Top quark pair production at threshold: Complete next-to-next-to-leading order relativistic corrections

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The complete next-to-next-to-leading order [i.e., $\mathcal{O}(v^2)$, $\mathcal{O}(v\,\alpha_s)$, and $\mathcal{O}(\alpha_s^2)$] relativistic corrections to the total photon mediated $t\bar{t}$ production cross section at threshold are presented in the framework of nonrelativistic quantum chromodynamics. The results are obtained using semianalytic methods and ''direct matching.'' The size of the next-to-next-to-leading order relativistic corrections is found to be comparable to the size of the next-to-leading order ones. [S0556-2821(98)05621-5]

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Because of its large mass the top quark decays dominantly through the weak channel $t\rightarrow b\,W$ resulting in a partial width $\Gamma(t\rightarrow b\,W)\approx 1.5\,\mathrm{GeV}$ which is much larger than the typical hadronization scale. As a consequence the weak decay of the top quark provides a natural infrared cutoff which almost entirely suppresses hadronization effects in top quark production and decay processes. This particular feature makes is possible to study $t\bar{t}$ production close to threshold in lepton pair collisions using perturbative QCD [1]. With this motivation in mind a considerable number of studies have been carried out in the past in order to calculate $t\bar{t}$ production observables [2–6] and explore their potential for measurements of the top quark mass[1] M_t and the strong coupling α_s at future experiments such as the LC (Linear Collider) [7] or the FMC (First Muon Collider) [8]. In view of the high precision which might be achieved for the QCD calculations as well as for $t\bar{t}$ production measurements even relatively small effects coming from a light Higgs boson [2,9] have been investigated. However, the present day analyses only include QCD effects up to next-to-leading order (NLO) in form of the one-loop corrections to the QCD potential [10,11] and various $\mathcal{O}(\alpha_s)$ short-distance corrections. A complete next-to-next-to-leading order (NNLO) calculation, which would be necessary to study the reliability of the present day analyses and to make the consideration of small effects from beyond QCD at all feasible, has been missing so far.

In this work we present the complete NNLO relativistic corrections to the total photon mediated $t\bar{t}$ production cross section. As NNLO we count all corrections of order v^2, $v\,\alpha_s$ and α_s^2 relative to the cross section in the nonrelativistic limit, v being the c.m. velocity of the top quarks. As relativistic, on the other hand, we count the corrections coming from the top quark kinetic energy, the $t\bar{t}$ production and annihilation process including the short-distance corrections, and the $t\bar{t}$ interaction potentials. We use nonrelativistic QCD (NRQCD) [12,13] to conveniently parametrize calculations and results in a systematic manner following the approach proposed in Ref. [14]. The calculations are carried out using semianalytic methods and the ''direct matching'' procedure introduced in Ref. [15]. We would like to point out that NNLO corrections involving the top quark decay are not determined here. The latter effects would include $\mathcal{O}(\alpha_s^2)$ two-loop corrections to the free top quark width and a consistent treatment of the effects from the off-shellness of the decaying top quarks, the time dilatation, and the interactions among the decay products and the other top quark (if it is not decayed yet). Although the size and the interplay of all these effects have been studied at various places in the literature (see, e.g., Refs. [5,16,17]), their consistent treatment at NNLO still remains an open problem. As far as the NNLO relativistic corrections discussed in this work are concerned we will use the naive replacement

$$E\equiv\sqrt{s}-2M_t \;\rightarrow\; \tilde{E}=E+i\,\Gamma_t \tag{1}$$

in the spirit of Ref. [1] in order to examine their size and properties, where Γ_t represents a constant which is not necessarily the decay width of a free top quark. We also would like to emphasize that we treat *all interactions* purely perturbatively and that nowhere in this work the confining long-range contributions to the QCD potential or other nonperturbative effects are taken into account. This is somewhat contrary to the standard present day approach used to describe $t\bar{t}$ production at threshold (see Refs. [2–5]), but we take the position that nonperturbative effects might be added later as a correction.

[1]Throughout this paper M_t is understood as the top quark pole mass.

