Padé improvement of QCD running coupling constants, running masses, Higgs decay rates, and scalar channel sum rules

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We discuss Padé improvement of known four-loop order results based upon an asymptotic three-parameter error formula for Padé approximants. We derive an explicit formula estimating the next-order coefficient R_4 from the previous coefficients in a series $1+R_1x+R_2x^2+R_3x^3$. We show that such an estimate is within 0.18% of the known five-loop order term in the β function for single-component scalar field theory, and within 10% of the known five-loop term in the corresponding anomalous mass-dimension function $\gamma_m(g)$ for scalar field theory. We apply the same formula to generate a [2|2] Padé summation of the QCD β function and anomalous mass dimension in order to demonstrate both the relative insensitivity of the evolution of $\alpha_s(\mu)$ and the running quark masses to higher order corrections, as well as a somewhat increased compatibility of the present empirical range for $\alpha_s(m_\tau)$ with the range anticipated via evolution from the present empirical range for $\alpha_s(M_z)$. For $3 \leq n_f \leq 6$ we demonstrate that positive zeros of any [2|2] Padé-summation estimate of the all-orders β function which incorporates known two-, three-, and four-loop contributions necessarily correspond to *ultraviolet* fixed points, *regardless* of the unknown five-loop term. Padé improvement of higher-order perturbative expressions is presented for the decay rates of the Higgs boson into two gluons and into a $b\bar{b}$ pair, and is used to show the relative insensitivity of these rates to higher order effects. However, Padé improvement of the purely perturbative component of scalar/pseudoscalar current correlation functions is indicative of large theoretical uncertainties in QCD sum rules for these channels, particularly if the continuum-threshold parameter s_0 is near 1 GeV^2. [S0556-2821(98)03021-5]

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I. INTRODUCTION

A recent body of work [1–3] has demonstrated how higher-order terms in a number of field-theoretical perturbative series can be estimated by Padé-approximant techniques. Of particular interest are applications to QCD quantities, particularly β and γ functions now known to four-loop order in α_s [4–6]. Padé-approximant methods have already addressed the n_f (flavor-number) dependence of higher order terms in the QCD β and γ functions [2,3]. Near cancellations between coefficients of successive powers of n_f, however, can lead to a large uncertainty in the estimated overall size of such higher-loop contributions—small errors in fitted coefficients of n_f^k have been seen to lead to much larger errors in the aggregate (now known) four-loop contribution to the β function [2].

In the present paper, our focus will be on using Padé-approximant methods to estimate the magnitude of higher-order corrections to quantities already calculated to three-and four-loop order in QCD. We assess the theoretical uncertainty of such calculations by seeing how closely they coincide with their own Padé improvements, as well as whether successive orders of perturbation theory exhibit convergence toward Padé-summation estimates of the full perturbative series.

The present paper is phenomenologically oriented, specifically aimed at developing Padé-improved estimates of what are hoped to be computationally and/or experimentally accessible quantities. The particular items of interest considered are the β function and anomalous mass dimension for massive $O(N)$-symmetric scalar field theory (Sec. II), the running QCD coupling constant (Sec. III), the running quark masses (Sec. IV), the Higgs-boson decay rates into two gluons and into a $b\bar{b}$ pair (Sec. V), and the purely perturbative content of QCD-sum rules based upon scalar and pseudoscalar current correlation functions (Sec. VI). In Sec. III, we also discuss some general implications of Padé summation as an approximation to all orders of perturbation theory. In particular, we analyze the fixed point structure of the most general [2|2] Padé-summation estimates of the full content of QCD β functions for $n_f=\{3,4,5,6\}$ whose Maclaurin expansions coincide with presently known perturbative contributions.

In the section that immediately follows, we discuss how Padé-approximant methods can be used to estimate higher-order corrections to perturbative series in which the leading four terms are known. Although this methodology appears in other work [3], the presentation leading to Eq. (2.12) [which has not appeared elsewhere] will, it is hoped, be of some value to those unfamiliar with Padé-approximant methods. We also demonstrate the reasonable agreement of predictions based on Eq. (2.12) with now-known five-loop terms in β and γ functions for $O(N)$-symmetric scalar field theory.

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II. THE APAP ALGORITHM: PADÉ-IMPROVEMENT FOR PEDESTRIANS

We consider the general problem of developing a Padé improvement of the series

$$S \equiv 1 + R_1 x + R_2 x^2 + R_3 x^3 + \dots, \tag{2.1}$$

where $\{R_1, R_2, R_3\}$ are known and $\{R_4, R_5, \dots\}$ are not known, through use of the asymptotic error formula for estimating $R_{N+M+1} x^{N+M+1}$ via the Padé approximant

$$\begin{aligned} S_{[N|M]} &\equiv \frac{1 + a_1 x + a_2 x^2 + \dots a_N x^N}{1 + b_1 x + b_2 x^2 + \dots + b_M x^M} \\ &= 1 + R_1 x + R_2 x^2 + R_3 x^3 + \dots \\ &\quad + R_{N+M+1} x^{N+M+1} + \dots \; . \end{aligned} \tag{2.2}$$

Let $R_{N+M+1}^{\text{Padé}}$ be the prediction one would obtain from the $[N|M]$ Padé approximant, and let R_{N+M+1} be the true value of the coefficient. The structure of the asymptotic error formula is given by [2]

$$\frac{R_{N+M+1}^{\text{Padé}} - R_{N+M+1}}{R_{N+M+1}} = -\frac{M! A^M}{[N + M + aM + b]^M} \tag{2.3}$$

with numbers $\{A, a, b\}$ (independent of N, M) to be determined.

This error formula simplifies considerably if M is always chosen to be 1 [3]: the right-hand side of Eq. (2.3) becomes $-A/[N + 1 + (a+b)]$, with only the two numbers $\{A, a+b\}$ to be determined. These can be determined explicitly for the series S in Eq. (2.1) given knowledge of the three coefficients $\{R_1, R_2, R_3\}$.

Given knowledge of R_1 only, the $[0|1]$ Padé approximant

$$S_{[0|1]} = \frac{1}{1 + b_1 x} = 1 - b_1 x + b_1^2 x^2 \dots = 1 + R_1 x + R_2^{\text{Padé}} x^2 \tag{2.4}$$

predicts $R_2^{\text{Padé}} = R_1^2$. Consequently, we see from the asymptotic error formula (2.3) that

$$\frac{R_2^{\text{Padé}} - R_2}{R_2} = \frac{R_1^2 - R_2}{R_2} = \frac{-A}{1 + (a+b)} \equiv \delta_2 . \tag{2.5}$$

Given knowledge of only R_1 and R_2, the $[1|1]$ Padé approximant

$$\begin{aligned} S_{[1|1]} &= \frac{1 + a_1 x}{1 + b_1 x} = 1 + (a_1 - b_1) x + b_1 (b_1 - a_1) x^2 \\ &\quad + b_1^2 (a_1 - b_1) x^3 + \dots \\ &= 1 + R_1 x + R_2 x^2 + R_3^{\text{Padé}} x^3 \dots \end{aligned} \tag{2.6}$$

predicts $R_3^{\text{Padé}} = R_2^2 / R_1$. The asymptotic error formula (2.3) for this case implies that

$$\frac{R_2^2/R_1 - R_3}{R_3} = \frac{-A}{2 + (a+b)} \equiv \delta_3 . \tag{2.7}$$

Given knowledge of $\{R_1, R_2, R_3\}$ in the series (2.1), the relative errors δ_2 and δ_3 are specified completely by the left-hand sides of Eqs. (2.5) and (2.7). These two equations may be regarded as two equations in the two unknowns A and $(a+b)$ characterizing the asymptotic error formula (2.3) when $M = 1$. The solution to these two equations is

$$A = [1/\delta_2 - 1/\delta_3]^{-1}, \tag{2.8}$$

$$(a+b) = \frac{\delta_2 - 2\delta_3}{\delta_3 - \delta_2}. \tag{2.9}$$

This information is sufficient to generate an asymptotic Padé approximant (APAP) estimate [2,3] of the unknown coefficient R_4 in the series (2.1). Consider the $[2|1]$ Padé approximant

$$\begin{aligned} S_{[2|1]} &= \frac{1 + a_1 x + a_2 x^2}{1 + b_1 x} \\ &= 1 + (a_1 - b_1) x + [a_2 - b_1 (a_1 - b_1)] x^2 \\ &\quad + [-b_1 [a_2 - b_1 (a_1 - b_1)]] x^3 \\ &\quad + [b_1^2 [a_2 - b_1 (a_1 - b_1)]] x^4 + \dots \\ &= 1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4^{\text{Padé}} x^4 . \end{aligned} \tag{2.10}$$

The three known values of $\{R_1, R_2, R_3\}$ completely determine the three parameters $\{a_1, a_2, b_1\}$ characterizing $S_{[2|1]}$. We see from Eq. (2.10) that $R_4^{\text{Padé}} = R_3^2 / R_2$. However, the asymptotic error formula (2.3) suggests that a more accurate estimate of the true value R_4 differs from $R_4^{\text{Padé}}$ by a predictable relative error:

$$\frac{R_3^2/R_2 - R_4}{R_4} \equiv \delta_4 = \frac{-A}{3 + (a+b)}, \tag{2.11}$$

in which case we find from Eqs. (2.11), (2.8), and (2.9) that

$$\begin{aligned} R_4 &= \frac{R_3^2/R_2}{1 + \delta_4} \\ &= \frac{R_3^2 (\delta_3 - 2\delta_2)}{R_2 (\delta_3 - 2\delta_2 - \delta_2 \delta_3)} \\ &= \frac{R_3^2 (R_2^3 + R_1 R_2 R_3 - 2 R_1^3 R_3)}{R_2 (2 R_2^3 - R_1^3 R_3 - R_1^2 R_2^2)} . \end{aligned} \tag{2.12}$$

As an example, we test the applicability of Eq. (2.12) by comparing its prediction to the known $\mathcal{O}(g^6)$ coefficient of the β function for single-component massive ϕ^4 scalar field theory [7]:

$$\begin{aligned} \beta^{(1)}(g) &= 1.5 g^2 [1 - (17/9) g + 10.8499 g^2 \\ &\quad - 90.5353 g^3 + 949.523 g^4 + \mathcal{O}(g^5)] . \end{aligned} \tag{2.13}$$

TABLE I. Comparison of β-function and γ-function coefficients for $O(N)$-symmetric scalar field theory obtained via Padé estimates to those obtained via exact calculation. The ''β_4 via Eq. (2.12)'' column displays estimates obtained via the APAP algorithm (2.12), based on β_{0-3} listed in [7]. The ''β_4^{true}'' column displays the results for β_4 obtained in [7] by explicit calculation. The ''β_4 via [2]'' column lists the Padé estimates given in Table 3 of Ref. [2], involving knowledge of the N^4 dependence and a fit of the overall N dependence of β_4, as well as a simplified asymptotic error formula. The ''(γ_4/γ_0) via Eq. (2.12)'' column displays estimates of γ_4/γ_0 in the anomalous mass dimension γ function obtained via the APAP algorithm (2.12), based on prior coefficients listed in [7]. The final column, ''$(\gamma_4/\gamma_0)^{true}$,'' are coefficients explicitly calculated in [7].

