Generating branes via sigma models

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Starting with the D-dimensional Einstein-dilaton-antisymmetric form equations and assuming a block-diagonal form of a metric we derive a $(D-d)$-dimensional σ model with the target space $\mathrm{SL}(d,R)/\mathrm{SO}(d)\times\mathrm{SL}(2,R)/\mathrm{SO}(2)\times R$ or its noncompact form. Various solution-generating techniques are applied to reproduce some known and to construct some new p-brane solutions. It is shown that the p-brane Harrison transformation belonging to the SL(2,R) subgroup generates black p-branes from the seed Schwarzschild solution. A flux-brane generalizing the Bonnor-Melvin-Gibbons-Maeda solution is constructed as well as a nonlinear superposition of the flux-brane and a spherical black hole. A new simple way to endow branes with additional internal structure such as plane waves is suggested. Applying the harmonic maps technique we generate new solutions with a nontrivial shell structure in the transverse space (''matrioshka'' p-branes). Bonnor-type symmetry relating the four-dimensional vacuum SL(2,R) to the corresponding sector of the above global symmetry group is used to construct a new magnetic six-brane with a dipole moment in the ten-dimensional type IIA theory. A similar σ model is constructed for the intersecting branes. It is shown that the intersection rules have a simple geometric interpretation as conditions ensuring the symmetric space property of the target space. The null-geodesic method is used to find intersecting ''matrioshka'' p-branes in type IIA supergravity. [S0556-2821(98)03120-8]

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I. INTRODUCTION

Investigation of classical p-brane solutions to supergravities in various dimensions has led to considerable progress in understanding relations between different string theories. Of particular interest are type IIA and IIB supergravities in ten dimensions which may be regarded as the low-energy limits of the corresponding superstring theories, and eleven-dimensional supergravity, which is supposed to be the low-energy limit of M theory. Recent progress in string theory is also connected with the discovery of nonperturbative objects called D-branes. In the low-energy region they correspond to solutions of appropriate classical equations. There are several types of p-branes which are important in the context of superstrings. The major role is played by bosonic solutions preserving a part of the initial supersymmetry, the so called Bogomol'nyi-Prasad-Sommerfield (BPS) states (for a review see Refs. [1,2]). Another family consists of nonsaturating Bogomol'nyi bound *black* p-branes possessing a regular event horizon. Both families may form intersecting multiple-brane structures. The solutions may be also endowed with additional structures such as traveling waves.

Solving highly nonlinear bosonic equations in multidimensional supergravities constitutes a formidable technical task. In most cases p-brane solutions were obtained using some special *Ansätze* for the metric and matter fields [3–6]. To find BPS-saturated solutions one can also use the first order Bogomol'nyi type equations instead of directly solving the equations of motion [7,8]. However, this method is efficient mostly in eleven- and ten-dimensional supergravities where the Killing spinor equations are relatively simple. Once such solutions are found, certain lower-dimensional solutions may be obtained via appropriate compactification schemes. Some *ad hoc* prescriptions are also known which allow one to perform ''blackening'' deformations of p-branes from extremal configurations [9–11]. It can be noted that the diversity of techniques and *Ansätze* makes it rather difficult to understand mutual relationships between different classes of multidimensional solutions. Also, because of the absence of the uniqueness theorems such as those for four-dimensional black holes, it is unclear to which extent the known explicit solutions are general and exhaustive. Therefore the search for new schemes joining solutions into classes and simplifying their generation seems to be desirable.

One such approach consists in the use of ''hidden'' symmetries (dualities) arising in dimensionally reduced theories. Duality symmetries can be used to generate new nontrivial solutions from known ones, as well as to suggest new algorithms for integration of the field equations. Such a technique is well known in four-dimensional general relativity, where it achieved a high level of sophistication. For a class of vacuum solutions effectively depending on three, rather than four coordinates, the hidden symmetry group is SL(2,R) which acts nonlinearly on the space of two moduli. One of the group transformations (Ehlers transformation) is nontrivial and may be used as generating symmetry. $N=2$ supergravity in four dimensions being restricted to the set of solutions possessing a non-null Killing vector field leads to the famous Kramer-Neugebauer-Kinnersley group SU(1,2) [12,13], while the (truncated to one vector) $N=4$ supergravity generates the Sp(4,R) symmetry [14]. The crucial role of *three dimensions* is due to the fact that the vector fields then can be traded for scalars allowing for a description of the system as a nonlinear σ model.

Global symmetries of supergravities reduced to dimen-

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sions higher that 3, generically involve vector and antisymmetric form fields. It is well known, for example, that type IIB supergravity, compactified to nine dimensions, exhibits the SL(2,R) symmetry (S duality) mixing the NS and RR fields. Also there is a correspondence between type IIA and IIB supergravities reduced to nine dimensions, which is called T duality [15]. Using these dualities, accompanied by appropriate boosts (or by the dimensional reduction and uplifting), it is possible to construct a variety of new solutions from the known ones [16–18]. However, in order to exploit global symmetries to full extent one has to construct explicit nonlinear realizations of these duality groups on the space of physical variables, which is generally a highly nontrivial problem. Meanwhile, if one chooses a self-consistent *Ansatz* in which vector and higher form fields are parametrized by scalars, one can find purely scalar σ models defined on spacetime of dimension higher than 3. Although such models do not include all degrees of freedom of the initial action, they still can be efficient for solution generating purposes.

The purpose of this paper is to formulate such an approach in the multidimensional model describing gravity coupled to dilaton and antisymmetric tensor field. Starting with the D-dimensional Einstein-dilaton-antisymmetric form equations we assume that spacetime possesses d commuting hypersurface-orthogonal Killing vectors, which corresponds to a block-diagonal form of a metric. Assuming also that the differential form has only one nontrivial component, we construct a σ model on the transverse space *of any dimension* with the target space

$$\mathrm{SL}(d,R)/\mathrm{SO}(1,d-1)\times \mathrm{SL}(2,R)/\mathrm{SO}(1,1)\times R.$$

The first coset here corresponds to Kaluza-Klein moduli (with an overall scale factor removed), the rest is the Ehlers-Harrison type coset involving the above scale factor, the dilaton, and the nonzero component of the form field. The SL(2,R) group contains a Harrison-type transformation which generates solutions endowed with Page charges from uncharged solutions.

Using this σ-model representation we explore different solution generating techniques known in general relativity. The most direct application consists in using the symmetries acting on the coset variables to construct new solutions from already known ones. It turns out that the standard single p-brane solutions are related to multidimensional Schwarzschild metric by the p-brane Harrison transformation in exactly the same way as the four-dimensional Reissner-Nordström solution of the Einstein-Maxwell theory is related to the vacuum Schwarzschild solution by the original Harrison transformation. The resulting p-brane is *black* (nonextremal), so our procedure gives a justification of the so-called blackening prescription producing nonextremal p-branes from extremal ones. In a similar way, just repeating the derivation of the Bonnor-Melvin solution from flat space via Harrison transformation, we construct ''flux-branes'' supported by antisymmetric forms of arbitrary rank. These solutions for higher rank forms are new.

Our second approach consists in an exponentiation of the coset generators under some additional simplifying assumptions, this is known as the harmonic maps technique. Using harmonic maps onto geodesic subspaces we derive new solutions with ''matrioshka'' type structure of singularities. The third method we use is based on the Bonnor-type map between similar cosets involved in physically distinct theories. In this way we obtain ''dressed'' p-branes endowed with dipole moments. Our σ-model approach gives especially nontrivial results in the case of intersecting branes. We give an alternative interpretation of the so-called intersection rules as conditions assuring the symmetric space property of the target space. We also derive new ''matrioshka'' type intersecting p-brane solutions. Other possible applications of our σ model are also briefly discussed.

