# Decay constants of heavy-light mesons in Bakamjian-Thomas heavy quark models

V. Morénas\*

Laboratoire de Physique Corpusculaire, Université Blaise Pascal-CNRS/IN2P3 F-63000 Aubière Cedex, France

A. Le Yaouanc,<sup>†</sup> L. Oliver,<sup>‡</sup> O. Pène,<sup>§</sup> and J.-C. Raynal

Laboratoire de Physique Théorique et Hautes Énergies, Université de Paris XI, Bâtiment 210, F-91405 Orsay Cedex, France

(Received 7 October 1997; published 29 October 1998)

In the quark models of the Bakamjian-Thomas type that yield covariance and Isgur-Wise scaling of form factors in the heavy quark limit we compute the decay constants  $f^{(n)}$  and  $f_{1/2}^{(n)}$  of *S*- and *P*-wave mesons composed of heavy and light quarks. Lorentz invariance of the decay constants and heavy quark limit scaling  $\sqrt{M}f$  = const is obtained, and it is shown that this class of models satisfies the sum rules involving decay constants and Isgur-Wise functions recently formulated by us in the heavy quark limit of QCD. Moreover, the model also satisfies selection rules of the type  $f_{3/2}^{(n)} = 0$ , which must hold in this limit. We discuss different *Ansätze* for the dynamics of the mass operator at rest. For nonrelativistic kinetic energies  $p^2/2m$ , the decay constants are finite even if the potential V(r) has a Coulomb part. For the relativistic form  $\sqrt{p^2 + m^2}$ , the *S*-wave decay constants diverge if there is a Coulomb singularity. Using phenomenological models of the spectrum with relativistic kinetic energy and a regularized short-distance part (Godfrey-Isgur model or Richardson potential of Colangelo *et al.*) that yield  $\rho^2 \cong 1$  for the elastic Isgur-Wise function, we compute the decay constants in the heavy quark limit and obtain  $f_B \cong 300$  MeV, of the same order although slightly smaller than in the static limit of lattice QCD. We find the decay constants of  $D^{**}$  with  $j = \frac{1}{2}$  of the same order of magnitude. The convergence of the heavy quark limit sum rules is also studied. [S0556-2821(98)04919-4]

PACS number(s): 12.39.Hg, 13.20.-v

#### I. INTRODUCTION

In a recent paper [1] we have shown that quark models of hadrons with a fixed number of constituents, based on the Bakamjian-Thomas (BT) formalism [2], yield form factors that are covariant and satisfy Isgur-Wise (IW) scaling [3] in the heavy mass limit for one of the quarks. In this class of models a lower bound is predicted for the slope of the heavy meson elastic IW function  $\rho^2 > \frac{3}{4}$ . Moreover, the model satisfies the Bjorken-Isgur-Wise sum rule [4], which relates the slope of the IW function to the *P*-wave form factors  $\tau_{1/2}(w)$ ,  $\tau_{3/2}(w)$  at zero recoil [5]. We have also explicitly computed the P-wave meson wave functions and the corresponding inelastic IW functions [6] within the model, and we have made a numerical study of  $\rho^2$  in a wide class of models of the meson spectrum (each of them characterized by an Ansatz for the mass operator M, i.e., the dynamics of the system at rest) [7], and a phenomenological study [8] of the elastic and inelastic IW functions and the corresponding  $d\Gamma/dw$  for  $B \rightarrow D, D^*, D^{**}l\nu$ .

The purpose of the present paper is to study the decay constants of heavy mesons within the same approach.

## II. DECAY CONSTANTS IN THE BT SCHEME: HEAVY QUARK SCALING

We want to compute in the model the decay constants of mesons  $Q\bar{q}$ , where Q and q are a heavy and a light quark, defined by

$$\langle P(0^{-})(v) | A_{\mu}^{qQ} | 0 \rangle = N f_{P} \sqrt{M} v_{\mu},$$
  
$$\langle V(1^{-})(v,\varepsilon) | V_{\mu}^{qQ} | 0 \rangle = N f_{V} \sqrt{M} \varepsilon_{\mu}^{*}, \qquad (1)$$

for S states, where  $N=1/\sqrt{2v^0}$  and M is the mass of the corresponding bound state  $(M=M_P \text{ or } M=M_V)$ , and similarly