	β_4 via Eq. (2.12)	β_4^{true}	β_4 via [2]	(γ_4/γ_0) via Eq. (2.12)	$(\gamma_4/\gamma_0)^{true}$
$N=1$	1421.7	1424.3	1432.3	135.1	150.76
$N=2$	1941.7	1922.3	1943.8	168.4	191.89
$N=3$	2555.9	2499.3	2540.3	203.1	236.94
$N=4$	3267.9	3158.8	3225.6	239.2	285.94

Identifying $R_1=-17/9$, $R_2=10.8499$ and $R_3=-90.5353$, we find from Eq. (2.12) that $R_4=947.8$ in startlingly close agreement to the next (g^4) term within Eq. (2.13). Although APAP improvement has also been applied elsewhere [2] to $\mathcal{O}(g^6)$ terms in $O(N)$ scalar field theory β functions, the result obtained here relies on direct and explicit use of the full asymptotic error formula (2.3). A comparison of Eq. (2.12) predictions and exact values of β_4, the $O(N)$ scalar field theory β-function coefficient of g^6, is presented in Table I. We emphasize that these predictions are *not* obtained by a fitting of the N dependence or any knowledge of the N^4 dependence of $\beta^{(4)}$, as is the case in Table 3 of Ref. [2]; for comparative purposes, the predictions of Ref. [2] are also listed in Table I.

We can use Eq. (2.12) to predict the known R_4 coefficient within the $N=1$ scalar field theory's anomalous mass dimension [7], as well:

$$\gamma_m(g)=(g/2)[1-0.8333g+3.500g^2 -19.96g^3+150.8g^4+\mathcal{O}(g^5)]. \tag{2.14}$$

Equation (2.12) predicts $R_4=135.1$, a result only 10% off the 150.8 value given in Eq. (2.14). Table I shows that predicted R_4 coefficients for $\gamma_m(g)$ within $O(2)$, $O(3)$, and $O(4)$-symmetric cases also remain within 20% of their true values, as given in [7]. These results provide a reasonable basis for applying Eq. (2.12) [and its concomitant asymptotic error formula (2.3)] to the β and γ functions of QCD, as we will do in Secs. III and IV.

A final improvement of the series (2.1) is possible by expressing this series as a [2|2] diagonal approximant—this is a more accurate representation of the infinite series S than one would obtain by arbitrarily truncating the series after the R_4x^4 term. Given known values of $\{R_1,R_2,R_3\}$ and using the APAP estimate (2.12) for R_4, the approximant $S_{[2|2]}$ of the infinite series S is fully determined:

$$S\rightarrow S_{[2|2]}=\frac{1+a_1x+a_2x^2}{1+b_1x+b_2x^2}, \tag{2.15}$$

$$b_1=\frac{R_1R_4-R_2R_3}{R_2^2-R_1R_3}, \tag{2.16}$$

$$b_2=\frac{R_3^2-R_2R_4}{R_2^2-R_1R_3}, \tag{2.17}$$

$$a_1=R_1+b_1, \tag{2.18}$$

$$a_2=R_2+b_1R_1+b_2. \tag{2.19}$$

Equation (2.15), as determined from Eqs. (2.12) and (2.16)–(2.19), constitutes the procedure we denote as ''Padé-summation'' of the series S in Eq. (2.1).

III. PADÉ-IMPROVEMENT OF THE QCD COUPLING

The QCD minimal subtraction (MS) or modified MS ($\overline{\text{MS}}$) renormalization-group functions $\beta(x)$ and $\gamma(x)$ are now known to four-loop order [4,5,6]. Prior work involving Padé-improvement methods has attempted to predict the flavor dependence of these functions to four- and five-loop order [2,3]. $\beta_3^{(n_f)}$ and $\beta_4^{(n_f)}$, the four- and five-loop order corrections to the β function, are respectively third- and fourth-degree polynomials in n_f, and Padé methods have already shown some success in predicting the polynomial coefficients now known for $\beta_3^{(n_f)}$ [2]. However, an accurate determination of the polynomial coefficients within β_3 is not reflected in the accuracy with which β_3 is itself determined. Thus the overall Padé-driven estimate of β_3 presented in [2] for $n_f=3$ [$\beta_3^{(3)}=(7.6\pm0.1)\times10^3/256=30\pm1$, using normalization conventions appropriate for Eq. (3.1) below] is substantially below the true value ($\beta_3^{(3)}=47.23$), even after allowances are made for claimed uncertainties arising from quadratic Casimir contributions [3]. It should also be noted that this estimate, once disentangled from a fitting procedure aimed at ascertaining the explicit n_f dependence of $\beta_3^{(n_f)}$, follows from a simplified version of the asymptotic error formula (2.3), in which the denominator $(N+M+aM+b)^M$ is taken to be N^M [2]. Indeed, it is difficult to tell at this stage whether the discrepancy between Padé estimates of β_3 and the true value arises primarily from quadratic-Casimir contributions not occurring in lower orders, or alternatively, arises from the error involved in simplifying the asymptotic error formula in order to make an estimate of β_3

possible. Without such a simplification, the error formula (2.3) has (in principle) three arbitrary constants (A,a,b) instead of one (A).

Consequently, in this section we will predict β_4 directly using the APAP algorithm (2.12) following from the full asymptotic error formula (2.3). We will not prejudice these predictions of β_4 by attempting a fit of the polynomial dependence on n_f, nor will we attempt to disentangle quadratic-and-higher Casimir contributions from β_3 and β_4. Rather, we will make distinct predictions of $\beta_4^{(n_f)}$ via Eq. (2.12) for $n_f=3,4,5,6$. The validity of such an approach, particularly the possibility that the full asymptotic error formula is inclusive of higher-order Casimir contributions to $\beta(x)$, would be best established by comparison to an exact calculation of $\beta_4^{(n_f)}$, when available.

We define the β function as in [4]:

$$\mu^2 \frac{d}{d\mu^2} x = -x^2 \sum_{i=0}^{\infty} \beta_i x^i$$

$$= -\beta_0 x^2 \sum_{i=0}^{\infty} R_i x^i, \tag{3.1a}$$

with $x \equiv \alpha_s(\mu)/\pi$ and

$$R_i \equiv \beta_i/\beta_0. \tag{3.1b}$$

Known values of $\beta_0-\beta_3$ are then found to be [5]

$$\beta_0^{(n_f)} = (11-2n_f/3)/4, \tag{3.2a}$$

$$\beta_1^{(n_f)} = (102-38n_f/3)/16, \tag{3.2b}$$

$$\beta_2^{(n_f)} = (2857/2-5033n_f/18+325n_f^5/54)/64, \tag{3.2c}$$

$$\beta_3^{(n_f)} = 114.23033-27.133944n_f+1.5823791n_f^2 + 5.85669582\times 10^{-3} n_f^3. \tag{3.2d}$$

We use Eqs. (3.2), (3.1b), and (2.12) to predict the following values for β_4:

$$n_f=3: \quad R_4=-849.74, \ \beta_4^{(3)}=-1911.9; \tag{3.3}$$

$$n_f=4: \quad R_4=40.203, \ \beta_4^{(4)}=83.7563; \tag{3.4}$$

$$n_f=5: \quad R_4=70.203, \ \beta_4^{(5)}=134.56; \tag{3.5}$$

$$n_f=6: \quad R_4=-239.22, \ \beta_4^{(6)}=-418.64. \tag{3.6}$$

The large negative values for $n_f=3$ and $n_f=6$ reflect the near cancellation of the factor $(2R_2^3-R_1^3R_3-R_1^2R_2^2)$ in the denominator of Eq. (2.12). Since a change of sign can easily occur if this cancellation is over- or underestimated, the safest interpretation of Eqs. (3.3) and (3.6) is to predict a relatively large magnitude for β_4, with the sign uncertain.

Corresponding results from Table III of Ref. [3] with quadratic Casimir contributions and a 1/1024 normalization factor appropriate to Eq. (3.1a) included are $\beta_4=\{278(n_f=3),202(n_f=4),165(n_f=5),166(n_f=6)\}$. As stated earlier, these latter results are based upon a fit to the polynomial coefficients of n_f^k in β_4, with near cancellation of very large opposite-sign coefficients for the $k=0$ and $k=1$ terms. These latter results appear most consistent with those obtained above when $n_f=5$.