We start from the NRQCD Lagrangian

$$\mathcal{L}_{\rm NRQCD} = -\frac{1}{2}\,{\rm Tr}\,G^{\mu\nu}G_{\mu\nu} + \sum_{q=u,d,s,c,b} \bar{q}\, i\not{D}\, q + \psi^\dagger\left[iD_t + a_1\frac{\mathbf{D}^2}{2M_t} + a_2\frac{\mathbf{D}^4}{8M_t^3}\right]\psi + \cdots + \psi^\dagger\left[\frac{a_3\, g_s}{2M_t}\boldsymbol{\sigma}\cdot\mathbf{B} + \frac{a_4\, g_s}{8M_t^2}(\mathbf{D}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{D}) + \frac{a_5\, g_s}{8M_t^2}\, i\,\boldsymbol{\sigma}(\mathbf{D}\times\mathbf{E}-\mathbf{E}\times\mathbf{D})\right]\psi + \cdots . \quad (2)$$

The gluonic and light quark degrees of freedom are described by the conventional relativistic Lagrangian, whereas the top and antitop quark are described by the Pauli spinors ψ and χ, respectively. For convenience all color indices are suppressed. The straightforward antitop bilinears are omitted and only those terms relevant for the NNLO cross section are displayed. D_t and $\mathbf{D}$ are the time and space components of the gauge covariant derivative D_μ, and $E^i = G^{0i}$ and $B^i = \frac{1}{2}\epsilon^{ijk}G^{jk}$ the electric and magnetic components of the gluon field strength tensor (in Coulomb gauge). The short-distance coefficients $a_1,\ldots,a_5$ are normalized to one at the Born level. Because we use "direct matching" [15] the actual form of their higher order contributions is irrelevant for this work.

To formulate the normalized total $t\bar{t}$ production cross section (via a virtual photon) $R=\sigma({}^{e^+e^-}_{\mu^+\mu^-}\to\gamma^*\to t\bar{t})/\sigma_{pt}$ ($\sigma_{pt} = 4\pi\alpha^2/3s$) in the nonrelativistic region at NNLO in NRQCD we start from the fully covariant expression for the cross section

$$R(q^2) = \frac{4\,\pi\, Q_t^2}{q^2}\,{\rm Im}[-i\langle\, 0\,|\, T\,\tilde{j}_\mu(q)\,\tilde{j}^\mu(-q)\,|\,0\,\rangle\,], \quad (3)$$

where $Q_t = 2/3$ is the top quark electric charge. We then expand the electromagnetic current (in momentum space) $\tilde{j}_\mu(\pm q) = (\bar{\tilde{t}}\gamma^\mu\tilde{t})(\pm q)$ which produces or annihilates a $t\bar{t}$ pair with c.m. energy $\sqrt{q^2}$ in terms of ${}^3S_1$ NRQCD currents up to dimension 8 ($i=1,2,3$)

$$\tilde{j}_i(q) = b_1\,(\tilde{\psi}^\dagger\sigma_i\tilde{\chi})(q) - \frac{b_2}{6M_t^2}\left[\tilde{\psi}^\dagger\sigma_i\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\tilde{\chi}\right](q) + \cdots , \quad (4)$$

where the constants b_1 and b_2 are short-distance coefficients normalized to one at the Born level. The expansion of $\tilde{j}_i(-q)$ is obtained from Eq. (4) via charge conjugation symmetry. It should be noted that only the spatial components of the currents contribute at NNLO. Inserting expansion (4) back into Eq. (3) leads to the NRQCD expression of the nonrelativistic cross section at the NNLO level

$$R^{\rm thr}_{\rm NNLO}(\tilde{E}) = \frac{\pi\, Q_t^2}{M_t^2}\, C_1(\mu_{\rm hard},\mu_{\rm fac})\,{\rm Im}[\,\mathcal{A}_1(\tilde{E},\mu_{\rm soft},\mu_{\rm fac})\,] - \frac{4\,\pi\, Q_t^2}{3M_t^4}\, C_2(\mu_{\rm hard},\mu_{\rm fac}) \times {\rm Im}[\,\mathcal{A}_2(\tilde{E},\mu_{\rm soft},\mu_{\rm fac})\,] + \cdots , \quad (5)$$