$$\langle ^{1/2}0^{+}(v)|V^{qQ}_{\mu}|0\rangle = Nf^{(1/2)}\sqrt{M}v_{\mu},$$
  
$$\langle ^{1/2}1^{+}(v,\varepsilon)|A^{qQ}_{\mu}|0\rangle = Ng^{(1/2)}\sqrt{M}\varepsilon^{*}_{\mu},$$
  
$$\langle ^{3/2}1^{+}(v,\varepsilon)|A^{qQ}_{\mu}|0\rangle = Ng^{(3/2)}\sqrt{M}\varepsilon^{*}_{\mu},$$
 (2)

for *P* states. The spin of the light degrees of freedom combines with orbital angular momentum adding to total conserved j = 1/2, 3/2, indicated in the preceding equations. The model satisfies the heavy quark limit relations (we use the phase convention for states of Ref. [6])

$$f_P = f_V = f,\tag{3}$$

$$g^{(1/2)} = -f^{(1/2)}, \quad g^{(3/2)} = 0.$$
 (4)

In terms of the internal wave function, i.e., the eigenfunction of the mass operator M which commutes with the global

<sup>\*</sup>Email address: morenas@clermont.in2p3.fr

<sup>&</sup>lt;sup>†</sup>Email address: leyaouan@qcd.th.u-psud.fr

<sup>&</sup>lt;sup>‡</sup>Email address: oliver@qcd.th.u-psud.fr

<sup>&</sup>lt;sup>§</sup>Email address: pene@qcd.th.u-psud.fr

Poincaré generators [1], we get [the superscript (n) means any radial excitation], for S-wave mesons:

$$\sqrt{M}f^{(n)} = \sqrt{N_c}\sqrt{2} \\ \times \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \sqrt{(p_2 \cdot v)(p_2 \cdot v + m_2)}\varphi^{(n)}(\mathbf{k}_2),$$
(5)

where the wave functions at rest of  $0^-$  mesons are given by

$$\varphi_{s_{1},s_{2}}^{(n)}(\mathbf{k}_{2}) = \frac{i}{\sqrt{2}} (\sigma^{2})_{s_{1},s_{2}} \varphi^{(n)}(\mathbf{k}_{2}),$$

$$\int \frac{d\mathbf{p}}{(2\pi)^{3}} |\varphi^{(n)}(\mathbf{p})|^{2} = 1,$$
(6)

and, for P wave mesons,

$$\sqrt{M} f_{1/2}^{(n)} = \sqrt{\frac{2}{3}} \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{p_2 \cdot v}}{\sqrt{p_2 \cdot v + m_2}} \\ \times \varphi_{1/2}^{(n)}(\mathbf{k}_2) [(p_2 \cdot v)^2 - m_2^2], \tag{7}$$

where the wave function at rest of  $0^+$  mesons is given by [5]

$$\varphi_{s_{1},s_{2}}^{(n)}(\mathbf{k}_{2}) = -\frac{i}{\sqrt{6}} \left[ (\boldsymbol{\sigma} \cdot \mathbf{k}_{2}) \boldsymbol{\sigma}^{2} \right]_{s_{1},s_{2}} \varphi_{1/2}^{(n)}(\mathbf{k}_{2}),$$

$$\int \frac{d\mathbf{p}}{(2\pi)^{3}} \frac{p^{2}}{3} |\varphi_{1/2}^{(n)}(\mathbf{p})|^{2} = 1.$$
(8)

In these expressions,

$$p_2 = B_v k_2, \tag{9}$$

where  $B_v$  is the boost operator. The expressions (5) and (7) are Lorentz invariant since, due to the *rotational invariance* of  $\varphi^{(n)}(\mathbf{k}_2)$ ,  $\varphi_{1/2}^{(n)}(\mathbf{k}_2)$ , they are functions of  $v^2 = 1$ . One can take simply the rest frame  $v = (1, \mathbf{0})$ .