To get a feeling of the magnitude of these Padé estimates of five-loop effects, we generate the [2|2] approximant (2.15) from the known values for R_1, R_2, and R_3 and our estimate of R_4, and incorporate this approximant directly into the β function $[\mu^2 dx/d\mu^2 \equiv \beta(x)]$:

$$n_f=3: \quad \beta^{(3)}(x) = -\frac{9x^2}{4}\left[\frac{1+94.383x-75.605x^2}{1+92.606x-244.71x^2}\right], \tag{3.7a}$$

$$n_f=4: \quad \beta^{(4)}(x) = -\frac{25x^2}{12}\left[\frac{1-5.8963x-4.0110x^2}{1-7.4363x+4.3932x^2}\right], \tag{3.7b}$$

$$n_f=5: \quad \beta^{(5)}(x) = -\frac{23x^2}{12}\left[\frac{1-5.9761x-6.9861x^2}{1-7.2369x-0.66390x^2}\right]. \tag{3.7c}$$

We can use these Padé approximations to the full β function to evolve $\alpha_s^{(n_f)}(\mu)=\pi x^{(n_f)}(\mu)$ down to $\mu=1$ GeV from an initial condition $\alpha_s^{(5)}(M_z)=0.118$ [8] through use of the following (four- and five-) flavor threshold matching conditions with $m(\mu_{th})=\mu_{th}$ [9]:

$$x^{(n_f-1)}(\mu_{th}) = x^{(n_f)}(\mu_{th})[1+0.1528(x^{(n_f)}(\mu_{th}))^2+\{0.9721-0.0847(n_f-1)\}(x^{(n_f)}(\mu_{th}))^3]. \tag{3.8}$$

In Fig. 1 we display the evolution of $\alpha_s(\mu)$ to low energies through use of two-loop, three-loop, four-loop, and the Padé-improved four-loop β functions of Eq. (3.7). All of these curves are generated from the initial condition $\alpha_s^{(5)}(M_z)=0.118$, with $\mu_{th}=4.3$ GeV [i.e., $m_b(m_b)=4.3$ GeV] identified as the $n_f=5$ flavor threshold, and with

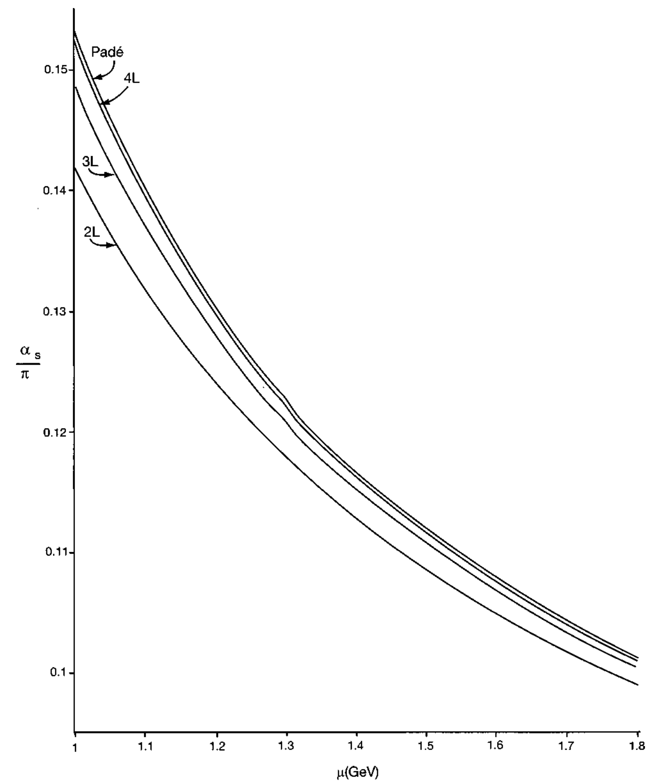

FIG. 1. A comparison of the evolution of $x(\mu)\equiv\alpha_s(\mu)/\pi$ for $m_\tau \geqslant \mu \geqslant 1$ GeV obtained from the evolution of two-loop (2L), three-loop (3L), four-loop (4L), and [2|2] Padé-summation β functions. All curves are generated from the initial condition $\alpha_s(M_z)=0.118$, with five-and four-flavor threshold matchings occurring at 4.3 GeV and 1.3 GeV, respectively.

$\mu_{th}=1.3$ GeV identified as the $n_f=4$ flavor threshold. Equation (3.8) is utilized in full in both the four-loop and Padé-improved calculations; it is utilized to $\mathcal{O}(x^2)$ to generate flavor-threshold initial conditions in the three-loop calculation, and the matching condition is trivial for the two-loop calculation. It is evident from the figure that curves from successive orders of the β function appear to converge from below to that generated via Eq. (3.7), the Padé summation approximating all orders. The gaps between curves of successive order clearly narrow as the order increases. Figure 1 shows that the Padé summation leads to a curve for $\alpha_s(\mu)$ that exceeds the unimproved four-loop curve by less than 1%. Such a difference is inconsequential compared to the estimated uncertainties in $\alpha_s(M_z)$ and μ_{th} for four- and five-flavor thresholds. Both the four-loop and the Padé-improved curve are indicative of benchmark values $\alpha_s(1\ \text{GeV})=0.48$ and $\alpha_s(m_\tau)=0.32$. However, if the flavor thresholds and $\alpha_s(M_z)$ are assigned their accepted [8] lower-bound values $[\alpha_s^{(5)}(M_z)=0.115,\ \mu_{th}^{(n_f=5)}=4.1\ \text{GeV},\ \mu_{th}^{(n_f=4)}=1.0\ \text{GeV}]$, the values we obtain at $\mu=1$ GeV and $\mu=m_\tau$ from either four-loop or Padé-summation β functions are $\alpha_s^{(4)}(1\ \text{GeV})=0.41$ and $\alpha_s^{(4)}(m_\tau)=0.29$. Corresponding upper-bound values $[\alpha_s^{(5)}(M_z)=0.121,\ \mu_{th}^{(n_f=5)}=4.5\ \text{GeV},\ \mu_{th}^{(n_f=4)}=1.6\ \text{GeV}]$ lead to $\alpha_s^{(3)}(1\ \text{GeV})=0.57$ and $\alpha_s^{(4)}(m_\tau)=0.35$. In view of these (much-) larger-than-1% uncertainties in low-energy values, the best possible test at present of Padé improvement would be nonempirical, i.e., a comparison to an explicit five-loop *calculation* of the β function.

We note, however, that higher-order effects do appear to increase the overlap between the present (somewhat large) empirical range for $\alpha_s(m_\tau)$ (0.370 ± 0.033 [8]) and the range predicted via evolution down from the present empirical range for $\alpha_s(M_z)$ (0.118 ± 0.003 [8]). Taking into account the present uncertainty in the five-flavor threshold ($\mu_{th}=4.3\pm0.3$ GeV [8]) and incorporating the matching condition (3.8) for $\alpha_s(\mu)$ below and above μ_{th}, we find the following *predicted* ranges for $\alpha_s(m_\tau)$ for two-loop, three-loop, four-loop and Padé-summation β functions:

Two-loop:

$$0.2910\leqslant\alpha_s(m_\tau)\leqslant0.3391, \tag{3.9a}$$

$$[\alpha_s(m_\tau)]_{cv}=0.3137, \tag{3.9b}$$

Three-loop:

$$0.2944\leqslant\alpha_s(m_\tau)\leqslant0.3451, \tag{3.10a}$$

$$[\alpha_s(m_\tau)]_{cv}=0.3182, \tag{3.10b}$$

Four-loop:

$$0.2957\leqslant\alpha_s(m_\tau)\leqslant0.3477, \tag{3.11a}$$

$$[\alpha_s(m_\tau)]_{cv}=0.3200, \tag{3.11b}$$

Padé-improved:

$$0.2963\leqslant\alpha_s(m_\tau)\leqslant0.3489, \tag{3.12a}$$

$$[\alpha_s(m_\tau)]_{cv}=0.3208. \tag{3.12b}$$

The ranges listed above progressively overlap the low end of the present experimental range $0.337\leqslant\alpha_s(m_\tau)\leqslant0.403$. The central values (*cv*) displayed above are evolved down from $\alpha_s(M_z)=0.118$ with a five-flavor threshold at $\mu_{th}=4.3$ GeV. The lower bounds evolve from $\alpha_s(M_z)=0.115$ with $\mu_{th}=4.0$ GeV, and the upper bounds evolve from $\alpha_s(M_z)=0.121$ with $\mu_{th}=4.6$ GeV.

There are also some unexpected theoretical consequences arising from estimating the summation of the full β-function series through use of an appropriately chosen Padé approximant [1]. The most general [2|2] approximant must yield $1+R_1x+R_2x^2+R_3x^3+R_4x^4$ as the first five terms of its Maclaurin expansion. For optimal generality, we require that R_1, R_2, and R_3 be given by Eqs. (3.1b) and (3.2), but allow R_4 to be *arbitrary*. We then find that

$$\beta(x)=-\beta_0x^2\left[\frac{1+a_1x+a_2x^2}{1+b_1x+b_2x^2}\right], \tag{3.13}$$

with $\{a_1,a_2,b_1,b_2\}$ linear in R_4 as follows:
$n_f=3$:

$$a_1 = 7.1945 - 0.10261 R_4, \tag{3.14a}$$

$$a_2 = -11.329 + 0.075643 R_4, \tag{3.14b}$$

$$b_1 = 5.4168 - 0.10261 R_4, \tag{3.14c}$$

$$b_2 = -25.430 + 0.25806 R_4; \tag{3.14d}$$

$n_f = 4$:

$$a_1 = 4.8401 - 0.11068 R_4, \tag{3.15a}$$

$$a_2 = -8.1842 + 0.10836 R_4, \tag{3.15b}$$

$$b_1 = 3.3001 - 0.11068 R_4, \tag{3.15c}$$

$$b_2 = -16.314 + 0.21904 R_4; \tag{3.15d}$$

$n_f = 5$:

$$a_1 = 2.6793 - 0.12329 R_4, \tag{3.16a}$$

$$a_2 = -6.19671 - 0.011245 R_4, \tag{3.16b}$$

$$b_1 = 1.4184 - 0.12329 R_4, \tag{3.16c}$$

$$b_2 = -9.4599 + 0.14421 R_4; \tag{3.16d}$$

$n_f = 6$:

$$a_1 = 0.61085 - 0.18424 R_4, \tag{3.17a}$$

$$a_2 = -6.6275 - 0.24181 R_4, \tag{3.17b}$$

$$b_1 = -0.31772 - 0.18424 R_4, \tag{3.17c}$$

$$b_2 = -6.0423 - 0.057574 R_4. \tag{3.17d}$$

For all n_f values listed above, the first positive zero of $1 + a_1 x + a_2 x^2$ in Eq. (3.13) is found to be above the first positive zero of $1 + b_1 x + b_2 x^2$, *regardless* of the choice for R_4. Consequently, the smallest positive zero of $\beta(x)$, *if given by Eq. (3.13),* is necessarily an ultraviolet fixed point and *not* an infrared fixed point, an inescapable result of the denominator sign change for x between 0 and the first positive zero of Eq. (3.13). Moreover, for those values of R_4 for which a second positive zero of $1 + a_1 x + a_2 x^2$ is possible, we find from Eqs. (3.14)–(3.17) that a second positive zero of $1 + b_1 x + b_2 x^2$ will also occur at some value of x between the two positive zeros of $1 + a_1 x + a_2 x^2$. This ensures that *neither* positive zero of $1 + a_1 x + a_2 x^2$ corresponds to an infrared fixed point.