where

$$\mathcal{A}_1 = i\,\langle\, 0\,|\,(\tilde{\psi}^\dagger\vec{\sigma}\,\tilde{\chi})\;(\tilde{\chi}^\dagger\vec{\sigma}\,\tilde{\psi})\,|\,0\,\rangle , \quad (6)$$

$$\mathcal{A}_2 = \tfrac{1}{2}\, i\left\langle\, 0\left|\,(\tilde{\psi}^\dagger\vec{\sigma}\,\tilde{\chi})\left[\tilde{\chi}^\dagger\vec{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\tilde{\psi}\right] + {\rm H.c.}\right|\, 0\right\rangle . \quad (7)$$

Using the equations of motion for the top quark fields one can show that

$$\mathcal{A}_2 = M_t\,\tilde{E}\,\mathcal{A}_1 . \quad (8)$$

We have used relation (8) to obtain the factor $-4/3$ in the second line of Eq. (5).

The right-hand side of Eq. (5) represents just an application of the factorization formalism proposed in Ref. [13]. The cross section is expanded in terms of a sum of absorptive parts of nonrelativistic current correlators (containing long-distance physics[2]) multiplied by short-distance coefficients C_i ($i=1,2,\ldots$). In Eq. (5) we have also shown the dependences on the various renormalization scales: the soft scale $\mu_{\rm soft}$ and the hard scale $\mu_{\rm hard}$ are governing the perturbative expansions of the correlators and the short-distance coefficients, respectively, and arise from the light degrees of freedom in $\mathcal{L}_{\rm NRQCD}$,[3] whereas the factorization scale $\mu_{\rm fac}$ essentially represents the boundary between hard (i.e., of order M_t) and soft momenta. This boundary is not uniquely defined and therefore both, the correlators and the short-distance coefficients, in general depend on $\mu_{\rm fac}$. (This leads to anomalous dimensions of the NRQCD currents [13]. However, in this paper this fact is not used to carry out any resummation of logarithms involving the factorization scale.) Because the term in the second line in Eq. (5) is already of NNLO (i.e., suppressed by v^2) we can set $C_2=1$. It should be noted that the short-distance coefficients and the correlators are individually invariant (up to terms beyond NNLO order) with respect to changes in the hard and the soft scale, respectively. The calculation of all terms in expression (5) proceeds in two basic steps.

Step 1: Calculation of the nonrelativistic correlators. The nonrelativistic correlators are calculated in Coulomb gauge using tools known from QED bound state calculations [12,18]. In Coulomb gauge the gluon propagator is split into

[2] In the context of this paper "long distance" is not equivalent to "nonperturbative."

[3] Throughout this work we use the convention $\alpha_s = \alpha_s^{(n_l=5)}$ in the modified minimal subtraction $\overline{\rm MS}$ scheme.