Let us give a brief outline of the calculation. Within the model [1], the internal wave function in motion of an S-wave meson with center-of-mass momentum  $\mathbf{P}$  (*n* denotes the radial excitation) is given in terms of

$$\Psi_{s_1s_2}^{(n)}(\mathbf{P}-\mathbf{p}_2,\mathbf{p}_2) = \sqrt{\frac{\Sigma p_j^0}{M_0}} \prod_{i=1,2} \frac{\sqrt{k_i^0}}{\sqrt{p_i^0}} \sum_{\{s_i'\}} \prod_{i=1,2} \left[ D_i(\mathbf{R}_i) \right]_{s_is_i'} \varphi_{s_1's_2'}^{(n)}(\mathbf{k}_2), \tag{10}$$

where  $\varphi_{s_1s_2}^{(n)}(\mathbf{k}_2)$  is the internal wave function at rest Eq. (5), and  $\mathbf{R}_i$  are the Wigner rotations:

$$\mathbf{R}_i = \mathbf{B}_{p_i}^{-1} \mathbf{B}_{\Sigma p_j} \mathbf{B}_{k_i}.$$
 (11)

The  $\bar{q}Q$  to vacuum matrix element of a current, e.g.,  $J = \bar{q} \gamma_{\mu} \gamma_5 Q$ , reads

$$O_{s_1s_2}(\mathbf{P} - \mathbf{p}_2, \mathbf{p}_2) = \langle 0 | J | \mathbf{P} - \mathbf{p}_2 s_1; \mathbf{p}_2 s_2 \rangle = [u_{s_1}(\mathbf{P} - \mathbf{p}_2)]^t i \gamma^2 \gamma^0 \gamma_\mu \gamma_5 u_{s_2}(\mathbf{p}_2).$$
(12)

The matrix element of interest to us,

$$\langle 0|A^{\mu}|P^{(n)}\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sum_{\{s_i\}} O_{s_1s_2}(\mathbf{P} - \mathbf{p}_2, \mathbf{p}_2) \Psi^{(n)}_{s_2s_1}(\mathbf{P} - \mathbf{p}_2, \mathbf{p}_2),$$
(13)

can be written, after some algebra, using Eqs. (10)-(12),

$$\langle 0|A^{\mu}|P^{(n)}\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sqrt{\frac{\Sigma p_j^0}{M_0}} \prod_{i=1,2} \frac{\sqrt{k_i^0}}{\sqrt{p_i^0}} \sqrt{\frac{m_1}{p_1^0}} \sqrt{\frac{m_2}{p_2^0}} \varphi^{(n)}(\mathbf{k}_2) \times \frac{1}{\sqrt{2}} \operatorname{Tr} \left\{ \frac{1+\gamma^0}{2} \mathbf{B}_{p_1}^{-1} \gamma^{\mu} \mathbf{B}_{p_2} \frac{1+\gamma^0}{2} \mathbf{B}_{p_2}^{-1} \mathbf{B}_{\Sigma p_j} \mathbf{B}_{k_2} \mathbf{B}_{k_1}^{-1} \mathbf{B}_{\Sigma p_j}^{-1} \mathbf{B}_{p_1} \right\},$$
(14)

and since the projector  $(1 + \gamma^0)/2$  commutes with the Wigner rotations (11), one can rewrite

$$\langle 0|A^{\mu}|P^{(n)}\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sqrt{\frac{\sum p_j^0}{M_0}} \prod_{i=1,2} \frac{\sqrt{k_i^0}}{\sqrt{p_i^0}} \sqrt{\frac{m_1}{p_1^0}} \sqrt{\frac{m_2}{p_2^0}} \varphi^{(n)}(\mathbf{k}_2) \frac{1}{\sqrt{2}} \operatorname{Tr} \left\{ \gamma^{\mu} \mathbf{B}_u \mathbf{B}_{k_2} \frac{1+\gamma^0}{2} \mathbf{B}_u^{-1} \mathbf{B}_u \frac{1+\gamma^0}{2} \mathbf{B}_{k_1}^{-1} \mathbf{B}_u^{-1} \right\},$$
(15)

where  $u = (p_1 + p_2)/\sqrt{(p_1 + p_2)^2}$ . Using the identities

$$\mathbf{B}_{u}\mathbf{B}_{k_{2}}\frac{1+\gamma^{0}}{2}\mathbf{B}_{u}^{-1} = \frac{m_{2}+\not{p}_{2}}{\sqrt{2m_{2}(k_{2}^{0}+m_{2})}}\frac{1+\not{u}}{2}, \quad \mathbf{B}_{u}\frac{1+\gamma^{0}}{2}\mathbf{B}_{k_{1}}^{-1}\mathbf{B}_{u}^{-1} = \frac{1+\not{u}}{2}\frac{m_{1}+\not{p}_{1}}{\sqrt{2m_{1}(k_{1}^{0}+m_{1})}}, \tag{16}$$