In Fig. 2, a schematic diagram is presented showing different branches for the evolution of $x(\mu)$ anticipated from a β function (3.13) with the above-described alternation of positive denominator and numerator zeros, with the smallest positive zero occurring for the denominator. Zeros of $1 + b_1 x + b_2 x^2$ represent values of x for which $x(\mu)$ has infinite slope. Zeros of $1 + a_1 x + a_2 x^2$ are fixed-point values of x. As is evident from the figure, all such fixed points are necessarily ultraviolet, and the infrared region is *inaccessible* for values of μ less than those corresponding to zeros of the denominator. Such behavior suggests (1) the possible existence of a strong phase of QCD at short distances, reflective of a nonzero *ultraviolet* fixed point, and (2) the inapplicability of a perturbative theory of quarks and gluons to the infrared region, specifically the region excluded from the domain of the first positive branch of $x(\mu)$ (Fig. 2).

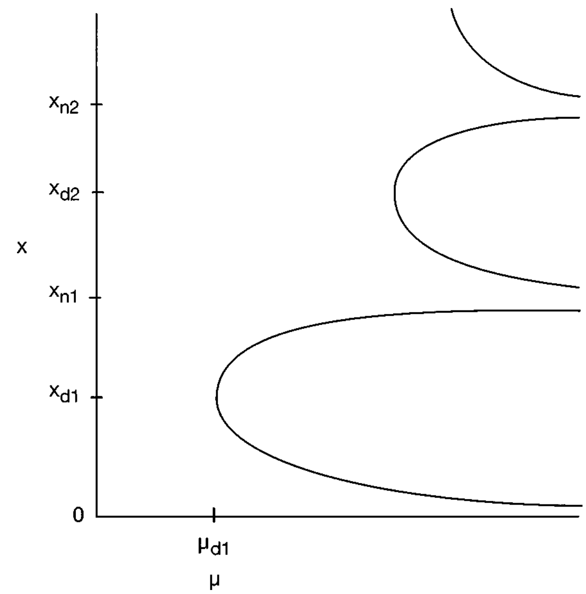

FIG. 2. Schematic behavior of $x(\mu)$ obtained from a [2|2] Padé-summation estimate of the β function whose positive numerator and denominator zeros alternate. The denominator zeros are denoted by x_{d1} and x_{d2}, the numerator zeros are denoted by x_{n1} and x_{n2}, and the alternation of zeros is consistent with the smallest positive zero being a zero of the denominator: $0 < x_{d1} < x_{n1} < x_{d2} < x_{n2}$. The value μ_{d1} is defined such that $x(\mu_{d1}) = x_{d1}$; values of $\mu < \mu_{d1}$ are outside the domain of $x(\mu)$.

The first statement above, which has applicability to supersymmetric gluodynamics as well [10], may also have ramifications for scenarios of dynamical electroweak symmetry breaking that usually involve a distinct technicolor group. The second statement parallels old infrared slavery ideas, except that the inapplicability of perturbation theory to low energies is not seen to follow from the coupling constant growing infinite (or nonperturbatively large), as in infrared slavery, but from an explicit decoupling of the infrared region from the ultraviolet by virtue of β-function singularities alternating with β-function zeros (Fig. 2). We have verified that this alternation occurs in the most general [2|2] Padé summation of the β function even when $n_f = 0$.

IV. PADÉ-IMPROVEMENT OF THE RUNNING MASS

The running mass $m(\mu)$ satisfies the differential equation

$$\frac{dm}{dx} = m \frac{\gamma(x)}{\beta(x)} \tag{4.1}$$

where $x\equiv\alpha_s(\mu)/\pi$, as in the previous section, and where $\beta(x)(\equiv\mu^2 dx/d\mu^2)$ is given by Eqs. (3.1a) and (3.2). The QCD MS anomalous mass dimension function $\gamma(x)$ has been calculated to four-loop order [4,6]:

$$\gamma(x)=-x\left[1+\sum_{i=1}\gamma_i x^i\right], \tag{4.2a}$$

$$\gamma_1=4.20833-0.138889n_f, \tag{4.2b}$$

$$\gamma_2=19.5156-2.28412n_f-0.0270062n_f^2, \tag{4.2c}$$

$$\gamma_3=98.9434-19.1075n_f+0.276163n_f^2+0.00579322n_f^3. \tag{4.2d}$$

Padé improvement of the square-bracketed expression within Eq. (4.2a) is straightforward via the methods of Sec. II by identifying R_1, R_2 and R_3 in Eq. (2.1) with γ_1, γ_2 and γ_3. Using Eq. (2.12) we obtain the following APAP estimates for γ_4:

$$n_f=3:\quad \gamma_4^{(3)}=162.987, \tag{4.3}$$

$$n_f=4:\quad \gamma_4^{(4)}=75.2349, \tag{4.4}$$

$$n_f=5:\quad \gamma_4^{(5)}=12.5550, \tag{4.5}$$

$$n_f=6:\quad \gamma_4^{(6)}=12.1820. \tag{4.6}$$

One can obtain a solution for $m[x(\mu)]$ which includes the $\mathcal{O}(x^4)$ Padé improvement of $\beta(x)$ and $\gamma(x)$ by expressing $\gamma(x)/\beta(x)$ as a Maclaurin series in x, using APAP estimates (3.3)–(3.6) for β_4 and (4.3)–(4.6) for γ_4. By truncating this series after x^4, the differential equation (4.1) can be approximated by

$$\frac{x}{m}\frac{dm}{dx}=[\beta_0^{-1}+d_1x+d_2x^2+d_3x^3+d_4x^4], \tag{4.7}$$

with β_0 given by Eq. (3.2a), and with d_i given as follows:

$$n_f=3:\quad d_1=0.895063,\quad d_2=1.94172,\quad d_3=2.88956,\quad d_4=\underline{417.493}; \tag{4.8}$$

$$n_f=4:\quad d_1=1.014131,\quad d_2=1.74994,\quad d_3=0.0880435,\quad d_4=\underline{-3.93256}; \tag{4.9}$$

$$n_f=5:\quad d_1=1.17549,\quad d_2=0.809817,\quad d_3=-1.05016,\quad d_4=\underline{-10.0138}; \tag{4.10}$$

$$n_f=6:\quad d_1=1.39796,\quad d_2=1.63266,\quad d_3=-6.84005,\quad d_4=\underline{142.769}. \tag{4.11}$$

The values for d_1, d_2, and d_3 are exactly determined by four-loop calculations of $\beta^{(n_f)}(x)$ and $\gamma^{(n_f)}(x)$. The value of d_4 is underlined to emphasize that it is determined from APAP estimates. The large values for d_4 when $n_f=3$ and $n_f=6$ reflect correspondingly large values for $\beta_4^{(3)}$ and $\beta_4^{(6)}$ that are discussed in the previous section.

The solution to Eq. (4.7) can be expressed in terms of $x(\mu)$ evaluated at two different values of μ: $x(\mu_1)\equiv x_1$, $x(\mu_2)\equiv x_2$, where $x(\mu)$ is the running coupling whose evaluation is discussed in the previous section. This solution to Eq. (4.7) is [4]

$$m(x_2)=m(x_1)c(x_2)/c(x_1), \tag{4.12a}$$

where

$$c(x)=x^{1/\beta_0}\{1+d_1x+[(d_1^2+d_2)/2]x^2+[(d_1^3+2d_3+3d_1d_2)/6]x^3+[(d_1^4+3d_2^2+6d_4+6d_1^2d_2+8d_1d_3)/24]x^4\}. \tag{4.12b}$$

Coefficients of x, x^2, and x^3 are determined in full by known coefficients in the four-loop β and γ functions. The x^4 term is the lowest-degree term sensitive to Padé-driven estimates of β_4 and γ_4. We find the following set of expressions for $c(x)$ from the d_i in Eqs. (4.8)–(4.11):

$$n_f=3:\quad c^{(3)}(x)=x^{4/9}[1+0.895063x+1.37143x^2+1.95168x^3+\underline{106.122x^4}], \tag{4.13}$$

$$n_f=4:\quad c^{(4)}(x)=x^{12/25}[1+1.01413x+1.38920x^2+1.09052x^3-\underline{0.0765827x^4}], \tag{4.14}$$

$$n_f=5:\quad c^{(5)}(x)=x^{12/23}[1+1.17549x+1.50071x^2+0.172486x^3-\underline{10.2813x^4}], \tag{4.15}$$

$$n_f=6:\quad c^{(6)}(x)=x^{4/7}[1+1.39796x+1.79347x^2-0.683486x^2+\underline{33.7949x^4}]. \tag{4.16}$$

These same expressions are obtained to $\mathcal{O}(x^3)$ in Ref. [4]; the effects of Padé improvement reside entirely in the x^4 terms.

It is important to recognize that these results ultimately derive from applying the asymptotic error formula (2.3) to the perturbative field-theoretical calculation of $\beta(x)$ and

$\gamma(x)$, as argued in [2] and [3]. As in the previous section, the results (4.3)–(4.6) differ from those one would obtain using the fits of Ref. [3] to the coefficients of n_f^k within γ_4, particularly as γ_4 so extracted involves the near cancellation of large terms from successive values of k: from Table X of Ref. [3] one finds for $n_f=5$ that $\gamma_4^{(5)}=530-(143)\cdot 5+(6.67)\cdot 5^2+(0.037)\cdot 5^3-(8.54\cdot 10^{-5})\cdot 5^4=-13.7$. Small variations in these Padé-estimated coefficients can easily lead to positive values comparable to that of Eq. (4.5).