a longitudinal and a transverse piece. The longitudinal gluon propagator is energy independent and can be written as an instantaneous potential if the longitudinal gluon is exchanged between the $t\bar{t}$ pair. (For production through a virtual photon or Z only $t\bar{t}$ in a color singlet state needs to be considered.) The longitudinal gluon exchange between the $t\bar{t}$ pair leads to the Coulomb interaction at LO in the nonrelativistic expansion and the Darwin and spin-orbit interactions at NNLO [see all interactions involving the top quarks and the **E** field in Eq. (2)]. For the transverse gluon the situation is more subtle because its propagator has an energy dependence and represents, in general, a temporally retarded interaction. At NNLO the transverse gluon leads to one- and two-loop corrections to the instantaneous Coulomb interaction (if at least one side of the transverse gluon is connected to another gluon) and to the ''hyperfine'' interactions (if the transverse gluon is exchanged directly between the $t\bar{t}$ pair). In the first case the full energy dependence of the transverse gluons has to be taken into account in the loop integrations, whereas in the second case the energy dependence can be neglected. This can be understood from the fact that the typical energy flowing through a transverse gluon which is exchanged between the $t\bar{t}$ pair is of order $M_t v^2$, the kinetic energy of the $t\bar{t}$ pair in the c.m. frame. The three momentum can either be of order $M_t v$, the relative three momentum of the $t\bar{t}$ pair, or also of order $M_t v^2$. For the energy-momentum configuration $(M_t v^2, M_t v^2)$ the transverse gluon is essentially real and, therefore, needs an additional phase space factor v to exist. Because the coupling of transverse gluons to the top quark is suppressed by v [see all interactions involving the top quarks and the **B** field in Eq. (2)] this configuration does not contribute at NNLO.[4] For the energy-momentum configuration $(M_t v^2, M_t v)$, on the other hand, the transverse gluon is far off-shell and the energy dependence can be neglected at NNLO. In the case of hydrogen it is this energy-momentum configuration for transverse photons which constitutes the hyperfine interactions. For $t\bar{t}$ production it leads to instantaneous interactions at NNLO. In fact, this feature is well known from classic positronium calculations[19,20] and has been shown recently from formal NRQCD/NRQED counting rules [21–23]. In other words, as far as the calculation of the nonrelativistic correlators in Eq. (5) at NNLO is concerned, NRQCD reduces to a two-body (top-quark–top-antiquark) Schrödinger theory where retardation (Lamb-shift type) effects can be ignored and all interactions can be expressed in terms of instantaneous potentials. The potentials in the resulting Schrödinger equation are determined by considering color singlet $t\bar{t}\to t\bar{t}$ one gluon exchange t-channel scattering amplitudes in NRQCD. To NNLO (i.e., including potentials suppressed by at most α_s^2, α_s/M_t, or $1/M_t^2$ relative to the Coulomb potential) the relevant potentials read [$a_s \equiv \alpha_s(\mu_{\rm soft})$, $C_A=3$, $C_F=4/3$, $T=1/2$, $\tilde{\mu}\equiv e^{\gamma}\mu_{\rm soft}$, $r\equiv|\vec{r}|$]

$$V_c(\vec{r}) = -\frac{C_F a_s}{r}\Bigg\{1+\left(\frac{a_s}{4\pi}\right)[2\beta_0\ln(\tilde{\mu}r)+a_1] + \left(\frac{a_s}{4\pi}\right)^2\left[\beta_0^2\left(4\ln^2(\tilde{\mu}r)+\frac{\pi^2}{3}\right) + 2(2\beta_0 a_1+\beta_1)\ln(\tilde{\mu}r)+a_2\right]\Bigg\}, \tag{9}$$

$$\begin{aligned} V_{\rm BF}(\vec{r}) = {} & \frac{C_F a_s \pi}{M_t^2}\left[1+\frac{8}{3}\vec{S}_t\vec{S}_{\bar{t}}\right]\delta^{(3)}(\vec{r}) + \frac{C_F a_s}{2M_t^2 r}\left[\vec{\nabla}^2+\frac{1}{r^2}\vec{r}(\vec{r}\vec{\nabla})\vec{\nabla}\right] \\ & - \frac{3C_F a_s}{M_t^2 r^3}\left[\frac{1}{3}\vec{S}_t\vec{S}_{\bar{t}} - \frac{1}{r^2}(\vec{S}_t\vec{r})(\vec{S}_{\bar{t}}\vec{r})\right] + \frac{3C_F a_s}{2M_t^2 r^3}\vec{L}(\vec{S}_t+\vec{S}_{\bar{t}}), \end{aligned} \tag{10}$$

$$V_{\rm NA}(\vec{r}) = -\frac{C_A C_F a_s^2}{2M_t r^2}, \tag{11}$$

where $\vec{S}_t$ and $\vec{S}_{\bar{t}}$ are the top quark and top antiquark spin operators and $\vec{L}$ is the angular momentum operator and ($n_l = 5$)

$$\begin{aligned} \beta_0 &= \frac{11}{3}C_A - \frac{4}{3}T n_l, \\ \beta_1 &= \frac{34}{3}C_A^2 - \frac{20}{3}C_A T n_l - 4C_F T n_l, \\ a_1 &= \frac{31}{9}C_A - \frac{20}{9}T n_l, \\ a_2 &= \left(\frac{4343}{162}+6\pi^2-\frac{\pi^4}{4}+\frac{22}{3}\zeta_3\right)C_A^2 - \left(\frac{1798}{81}+\frac{56}{3}\zeta_3\right)C_A T n_l \\ &\quad - \left(\frac{55}{3}-16\zeta_3\right)C_F T n_l + \left(\frac{20}{9}T n_l\right)^2. \end{aligned} \tag{12}$$