The paper is organized as follows. In Sec. II we describe the derivation of the σ model starting with the D-dimensional gravity coupled system of the d-form field and the dilaton. Using the block-diagonal *Ansatz* for the metric we derive the corresponding σ-model action and examine its symmetries. Section III is devoted to generation of the general black p-brane solution by applying Harrison transformation. We argue that the prescription of ''blackening'' the extremal p-branes is a manifestation of the target space isometries. In Sec. IV using the σ-model transformations we generate a flux-brane, which is a multidimensional analogue of the Bonnor-Melvin universe. We also find the nonlinear superposition of the flux-brane and a black hole. In Sec. V we apply the technique of harmonic maps to obtain new solutions of the p-brane type and study their properties. Also we use the null geodesic method to generate the Brinkmann wave and to demonstrate its independence of the p-brane structure. In Sec. VI we discuss a Bonnor-type map relating four-dimensional solutions of the vacuum Einstein equations to multidimensional p-brane type solutions and derive an apparently new p-brane solution to the type IIA supergravity in ten dimensions endowed with a dipole moment. In Sec. VII we discuss the *intersecting* p-branes in the σ-model terms. It is shown that the well-known intersection rules restricting dimensionalities and the coupling constants for known classes of composite p-branes are equivalent to the symmetric space condition for the target space. In this case the coset models can be formulated which opens a way to construct more general classes of intersecting branes, an example is given for the case of type IIA supergravity. In conclusion we discuss supersymmetry properties of new obtained solutions and give some remarks on further perspectives of the suggested approach.

II. SIGMA-MODEL REPRESENTATION

We will consider the model theory with the following action in the D-dimensional spacetime:

$$S=\frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}\left(R-\frac{1}{2}(\nabla\phi)^2-\frac{e^{-\alpha\phi}}{2(d+1)!}F_{d+1}^2\right), \tag{1}$$

where F_{d+1} is a $(d+1)$-differential form, $F_{d+1}=d\mathcal{A}_d$, ϕ is a dilaton. The corresponding equations of motion are

$$R_{MN}-\frac{1}{2}G_{MN}R=e^{-\alpha\phi}T^{(F)}_{MN}+T^{(\phi)}_{MN}, \tag{2}$$

$$\partial_M(e^{-\alpha\phi}\sqrt{-G}F^{MM_1\cdots M_d}_{d+1})=0, \tag{3}$$

$$\frac{1}{\sqrt{-G}}\partial_M(\sqrt{-G}G^{MN}\partial_N\phi)+\frac{\alpha}{2(d+1)!}\,e^{-\alpha\phi}F^2_{d+1}=0. \tag{4}$$

The energy-momentum tensors for the matter fields have the form

$$T^{(F)}_{MN}=\frac{1}{2d!}\left(F_{MM_1\cdots M_d}F_N{}^{M_1\cdots M_d}-\frac{G_{MN}}{2(d+1)}F^2_{d+1}\right), \tag{5}$$

$$T^{(\phi)}_{MN}=\frac{1}{2}\left(\partial_M\phi\partial_N\phi-\frac{1}{2}G_{MN}(\nabla\phi)^2\right). \tag{6}$$

Let us suppose that the space-time has d commuting hypersurface orthogonal Killing vectors, one of which is timelike (the case with only spacelike Killing vectors will be discussed later). Then we can use the following *Ansatz* for the metric:

$$ds^2=g_{\mu\nu}(x)dy^\mu dy^\nu+(\sqrt{-g})^{-2/s}h_{\alpha\beta}(x)dx^\alpha dx^\beta, \tag{7}$$

where $g_{\mu\nu}$ and $h_{\alpha\beta}$ are arbitrary d- and $(s+2)$-dimensional metrics, with μ, ν running from 0 to $d-1$, and α, β running from 1 to $s+2$, $s\geqslant 1$, $D=d+s+2$, $g=\det(g_{\mu\nu})$. The factor $(\sqrt{-g})^{-2/s}$ is introduced for future convenience. Both metric tensors depend only on (transverse) coordinates x^α.

For the antisymmetric form we assume either electric or magnetic *Ansätze*. In the electric case the d form has only one nontrivial component

$$A_{01\cdots d-1}=v(x). \tag{8}$$

With this choice of the metric and the d form one obtains a reduced theory in $(s+2)$-dimensional space. Clearly this reduction is not the most general one, namely, we have tacitly assumed that all Kaluza-Klein vectors as well as the lower-dimensional antisymmetric forms arising in the full dimensional reduction are not excited. However, this truncated theory is still reach enough to be explored in details.

In terms of the functions $g_{\mu\nu}$, $h_{\alpha\beta}$, ϕ, and v the equations of motion read

$$\frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta g_{\mu\lambda}g^{\lambda\sigma})g_{\sigma\nu}=\frac{s}{d+s}\,e^{-\psi}g_{\mu\nu}h^{\alpha\beta}\partial_\alpha v\,\partial_\beta v, \tag{9}$$

$$\partial_\alpha(\sqrt{h}h^{\alpha\beta}e^{-\psi}\partial_\beta v)=0, \tag{10}$$

$$\frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta\phi)=\frac{\alpha}{2}\,e^{-\psi}h^{\alpha\beta}\partial_\alpha v\,\partial_\beta v, \tag{11}$$

$$R^{(h)}_{\alpha\beta}=\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi+\frac{1}{s}\partial_\alpha(\ln\sqrt{-g})\partial_\beta(\ln\sqrt{-g})+\frac{1}{4}g^{\mu\lambda}\partial_\alpha g_{\lambda\nu}g^{\nu\sigma}\partial_\beta g_{\sigma\mu}-\frac{1}{2}\,e^{-\psi}h^{\alpha\beta}\partial_\alpha v\,\partial_\beta v, \tag{12}$$

where

$$\psi=\alpha\phi+2\ln\sqrt{-g}. \tag{13}$$

It is straightforward to check that the field equations (9)–(12) can be obtained from the new action of the σ-model type

$$S=\frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{h}\left\{R^{(h)}-h^{\alpha\beta}\left(\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi+\frac{1}{s}\partial_\alpha(\ln\sqrt{-g})\partial_\beta(\ln\sqrt{-g})+\frac{1}{4}g^{\mu\lambda}\partial_\alpha g_{\lambda\nu}g^{\nu\sigma}\partial_\beta g_{\sigma\mu}-\frac{1}{2}\,e^{-\psi}\partial_\alpha v\,\partial_\beta v\right)\right\}. \tag{14}$$

A similar action can be obtained assuming the purely magnetic *Ansatz* for the d form

$$F^{\alpha_1\cdots\alpha_{s+1}}=\frac{1}{\sqrt{-G}}\,e^{\alpha\phi}\epsilon^{\alpha_1\cdots\alpha_{s+1}\beta}\partial_\beta u(x), \tag{15}$$

in this case one has to set in the metric $s=d$. The Maxwell equations (3) are trivially satisfied, while the equation for u follows from the Bianchi identity. In this case the σ-model action still has the form (14) with the replacement of v on u and reversing the sign of α. This fact is a manifestation of the electric-magnetic duality. In what follows we consider explicitly an electric case, the corresponding magnetic solutions can be obtained by the above dualization procedure.

