PHYSICAL REVIEW D, VOLUME 58, 121501

Hoop conjecture for colliding black holes

Daisuke Ida* and Ken-ichi Nakao†
Department of Physics, Kyoto University, Kyoto 606-8502, Japan

Masaru Siino‡ and Sean A. Hayward§
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

(Received 6 May 1998; published 4 November 1998)

We study the collision of black holes in the Kastor-Traschen space-time, at present the only such analytic solution. We investigate the dynamics of the event horizon in the case of the collision of two equal black holes, using the ray-tracing method. We confirm that the event horizon has trouser topology and show that its set of past end points (where the horizon is nonsmooth) is a spacelike curve resembling a seam of trousers. We show that this seam has a finite length and argue that twice this length be taken to define the minimal circumference C of the event horizon. Comparing with the asymptotic mass M, we find the inequality $C<4\pi M$ supposed by the hoop conjecture, with both sides being of similar order, $C\sim 4\pi M$. This supports the hoop conjecture as a guide to general gravitational collapse, even in the extreme case of head-on black-hole collisions. [S0556-2821(98)50322-0]

PACS number(s): 04.70.Bw

The Kastor-Traschen (KT) space-time is a family of solutions to the Einstein-Maxwell equations with a positive cosmological constant, Λ [1]. This space-time represents a multi-black-hole cosmos and each black hole has a $U(1)$ charge whose absolute value is equal to its mass parameter. The remarkable property of this solution is to describe the collisions of black holes. Hence, in order to understand the dynamical process of the collision between compact objects and the following formation of black holes without spherical symmetry, it is important to make both the local and global structures of this space-time clear.

After this space-time was discovered, Brill *et al.* [2–4] investigated, in part, the occurrence of the event horizon of this space-time and its global structure. The dynamics of marginally trapped surfaces in this space-time has been investigated by Nakao *et al.* [5] for the case of two black holes with an identical mass parameter. These researches revealed that the KT space-time really describes the merging process of black holes if the Abbott-Deser (AD) mass [6] of the system is smaller than or equal to the critical value, $M_* \equiv \sqrt{3/16\Lambda}$.

In this article, we numerically investigate the location of the event horizon of the KT space-time in the case of two black holes with an identical mass parameter by the ray-tracing method [7,8]. By this investigation, the evolution of the topology of the event horizon in this space-time is clarified.

Recently, numerical research has shown that a section of the event horizon by a spacelike hypersurface might have a nonspherical topology [7,9]. In connection with topological censorship about the event horizon [10,11], there were several theoretical studies of the topology of the event horizon [12–14]. Shapiro *et al.* [13] have shown that the numerical result given by Hughes *et al.* [7] does not conflict with the topological censorship theorem, using a simple flat space-time example.

More recently, several important features of the topology of event horizons were revealed by Siino [15]. This reference discussed the set composed of endpoints of null geodesic generators of an event horizon, which in this article will be called the *seam* of the event horizon. For an event horizon with trouser topology such as the one considered here, this reflects the common experience that trousers must have a seam. (One might call this a tailor's theorem.) A remarkable fact pointed out in Ref. [15] is that the seam is spacelike (which was pointed out in Ref. [17] in axisymmetric cases) and connected, with the assumption of a noneternal black hole. Similarly, one can easily comprehend that the spacelike nature holds even for an eternal black hole like the KT space-time. This means that one can choose a spacelike hypersurface which crosses the seam arbitrarily often. This leads to the further important consequence that the number of black holes, defined as the number of connected components of a section of an event horizon by a spacelike hypersurface, depends on the choice of the time slicing. In other words, the number of black holes is a gauge-dependent notion, in the case of general nonspherically symmetric space-times [16]. Thus, for example, the statement of *a two-black-hole collision* is not gauge invariant. We can imagine that two massive stars which are very distant from each other collapse into ''two black holes'' and later they coalesce with each other to form one final black hole. However, if we adopt certain time slicings, the picture of collapse is drastically altered: two massive stars collapse into a single highly deformed black hole (a spindle black hole). Furthermore, by adopting another kind of time slicing, where the seam repeatedly crosses the spacelike hypersurface, it is possible to regard this space-time as, for example, that of a fifteen-black-hole collision. Similarly, if we adopt a time slicing on which the two stars form *instantaneously* a single highly deformed

*Email address: ida@tap.scphys.kyoto-u.ac.jp
†Email address: nakao@tap.scphys.kyoto-u.ac.jp
‡Email address: siino@yukawa.scphys.kyoto-u.ac.jp
§Email address: hayward@yukawa.kyoto-u.ac.jp

black hole, the question arises whether the length scale of the black hole on this hypersurface is of the order of the Schwarzschild radius determined by its gravitational mass.