It is also important to note that the application of the APAP algorithm at the field-theoretical level—i.e., to $\beta(x)$ and $\gamma(x)$—is *not* equivalent to applying it to ''perturbative'' expressions which are obtained by integrating over these functions. One could question, for example, whether the x^4 terms appearing in Eqs. (4.13)–(4.16) might be obtainable by direct application of the APAP algorithm (2.12) to lower-degree terms in x. If we apply Eq. (2.12) directly to Eqs. (4.13)–(4.16) using the explicit coefficients of x, x^2, and x^3 in order to estimate the coefficients of x^4, the x^4 coefficients we obtain are very different from those listed. Instead we obtain, respectively, for $n_f=\{3,4,5,6\}$: $2.683x^4$, $0.7426x^4$, $0.01839x^4$, and $0.2850x^4$. This discrepancy is indicative of the inapplicability of the error formula (2.3) to the series in Eq. (4.12b), assuming Eq. (2.3) *is* applicable to the perturbative field-theoretical series (3.1a) and (4.2a). Such applicability is suggested by the predictions of five-loop terms for $O(N)$ scalar field theory β and γ functions already noted in Sec. II.

Figures 3a–3d display the relative impact on running quark masses of higher order corrections both augmented and unaugmented by Padé improvement. Given an initial value $m_b(4.3\text{ GeV})=4.3$ GeV [8], we evolve $m_b^{(5)}(\mu)$ up to $\mu=175$ GeV. Figure 3a indicates the evolution obtained via Eq. (4.1) from three-loop β and γ functions—i.e., from truncation of the series (3.1a) and (4.2a) after $i=2$. Figure 3b displays the relative effects of higher-order corrections augmented and unaugmented by Padé improvement. We first consider the unaugmented four-loop case. The upper curve in Fig. 3b is the ratio of m_b obtained from four-loop β and γ functions to m_b obtained from three-loop β and γ functions (Fig. 3a). As μ increases from 4 GeV to 175 GeV, the relative decrease in $m_b^{(5)}(\mu)$ from use of four-loop information is seen to be less than 0.1%.

Padé improvement of four-loop results is displayed in the lower curve of Fig. 3b. This curve is the ratio of a fully Padé-improved estimate of $m_b^{(5)}(\mu)$ to the three-loop calculation of $m_b^{(5)}(\mu)$ displayed in Fig. 3a. The full Padé improvement is obtained via an APAP-algorithm determination of $\beta_4^{(5)}$ and $\gamma_4^{(5)}$, which is then utilized to construct Padé-summation [2|2] approximants as estimates of the aggregate effect of *all* higher-order terms in β and γ. The [2|2] approximant for $\beta^{(5)}(x)$ is given by Eq. (3.7c); the [2|2] approximant within

$$\gamma^{(5)}(x)=-x\left[\frac{1+1.19485x+1.02765x^2}{1-2.31904x+1.75664x^2}\right] \tag{4.17}$$

is obtained via Eqs. (2.15)–(2.19) using the $n_f=5$ values of γ_1, γ_2, and γ_3 as well as the APAP-algorithm value for $\gamma_4^{(5)}$ given in Eq. (4.5). The full [2|2] approximants are then used to evaluate $c(x)$ in Eq. (4.12a): $c^{(5)}(x)=\exp[\int(\gamma^{(5)}(x)/\beta^{(5)}(x))dx]$. It is evident from Fig. 3b that Padé-improvement does not significantly alter $m_b(\mu)$ beyond a correction comparable to five-loop expectations; the relative change from such Padé improvement is, respectively, within 0.1% (Fig. 3b) and 0.01% of the unimproved three- and four-loop results.

Figure 3c displays the corresponding evolution of the charmed quark mass $m_c(\mu)$ for $1.3\text{ GeV}\leq\mu\leq 20\text{ GeV}$, as obtained from β and γ functions evaluated to two-, three-, and four-loop order as well as from Padé improvement of the four-loop β and γ functions, both below and above the five-flavor threshold. The initial value is taken to be $m_c(1.3\text{ GeV})=1.3$ GeV for all four curves, and the five-flavor threshold is assumed to occur at 4.3 GeV [8]. The ''Padé'' curve is obtained by direct substitution of appropriate [2|2] Padé-summation β and γ functions into Eq. (4.1). At the five-flavor threshold, we utilize the threshold-matching constraint [9]

$$m_q^{(n_f)}(\mu_{th})=m_q^{(n_f-1)}(\mu_{th})[1+0.2060(\alpha_s^{(n_f)}(\mu_{th})/\pi)^2+(1.8229+0.0247n_f)(\alpha_s^{(n_f)}(\mu_{th})/\pi)^3]^{-1} \tag{4.18}$$

with $n_f=5$ and $\mu_{th}=4.3$ GeV to generate the above-threshold initial condition for m_c. Thus, the above-threshold portion of the Padé curve in Fig. 3c is obtained from this initial condition via substitution of Eqs. (3.7c) and (4.17) into the differential equation (4.1). The below threshold portion of the Padé curve is obtained from the initial condition $m_c(1.3\text{ GeV})=1.3$ GeV via substitution of the Padé-summation four-flavor β function (3.7b) and four-flavor γ function

$$\gamma^{(4)}(x)=-x\,\frac{[1-0.4541x-1.3454x^2]}{[1-4.1069x+3.7090x^2]} \tag{4.19}$$

into the differential equation (4.1). The constraint (4.18) is also utilized to generate the four-loop curve in Fig. 3c, and [when taken to order-$(\alpha_s^{(n_f)}(\mu_{th}))^2$] to generate the three-loop curve. As in Fig. 1, curves of successive order appear to converge (at least qualitatively) to the Padé estimate, which is almost indistinguishable from the four-loop curve.

Padé-improvement effects are somewhat larger for light quarks (*u*,*d*,*s*). The evolution of light quarks from an initial value (normalized to unity) at $\mu=1$ GeV is displayed in Fig. 3d. This latter set of curves is obtained via utilization of Eq. (4.18) at both $n_f=4$ and $n_f=5$ flavor thresholds, which are, respectively, taken to be at 1.3 GeV and 4.3 GeV [8]. Below the 1.3 GeV four-flavor threshold, the Padé curve is generated via Eq. (4.1) using the $n_f=3$ Padé-summation γ function

$$\gamma^{(3)}(x)=-x\,\frac{[1-1.2373x-1.8485x^2]}{[1-5.0289x+4.7993x^2]} \tag{4.20}$$

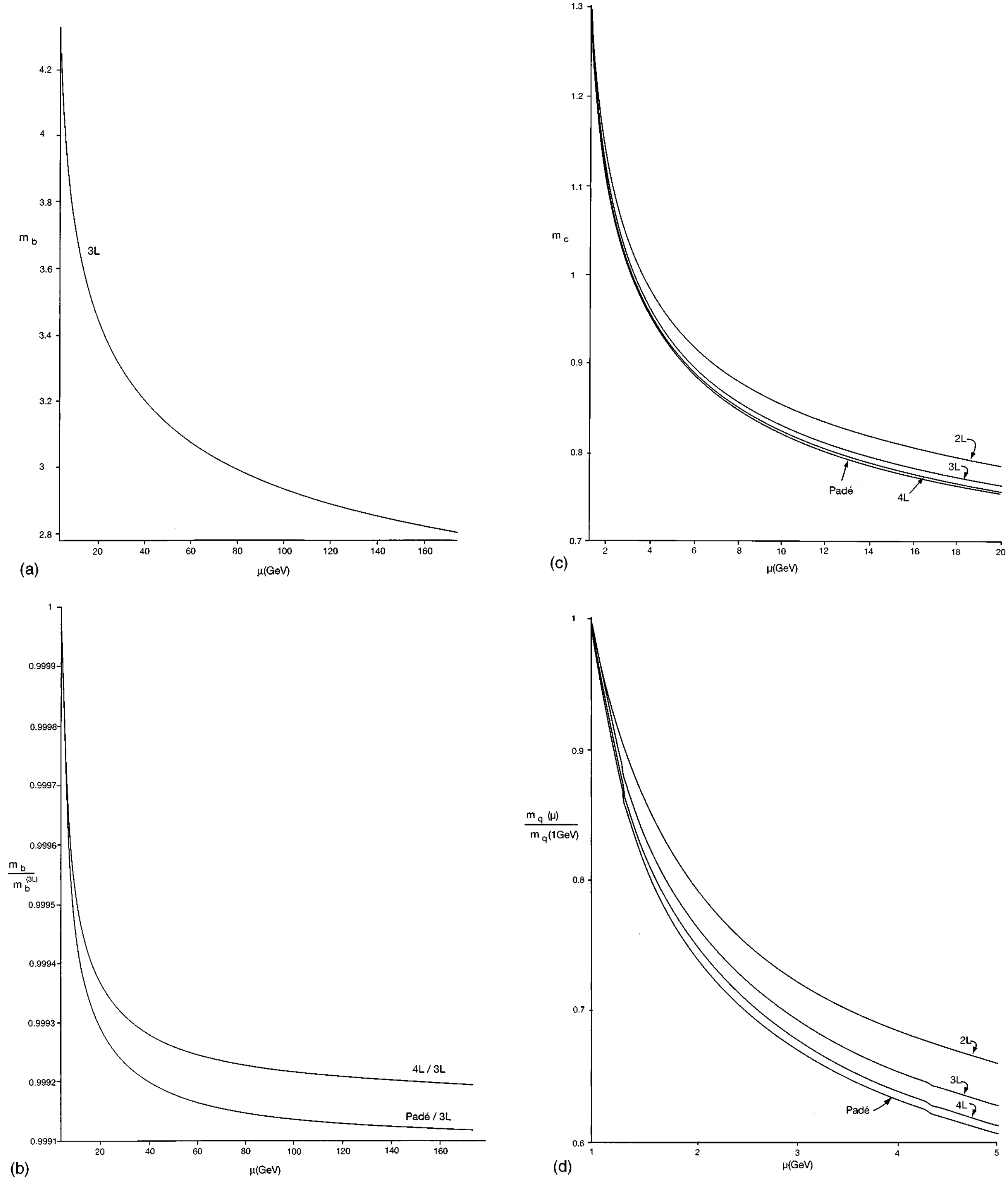

FIG. 3. (a) Evolution of the running mass $m_b(\mu)$ obtained from three-loop (3L) β and γ functions from the initial condition $m_b(4.3\ \text{GeV}) = 4.3\ \text{GeV}$. (b) A comparison of the ratio of $m_b(\mu)$ obtained from four-loop (4L) and [2|2] Padé-summation β and γ functions to $m_b(\mu)$ obtained from three-loop (3L) β and γ functions, as shown in (a), given the initial condition $m_b(4.3\ \text{GeV}) = 4.3\ \text{GeV}$. (c) A comparison of the evolution of $m_c(\mu)$ obtained from evolution of two-loop (2L), three-loop (3L), four-loop (4L) and [2|2] Padé-summation β and γ functions. (d) Masses of light (u,d,s) quarks from two-loop, three-loop, four-loop, and [2|2] Padé-summation β and γ functions. Masses at $\mu = 1$ GeV are normalized to unity.

and the $n_f=3$ β function (3.7a). Between the four- and five-flavor thresholds we utilize Eqs. (4.19) and (3.7b), and above the five-flavor threshold, we utilize Eqs. (4.17) and (3.7c) as before. The bumps in Fig. 3d occur at the four- and five-flavor thresholds (1.3 GeV and 4.3 GeV, respectively), and are anticipated from the threshold matching condition (4.18). At $\mu=5$ GeV, Fig. 3d shows that there is a 1% difference between running masses obtained via unimproved and Padé-improved four-loop β and γ functions. Once again, however, the distance between curves of successive order decreases as the order increases, giving the appearance of convergence towards the Padé-improved curve.