For further analysis of the action (14) it is convenient to rescale the world-volume metric $g_{\mu\nu}$ introducing the matrix

$$\tilde{g}_{\mu\nu}=(\sqrt{-g})^{-2/d}g_{\mu\nu}, \tag{16}$$

such that $\det(\tilde{g}_{\mu\nu})=-1$. Then the action will read

$$S=\frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{h}\left\{R^{(h)}-h^{\alpha\beta}\left(\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi+\frac{s+d}{sd}\partial_\alpha(\ln\sqrt{-g})\partial_\beta(\ln\sqrt{-g})+\frac{1}{4}\tilde{g}^{\mu\lambda}\partial_\alpha\tilde{g}_{\lambda\nu}\tilde{g}^{\nu\sigma}\partial_\beta\tilde{g}_{\sigma\mu}-\frac{1}{2}\,e^{-\psi}\partial_\alpha v\,\partial_\beta v\right)\right\}. \tag{17}$$

Note, that the matrix $\tilde{g}_{\mu\nu}$ now is decoupled from the rest of the σ-model variables, interacting with them only through the gravitational field $h_{\alpha\beta}$. Since $\tilde{g}_{\mu\nu}$ is a symmetric matrix with (minus) unit determinant, this matrix parametrizes a coset $\mathrm{SL}(d,R)/\mathrm{SO}(1,d-1)$. Therefore the metric on the world-volume of the p-brane is to high extent independent of the other σ-model variables, which only influence its overall scale.

To simplify the rest of the action we introduce together with Eq. (13) another variable

$$\xi = sd\phi - \alpha(s+d)\ln\sqrt{-g}, \tag{18}$$

so that the inverse transformations read

$$\phi = \frac{1}{\Delta}\left(\alpha\psi + \frac{2\xi}{(s+d)}\right), \tag{19}$$

$$\ln\sqrt{-g} = \frac{1}{\Delta(s+d)}(sd\psi - \alpha\xi), \tag{20}$$

where $\Delta = \alpha^2 + 2sd/(s+d)$.

In the new variables the part of the action not including the matrix $\tilde{g}_{\mu\nu}$ will read

$$S = \frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{h}\left\{R^{(h)} - h^{\alpha\beta}\times\left(A\partial_\alpha\xi\partial_\beta\xi + B\partial_\alpha\psi\partial_\beta\psi - \frac{1}{2}e^{-\psi}\partial_\alpha v\partial_\beta v\right)\right\}, \tag{21}$$

where

$$A = \frac{1}{\alpha^2 sd(s+d) + 2s^2d^2}, \quad B = \frac{s+d}{2\alpha^2(s+d) + 4sd}. \tag{22}$$

Now the ξ part is also decoupled. The remaining fields ψ and v parametrize a coset SL(2,R)/SO(1,1). Therefore the action (17) corresponds to a nonlinear σ model with the target space SL(d,R)/SO(1,$d-1$)$\times$SL(2,R)/SO(1,1)$\times R$.

Let us recall that the possibility of description of the gravitating systems in three dimensions as σ models on homogeneous target spaces exists for many dimensionally reduced theories. The four-dimensional Einstein-Maxwell theory in the presence of one non-null Killing vector field is equivalent to the SU(2,1)/S(U(2)$\times$U(1)) σ model [12,13]. A more complicated example is given by the dilaton-axion coupled Einstein-Maxwell theory, in which case one has the coset target space Sp(4,R)/U(2) [14]. Several σ models were also derived in multidimensional supergravities [4,19,20], but the geometric structure of the target spaces was not studied.

Let us discuss our σ model (17) in detail. Since the potential space is the direct product of three independent cosets, one can analyze each of them separately. The transverse SL(2,R)/SO(1,1) part can be conveniently described by an analogue of the Ernst potentials [21]

$$\Phi = \frac{v}{2\sqrt{2B}}, \quad \mathcal{E} = e^{\psi} - \frac{v^2}{8B}, \tag{23}$$

using which the target space metric can be rewritten in a familiar form [13]

$$dl^2 = \frac{1}{2}F^{-2}(d\mathcal{E} + 2\Phi d\Phi)^2 - 2F^{-1}d\Phi d\Phi, \tag{24}$$

$$F = \mathcal{E} + \Phi^2. \tag{25}$$

The action of SL(2,R) on the potentials is realized nonlinearly. It can be presented in terms of the following three one-parametric subgroups:

$$\text{(I)}\quad \mathcal{E} = a^2\mathcal{E}_0, \quad \Phi = a\Phi_0, \tag{26}$$

$$\text{(II)}\quad \mathcal{E} = \mathcal{E}_0 - 2b\Phi_0 - b^2, \quad \Phi = \Phi_0 + b, \tag{27}$$

$$\text{(III)}\quad \mathcal{E}' = \frac{\mathcal{E}}{1 - 2c\Phi - c^2\mathcal{E}}, \quad \Phi' = \frac{\Phi + c\mathcal{E}}{1 - 2c\Phi - c^2\mathcal{E}}, \tag{28}$$

where a, b, and c are parameters. Transformation (I) changes the scale of the solution. Note, that in order to have asymptotical flatness (in the case of *time* coordinate reduced), one has to impose the condition $\mathcal{E}(\infty) = 1$. The transformation (II) is pure gauge, it changes an asymptotic value of the antisymmetric form field. The third, Harrison transformation, acts on the space-time variables and matter fields nontrivially.

Similarly one can consider the symmetry transformations realized on the variables ξ and $\tilde{g}$. Subgroup R acts only on ξ:

$$\xi \to \xi + a.$$

In terms of initial fields it corresponds to the shift of the dilaton on a constant accompanied by the rescaling of the metric. The matrix $\tilde{g}$ parametrizes the coset SL(d,R)/SO(1,$d-1$), the representation of the group SL(d,R) is realized in a natural way,

$$\tilde{g} \to U^{-1}\tilde{g}U,$$

where U is a constant element of SL(d,R).

So far we have considered the equations of motion (2)–(4) assuming that the space-time metric admits d commuting Killing vectors one of which is timelike. One can also investigate the case when Killing vectors are *all spacelike*. In this case the σ-model action will read

$$S = \frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{-h}\times\left\{R^{(h)} - h^{\alpha\beta}\left(A\partial_\alpha\xi\partial_\beta\xi + \frac{1}{4}\tilde{g}^{\mu\lambda}\partial_\alpha\tilde{g}_{\lambda\nu}\tilde{g}^{\nu\sigma}\partial_\beta\tilde{g}_{\sigma\mu} + B\partial_\alpha\psi\partial_\beta\psi + \frac{1}{2}e^{-\psi}\partial_\alpha v\partial_\beta v\right)\right\}, \tag{29}$$

where A and B are the same as in Eq. (22). This action differs from the previous one by the sign of the last term. As a result, the metric on the target space is positively definite so we deal with the coset SL(d,R)/SO(d)$\times$SL(2,R)/SO(2)$\times R$. Introducing modified Ernst potentials for the SL(2,R)/SO(2) sector

$$\Phi=\frac{v}{2\sqrt{2B}},\quad \mathcal{E}=-e^{\psi}-\frac{v^2}{8B}, \tag{30}$$

we come back to Eq. (24). Hence the Harrison transformation again is given by Eq. (28).

The above symmetries can be used to generate new solutions from the old ones. Obviously, we are restricted by the chosen *Ansatz* for the metric (7) and antisymmetric form (8), where we can take $g_{\mu\nu}(x)$ to be of Minkowski or Euclidean signature. This means that the seed solutions should have d commuting Killing vectors orthogonal to hypersurfaces and only one nontrivial component of the d form. New solutions obtained by application of σ-model symmetries will have the same properties.