This question may be addressed in terms of the so-called *hoop conjecture* (HC) proposed by Thorne [18], which states that *black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is* $C \lesssim 4\pi M$ [19]. (In this article, we follow the notation of Misner, Thorne, and Wheeler [19] and adopt geometrized units). This is based on the physical consideration that one expects the formation of a black hole to occur when the matter or inhomogeneity of the space-time itself (nonlinear gravitational waves) is concentrated into a sufficiently ''small'' region.

Various research has supported the HC as catching an important physical feature of gravitational collapse, including work on the initial data for gravitational collapse [20–22] and on the dynamics of spindle gravitational collapses of collisionless matter [23]. The results of this paper suggest that the HC may also be applied to the case of head-on black-hole collisions.

Since there is a positive cosmological constant, Λ, in the KT space-time, this space-time is not asymptotically flat, but is asymptotically de Sitter. Although the hoop conjecture is usually proposed for an asymptotically flat space-time, if the mass scale of black holes is much smaller than the cosmological horizon scale, $\sqrt{3/\Lambda}$, this conjecture should hold also in an asymptotically de Sitter space-time. Furthermore, it has been suggested that the hoop conjecture holds even for the case of gravitational collapse with a large mass scale in such an asymptotically de Sitter space-time [24].

As mentioned, in this article we focus on the KT space-time of two black holes with an identical mass parameter. The line element and the electromagnetic potential one-form in the contracting cosmological chart are given in the form

$$ds^2 = -U^{-2}d\tau^2 + U^2(dx^2+dy^2+dz^2), \tag{1}$$

$$A = -U^{-1}d\tau, \tag{2}$$

where

$$U = -H\tau + \frac{m}{r_+} + \frac{m}{r_-}, \tag{3}$$

$$r_\pm = \sqrt{x^2+y^2+(z\mp a)^2}, \tag{4}$$

and H, m and a are constant. The constant H is related to the cosmological constant via $H=\sqrt{\Lambda/3}$. The above line element represents two point sources, each with mass parameter m and charge $|e|=m$, located at $\mathbf{x}=(0,0,\pm a)$ in the background Euclidean 3-space E^3. The AD mass M of this system is given by $2m$ which must be less than the critical value $M_* = 1/4H$, so that the space-time describes the merging of two black holes. Then the event horizon is located in the range $\tau<0$. Concerning what region of the space-time is covered by the above cosmological chart, see Refs. [2] and [5].

Here we should comment on the procedure to find the event horizon. The definition of the event horizon given by Hawking is the boundary of the causal past of future null infinity in the case of an asymptotically flat space-time [25]. For an asymptotically de Sitter space-time, the same definition is applicable by virtue of the existence of well-defined future null infinity [26]. An important consequence from the definition of the event horizon is that both in the asymptotically flat space-time and the asymptotically de Sitter one, the trapped regions are located inside event horizons. The method of Hughes *et al.* [7] is to solve large numbers of future directed null geodesics starting from each space-time point to various directions and then to investigate whether those null geodesics go into the trapped region or not. The starting points from which the null geodesics enter into the trapped region are placed inside the event horizon and the others are outside of it. On the other hand, Anninos *et al.* [8] adopt a different method to simplify the numerical calculation drastically. The latter method uses the effect that the event horizon will be the attractor of null hypersurfaces generated by past directed ingoing null geodesics near the event horizon, and hence the equations for large numbers of null geodesics going to various directions need not be integrated. Here we essentially follow the latter method in the sense that the event horizon is determined backward in time. Nevertheless, we do not treat the equation for the null hypersurface generated by past directed ingoing null geodesics, but directly solve the equations for the null geodesic generators themselves. In order to determine whether the null geodesic generators approach an attractor which will be the event horizon, we investigate the area of a section of a null hypersurface by the spacelike hypersurface labeled by τ, since the coordinate values of the null geodesic trajectories do not have any invariant meaning. More detailed analysis will be reported in our forthcoming work [27].