V. APPLICATION TO HIGGS BOSON DECAYS

Although the Higgs particle has yet to be directly observed, expressions for its decay into either two-gluons [11] or a $b\bar{b}$ pair [12] have been worked out with precision in perturbation theory. In much the same way, knowledge of the Z^0 decay widths, whose precise values are important for bounds on standard-model parameters, preceded the discovery of the Z^0 itself. In this section, we apply Padé improvement to the decay processes $H\rightarrow$two gluons and $H\rightarrow b\bar{b}$, and examine whether such improvement leads to detectable changes from the calculated rates obtained without such Padé improvement.

A. Higgs boson→two gluons

The decay rate $H\rightarrow gg$ has been calculated to three-loop order in perturbation theory [11]:

$$\Gamma(H\rightarrow gg)=\frac{G_F M_H^3 x_H^2}{36\pi\sqrt{2}}\times[1+17.9167x_H + (156.808-5.70833\ln(m_t^2/M_H^2))x_H^2 + \mathcal{O}(x_H^3)], \quad (5.1)$$

where $x_H=x(M_H)=\alpha_s^{(5)}(M_H)/\pi$, and where M_H is assumed to be less than m_t. Padé improvement can enter this expression both in the actual value of $\alpha_s^{(5)}(M_H)$ evolving from a Padé-improved β function, as well as in a Padé-driven estimate of the $\mathcal{O}(x_H^3)$ contribution to the square bracketed expression in Eq. (5.1). One cannot apply the APAP algorithm of Sec. II to estimate this term because for a given value of M_H, only the coefficient of $x(R_1=17.9167)$ and $x^2[R_2=156.808-5.70833\ln(m_t^2/M_H^2)]$ are known; the coefficient R_3 of x^3 is *not* known. One way to estimate R_3 is to express $1+R_1x+R_2x^2$ as a [1|1] Padé approximant, which upon expansion yields $R_3=R_2^2/R_1$. A refinement on this estimate that is actually utilized to approximate β_3 in Ref. [2] is to assume in the asymptotic error formula (2.3) that $a+b$ is small compared to $N+1$. One can then argue from Eqs. (2.5) and (2.7) that $\delta_3\simeq\delta_2/2$, where $\delta_2=(R_1^2-R_2)/R_2$ is determined in full by R_1 and R_2. Equation (2.7) can then be rearranged to yield the following estimate of R_3:

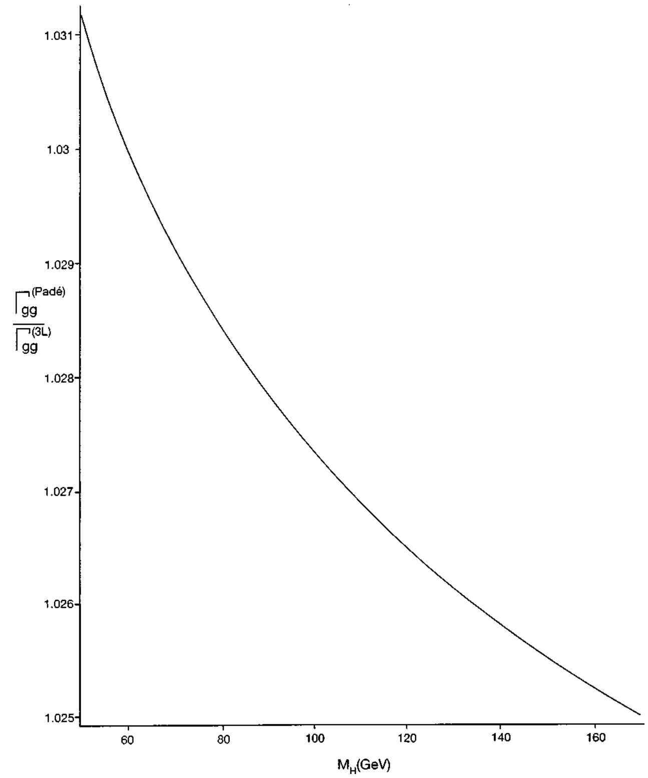

FIG. 4. A comparison of the ratio of the Higgs decay rate into two gluons obtained via Padé improvement discussed in the text to the three-loop (3L) expression for the same rate.

$$R_3=\frac{R_2^2/R_1}{1+\delta_3}=\frac{2R_2^3}{R_1^3+R_1R_2}. \quad (5.2)$$

For different M_H values we plot in Fig. 4 the ratio of the Padé-improved $H\rightarrow gg$ rate to the rate obtained directly from Eq. (5.1), using the three-loop β function to obtain x_H from the initial condition $x(M_z)=0.118/\pi$ [8]. The Padé improvement of the $H\rightarrow gg$ rate is obtained for a given choice of M_H first by evolving x_H from the same initial condition via the [2|2] approximant (3.7c) for the β function, then by using Eq. (5.2) to estimate the $\mathcal{O}(x_H^3)$ term in Eq. (5.1), and finally by replacing the now-known cubic $1+R_1x_H+R_2x_H^2+R_3x_H^3$ in Eq. (5.1) with its appropriate [2|1] Padé summation:

$$1+R_1x+R_2x^2+R_3x^3 \rightarrow \frac{1+(R_1-R_3/R_2)x+(R_2-R_3R_1/R_2)x^2}{1-(R_3/R_2)x}. \quad (5.3)$$

Figure 4 shows that such Padé improvement yields a 2.5%–3% increase in the $H\rightarrow gg$ rate, with very little sensitivity to the Higgs boson mass. Such improvement is best understood to be an estimate of (unknown) higher-order corrections to Eq. (5.1) that should be eventually testable against both experimental and future higher-order calculations of the $H\rightarrow gg$ rate.

B. Higgs boson→$b\bar{b}$

The decay rate Higgs→$b\bar{b}$ has been calculated to four-loop order in perturbation theory [12]:

$$\Gamma(H\to b\bar{b}) = [3G_F M_H m_b^2(M_H)/(4\pi\sqrt{2})] \times\{[1+(17/3)x_H+29.1467x_H^2+41.7581x_H^3] -(6m_b^2(M_H)/M_H^2)[1+(20/3)x_H+14.62x_H^2]\}. \quad (5.4)$$

Full Padé improvement of this expression for a given value of M_H (with $M_H < m_t$) entails (1) determination of $x(M_H)$ through use of Eq. (3.7), the [2|2] Padé summation of the β function, to evolve $x(\mu)$ from an appropriate initial condition, e.g., $\alpha_s^{(5)}(M_z)=0.118$ [8], (2) determination of $m_b(M_H)$ through substitution into Eq. (4.1) of Eqs. (3.7c) and (4.17), the [2|2] Padé summations for $\beta^{(5)}(x)$ and $\gamma^{(5)}(x)$, so as to evolve $m_b^{(5)}(\mu)$ from an appropriate initial condition, e.g., $m_b(4.3\text{ GeV})=4.3\text{ GeV}$ [8], (3) Padé improvement and [2|2] Padé summation of the cubic expression in Eq. (5.4),

$$1+(17/3)x+29.1467x^2+41.7581x^3 \to \frac{[1+4.30262x+21.0641x^2]}{[1-1.36405x-0.352971x^2]}, \quad (5.5)$$

where Eq. (2.12) is used to generate an estimate of the x^4 coefficient [67.2472], and where Eqs. (2.15)–(2.19) are used to generate the [2|2] approximant in Eq. (5.5), and (4) Padé-improvement and [2|1] Padé summation of the quadratic expression in Eq. (5.4),

$$1+(20/3)x+14.62x^2 \to \frac{[1+5.581x+7.382x^2]}{[1-1.086x]}, \quad (5.6)$$

where Eq. (5.2) is used to generate an estimate of the x^3 coefficient [15.87], and where Eq. (5.3) is used to generate the [2|1] approximant in Eq. (5.6).

The relative size of all these corrections, referenced to the rate calculated to the next-to-highest-known [three-loop] order in perturbation theory, is displayed in Fig. 5. The top curve is the ratio $[\Gamma(H\to bb)]_{4\text{-}loop}/[\Gamma(H\to bb)]_{3\text{-}loop}$, as a function of M_H. The four-loop rate is obtained directly from Eq. (5.4), with $m_b(M_H)$ and $x_H[=x(M_H)]$ obtained from β and γ functions that are truncated to zero after their β_3 and γ_3 contributions. The three-loop rate is obtained by truncating off the highest-order terms in Eq. (5.4)—specifically the $\mathcal{O}(x^3)$ term on the left-hand side of Eq. (5.5) and the $\mathcal{O}(x^2)$ term on the left-hand side of Eq. (5.6)—and by obtaining x_H and $m_b(M_H)$ from β and γ functions that are truncated to zero after β_2 and γ_2 contributions. The top curve shows a change of 0.02% to 0.09% in going from three- to four-loop order.

