q)^2 - V(Q) + L_M \right], \quad (13)$$

where Φ is the BD scalar field related to the effective (Planck mass squared) value, ω_0 is the BD parameter, and $V(\Phi)$ is a scalar potential associated to the BD field. As in the Einstein case, the matter Lagrangian L_M is considered to be dominated by dust, with the equation of state $p_M = 0$. We also keep the quintessence component described by the scalar field Q .

When the FRW closed metric is introduced into the action (13), together with the assumptions that the different scalar fields are time-dependent quantities only, the following set of field equations are obtained

$$H^2 + H \left(\frac{\dot{\Phi}}{\Phi} \right) = \frac{\omega_0}{6} \left(\frac{\dot{\Phi}}{\Phi} \right)^2 + \frac{8\pi}{3\Phi} (\rho_M + \rho_Q) - \frac{1}{a^2} + \frac{V(\Phi)}{6\Phi},$$

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{\Phi^3}{2\omega_0 + 3} \frac{d}{d\Phi} \left(\frac{V(\Phi)}{\Phi^2} \right) = \frac{8\pi}{2\omega_0 + 3} [\rho_M + (1-3w)\rho_Q], \quad (14)$$

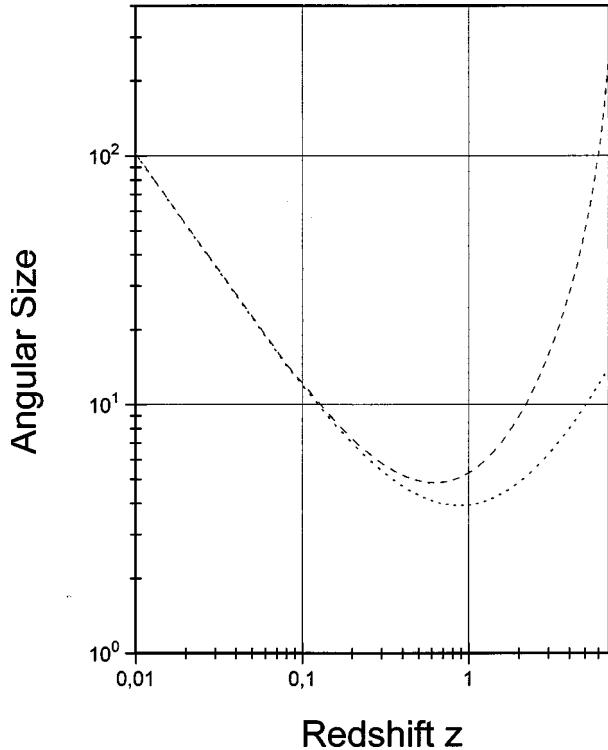


FIG. 1. We plot the angular size (in units of lH_0) as a function of the redshift, in Einstein theory. The dotted curve corresponds to a flat $\Omega_0 = 1$ universe. The solid curve represents a closed universe, with $\Omega_0 = \frac{3}{2}$.

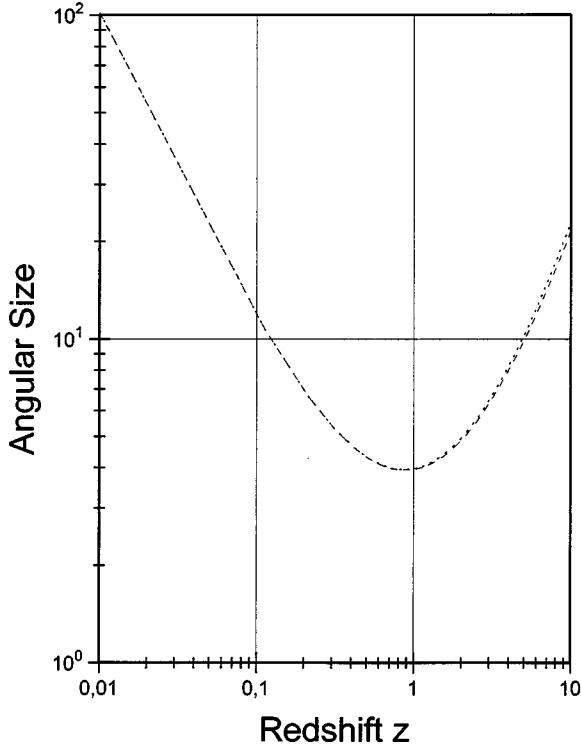


FIG. 2. This plot shows how the angular size in the Einstein (dashed line) and Brans-Dicke (dotted curve) theories depend on the redshift for a flat universe ($\Omega_0=1$). We have used the value $\omega_0=500$ for the BD parameter.

and

$$\ddot{Q} + 3H\dot{Q} = -\frac{\partial V(Q)}{\partial Q}.$$

As before, we have taken the Q component with equation of state $p_Q = w\rho_Q$, where w will be determined later on.