We numerically integrate equations for the null geodesics backward in time from a sufficiently late time $\tau=-\epsilon$, where ϵ is a sufficiently small positive parameter. In the contracting cosmological chart, (1), the marginally trapped surface enclosing both black holes goes to infinity, $r\equiv\sqrt{x^2+y^2+z^2}\rightarrow+\infty$, in the limit of $\epsilon\rightarrow 0$ [5]. Hence, at $\tau=-\epsilon$, the marginally trapped surface enclosing both holes is sufficiently spherical since the higher multipole moments are much smaller than the monopole term in the line element (1) in the limit of $r\rightarrow+\infty$. The vicinity of the marginally trapped surface is well approximated by one-black-hole solution, i.e., the Reissner–Nordström–de Sitter (RNdS) space-time [28]. Moreover, by this reason, the event horizon is very close to the marginally trapped surface enclosing both holes at $\tau=-\epsilon$. Therefore we integrate the equations for the past directed ingoing null geodesics normal to the marginally trapped sphere at $\tau=-\epsilon$, which is easily derived as

$$r = \frac{(1-\sqrt{1-8Hm})/2H - 2m}{H\epsilon}. \tag{5}$$

Here the ingoing direction is determined by the usual sense with respect to the radial coordinate r. The above expression agrees with the location of the event horizon in the RNdS

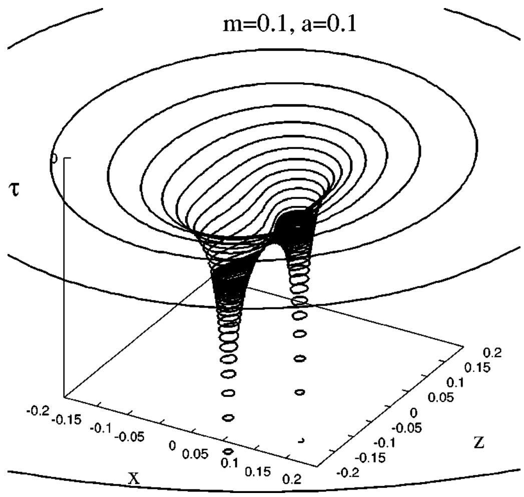

FIG. 1. The shape of the EH of the KT space-time with parameters $m=0.1$, $a=0.1$. A closed curve represents the section of the EH by the $\tau=$constant hypersurface.

solution with a mass parameter $2m$ [3]. If we choose sufficiently small ϵ, the deviation of the spacelike two-sphere defined by Eq. (5) from the event horizon should be smaller than the error due to the numerical integration.

Figure 1 depicts the time evolution of the event horizon in the contracting cosmological chart, (1), which shows the usual picture of a two black-hole collision. Adopting a normalization $H=1$, Fig. 1 corresponds to the case with parameters $m=0.1$ and $a=0.1$. Then the AD mass of this system is $M=0.8M_*$. As can be seen from this figure, the seam (set of past endpoints of the null geodesic generators) of the event horizon is a curve. In order to investigate whether the seam is spacelike or not, we have calculated the following quantity:

$$N=g_{\mu\nu}m^\mu m^\nu, \tag{6}$$

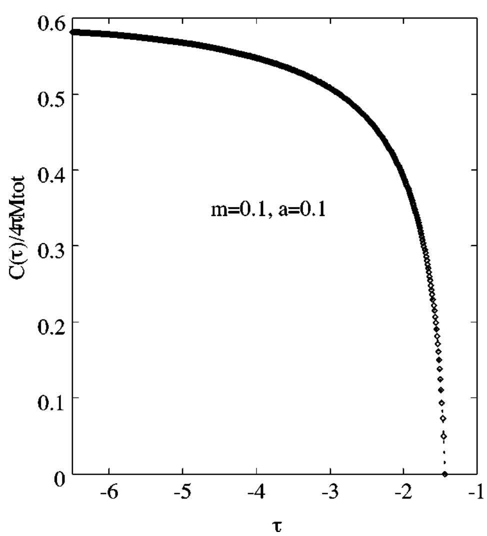

FIG. 2. $C(\tau):=2l(\tau)$ which is analogous to the circumference of the black hole is shown as a function of τ with parameters $m=0.1$, $a=0.1$.