The bottom curve compares the ratio of $\Gamma(H\to b\bar{b})$, obtained by full Padé improvement of $[\Gamma(H\to bb)]$, as described above, to $[\Gamma(H\to bb)]_{3\text{-}loop}$. It is evident from the figure that the two ratios are within 0.0001 of each other for all values of M_H below m_t. In other words, Padé improvement reduces the four-loop result for $\Gamma(H\to bb)$ by *at most* 0.01%. The effect that *is* seen seems to derive wholly from the Padé improvement of $m_b^{(5)}(\mu)$, as is evident from comparison of Fig. 5 to Fig. 3b. Of course, such close agreement between Eq. (5.4) and its fully Padé-improved version suggests that the expression (5.4) is more than adequate for future comparison to experiment. Thus the purpose of the analysis presented here is really to demonstrate the robustness of Eq. (5.4) against Padé estimates of higher-order corrections.

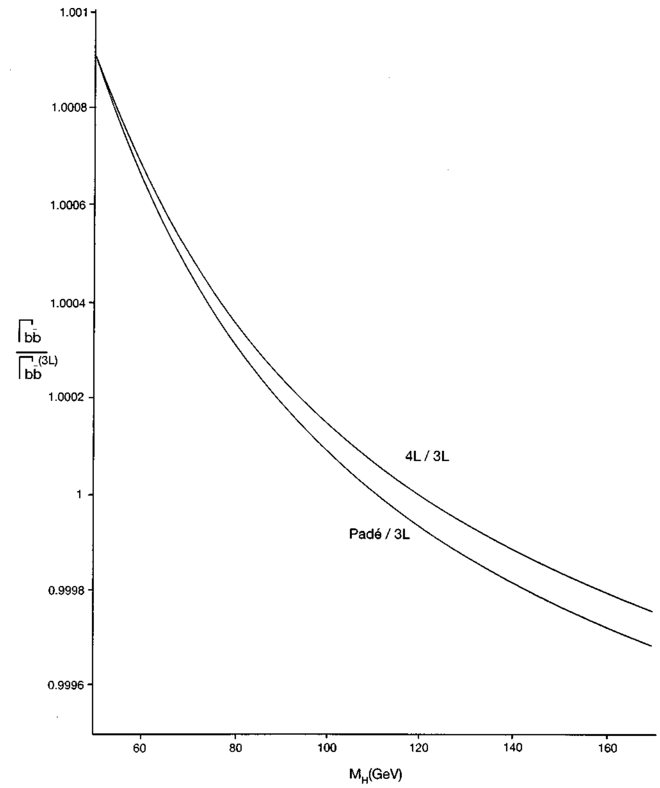

FIG. 5. A comparison of the ratio of the Higgs decay rate into a $b\bar{b}$ pair obtained to four-loop (4L) order and through subsequent Padé improvement (as described in the text) to the same rate obtained to three-loop (3L) order.

The small size of corrections past even the three-loop order is partly a consequence of the small size of $x(M_H)$ characterizing Higgs decay rates. Padé corrections are of much more interest when the magnitude of x is larger, suggesting their usefulness in assessing the perturbative content of low-energy QCD—i.e., QCD sum rules. In the section that follows we will address how Padé improvement can be utilized to estimate substantial higher-order corrections to sum rules relevant to scalar- and pseudoscalar-meson static properties.

VI. PERTURBATIVE CONTENT OF SCALAR/PSEUDOSCALAR QCD SUM RULES

The resonance content of finite-energy $\mathcal{F}_k$ and Laplace $\mathcal{R}_k$ QCD sum rules [13,14] is obtained from integrals over the imaginary part of current correlation functions $\Pi(s,\mu^2)$ in the subcontinuum region ($s<s_0$):

$$\mathcal{F}_k(s_0)=\frac{1}{\pi}\int_0^{s_0}\text{Im}\ \Pi(s,s_0)s^k ds, \tag{6.1}$$

$$\mathcal{R}_k(\tau,s_0)=\frac{1}{\pi}\int_0^{s_0}\text{Im}\ \Pi(s,1/\tau)s^k e^{-s\tau}ds. \tag{6.2}$$

We consider here the purely perturbative content of the correlation function for scalar currents, which is presently known to four-loop order [12]:

$$\frac{1}{\pi}\text{Im}\ \Pi(s,\mu^2)=\frac{3s}{8\pi^2}\left\{1+\left(\frac{\alpha_s(\mu)}{\pi}\right)\left[a_0+a_1\ \ln\left(\frac{\mu^2}{s}\right)\right]+\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[b_0+b_1\ \ln\left(\frac{\mu^2}{s}\right)+b_2\left(\ln\left(\frac{\mu^2}{s}\right)\right)^2\right]+\left(\frac{\alpha_s(\mu)}{\pi}\right)^3\right.$$
$$\left.\times\left[c_0+c_1\ \ln\left(\frac{\mu^2}{s}\right)+c_2\left(\ln\left(\frac{\mu^2}{s}\right)\right)^2+c_3\left(\ln\left(\frac{\mu^2}{s}\right)\right)^3\right]+\left(\frac{\alpha_s(\mu)}{\pi}\right)^4 R_4+\dots\right\}. \tag{6.3}$$

The problem we will address in this section is the computation of R_4, which is necessary for the determination of $\mathcal{O}(\alpha_s^4)$ contributions to $\mathcal{F}_k$ and $\mathcal{R}_k$. For three flavors Chetyrkin [12] has found that

$$a_0=17/3,\ \ a_1=2, \tag{6.4a}$$

$$b_0=31.8640,\ \ b_1=31.6667,\ \ b_2=17/4, \tag{6.4b}$$

$$c_0=89.1564,\ \ c_1=297.596,$$

$$c_2=229/2,\ \ c_3=9.20833. \tag{6.4c}$$

If we define $w\equiv s/\mu^2$, we can estimate $R_4[w]$ directly by substituting

$$R_1[w]=a_0-a_1\ \ln\ w, \tag{6.5a}$$

$$R_2[w]=b_0-b_1\ \ln\ w+b_2(\ln\ w)^2, \tag{6.5b}$$

$$R_3[w]=c_0-c_1\ \ln\ w+c_2(\ln\ w)^2-c_3(\ln\ w)^3 \tag{6.5c}$$

directly into Eq. (2.12). This enables one to determine explicitly $\mathcal{O}(\alpha_s^4)$ corrections to the sum-rules (6.1) and (6.2), even though $R_4[w]$ determined in this way is manifestly not a fourth-order polynomial in ln(w). In particular, we easily find the $\mathcal{O}(\alpha_s^4)$ contribution to $\mathcal{F}_k(s_0)$ to be

$$\Delta\mathcal{F}_0(s_0)=\frac{3s_0^2}{16\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4\int_0^1 2R_4[w]w\,dw$$
$$=\frac{3s_0^2}{16\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4(2059.4), \tag{6.6a}$$

$$\Delta\mathcal{F}_1(s_0)=\frac{s_0^3}{8\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4\int_0^1 3R_4[w]w^2dw$$
$$=\frac{s_0^3}{8\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4(1158.4), \tag{6.6b}$$

$$\Delta\mathcal{F}_2(s_0)=\frac{3s_0^4}{32\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4\int_0^1 4R_4[w]w^3dw$$
$$=\frac{3s_0^4}{32\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4(833.47). \tag{6.6c}$$

The integrals in Eqs. (6.6) have been evaluated numerically. This approach, however, ignores the known structural dependence of R_4 on the variable w,

$$R_4[w]=d_0-d_1\ \ln\ w+d_2(\ln\ w)^2-d_3(\ln\ w)^3+d_4(\ln\ w)^4, \tag{6.7}$$

which may be important when one integrates over the w variable, as in Eqs. (6.6). The $\mathcal{O}(\alpha_s^4)$ corrections to the first three finite energy sum rules are easily determined in terms of the constants d_i by substitution of Eq. (6.7) into the integrand of Eq. (6.1) via Eq. (6.3):

$$\Delta\mathcal{F}_0(s_0)=\frac{3s_0^2}{16\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4\times\left(d_0+\frac{d_1}{2}+\frac{d_2}{2}+\frac{3d_3}{4}+\frac{3d_4}{2}\right), \tag{6.8}$$

$$\Delta\mathcal{F}_1(s_0)=\frac{s_0^3}{8\pi^2}\left(\frac{\alpha_s(s_0^{1/2})}{\pi}\right)^4\times\left(d_0+\frac{d_1}{3}+\frac{2d_2}{9}+\frac{2d_3}{9}+\frac{8d_4}{27}\right), \tag{6.9}$$

$$\Delta\mathcal{F}_2(s_0)=\frac{3s_0^4}{32\pi^2}\left(\frac{\alpha_2(s_0^{1/2})}{\pi}\right)^4\times\left(d_0+\frac{d_1}{4}+\frac{d_2}{8}+\frac{3d_3}{32}+\frac{3d_4}{32}\right). \tag{6.10}$$

We can use the Padé algorithm (2.12) to estimate the coefficients d_i. To do so, we let R_1, R_2, and R_3 be given by Eqs. (6.5) for five representative values of w between zero and one: $w=\{1,e^{-1/4},e^{-1/2},e^{-1},e^{-2}\}$. When $w=1$, corresponding to $s=s_0$ in the finite-energy sum rule integrand (6.1), we see from Eqs. (6.5) that $R_1=a_0$, $R_2=b_0$, $R_3=c_0$. Using the APAP algorithm (2.12), we find that $R_4[1]=251.442=d_0$. When $w=e^{-1/4}(s=0.779s_0)$, we find from Eqs. (6.5) that $R_1=37/6$, $R_2=40.0463$, $R_3=170.8554$. Using Eqs. (2.12) and (6.7), respectively, we then find that

$$R_4[e^{-1/4}]=699.398=d_0+d_1/4+d_2/16+d_3/64+d_4/256. \quad (6.11)$$

Similarly we find the following results when $w=e^{-1/2}(s=0.606s_0)$, $w=e^{-1}(s=0.368s_0)$, and $w=e^{-2}(s=0.135s_0)$:

$$R_4[e^{-1/2}]=1389.82=d_0+d_1/2+d_2/4+d_3/8+d_4/16, \quad (6.12)$$

$$R_4[e^{-1}]=3652.36=d_0+d_1+d_2+d_3+d_4, \quad (6.13)$$

$$R_4[e^{-2}]=12804.9=d_0+2d_1+4d_2+8d_3+16d_4. \quad (6.14)$$

We solve the four linear equations (6.11)–(6.14) for the four unknowns d_1,d_2,d_3,d_4 using the value already obtained for $d_0(=251.422)$, and we obtain $d_1=1357.84$, $d_2=1634.53$, $d_3=404.630$, $d_4=3.9097$. Substitution of these numbers into Eqs. (6.8)–(6.10) yields results remarkably close to those obtained in Eqs. (6.6). These results are listed in the underlined highest-order terms given below for the Padé-improved perturbative content of the first three finite energy sum rules $[x\equiv\alpha_s(s_0^{1/2})/\pi]$:

$$\mathcal{F}_0(s_0)=\frac{3s_0^2}{16\pi^2}[1+6.66667x+49.8223x^2+302.110x^3+\underline{2057.0x^4}], \quad (6.15a)$$

$$\mathcal{F}_1(s_0)=\frac{s_0^3}{8\pi^2}[1+6.33333x+43.3640x^2+215.846x^3+\underline{1158.4x^4}], \quad (6.15b)$$

$$\mathcal{F}_2(s_0)=\frac{3s_0^4}{32\pi^2}[1+6.16667x+40.3119x^2+178.731x^3+\underline{833.52x^4}]. \quad (6.15c)$$