one obtains

$$\langle 0|A^{\mu}|P^{(n)}\rangle = \frac{\sqrt{N_c}}{8} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} F(\mathbf{p}_2, \mathbf{P}) \operatorname{Tr}[\gamma^{\mu}(m_2 + \not p_2)(1 + \not u)(m_1 + \not p_1)]\varphi^{(n)}(\mathbf{k}_2), \tag{17}$$

with  $k^2 = \mathbf{B}_u^{-1} p^2$  and

$$F(\mathbf{p}_2, \mathbf{P}) = \sqrt{2} \, \frac{\sqrt{u^0}}{p_1^0} \, \frac{\sqrt{k_1^0}}{\sqrt{k_1^0 + m_1}} \, \frac{\sqrt{k_2^0}}{\sqrt{k_2^0 + m_2}}.$$
(18)

Finally, in the heavy mass limit [1],

$$u \to v, \quad \frac{p_1}{m_1} \to v, \quad \frac{k_1^0}{m_1} \to 1, \quad k_2^0 \to (\mathbf{B}_v^{-1} p_2)^0 = p_2 \cdot v,$$
 (19)

one obtains

$$\langle 0|A^{\mu}|P^{(n)}\rangle = \frac{1}{\sqrt{2v^0}} \frac{\sqrt{N_c}}{2\sqrt{2}} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{p_2 \cdot v}}{\sqrt{p_2 \cdot v + m_2}}, \quad \mathrm{Tr}[\gamma^{\mu}(m_2 + \not\!\!\!\!/ p_2)(1 + \not\!\!\!\!/ v)]\varphi^{(n)}(\mathbf{k}_2), \tag{20}$$

where  $k_2 = \mathbf{B}_v^{-1} p_2$  and  $\varphi^{(n)}(\mathbf{k}_2)$  depends on  $\mathbf{k}_2^2 = (p_2 \cdot v)^2 - m_2^2$  because of rotational invariance.

Thus the integrand (20) must be proportional to  $v^{\mu}$ , yielding Eq. (5) for the decay constant. This expression exhibits the expected heavy quark limit scaling. Moreover, varying the current, one can easily show relations (3). It is also worth noticing that expression (5) becomes the well-known nonrelativistic expression [9] if we assume that the light quark is also nonrelativistic.

Let us give some details of the calculation of Eqs. (6) and the corresponding relations (4). Mutatis mutandis, we consider the matrix element  $\langle 0|V^{\mu}|0^{+}\rangle$  with the  $0^{+}$  wave functions given by Eqs. (8). After some algebra, the matrix element equivalent to Eq. (15) is, in the present case,

$$\langle 0|V^{\mu}|0^{+}\rangle = \sqrt{N_{c}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \sqrt{\frac{\Sigma p_{j}^{0}}{M_{0}}} \prod_{i=1,2} \frac{\sqrt{k_{i}^{0}}}{\sqrt{p_{i}^{0}}} \sqrt{\frac{m_{1}}{p_{1}^{0}}} \sqrt{\frac{m_{2}}{p_{2}^{0}}} \varphi_{1/2)}^{(n)}(\mathbf{k}_{2})$$

$$\times \frac{1}{\sqrt{6}} \operatorname{Tr} \left\{ \gamma^{\mu} \gamma_{5} \mathbf{B}_{u} \mathbf{B}_{k_{2}} \frac{1+\gamma^{0}}{2} \mathbf{B}_{u}^{-1} \mathbf{B}_{u} (\boldsymbol{\sigma} \cdot \mathbf{k}_{2}) \mathbf{B}_{u}^{-1} \mathbf{B}_{u} \frac{1+\gamma^{0}}{2} \mathbf{B}_{k_{1}}^{-1} \mathbf{B}_{u}^{-1} \right\}.$$

$$(21)$$

Using Eqs. (16) and taking the limit (19), we obtain

$$\langle 0|V^{\mu}|0^{+}\rangle = \frac{\sqrt{N_{c}}}{4\sqrt{6}} \frac{1}{\sqrt{2v^{0}}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \frac{1}{p_{2}^{0}} \frac{\sqrt{p_{2} \cdot v}}{\sqrt{p_{2} \cdot v + m_{2}}} \varphi_{1/2}^{(n)}(\mathbf{k}_{2}) \operatorname{Tr}[-\gamma^{\mu}(m_{2} + \not{p}_{2})(1 + \not{v})(-\not{v}\not{p}_{2} + p_{2} \cdot v)(1 - \not{v})], \quad (22)$$

and using the fact that this expression must be proportional to  $v^{\mu}$ , we obtain expression (7), scaling invariant in the heavy quark limit. Moreover, varying the current and considering instead the matrix element  $\langle 0|A^{\mu}|1_{1/2}^{+}\rangle$ , one can easily verify the first relation of Eqs. (4).