The generating procedure is as follows. Choosing the seed solution with the above properties, one has to find the corresponding potentials v, ψ, and ξ, which are given by Eqs. (8), (13), and (18). The next step is to construct the Ernst potentials (23) and apply the transformations (27), (26), and (28). The new Ernst potentials should be converted into the explicit solution by using the Eqs. (23), (19), (20). As we have mentioned, only the third transformation (28) is nontrivial, but two others can help to fix the scale in order to ensure asymptotic flatness, as well as to set zero an asymptotic value of the form potential. The physical meaning of the Harrison transformation is recharging of the solution (in the case of one timelike Killing vector in the reduction scheme) or adding flux of the differential form (in the case of spacelike Killing vectors). Thus it gives us the possibility to get (charged) p-branes and flux-branes from vacuum configurations. Some concrete examples of this procedure will be given in Secs. 3 and 4.

III. GENERATION OF BLACK *p*-BRANES

As the first application of the above technique we consider the generation of black p-branes. A black p-brane is a multidimensional generalization of the Reissner-Nordström black hole. The simplest case $p=1$ corresponds to the black string [22]. Another important example is the black membrane of $D=11$ supergravity [23]. Black p-branes for general dimensions were constructed in Refs. [9,24]. Also there is a number of papers treating intersections of black p-branes (see, e.g., Refs. [10,11,25], and references therein).

Black p-branes are usually treated as suitable deformations of the corresponding extremal p-branes, the latter being specified by the one-center harmonic functions. The process of deformation is called ''blackening,'' the relevant prescription was given in Refs. [9,10]. Also it is known that black p-brane solutions can be obtained from Schwarzschild solution by sequences of boosts and dualities [26]. We prove here that the existence of such prescriptions is a manifestation of the hidden symmetry contained in the model. This symmetry is nothing but the SL$(2,R)$, which was considered in the previous section. In this section we use this symmetry for an explicit generation of a *single* black p-brane, while in Sec. VII we will explain the generation of *intersecting* black p-branes.

Let us start with the Schwarzschild solution in the D-dimensional spacetime corresponding to a ''neutral'' $(d-1)$-brane

$$ds^2=-\left(1-\frac{2M}{r^s}\right)dt^2+dy_1^2+\cdots+dy_{d-1}^2+\left(1-\frac{2M}{r^s}\right)^{-1}dr^2+r^2d\Omega_{s+1}. \tag{31}$$

Using Eqs. (13) and (18) we obtain

$$\psi_0=\ln\left(1-\frac{2M}{r^s}\right),\quad \xi_0=-\frac{1}{2}\alpha(s+d)\ln\left(1-\frac{2M}{r^s}\right),$$
$$v_0=0, \tag{32}$$

which corresponds to the following seed Ernst potentials:

$$\Phi_0=0,\ \ \mathcal{E}_0=1-\frac{2M}{r^s}. \tag{33}$$

The Harrison transformations (28) and the rescaling of potentials yield the new functions ψ and v:

$$\psi=\ln\left(1-\frac{2M}{r^s}\right)+\ln\left(1+\frac{2Q}{r^s}\right)^{-2},$$
$$v=2c\sqrt{2B}\left(1-\frac{2M}{r^s}\right)\left(1+\frac{2Q}{r^s}\right)^{-1}, \tag{34}$$

where

$$Q=\frac{Mc^2}{1-c^2}. \tag{35}$$

The function ξ remains the same. The resulting metric is

$$ds^2=\left(1+\frac{2Q}{r^s}\right)^{-\nu s}\left\{-\left(1-\frac{2M}{r^s}\right)dt^2+dy_1^2+\cdots+dy_{d-1}^2\right\}+\left(1+\frac{2Q}{r^s}\right)^{\nu d}\left\{\left(1-\frac{2M}{r^s}\right)^{-1}dr^2+r^2d\Omega_{s+1}\right\}, \tag{36}$$

where $\nu=4\Delta^{-1}(s+d)^{-1}$. This coincides with the metric of the black p-brane solution [9]. The corresponding dilaton field is given by

$$e^{-\alpha\phi}=\left(1+\frac{2Q}{r^s}\right)^{2\alpha^2/\Delta}. \tag{37}$$

Note that the extremal limit of this solution is $M\to0$, $c\to1$ so that Q is finite.

IV. GENERATION OF THE FLUX-BRANE

Assuming that all Killing vectors are space-like we can apply the same technique to obtain the *flux-brane* solution which is a multidimensional generalization of the Bonnor-Melvin universe [27] in the Einstein-Maxwell gravity. The

Bonnor-Melvin solution with a dilaton was constructed by Gibbons and Maeda [28], the flux-brane solutions to the multidimensional gravity with one-form field were considered in Ref. [29]. We give their generalization for the case of the arbitrary rank d-form of either electric or magnetic type.

Our starting point is a flat D-dimensional space-time written in the multicylindrical coordinates

$$ds^2 = -dt^2 + (\rho_1^2 d\varphi_1^2 + \cdots + \rho_d^2 d\varphi_d^2) + d\rho_1^2 + \cdots + d\rho_d^2 + dx_\alpha dx^\alpha, \tag{38}$$

where $\alpha = 1,\ldots,s+1-d$. This yields

$$\psi_0 = 2 \ln \rho_1 \cdots \rho_d, \;\; \xi_0 = -\alpha(s+d) \ln \rho_1 \cdots \rho_d, \;\; v_0 = 0, \tag{39}$$

so that the corresponding Ernst potentials are

$$\mathcal{E}_0 = -\rho_1^2 \cdots \rho_d^2, \quad \Phi_0 = 0. \tag{40}$$

Applying the electric Harrison transformation (28) we obtain

$$\psi = 2 \ln \rho_1 \cdots \rho_d - 2 \ln(1 + c^2 \rho_1^2 \cdots \rho_d^2),$$
$$v = -\frac{2c\sqrt{2B}\rho_1^2 \cdots \rho_d^2}{1 + c^2 \rho_1^2 \cdots \rho_d^2}, \tag{41}$$

with ξ remaining the same. As a result we get the following metric:

$$ds^2 = (1 + c^2\rho_1^2 \cdots \rho_d^2)^{-\nu s}(\rho_1^2 d\varphi_1^2 + \cdots + \rho_d^2 d\varphi_d^2) + (1 + c^2\rho_1^2 \cdots \rho_d^2)^{\nu d}(-dt^2 + d\rho_1^2 + \cdots + d\rho_d^2 + dx_\alpha dx^\alpha). \tag{42}$$

The corresponding dilaton field is

$$e^{-\alpha\phi} = (1 + c^2\rho_1^2 \cdots \rho_d^2)^{2\alpha^2/\Delta}, \tag{43}$$

while the d-form potential has the nonvanishing component

$$A_{\varphi_1 \cdots \varphi_d} = -\frac{2c\sqrt{2B}\rho_1^2 \cdots \rho_d^2}{1 + c^2 \rho_1^2 \cdots \rho_d^2}, \tag{44}$$

where the coefficient B is given by Eq. (22).