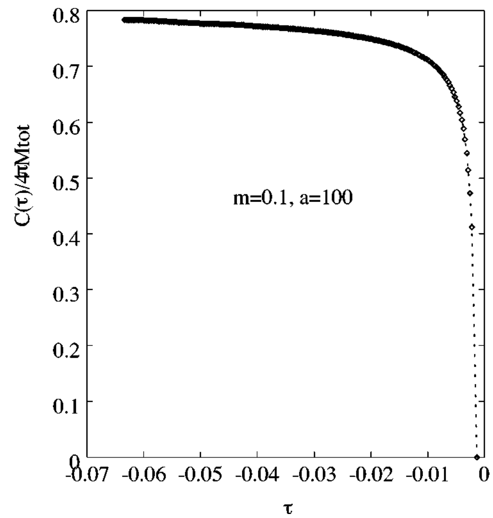

FIG. 3. Same to Fig. 3, but with a parameter $a=100$.

where $m^\mu=dx^\mu/ds$ is the tangent vector of the seam, and it has been confirmed that N is always positive, i.e., the seam is a spacelike curve, as expected from Ref. [15]. Then we calculate the proper length

$$l(\tau')=\int_{\tau>\tau'} N^{1/2}ds \tag{7}$$

of the seam from a given time. The result is shown in Fig. 2. It can be seen that the length of the seam converges to a finite value of order M. It is difficult to determine the limit precisely, since numerical instability eventually sets in. Figure 3 shows a similar result in the case $a=100$, with the other parameters as for Fig. 2. In fact, we obtained similar results for combinations of parameters $H=1$, $M/M_*=0.01$, 0.25, 0.5, 0.8, 0.99 and $a=0.01$, 1.0, 100, so that the property appears to be universal. More details will be given in our forthcoming work [27].

Since the black holes diverge in the past, their separation in the constant-τ hypersurfaces becoming infinite as $\tau\to-\infty$, it was unexpected that the length of the seam converges. Were this not so, however, a counterexample to the HC could be given, as follows. Since the seam is a spacelike

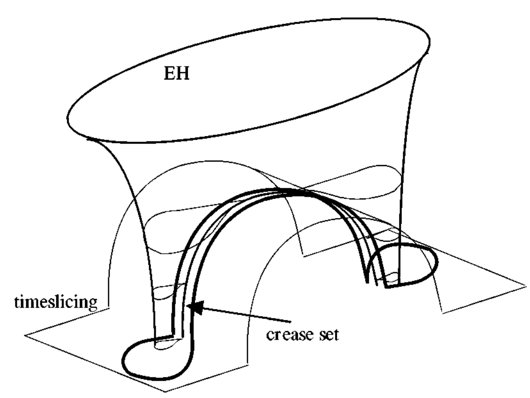

FIG. 4. Since the crease set is a spacelike curve, it may be possible to define the circumference of the black hole as the length of the crease set by taking a time slicing like this figure.

curve, we can always adopt a time slicing which includes a spacelike hypersurface as shown in Fig. 4. The shape of a section of the event horizon by that spacelike hypersurface is like a dumb-bell, with a long thin tube connecting two parts which, in a slightly earlier time slice, would be two disjoint, widely separated components of the event horizon. If this separation could be arbitrarily large for given parameters, this would be a clear violation of the HC for any reasonable definition of circumference of the event horizon.

Remarkably, however, the length is constrained in accordance with the HC. More precisely, we suggest defining the minimal circumference of the event horizon by

$$C = \lim_{\tau \to -\infty} 2l(\tau) \tag{8}$$

which is the limiting circumference of a loop shrunk around the seam. We imagine that this will be a lower bound for the circumference of hoops in the event horizon which encircle both legs. This is difficult to formulate precisely, reflecting the main problem in formulating the hoop conjecture anyway, the definition of circumference [29]. We simply take the above definition of C as the relevant length scale.