In order that the model mimics a flat universe, we impose the following conditions:

$$\frac{8\pi}{3\Phi} \rho_Q = \frac{1}{a^2} - \frac{1}{6\Phi} V(\Phi) \quad (15)$$

and

$$\rho_Q = \frac{\Phi^3}{8\pi(1-3w)} \frac{d}{d\Phi} \left(\frac{V(\Phi)}{\Phi^2} \right). \quad (16)$$

Under these restrictions, the BD field equations become equivalent to that of a flat universe, in which we assume a matter content dominated by dust.

It is known that the solutions of the scale factor $a(t)$ and the JBD field $\Phi(t)$ are given by $a(t) = a_0(t/t_0)^{2(1+\omega_0)/(4+3\omega_0)}$ and $\Phi(t) = \Phi_0(t/t_0)^{2/(4+3\omega_0)}$, respectively. These solutions together with the constraint equations (15) and (16) yield to the following expression for the quintessence matter field:

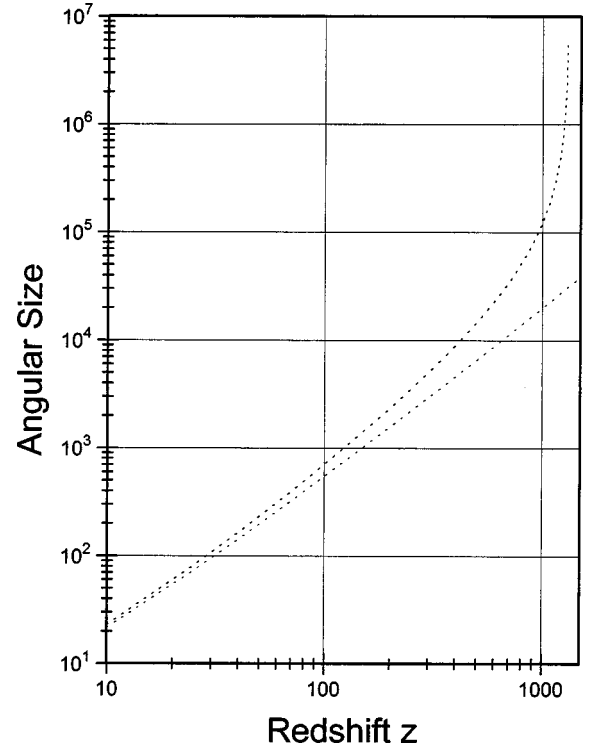


FIG. 3. This plot is the same as Fig. 2, but now the range for z is $10 \leq z \leq 1000$.

$$Q(t) = Q_0 \left(\frac{t}{t_0} \right)^{(3+\omega_0)/(4+3\omega_0)}, \quad (17)$$

where now Q_0 is defined by

$$Q_0 = \sqrt{\frac{3(1+w)(4+3\omega_0)^2(3+2\omega_0)}{(3+\omega_0)(9w+2\omega_0)} \frac{\Phi_0}{8\pi} \left(\frac{t_0}{a_0} \right)}$$

and where, as before, the quantities with the subscript 0 represent the actual values. Notice that this result reduces to Einstein solution [Eq. (6)], for $\omega_0 \rightarrow \infty$, together with the identification of the gravitational constant, $\Phi_0 = 1/G$.

Equation (17) together with equations (15) and (16) yields the potential associated to the BD field

$$V(\Phi) = \begin{cases} V(\Phi_0) \left(\frac{\Phi}{\Phi_0} \right)^{9w} & \text{if } 1+2\omega_0+9w=0, \\ V(\Phi_0) \left(\frac{\Phi}{\Phi_0} \right)^{1+2\omega_0} & \text{if } 1+2\omega_0+9w \neq 0, \end{cases} \quad (18)$$

where $V(\Phi_0)$ is given by

$$V(\Phi_0) = \begin{cases} 3(1-3w) \left(\frac{\Phi_0}{a_0^2} \right) & \text{if } 1+2\omega_0+9w=0, \\ -3 \left(\frac{1-3w}{2\omega_0+9w} \right) \left(\frac{\Phi_0}{a_0^2} \right)^{1+2\omega_0} & \text{if } 1+2\omega_0+9w \neq 0. \end{cases} \quad (19)$$

We shall consider the second case only, i.e., $1+2\omega_0+9w \neq 0$. The first case gives $w = -\frac{1}{9}(1+2\omega_0)$, and since $\omega_0 > 500$, in agreement with solar system gravity experiments, one obtains $w \ll -1$, which results inappropriated for describing the present astronomical observational data. Notice that the second case gives a lower bound for the parameter w , given by $w > -\frac{1}{9}(1+2\omega_0)$. However, the experiments motive us to only consider the range $-1 < w < 0$.