Applying a similar technique one can easily construct more complicated solutions. As an example let us derive the metric describing a six-dimensional dilatonic black hole in the magnetic field of the one-flux-brane. Now we start not with the flat space-time, but with the six-dimensional Schwarzschild solution writing the metric on the four-sphere in the form [30]

$$ds^2 = -\left(1 - \frac{2M}{r^3}\right)dt^2 + \left(1 - \frac{2M}{r^3}\right)^{-1} dr^2 + r^2(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\varphi_1^2 + \cos^2\theta \sin^2\psi d\varphi_2^2). \tag{45}$$

According to Eqs. (13) and (18),

$$\psi_0 = 2 \ln(r^2 \sin\theta \cos\theta \sin\psi),$$
$$\xi_0 = -4\alpha \ln(r^2 \sin\theta \cos\theta \sin\psi), \tag{46}$$

thus the seed Ernst potentials have the form

$$\mathcal{E}_0 = -r^4 \sin^2\theta \cos^2\theta \sin^2\psi, \;\; \Phi_0 = 0.$$

Using Harrison transformations (28) we obtain a new solution with the metric

$$ds^2 = \{1 + c^2 r^4 \sin^2\theta \cos^2\theta \sin^2\psi\}^{2/(\alpha^2+2)} \times \left\{ -\left(1 - \frac{2M}{r^3}\right)dt^2 + \left(1 - \frac{2M}{r^3}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \cos^2\theta d\psi^2 \right\} + \{1 + c^2 r^4 \sin^2\theta \cos^2\theta \sin^2\psi\}^{-2/(\alpha^2+2)} \times r^2(\sin^2\theta d\varphi_1^2 + \cos^2\theta \sin^2\psi d\varphi_2^2).$$

The corresponding dilaton field is

$$e^{-\alpha\phi} = \{1 + c^2 r^4 \sin^2\theta \cos^2\theta \sin^2\psi\}^{2\alpha^2/(\alpha^2+2)},$$

while the nonvanishing component of the two-form potential is given by

$$A_{\varphi_1\varphi_2} = -\frac{2c}{\alpha^2+2} \frac{r^4 \sin^2\theta \cos^2\theta \sin^2\psi}{\{1 + c^2 r^4 \sin^2\theta \cos^2\theta \sin^2\psi\}}.$$

It is easy to see that the obtained solution is indeed a nonlinear superposition of the black hole and the flux-brane. The limit $c \to 0$ returns us back to the Schwarzschild solution, while putting $M = 0$ we recover the flux-brane. Along the same lines one can construct the instanton describing the nucleation of p-branes by the homogeneous antisymmetric form.

V. HARMONIC MAPS

The sigma-model representation can be used for constructing new solutions in a different way not using directly the symmetries of the target space. In this section we will find solutions of the theory (1) by the harmonic maps technique which is based on direct exponentiation of the coset generators.

In order to apply this procedure we rewrite the σ-model action (17) in the following matrix form

$$S = \frac{1}{2\kappa^2} \int d^{s+2}x \sqrt{h} \left\{ R^{(h)} + h^{\alpha\beta} \times \left(2B \,\mathrm{Tr}\, \partial_\alpha M \partial_\beta M^{-1} + \frac{1}{4} \mathrm{Tr}\, \partial_\alpha \tilde{g} \partial_\beta \tilde{g}^{-1} \right) \right\}, \tag{47}$$

where the matrix M is built from the fields Ψ, v, and ξ as follows:

$$M=\exp\left(-\frac{\psi}{2}\right)\begin{pmatrix} 2 & \frac{v}{2\sqrt{2B}} & 0 \\ \frac{v}{2\sqrt{2B}} & -\frac{1}{2}\left(\exp\,\psi-\frac{v^2}{8B}\right) & 0 \\ 0 & 0 & \exp\left(\frac{\psi}{2}+\frac{\xi}{\sqrt{sd(s+d)}}\right). \end{pmatrix} \tag{48}$$

This representation is a convenient starting point for an application of the harmonic maps technique. In particular we will be interested here in constructing solutions corresponding to the null geodesics of the target space [31,32].

Consider the transverse part of the action (47),

$$S=\frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{h}[R^{(h)}+2Bh^{\alpha\beta}\mathrm{Tr}(\partial_\alpha M\partial_\beta M^{-1})], \tag{49}$$

where M is an element of the appropriate coset space G/H. The equations of motion read

$$\frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}M^{-1}\partial_\beta M)=0, \tag{50}$$

$$R^{(h)}_{\alpha\beta}=-2B\ \mathrm{Tr}(\partial_\alpha M\partial_\beta M^{-1}). \tag{51}$$

It was noticed [12] that if the matrix M depends on x coordinates through a single function, $M=M(\sigma(x))$, then $\sigma(x)$ can be chosen to be a harmonic function on the curved space with the metric h, i.e.,

$$\frac{1}{\sqrt{h}}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta\sigma)=0. \tag{52}$$

Equation (50) then reduces to the matrix equation

$$\frac{d}{d\sigma}\left(M^{-1}\frac{dM}{d\sigma}\right)=0, \tag{53}$$

whose solution can be expressed in the exponential form

$$M=M_0e^{K\sigma}, \tag{54}$$

where K belongs to the Lie algebra of the group G, and $M_0\in G/H$. Substituting this into the Einstein equations (51) one gets

$$R^{(h)}_{\alpha\beta}=2B\ \mathrm{Tr}(K^2)\partial_\alpha\sigma\partial_\beta\sigma. \tag{55}$$

It is clear that in the particular case $\mathrm{Tr}(K^2)=0$ the metric h is Ricci-flat (and hence can be chosen flat). This is a constructive way to build null-geodesic solutions to an arbitrary σ model. Let us apply it to our model with the target space $\mathrm{SL}(2,R)/\mathrm{SO}(1,1)\times R$. Here we are interested in the asymptotically flat solutions, so we choose the harmonic function σ such that $\sigma(\infty)=0$. According to the above general scheme, we present M in the form (54), where M_0 is an element of the coset $\mathrm{SL}(2,R)/\mathrm{SO}(1,1)\times R$ and the generator K belongs to the algebra $\mathrm{sl}(2,R)\times R$. M_0 has to be taken corresponding to an assumed asymptotic behavior, Eq. (48) gives

$$M_0=\mathrm{diag}\left(2,-\frac{1}{2},1\right). \tag{56}$$

A convenient parametrization of the matrix K is

$$K=\begin{pmatrix} a & c & 0 \\ d & -a & 0 \\ 0 & 0 & b \end{pmatrix}. \tag{57}$$

One has to distinguish two different cases: $\det K\neq 0$ and $\det K=0$.

(1) *Degenerate case:* $\det K=0$. The general constraint $\mathrm{Tr}\,K^2=0$ gives $2(a^2+cd)+b^2=0$. Together with the restriction $\det K=0$ this means $b=0$, $a^2+cd=0$. In terms of the matrix K this leads to $K^2=0$, so the exponentiation is essentially different from that in the nondegenerate case:

$$e^{K\sigma}=I+K\sigma. \tag{58}$$

Therefore for the matrix M one obtains

$$M=\begin{pmatrix} 2+2a\sigma & 2c\sigma & 0 \\ -\frac{1}{2}d\sigma & -\frac{1}{2}(1-a\sigma) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{59}$$

This matrix should be symmetric which gives an additional constraint on the parameters, so the resulting matrix depends on a single parameter a

$$M=\begin{pmatrix} 2+2a\sigma & a\sigma & 0 \\ a\sigma & -\frac{1}{2}(1-a\sigma) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{60}$$

Comparing this with Eqs. (48) and (13) we get

$$\psi = -2\ln(1+a\sigma), \quad \xi = 0, \quad v = 2\sqrt{2B}\left(1 - \frac{1}{1+a\sigma}\right). \tag{61}$$

Since σ is an arbitrary harmonic function, it is defined up to a scale parameter. Thus without loss of generality one can put $a=1$. The resulting metric is

$$ds^2 = (1+\sigma)^{-\nu s}(-dt^2 + dy_1^2 + \cdots + dy_{d-1}^2) + (1+\sigma)^{\nu d}(dx_1^2 + \cdots + dx_{s+2}^2). \tag{62}$$