In conclusion, our results confirm the inequality

$$C < 4\pi M \tag{9}$$

supposed by the HC. Moreover, the two scales are of the same order,

$$C \sim 4\pi M. \tag{10}$$

Thus one side of the HC is confirmed exactly, while the other side is not exact, but still a rough guide. Even this is remarkable given that we are applying a severe test of the HC in the case of head-on black-hole collisions.

We are grateful H. Sato for his encouraging discussion. This work was partly supported by the Grant-in-Aid for Creative Basic Research (09NP0801) from the Japanese Ministry of Education, Science, Sports and Culture.

[1] D. Kastor and J. Traschen, Phys. Rev. D **47**, 5370 (1993).

[2] D. R. Brill, G. T. Horowitz, D. Kastor, and J. Traschen, Phys. Rev. D **49**, 840 (1994).

[3] D. R. Brill and S. A. Hayward, Class. Quantum Grav. **11**, 359 (1994).

[4] D. R. Brill, in *Proceedings of Global structure and evolution in general relativity* (Springer, Berlin, 1996).

[5] K. Nakao, T. Shiromizu, and S. A. Hayward, Phys. Rev. D **52**, 796 (1995).

[6] L. Abbott and S. Deser, Nucl. Phys. **B195**, 76 (1982).

[7] S. A. Hughes *et al.*, Phys. Rev. D **49**, 4004 (1994).

[8] P. Anninos *et al.*, Phys. Rev. Lett. **74**, 630 (1995).

[9] A. M. Abrahams, G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D **49**, 5153 (1994).

[10] D. Gannon, J. Math. Phys. **16**, 2364 (1975).

[11] J. L. Friedman, K. Schleich, and D. M. Witt, Phys. Rev. Lett. **71**, 1486 (1993); **75**, 1872(E) (1995).

[12] T. Jacobson and S. Venkatarmani, Class. Quantum Grav. **12**, 1055 (1995).

[13] S. L. Shapiro, S. A. Teukolsky, and J. Winicour, Phys. Rev. D **52**, 6982 (1995).

[14] G. Galloway and E. Woolgar, Class. Quantum Grav. **14**, L1 (1997).

[15] M. Siino, Phys. Rev. D **58**, 104016 (1998).

[16] M. Siino, Kyoto University Report No. KUNS-1461, gr-qc/9709015, 1997.

[17] R. A. Matzner, H. E. Seidel, S. L. Shapiro, L. Smarr, W.-M. Suen, S. A. Teukolsky, and J. Winicour, Science **270**, 941 (1995).

[18] K. S. Thorne, in *Magic Without Magic; John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972), p. 231.

[19] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 867.

[20] T. Nakamura, S. L. Shapiro, and S. A. Teukolsky, Phys. Rev. D **38**, 2972 (1988).

[21] J. Wojtkiewicz, Phys. Rev. D **41**, 1867 (1990); Class. Quantum Grav. **9**, 1255 (1992).

[22] T. Chiba, T. Nakamura, K. Nakao, and M. Sasaki, Class. Quantum Grav. **11**, 431 (1994).

[23] S. L. Shapiro and S. A. Teukolsky, Phys. Rev. Lett. **66**, 994 (1991); Phys. Rev. D **45**, 2006 (1992).

[24] T. Chiba and K. Maeda, Phys. Rev. D **50**, 4903 (1994).

[25] S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972).

[26] T. Shiromizu, K. Nakao, H. Kodama, and K. Maeda, Phys. Rev. D **47**, R3099 (1993); In the case of the KT solution, the maximal extension is not analytic except for the very special cases [2]. Hence, the future null infinity is rather arbitrary, but as Brill pointed out, there is another manner to define the event horizon by the use of a set of future complete timelike curves [4].

[27] D. Ida, K. Nakao, M. Siino, and S. A. Hayward (in preparation).

[28] B. Carter, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (New York, Gordon and Breach, 1973).

[29] E. Flanagan, Phys. Rev. D **44**, 2409 (1991).