Padé corrections to the Laplace sum rules (6.2) are not listed, as they are complicated by the occurrence of two scale variables (s_0 and τ). However, such corrections are straightforward to obtain via integration of the $\mathcal{O}(\alpha_s^4)$ term of Eq. (6.3), which we have already obtained via APAP estimates of d_{0-4}:

$$R_4=251.44+1357.8\ln\left(\frac{\mu^2}{s}\right)+1634.5\left[\ln\left(\frac{\mu^2}{s}\right)\right]^2+404.63\left[\ln\left(\frac{\mu^2}{s}\right)\right]^3+3.9097\left[\ln\left(\frac{\mu^2}{s}\right)\right]^4. \quad (6.16)$$

As a cross-check on these results, we note that $m^2\,\mathrm{Im}\,\Pi^{(3)}(s,\mu^2)$, the $n_f=3$ correlator based on the renormalization-group invariant scalar current $m\bar{\psi}\psi$, satisfies the homogeneous renormalization-group equation

$$0=\left[\mu^2\frac{\partial}{\partial\mu}+\beta^{(3)}(x)\frac{\partial}{\partial x}+m\gamma^{(3)}(x)\frac{\partial}{\partial m}\right]m^2\,\mathrm{Im}\,\Pi^{(3)}(s,\mu^2), \quad (6.17)$$

where $x\equiv\alpha_s/\pi$ and where $\mathrm{Im}\,\Pi^{(3)}(s,\mu^2)$ is given by Eq. (6.3) with

$$R_4=d_0+d_1\ln(\mu^2/s)+d_2\ln^2(\mu^2/s)+d_3\ln^3(\mu^2/s)+d_4\ln^4(\mu^2/s). \quad (6.18)$$

When substituted into Eq. (6.17), the known four-loop coefficients of the $n_f=3\,\beta$ function (3.2) and γ function (4.2) are sufficient in themselves to determine the constants d_1, d_2, d_3, and d_4. These numbers are found, respectively, to be 1562.96, 1583.62, 356.036, and 20.143. We thus see that values of d_1, d_2, and d_3 determined by the renormalization group equation (6.17) differ, respectively, from the values in Eq. (6.16) via APAP methods by 13.1%, 3.2%, and 13.6%. Although the estimate for d_4 does not share this otherwise remarkable agreement, it should be noted that the very small APAP estimate for d_4 given in Eq. (6.16) follows from the near cancellation of much larger numbers. Both the APAP estimate for d_4 and the value obtained from Eq. (6.17) are quite small in magnitude compared to the other coefficients.

It is also worth noting that the underlined R_4 terms in Eqs. (6.15) are not very different from those one would obtain using either the R_3^2/R_2 estimate suggested by a [2|1] approximant, as discussed following Eq. (2.10), or the APAP algorithm (2.12) applied directly to the known (nonunderlined) terms of Eqs. (6.15). The increase in the size of coefficients with increasing powers of x suggests the utility of a [2|2] Padé summation for these three expressions as an improvement over truncating off what may be substantial $\mathcal{O}(x^5)$ corrections to Eqs. (6.15). By applying Eqs. (2.15)–(2.19) to the three equations (6.15), we obtain the following [2|2] approximants to the full perturbative content of the first three finite-energy sum rules:

$$\mathcal{F}_0(s_0)=\frac{3s_0^2}{16\pi^2}\left[\frac{1+3.8073x+6.8124x^2}{1-2.8594x-23.947x^2}\right], \quad (6.19a)$$

$$\mathcal{F}_1(s_0)=\frac{s_0^3}{8\pi^2}\left[\frac{1+2.3917x+11.308x^2}{1-3.9416x-7.0930x^2}\right], \quad (6.19b)$$

$$\mathcal{F}_2(s_0)=\frac{3s_0^4}{32\pi^2}\left[\frac{1+2.2174x+12.791x^2}{1-3.9492x-3.1670x^2}\right]. \quad (6.19c)$$

Both Eqs. (6.15a) and (6.19a) are indicative of a need for $s_0^{1/2}$ to be substantially larger than 1 GeV for finite-energy sum rules to be useful in the scalar and pseudoscalar channels. If $s_0^{1/2}=1$ GeV, we see from Fig. 1 that $x(1\text{ GeV})=0.153$. For this value of x, each successive term in the square brackets of Eq. (6.15a) is approximately unity, indicative of nonconvergence. This is reflected by a near-vanishing of the denominator of Eq. (6.19a), implying a divergent result for the summation of the full perturbative series. If $s_0=3.24\text{ GeV}^2$, we see from Fig. 1 that $x(s_0^{1/2})\cong 0.10$. The truncated series (6.15a) is then seen to yield a value that is only 87% of that obtained via the Padé summation (6.19a), indicative of the magnitude of the higher-order terms missing from Eq. (6.15a). Note that a choice for s_0 near or somewhat above 3 GeV^2 is suggested by Laplace sum-rule fits in both the pseudoscalar [15] and scalar [16] resonance channels.

The finite energy sum rule $\mathcal{F}_0(s_0)$ provides an example of how it is not enough just to have precise higher-order results. Even though Eq. (6.15a) includes four-loop effects as well as an APAP-algorithm estimate of five-loop effects, the five terms listed demonstrate only sluggish convergence for a realistic choice of s_0. There is found to be enough of a difference between the truncated series (6.15a) and its Padé-summation (6.19a) to suggest the advisability of using the latter.

VII. SUMMARY

Using a Padé-motivated algorithm (2.12), we have estimated in Sec. III the five-loop contributions to the β function for $n_f=\{3,4,5,6\}$, and we have compared the evolution of $\alpha_s(\mu)$ from $\mu=M_z$ obtained from two-loop, three-loop, four-loop, and Padé-summation estimates of the full β function. Low energy values of α_s obtained from the four-loop β functions with quoted flavor thresholds and appropriate threshold matching conditions are within 1% of those obtained from the Padé-summation β functions, a small effect compared to the much larger sensitivity of $\alpha_s(1\text{ GeV})$ and $\alpha_s(m_\tau)$ to present uncertainties in $\alpha_s(M_z)$ and c- and b-quark flavor thresholds.

We concluded Sec. III by extracting the most general set of [2|2] Padé-summation estimates of 3, 4, 5, and 6 flavor QCD β functions whose Maclaurin expansions yield known four-loop results for their first four terms. For positive values of α_s, these Padé-summation estimates of the β function were shown to alternate denominator and numerator zeros, *regardless* of the size of the (presently unknown) five-loop term serving as a free parameter in these [2/2]-approximants. Such alternation necessarily implies that all positive numerator zeros represent *ultraviolet* fixed points, behavior which, if applicable to the true β function, would decouple the (suitably defined) infrared region from perturbative QCD.

In Sec. IV, we applied Padé-improvement methods to the running quark mass by estimating five-loop contributions to the γ functions for 3, 4, 5, and 6 quark flavors. We then extracted an estimate for the $O(x^4)$ contribution to closed-form expressions for $m_q[\alpha_s(\mu)/\pi]$ that had been obtained earlier [4,6] to $O(x^3)$. We compared the evolution of three-loop, four-loop, and Padé-summation estimates of $m_b(\mu)$, once again finding very little relative difference (0.01%) between Padé-summation and four-loop determinations of $m_b(\mu)$ over the range $\mu<m_t$, given identical five-flavor-threshold initial conditions. Corresponding agreement was still seen to occur at the 1% level for light quarks.

In Sec. V we applied the results of the previous two sections to higher-loop calculations of the Higgs decay ratio $H\rightarrow gg$ and $H\rightarrow b\bar{b}$, rates which are sensitive to running couplings and running masses, as well as higher-loop corrections that are polynomial in $\alpha_s(m_H)$. The calculated three-loop $H\rightarrow gg$ rate is shown to be within 3% of our Padé-improvement estimate, given identical choices for M_H, $\alpha(M_z)$, and m_b-threshold initial conditions. Similarly, the calculated four-loop $H\rightarrow b\bar{b}$ rate is seen to differ from full Padé improvement by at most 0.01%.

All the results summarized up until this point are indicative of close agreement between known perturbation theory and Padé-approximant improvements intended to take into account higher-order effects. Consequently, the theoretical uncertainties associated with the truncation of any such calculations at the three-or-four-loop order are shown to be small. In Sec. VI, we considered quantities known to four-loop order for which this is *not* the case, the purely perturbative content of QCD sum rules in scalar/pseudoscalar-resonance channels. We constructed a Padé-algorithm estimate of the purely perturbative $O(\alpha_s^4)$ contribution to the imaginary part of scalar/pseudoscalar correlation functions, and we obtained [2|2] Padé-summation estimates of the all-orders content of the first three finite energy sum rules. We found the overall convergence of the primary sum rule to be doubtful for values of the QCD continuum threshold near $s_0=1$ GeV^2. Even for s_0 above 3 GeV^2, we found a greater-than-10% discrepancy between Padé-summation and four-loop-order contributions to this sum rule, suggesting the existence of substantial theoretical uncertainties from higher-than-four-loop contributions.

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