The constants β_0 and β_1 are the one- and two-loop coefficients of the QCD beta function and $\gamma = 0.577216\ldots$ is the Euler constant. V_c is the Coulomb (static) potential. Its $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ corrections have been determined in Refs. [10], [11] and [24], respectively. $V_{\rm BF}$ is the Breit-Fermi potential known from positronium. It describes the Darwin and the spin-orbit interactions mediated by longitudinal gluons and the hyperfine interactions mediated by transverse gluons in the instantaneous approximation. $V_{\rm NA}$ is a purely non-

[4] We would like to note that this argument only holds for the case of $t\bar{t}$ production because the large quark top width, through replacement (1), ensures that the scale $M_t v^2$ remains perturbative.

Abelian potential generated through non-analytic terms in the one-loop vertex corrections to the Coulomb potential involving the triple gluon vertex (see, e.g., Refs. [25,26]). The nonrelativistic correlators are directly related to the Green function of the Schrödinger equation

$$\left(-\frac{\vec{\nabla}^2}{M_t}-\frac{\vec{\nabla}^4}{4M_t^3}+V_c(\vec{r})+V_{\rm BF}(\vec{r})+V_{\rm NA}(\vec{r})-\tilde{E}\right)G(\vec{r},\vec{r}\,',\tilde{E})=\delta^{(3)}(\vec{r}-\vec{r}\,'),\quad (13)$$

where $V_{\rm BF}$ is evaluated for the ${}^3S_1$ configuration only. The correlator $\mathcal{A}_1$ reads

$$\mathcal{A}_1=6\,N_c\,[\lim_{|\vec{r}|,|\vec{r}\,'|\to 0}G(\vec{r},\vec{r}\,',\tilde{E})\,]\,.\quad (14)$$

Relation (14) can be easily inferred by taking into account that the Green function $G(\vec{r},\vec{r}\,',\tilde{E})$ describes the propagation of a top-quark–top-antiquark pair which is produced and annihilated at distances $|\vec{r}|$ and $|\vec{r}\,'|$, respectively [1,2]. Because the exact solution of Eq. (13) seems to be an impossible task, we rely on a numerical solution of the equation

$$\left(-\frac{\vec{\nabla}^2}{M_t}+V_c(\vec{r})-\tilde{E}\right)G_c(\vec{r},\vec{r}\,',\tilde{E})=\delta^{(3)}(\vec{r}-\vec{r}\,')\quad (15)$$

using techniques developed in Refs. [2,4]. The result for $G_c(0,0,\tilde{E})$ is then combined with the corrections to the leading order (LO) Coulomb Green function $G_c^{(0)}$ [27] [defined through Eq. (15) for $V_c(\vec{r})=-C_F a_s/r$] coming from the kinetic energy correction and the potentials $V_{\rm BF}$ and $V_{\rm NA}$. These corrections are calculated analytically using Rayleigh-Schrödinger time-independent perturbation theory,

$$\delta G(\vec{r},\vec{r}\,',\tilde{E})=\int d\vec{x}^3\,G_c^{(0)}(\vec{r},\vec{x},\tilde{E})\left[\frac{\vec{\nabla}^4}{4M_t^3}-V_{\rm BF}(\vec{x})-V_{\rm NA}(\vec{x})\right]G_c^{(0)}(\vec{x},\vec{r}\,',\tilde{E})\,.\quad (16)$$