This metric is nothing but the usual p-brane solution with the harmonic function $H = 1+\sigma$ [33]. The corresponding dilaton field is given by

$$e^{-\alpha\phi} = (1+\sigma)^{2\alpha^2/\Delta}. \tag{63}$$

This solution saturates the Bogomol'nyi bound

$$M = \frac{4sBQ\Omega_{s+1}}{\kappa^2}. \tag{64}$$

(2) *Nondegenerate case:* det $K \neq 0$. Once again we have a constraint Tr $K^2 = 0$, which implies $2(a^2+cd)+b^2=0$. Performing a direct exponentiation one obtains

$$M = \begin{pmatrix} 2\cos\frac{b\sigma}{\sqrt{2}} + \frac{2a\sqrt{2}}{b}\sin\frac{b\sigma}{\sqrt{2}} & \frac{2c\sqrt{2}}{b}\sin\frac{b\sigma}{\sqrt{2}} & 0 \\ -\frac{d\sqrt{2}}{2b}\sin\frac{b\sigma}{\sqrt{2}} & -\frac{1}{2}\cos\frac{b\sigma}{\sqrt{2}} + \frac{a\sqrt{2}}{2b}\sin\frac{b\sigma}{\sqrt{2}} & 0 \\ 0 & 0 & e^{b\sigma} \end{pmatrix}. \tag{65}$$

This matrix should be symmetric, so taking into account the constraints on the coefficients we obtain

$$M = \begin{pmatrix} 2\frac{\sin(\sigma+\varphi)}{\sin\varphi} & \frac{\sin\sigma}{\sin\varphi} & 0 \\ \frac{\sin\sigma}{\sin\varphi} & \frac{\sin(\sigma-\varphi)}{2\sin\varphi} & 0 \\ 0 & 0 & e^{\sqrt{2}\sigma} \end{pmatrix}, \tag{66}$$

where we put $b=\sqrt{2}$ because of the scaling freedom for the harmonic function and denoted

$$\sin\varphi = \frac{1}{\sqrt{a^2+1}}. \tag{67}$$

Equations (48) and (13) yield

$$\xi = \sigma\sqrt{2sd(s+d)}, \quad \psi = -2\ln\left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right],$$

$$v = \frac{2\sqrt{2B}\sin\sigma}{\sin(\sigma+\varphi)}. \tag{68}$$

Now it is easy to construct the whole metric:

$$ds^2 = \left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right]^{-\nu s} \times e^{-\sqrt{s(s+d)/2d}\,\nu\alpha\sigma}(-dt^2 + dy_1^2 + \cdots + dy_{d-1}^2) + \left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right]^{\nu d} e^{\sqrt{d(s+d)/2s}\,\nu\alpha\sigma}(dx_1^2 + \cdots + dx_{s+2}^2) \tag{69}$$

and the dilaton field

$$e^{-\alpha\phi} = \left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right]^{2\alpha^2/\Delta} e^{-\sqrt{sd(s+d)/2}\,\nu\alpha\sigma}. \tag{70}$$

The structure of this solution is similar to that of the usual p-brane, but the metric functions are essentially different (for the 0-brane case see Ref. [32]). The full r-range solution contains a sequence of compact singular transverse hypersurfaces lying between the subsequent roots r_k of the equation

$$\sigma(r_k) + \varphi = \pi k, \quad k = 1,2,\ldots \tag{71}$$

and forming a ''matrioshka''-type structure in the transverse space. Curvature invariants diverge at r_k. The outermost spacetime is asymptotically flat, and for it one can calculate the Arnowitt-Deser-Misner (ADM) mass and the Page charge [34]. It easy to check that the Bogomol'nyi bound is indeed saturated (as could be expected since the solution corresponds to a null geodesic in the target space) if the parameter φ satisfies the constraints

$$\sin(\varphi+\chi)=\sqrt{\frac{sd}{2(s+d)}},\quad \cos\chi=\alpha\sqrt{2B}. \tag{72}$$

As a realistic example let us take the type IIA supergravity, whose bosonic action in the Einstein frame is given by

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\times\left(R-\frac{1}{2}(\nabla\phi)^2-\frac{e^{-\phi}}{2\times 3!}F_3^2-\frac{e^{3\phi/2}}{2\times 2!}F_2^2-\frac{e^{\phi/2}}{2\times 4!}F_4'^2\right)-\frac{1}{4\kappa^2}\int F_4\wedge F_4\wedge A_2, \tag{73}$$

where

$$F_4'=dA_3+A_1\wedge F_3. \tag{74}$$

We will consider the Neveu-Schwarz (NS) part of the action consisting of the metric, dilaton, and two-form. The usual extremal one-brane solution corresponds to the elementary NS string and has the form

$$ds^2=H^{-3/4}(-dt^2+dy^2)+H^{1/4}(dx_1^2+\cdots+dx_8^2). \tag{75}$$

The ''matrioshka'' one-brane line element reads

$$ds^2=\left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right]^{-3/4}e^{-(\sqrt{3}/4)\sigma}(-dt^2+dy^2)+\left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right]^{1/4}e^{(1/4\sqrt{3})\sigma}(dx_1^2+\cdots+dx_8^2), \tag{76}$$

while the dilaton is

$$e^{-\phi}=\left[\frac{\sin(\sigma+\varphi)}{\sin\varphi}\right]^{1/2}e^{-(\sqrt{3}/2)\sigma}. \tag{77}$$

The null geodesic method can also be applied to the $SL(d,R)/SO(1,d-1)$ part of the initial σ model (17). The matrix $\tilde{g}$ should be taken in a form similar to Eq. (54),

$$\tilde{g}=\tilde{g}_0e^{K\sigma}, \tag{78}$$

where $\mathrm{Tr}\,K^2=0$, K belongs to an algebra $sl(d,R)$. Asymptotic flatness conditions imply

$$\tilde{g}_0=\mathrm{diag}(-1,1,\ldots). \tag{79}$$

In the simplest case $d=2$ the condition $\mathrm{Tr}\,K^2=0$ leads to $\det K=0$, i.e., the resulting matrix $\tilde{g}$ has the form

$$\tilde{g}=\begin{pmatrix}-(1+a\sigma) & -c\sigma\\ d\sigma & 1-a\sigma\end{pmatrix},\quad a^2+dc=0. \tag{80}$$

The matrix $\tilde{g}$ should be symmetric, so $c=-d=\pm a$, and after rescaling of the harmonic function σ

$$\tilde{g}=\begin{pmatrix}-(1+\sigma) & \pm\sigma\\ \pm\sigma & 1-\sigma\end{pmatrix}. \tag{81}$$

This metric can be rewritten in the light-cone coordinates as

$$ds^2=-du\,dv-\sigma du^2. \tag{82}$$

This corresponds to the well-known Brinkmann wave [35] and the decoupling of $\tilde{g}$ from the action (17) reflects the possibility of a superposition of p-branes and waves [36].

VI. BONNOR-TYPE MAP

One can obtain new nontrivial solutions from the old one using a map between similar cosets describing physically different theories. This idea traces back to the Bonnor construction of the metric of a magnetic dipole in general relativity using a correspondence of two $SL(2,R)/SO(1,1)$ describing stationary vacuum gravity and static electrovacuum. Since we have the same subspace in the p-brane case (17), one can use the same correspondence to generate new p-brane solutions.

For the vacuum Einstein theory in four dimensions the target space describing stationary solutions has the form

$$ds^2=\frac{1}{2f^2}(df^2+d\chi^2), \tag{83}$$

where $f=g_{tt}$ and χ is the twist potential. One can check that the correspondence between two σ models can be achieved only if $B=1/8$. Note that the appropriate map is complex:

$$\Psi=2\ln f,\quad v=i\chi, \tag{84}$$

so, in order to obtain real solutions in the Minkowski space, we should take the complexified seed solutions.