From Eq. (18), together with $V(Q) = \frac{1}{2}[(1-w)/(1+w)]\dot{Q}^2$ we obtain

$$V(Q) = V(Q_0) \left(\frac{Q_0}{Q} \right)^{2(1+2\omega_0)/(3+\omega_0)}, \quad (20)$$

where $V(Q_0)$ is defined by

$$V(Q_0) = \frac{3(1-w)(3+2\omega_0)}{9w+2\omega_0} \frac{\Phi_0}{16\pi} \frac{1}{a_0^2}.$$

When these solutions are plugged into the evolution of the Q -field equation of motion, we find that the parameter w is given by $w = -\frac{1}{3}[(2+\omega_0)/(1+\omega_0)]$, for this equation to be valid. Note also that if $w \rightarrow -\frac{1}{3}$ in the Einstein limit, $\omega_0 \rightarrow \infty$.

The corresponding angular size (in units of lH_0) for this kind of theory is found to be

$$\Theta = \frac{1}{a_0 H_0} \frac{1+z}{\sin\{(2/a_0 H_0) \alpha(\omega_0) [1 - (1+z)^{-\beta(\omega_0)/2}]\}}, \quad (21)$$

where

$$\alpha(\omega_0) = \frac{\sqrt{\omega_0^2 + (17/6)\omega_0 + 2}}{\omega_0 + 2},$$

$$\beta(\omega_0) = \frac{\omega_0 + 2}{\omega_0 + 1}, \quad \text{and} \quad H_0 = \sqrt{\frac{8\pi\rho_M^0}{3\phi_0}}.$$

In Fig. 2 we have plotted Θ as a function of z in the Einstein theory and Brans-Dicke theory with $\omega_0 = 500$. Note that at $z \sim 10$ or greater, they start to become different.

Since $\omega_0 \gg 1$ and if we take only the first-order term in $1/\omega_0$, we obtain that

$$\Theta^{\text{BD}} = \Theta^E + \frac{1+z}{(a_0 H_0)^2} \frac{\cos[2Z(z)/a_0 H_0]}{\sin^2[2Z(z)/(a_0 H_0)^2]} \times \left\{ 2[Z(z)+1] \ln[Z(z)+1] - \frac{7}{6} Z(z) \right\} \times \frac{1}{\omega_0} + \mathcal{O}\left(\frac{1}{\omega_0}\right)^2, \quad (22)$$

where $Z(z) = \sqrt{1+z} - 1$, and Θ^{BD} and Θ^E represent the angular size for Brans-Dicke and Einstein theories, respectively. At $z = z_{\text{LS}} \approx 1100$ the difference $\Delta\Theta \equiv \Theta^{\text{BD}} - \Theta^E$, becomes $\Delta\Theta \approx 147(1/\omega_0)$, which for $\omega_0 \sim 500$ becomes $\Delta\Theta \approx 0.3$. This difference for $z = 1$, with the same value of ω_0 , becomes $\Delta\Theta \approx 0.05$. Thus, we observe that this difference increases as z increases, i.e., as time becomes more remote the difference between the angular size in both theories becomes stronger. Figure 3 shows how this difference becomes more important at a redshift close to last scattering values. This difference clearly is hard to detect experimentally. However, they may be an indication that both theories become very different at z_{LS} , and it is probably necessary to search for another observable that might distinguish between these two possibilities. Perhaps, the spectrum due to different matter components may be the answer. They certainly have an effect on the cosmic background radiation, which in principle could be observable via temperature fluctuations.

IV. CONCLUSIONS

Assuming an effective equation of state for the Q field given by $p = w\rho$ with negative w , we have computed the form of the potential $V(Q)$ for the Q field in the model where a closed universe looks similar to a flat one at low redshifts. We have found it to vary as $V(Q) \sim Q^{-\alpha}$, where the parameter α becomes a function of the BD parameter ω_0 . This parameter has the correct Einstein limit, since for $\omega_0 \rightarrow \infty$ this parameter becomes equal to 4. We have also determined the angular size (in unit of lH_0) as a function of the redshifts, in both theories. Our conclusion is that the angular size at high redshift (close to last scattering values) could distinguish between Einstein and Brans-Dicke theories.

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