For the calculation of Eq. (16) we use techniques employed in Ref. [28], where the Abelian NNLO contributions have already been determined. (See also Ref. [29] for a more detailed presentation.) The final result for $\mathcal{A}_1$ at NNLO reads

$$\mathcal{A}_1=6\,N_c\,[G_c(0,0,\tilde{E})-G_c^{(0)}(0,0,\tilde{E})]+\frac{N_c\,C_F\,a_s\,M_t^2}{2\pi}\left(1+\frac{3\,C_A}{2\,C_F}\right)\left\{i\,\tilde{v}-C_F\,a_s\left[\ln\left(-i\,\frac{M_t\,\tilde{v}}{\mu_{\rm fac}}\right)+\gamma+\Psi\left(1-i\,\frac{C_F\,a_s}{2\,\tilde{v}}\right)\right]\right\}^2$$
$$+\frac{3\,N_c\,M_t^2}{2\,\pi}\left\{i\,\tilde{v}\left(1+\frac{5}{8}\tilde{v}^2\right)-C_F\,a_s\,(1+2\,\tilde{v}^2)\left[\ln\left(-i\,\frac{M_t\,\tilde{v}}{\mu_{\rm fac}}\right)+\gamma+\Psi\left(1-i\,\frac{C_F\,a_s\left(1+(11/8)\,\tilde{v}^2\right)}{2\,\tilde{v}}\right)\right]\right\},\quad (17)$$

where

$$\tilde{v}\equiv\sqrt{\frac{\tilde{E}}{M_t}},\quad (18)$$

and Ψ is the digamma function, $\Psi(z)\equiv(d/dz)\ln\Gamma(z)$. In the first line of Eq. (17) the LO Green function has been subtracted to avoid double counting of the LO contribution contained in the third line. It should be noted that the limit $|\vec{r}|,|\vec{r}\,'|\to 0$ in expression (16) causes UV divergences which are regulated using the short-distance cutoff $\mu_{\rm fac}$. Further, all power divergences $\propto\mu_{\rm fac}/M_t$ are subtracted [13] and $\mu_{\rm fac}$ is defined in a way that between the brackets in expression (17) all constants except the Euler number are absorbed. We also note that we have suppressed the factorization scale dependence in the first line of Eq. (17) because it is contained entirely in the real part which is irrelevant for the cross section in Eq. (5) [1,27]. The corresponding result in any other regularization scheme which suppresses power divergences could be obtained from the one presented here through a redefinition of the factorization scale. For $\mathcal{A}_2$ only the LO contribution in Eq. (17) is relevant and we arrive at

$$\mathcal{A}_2=\tilde{v}^2\frac{3\,N_c\,M_t^4}{2\,\pi}\left\{i\,\tilde{v}-C_F\,a_s\left[\ln\left(-i\,\frac{M_t\,\tilde{v}}{\mu_{\rm fac}}\right)+\gamma+\Psi\left(1-i\,\frac{C_F\,a_s}{2\,\tilde{v}}\right)\right]\right\}.\quad (19)$$

There are no non-Abelian contributions to $\mathcal{A}_2$.

Step 2: Matching calculation. The contributions to C_1 up to $\mathcal{O}(\alpha_s^2)$ are determined by carrying out the direct matching procedure [15]. For that we extract the Born, one- and two-loop contributions from Eq. (5) through an expansion in α_s up to order α_s^2 for stable quarks ($\Gamma_t=0$), and we set $\mu_{\rm soft}=\mu_{\rm hard}$.[5] To obtain the corresponding contributions $\propto\alpha_s^2$ originating from

[5]For $\alpha_s\ll v\ll 1$, i.e., far away from the threshold regime, a distinction between the soft and hard scale is irrelevant.