As an example let us consider the complexified Kerr-NUT (Newmann-Unti-Tamborino) solution of the Einstein theory taking pure imaginary rotation and NUT parameters $\tilde{a}=ia$, $\tilde{N}=iN$

$$ds^2=-\frac{\Delta+\tilde{a}^2\sin^2\theta}{\Sigma}(dt-\omega d\varphi^2)+\Sigma\left(\frac{dr^2}{\Delta}+d\theta^2+\frac{\Delta\sin^2\theta}{\Delta+\tilde{a}^2\sin^2\theta}d\phi^2\right), \tag{85}$$

where

$$\Delta=r^2-2Mr-\tilde{a}^2+\tilde{N}^2, \tag{86}$$

$$\Sigma=r^2+\delta^2,\quad \delta=-i(\tilde{a}\cos\theta+\tilde{N}), \tag{87}$$

$$\omega=\frac{2i}{\Delta+\tilde{a}^2\sin^2\theta}[\tilde{N}\Delta\cos\theta+\tilde{a}\sin^2\theta(Mr-\tilde{N}^2)]. \tag{88}$$

The potentials f and χ will be given by

$$f=\frac{\Delta+\tilde{a}^2\sin^2\theta}{\Sigma},\quad \chi=-\frac{2i(M\tilde{a}\cos\theta+M\tilde{N}-\tilde{N}r)}{\Sigma}. \tag{89}$$

Since the potential χ is pure imaginary, the complex transformation (84) will give a real solution.

Consider the type IIA supergravity (73). It contains a one-form which can be connected with electric black hole or with the magnetic D6-brane. We will construct the metric of the D6-brane. It is easy to check that in this case B is equal to 1/8, so we can use the above technique. The map (84) applied to the potentials (89) and the Eqs. (19),(20) lead to the following metric:

$$ds^2=f^{1/8}(-dt^2+dy_1^2+\cdots+dy_6^2)+f^{1/8}\frac{\Sigma}{\Delta}dr^2 + f^{1/8}\Sigma d\theta^2+f^{-7/8}\Delta\,\sin^2\theta d\varphi^2, \tag{90}$$

where f, Δ, and Σ are given by Eqs. (89), (86), and (87). This metric describes the magnetic D6-brane with the dipole moment which is generated by the parameter $\tilde{a}$. The nontrivial components of the one-form field strength are

$$F^{r\varphi}=-\frac{f^{-5/4}}{\Sigma\,\sin\,\theta}\partial_\theta u,\;\; F^{\theta\varphi}=\frac{f^{-5/4}}{\Sigma\,\sin\,\theta}\partial_r u, \tag{91}$$

where u is given by

$$u=\frac{2(M\tilde{a}\,\cos\,\theta+M\tilde{N}-\tilde{N}r)}{\Sigma}. \tag{92}$$

The corresponding dilaton field can be expressed through the function f as follows:

$$e^{(3/2)\phi}=f^{9/8}. \tag{93}$$

In the limit $\tilde{a}=0$, $M=N$ the configuration obtained reduces to the usual extremal magnetic D6-brane.

VII. INTERSECTING *p*-BRANES

In order to describe within the same approach the *intersecting p*-branes we have to change our basic *Ansatz* (7),(8). Now we assume that the d-form has more than one nontrivial component and the metric exhibits a block structure

$$ds^2=\sum_{i=1}^{N} g^{(i)}_{\mu_i\nu_i}(x)dy^{\mu_i}dy^{\nu_i} + \left(\prod_{i=1}^{N}\sqrt{|g^{(i)}|}\right)^{-2/s} h_{\alpha\beta}(x)dx^\alpha dx^\beta, \tag{94}$$

where $g^{(i)}_{\mu_i\nu_i}$, $g^{(i)}=\det g^{(i)}_{\mu_i\nu_i}$ and $h_{\alpha\beta}$ are arbitrary symmetric tensors, $\mu_i,\nu_i=1\cdots r_i$, $\alpha,\beta=1\cdots s+2$.

A convenient description of the d-form *Ansatz* is based on the incidence matrix approach [5]. The incidence matrix is an rectangular $N\times E$ matrix,

$$\Delta=(\Delta_{ai}),\;\; a=1,\dots,E,\;\; i=1,\dots,N, \tag{95}$$

where rows correspond to different components of the d-form and columns refer to the blocks in the metric (94). The entries of the incidence matrix are equal to 0 or 1, $\Sigma_{i=1}^N r_i\Delta_{ai}=d$ for each a. Since we intend to consider the usual p-branes we suppose that each nontrivial component of the d form has zero index $A_{0\cdots}$, i.e., it covers the timelike direction. In terms of the incidence matrix it means $\Delta_{a1}=1$. We assume that nontrivial components of the d form are

$$A_a=v_a(x)\prod_{\{i|\Delta_{ai}=1\}}\wedge\omega_i, \tag{96}$$

where ω_i are r_i forms

$$\omega_i=dy_i^1\wedge\cdots\wedge dy_i^{r_i}.$$

v_a are some functions of x variables, $a=1,\dots,E$.

Now, as in Sec. II, we substitute this *Ansatz* into the equations of motion and obtain the corresponding σ model. Introducing as in Eq. (16) the ''internal'' metrics $\tilde{g}^{(i)}$

$$g^{(i)}_{\mu_i\nu_i}=(\sqrt{|g^{(i)}|})^{2/r_i}\tilde{g}^{(i)}_{\mu_i\nu_i}, \tag{97}$$

it is easy to obtain the following. These renormalized metric tensors are decoupled from the rest of the σ model and we find them only in the sector $\frac{1}{4}\mathrm{Tr}\;\partial_\alpha\tilde{g}^{(i)}\partial_\beta\tilde{g}^{(i)^{-1}}$. The rest of the σ-model action reads

$$S=\frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{h}\Bigg\{R^{(h)}-h^{\alpha\beta}\bigg(\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi + \hat{G}_{ij}\partial_\alpha(\ln\sqrt{|g^{(i)}|})\partial_\beta(\ln\sqrt{|g^{(j)}|}) - \frac{1}{2}\sum_{a=1}^{E} e^{-\alpha\phi-2\Sigma_{i=1}^N\Delta_{ai}\ln\sqrt{|g^{(i)}|}}\partial_\alpha v_a\partial_\beta v_a\bigg)\Bigg\}, \tag{98}$$

where $\hat{G}_{ij}$ is a matrix with the following components:

$$\hat{G}_{ij}=\frac{1}{r_i}\delta_{ij}+\frac{1}{s}. \tag{99}$$

Now the target space is $(N+E+1)$ dimensional, it is parametrized by ϕ, $\ln\sqrt{|g^{(i)}|}$, and v_a.

The explicit solutions known for the intersecting branes were found only assuming certain conditions on the parameters (intersection rules). It turns out that these conditions correspond to the symmetric space property of the sigma-model target space. Let us remind the reader that the metric space is called symmetric if the Riemann tensor is covariantly constant, i.e.,

$$\nabla_a R_{bcde}=0. \tag{100}$$

Straightforward calculations yield the following. All nonzero components of the five-index tensor $\nabla_a R_{bcde}$ are proportional to

$$\exp\left(-\alpha\phi-2\sum_{i=1}^{N}\Delta_{ai}\ln\sqrt{|g^{(i)}|}\right)$$
$$\times\exp\left(-\alpha\phi-2\sum_{i=1}^{N}\Delta_{a'i}\ln\sqrt{|g^{(i)}|}\right)$$
$$\times\left(\frac{\alpha^2}{2}-\frac{d^2}{D-2}+\sum_{i=1}^{N}r_i\Delta_{ai}\Delta_{a'i}\right).$$

This means that the target space is symmetric when the incidence matrix satisfy the following condition:

$$\frac{\alpha^2}{2}-\frac{d^2}{D-2}+\sum_{i=1}^{N}r_i\Delta_{ai}\Delta_{a'i}=0, \tag{101}$$

showing that we deal with the usual p-brane intersection rule [5]. Thus, the σ-model approach gives a simple geometrical interpretation of the intersection rule (101): only when Eq. (101) is satisfied is the target space a symmetric (pseudo)Riemannian space.[1]

Let us consider the simplest case of two intersecting p-branes (with two nontrivial components of the d form, $E=2$, and three blocks in the metric: $N=3$, $r_1=q$, $r_2=r_3=r$). If the parameters of our configuration satisfy Eq. (101), the σ-model action (98) could be diagonalized and reduced to a simple form similar to Eq. (21):

$$S=\frac{1}{2\kappa^2}\int d^{s+2}x\sqrt{h}\Bigg\{R^{(h)}-h^{\alpha\beta}\Bigg(A_1\partial_\alpha\xi_1\partial_\beta\xi_1+A_2\partial_\alpha\xi_2\partial_\beta\xi_2+B_1\partial_\alpha\psi_1\partial_\beta\psi_1-\frac{1}{2}e^{-\psi_1}\partial_\alpha v_1\partial_\beta v_1+B_2\partial_\alpha\psi_2\partial_\beta\psi_2-\frac{1}{2}e^{-\psi_2}\partial_\alpha v_2\partial_\beta v_2\Bigg)\Bigg\}, \tag{102}$$

where

$$\psi_1=\alpha\phi+2\ln\sqrt{-g^{(0)}}+2\ln\sqrt{g^{(1)}}, \tag{103}$$

$$\psi_2=\alpha\phi+2\ln\sqrt{-g^{(0)}}+2\ln\sqrt{g^{(2)}}, \tag{104}$$

$$\xi_1=-\alpha s\phi-2(s-r)\ln\sqrt{-g^{(0)}}+2r\ln\sqrt{g^{(1)}}+2r\ln\sqrt{g^{(2)}}, \tag{105}$$

$$\xi_2=q(r+s)\phi-\alpha(q+2r+s)\ln\sqrt{-g^{(0)}}, \tag{106}$$

and the constants are

$$A_1=\frac{1}{4sr^2},\ \ A_2=\frac{1}{2(q+2r+s)r^2d},\ \ B_1=\frac{1}{4r},\ \ B_2=\frac{1}{4r}. \tag{107}$$

Thus we have obtained the σ model with the $SL(2,R)/SO(1,1)\times SL(2,R)/SO(1,1)\times R\times R$ target space. This structure means that two p-branes can be generated separately. As an example one can construct two intersecting black p-branes. The procedure is similar to that discussed in Sec. III, but now one has to apply Harrison transformations with different parameters to each $SL(2,R)/SO(1,1)$ component. Thus one obtains two intersecting nonextremal p-branes with different charges [10,11]. This derivation demonstrates that the existence of such configurations is a consequence of the σ-model target space symmetries.

In Sec. V we constructed some new solutions using the null geodesic method applied to the σ model (21). The same strategy applied to the σ model (102) leads to extremal intersecting p-branes with two charges (in the case of the degenerate matrix K) and to the intersecting ''matrioshka''-type p-branes (in the case of the nondegenerate matrix K). Thus we can speculate that ''matrioshka'' p-branes are subject to the usual intersection rule. As an example we exhibit the metric of two intersecting ''matrioshka''-type one-branes in type IIA supergravity:

$$ds^2=\left[\frac{\sin(\sigma_1+\varphi_1)}{\sin\varphi_1}\right]^{3/8}\left[\frac{\sin(\sigma_2+\varphi_2)}{\sin\varphi_2}\right]^{3/8}e^{-(7\sqrt{6}/48)\sigma_1-(\sqrt{2}/16)\sigma_2}\Bigg(\left[\frac{\sin(\sigma_1+\varphi_1)}{\sin\varphi_1}\right]^{-1}\left[\frac{\sin(\sigma_2+\varphi_2)}{\sin\varphi_2}\right]^{-1}e^{-(\sqrt{6}/6)\sigma_1+(\sqrt{2}/2)\sigma_2}\times(-dt^2+dy^2)+\left[\frac{\sin(\sigma_1+\varphi_1)}{\sin\varphi_1}\right]^{-1}e^{(\sqrt{6}/3)\sigma_1}(dz_1^2+dz_2^2)+\left[\frac{\sin(\sigma_2+\varphi_2)}{\sin\varphi_2}\right]^{-1}e^{(\sqrt{6}/3)\sigma_1}(dz_3^2+dz_4^2)+dx_\alpha dx^\alpha\Bigg). \tag{108}$$

VIII. DISCUSSION AND CONCLUSIONS

In this paper, we have focused on the technical aspects of getting p-brane solutions via the σ-model formulation of the simplest brane-containing theory. Although the idea of using dualities of dimensionally reduced theories is not new, we have shown that knowing an explicit nonlinear realization of dualities in terms of the target space variables one can exploit ''hidden symmetries'' more effectively. We have shown that under the restriction of the block-diagonal metrics and the corresponding *Ansätze* for an antisymmetric

[1] The related ideas were recently discussed in Ref. [37].

form a typical p-brane producing action reduces to the multidimensional σ-model on a symmetric target space. Among target space isometries there are Harrsion-type transformations generating Page charges, which relate uncharged uplifted black holes and genuine black p-branes.

Apart from the direct use of transformations to get new solutions from old ones, one can also apply various integration methods developed earlier in general relativity. In particular, using a technique of harmonic maps we have found new classes of p-branes with a nontrivial ''matrioshka''-type structure of the transverse space. We have shown that some p-brane ''rules,'' such as intersection rules for composite branes or ''blackening'' prescriptions, have a rather natural geometric interpretation in the σ-model terms. Since the main subgroup involved is $SL(2,R)$, one can effectively use solutions to other theories sharing the same group structure to get new p-brane solutions. This Bonnor-type correspondence is somewhat similar to duality between different theories which was widely discussed recently in the context of superstrings.

We have considered the purely bosonic problem and constructed solutions to the model action (1). However, in the most interesting cases, such as type IIA and IIB supergravities, we have supersymmetric actions. Bosonic fields of these theories are the same as in the problem we have discussed. Thus it is important to find whether or not the constructed solutions, being at the same time the solutions to supersymmetric theories, preserve a part of the initial supersymmetry. The corresponding analysis shows that, except for the usual extremal p-branes, our new solutions do not preserve supersymmetries. In particular, we have checked that flux-branes in type IIA supergravity are not supersymmetric. Recall that this is also true for the Melvin solution of the Einstein-Maxwell theory embedded into $N=2$ supergravity. Also, additional structure on extremal p-branes, such as dipole moment, leads to supersymmetry breaking. The situation with ''matrioshka''-type p-branes is more complicated. The outermost component of the corresponding spacetime is asymptotically flat, and one can check that the BPS bound is indeed saturated. However, both in type II ten-dimensional supergravities and in eleven-dimensional supergravity no Killing spinors exist on this background. There is no contradiction between these facts because the solution is singular and possesses an inner boundary which has to be taken into account when integrating the corresponding Nester forms.

We note in conclusion that our formulation also opens a way to apply techniques of integrable systems assuming that the target space variables depend only on two of the transverse coordinates. In four-dimensional theories the full space-time metric can be recovered once the solution of the corresponding integrable system is found. In multidimensional cases additional assumptions are needed about the structure of the transverse space to ensure complete solvability. More general Lagrangians including several antisymmetric forms and dilatons can also be investigated under the assumption of the block-diagonal metrics. However, in the nondiagonal cases one encounters serious technical complications while attempting to find an explicit nonlinear realization of ''hidden'' symmetries.

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