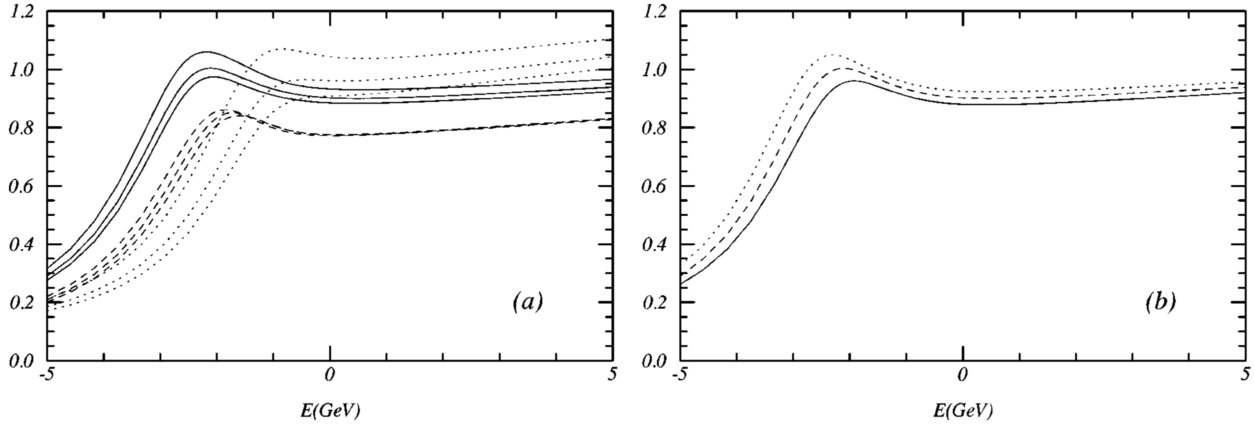

FIG. 1. (a) The total normalized photon-mediated $t\bar{t}$ cross section at LO (dotted lines), NLO (dashed lines), and NNLO (solid lines) for the soft scales $\mu_{\rm soft}=50$ (upper lines), 75 and 100 GeV (lower lines). (b) The NNLO cross section for $\alpha_s(M_z)=0.115$ (solid line), 0.118 (dashed line), and 0.121 (dotted line). More details and the other parameters are given in the text.

the one-loop corrections to the potential V_c we employ time-independent perturbation theory in analogy to Eq. (16). For the resulting expression we demand, for positive E, equality to the two-loop cross section calculated in full QCD and expanded in the velocity up to NNLO. Because the cross sections in NRQCD and in full QCD have the same infrared behavior, we can use this equality to determine C_1 at $\mathcal{O}(\alpha_s^2)$. The cross section in full QCD and expanded up to NNLO in the velocity reads $[a_h\equiv\alpha_s(\mu_{\rm hard}),\ v\equiv(E/M_t)^{1/2}]$

$$R^{\rm NNLO}_{2\text{ loop QCD}}=N_cQ_t^2\Bigg(\bigg[\frac{3}{2}v-\frac{17}{16}v^3\bigg]+\frac{C_Fa_h}{\pi}\bigg[\frac{3\pi^2}{4}-6v+\frac{\pi^2}{2}v^2\bigg]$$
$$+a_h^2\Bigg\{\frac{C_F^2\,\pi^2}{8\,v}+\frac{3}{2}C_F\Bigg[-2C_F+C_A\bigg(-\frac{11}{24}\ln\frac{4v^2M_t^2}{\mu_{\rm hard}^2}+\frac{31}{72}\bigg)+T\,n_l\bigg(\frac{1}{6}\ln\frac{4\,v^2\,M_t^2}{\mu_{\rm hard}^2}-\frac{5}{18}\bigg)\Bigg]$$
$$+\Bigg[\frac{49\,C_F^2\,\pi^2}{192}+\frac{3}{2}\kappa+\frac{C_F}{\pi^2}\bigg(\frac{11}{2}C_A-2\,T\,n_l\bigg)\ln\frac{M_t^2}{\mu_{\rm hard}^2}-C_F\bigg(C_F+\frac{3}{2}C_A\bigg)\ln\,v\Bigg]v\Bigg\}\Bigg), \qquad (20)$$

where

$$\kappa=C_F^2\bigg[\frac{1}{\pi^2}\bigg(\frac{39}{4}-\zeta_3\bigg)+\frac{4}{3}\ln 2-\frac{35}{18}\bigg]$$
$$-C_A\,C_F\bigg[\frac{1}{\pi^2}\bigg(\frac{151}{36}+\frac{13}{2}\zeta_3\bigg)+\frac{8}{3}\ln 2-\frac{179}{72}\bigg]$$
$$+C_F\,T\bigg[\frac{4}{9}\bigg(\frac{11}{\pi^2}-1\bigg)\bigg]+C_F\,T\,n_l\bigg[\frac{11}{9\,\pi^2}\bigg]. \qquad (21)$$

The Born and $\mathcal{O}(\alpha_s)$ [30] contributions are standard. The $\mathcal{O}(\alpha_s^2)$ contributions are sorted according to the SU(3) group theoretical factors C_F^2 [31], C_AC_F [32], C_FT_n [33,34], and C_FT [33,35], respectively. The result for C_1 reads

$$C_1=1-4\,C_F\frac{a_h}{\pi}+a_h^2\Bigg[\kappa+\frac{C_F}{\pi^2}\bigg(\frac{11}{3}C_A-\frac{4}{3}T\,n_l\bigg)\ln\frac{M_t^2}{\mu_{\rm hard}^2}$$
$$+C_F\bigg(\frac{1}{3}C_F+\frac{1}{2}C_A\bigg)\ln\frac{M_t^2}{\mu_{\rm fac}^2}\Bigg]. \qquad (22)$$

The consistency of the direct matching procedure ensures that C_1 does not contain any energy-dependent terms. We would like to point out that the factorization scale dependence of Im$[\mathcal{A}_1]$ is cancelled by the factorization scale dependence in C_1 up to a small term proportional to $C_F\,\alpha_s\,(\Gamma_t/M_t)\ln(M_t/\mu_{\rm fac})$ (see also Ref. [28]). This term remains as a consequence of our ignorance of a consistent treatment of the finite width effects at the NNLO level. Due to the small size of this contribution, however, the corresponding ambiguity can be ignored for the examination of the relativistic NNLO corrections determined in this paper. The LO cross section can be recovered from the NNLO one in Eq. (5) by taking into account only the dominant contributions in the third line of Eq. (17) and setting $C_1=1$, $C_2=0$, whereas the NLO cross section can be obtained by incorporating also the $\mathcal{O}(\alpha_s)$ corrections to the Coulomb potential and to the constant C_1.

In Fig. 1(a) the LO (dotted lines), NLO (dashed lines), and NNLO (solid lines) normalized cross sections are plotted versus E in the range -5 GeV$<E<5$ GeV for $M_t=175$ GeV, $\alpha_s(M_z)=0.118$, and $\Gamma_t=1.43$ GeV. For the scales the choices $\mu_{\rm soft}=50$ (upper lines), 75 and 100 GeV (lower lines), and $\mu_{\rm hard}=\mu_{\rm fac}=M_t$ have been made and two-loop running of the strong coupling has been used. It is evident that the NNLO corrections are large. Compared to the NLO cross section, the $1S$ peak is shifted towards smaller energies by several hundred MeV and the large negative NLO corrections for positive energies are compensated to some extent. Whereas the location of the $1S$ peak is quite insensitive to changes in the soft scale, the residual dependence of the normalization of the NNLO cross section on the soft scale $\mu_{\rm soft}$ is not improved at all compared to the NLO cross section. In fact, for energies above the $1S$ peak it is worse for the NNLO cross section than for the NLO one. The dependence of the NNLO cross section on the hard scale $\mu_{\rm hard}$ and the factorization scale $\mu_{\rm fac}$ are, on the other hand, much smaller and, therefore, not displayed here. The behavior of the NNLO corrections clearly indicates that the convergence of the perturbative series for the $t\bar{t}$ cross section is much worse than expected from the general arguments given by Fadin and Khoze [1]. For the normalization of the cross section we estimate, at least at the present stage, a theoretical uncertainty at the level of five to ten percent. For comparison, in Fig. 1(b) the NNLO cross section is displayed for $\alpha_s(M_z)=0.115$ (solid line), 0.118 (dashed line) and 0.121 (dotted line), and $\mu_{\rm soft}=75$ GeV. The other parameters are chosen as before. A more detailed examination of the NNLO contributions will be carried out in a future publication.

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