The result  $g^{(3/2)}=0$ , which must hold in the heavy quark limit [10], deserves a few details. The matrix element to be considered is (*m* denotes the polarization of the state 1<sup>+</sup> and *n* the radial excitation)

$$\langle 0|A^{\mu}|1_{3/2}^{+},m\rangle = \sqrt{N_{c}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \sqrt{\frac{\Sigma p_{j}^{0}}{M_{0}}} \prod_{i=1,2} \frac{\sqrt{k_{i}^{0}}}{\sqrt{p_{i}^{0}}} \sqrt{\frac{m_{1}}{p_{1}^{0}}} \sqrt{\frac{m_{2}}{p_{2}^{0}}} \\ \times \operatorname{Tr} \left\{ \left( \frac{1+\gamma^{0}}{2} \sigma_{2} \mathbf{B}_{p_{1}}^{-1} i \gamma^{0} \gamma_{5} \gamma^{0} \gamma^{\mu} \gamma_{5} \mathbf{B}_{p_{2}} \frac{1+\gamma^{0}}{2} \right) [D(\mathbf{R}_{2})] [\varphi^{(n)}(\mathbf{k}_{2},m)]^{t} [D^{t}(\mathbf{R}_{1})] \right\},$$
(23)

where the wave function is, for the  $1^+ j = \frac{3}{2}$  states [6],

$$\varphi_{s_1,s_2}^{(n)}(\mathbf{k}_2,m) = \frac{i}{\sqrt{2}} \, \mathbf{e}^{(m)} \cdot \left[ \sqrt{\frac{2}{3}} \, \mathbf{k}_2 - \frac{i}{\sqrt{6}} \, (\mathbf{k}_2 \times \boldsymbol{\sigma}) \right] \sigma_2 \varphi_{3/2}^{(n)}(\mathbf{k}_2), \tag{24}$$

where  $\mathbf{e}^{(m)}$  is a unit vector and the rotational invariant function  $\varphi_{3/2}^{(n)}(\mathbf{k}_2)$  is normalized according to Eqs. (8). The equivalent expression to Eq. (21) will be

$$\langle 0|A^{\mu}|1_{3/2}^{+},m\rangle = \sqrt{N_{c}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \sqrt{\frac{\Sigma p_{j}^{0}}{M_{0}}} \prod_{i=1,2} \frac{\sqrt{k_{i}^{0}}}{\sqrt{p_{i}^{0}}} \sqrt{\frac{m_{1}}{p_{1}^{0}}} \sqrt{\frac{m_{2}}{p_{2}^{0}}} \frac{\varphi_{3/2}^{(n)}(\mathbf{k}_{2})}{\sqrt{2}} \\ \times \operatorname{Tr} \left\{ \gamma^{\mu} \mathbf{B}_{u} \mathbf{B}_{k_{2}} \frac{1+\gamma^{0}}{2} \mathbf{e}^{(m)} \cdot \left[ \sqrt{\frac{2}{3}} \mathbf{k}_{2} + \frac{i}{\sqrt{6}} (\mathbf{k}_{2} \times \boldsymbol{\sigma}) \right] \frac{1+\gamma^{0}}{2} \mathbf{B}_{k_{1}}^{-1} \mathbf{B}_{u}^{-1} \right\}.$$
 (25)

We need to compute the expression  $\mathbf{B}_{u}\mathbf{e}^{(m)} \cdot \left[\sqrt{\frac{3}{2}}\mathbf{k}_{2} + (i/\sqrt{6})(\mathbf{k}_{2} \times \boldsymbol{\sigma})\right]\mathbf{B}_{u}^{-1}$  in the heavy quark limit. After some algebra, one gets

$$\mathbf{B}_{u}\mathbf{e}^{(m)} \cdot \left[ \sqrt{\frac{2}{3}} \mathbf{k}_{2} + \frac{i}{\sqrt{6}} \left( \mathbf{k}_{2} \times \boldsymbol{\sigma} \right) \right] \mathbf{B}_{u}^{-1} \rightarrow \frac{1}{\sqrt{6}} \left[ -\boldsymbol{\epsilon}_{v}^{(m)}\boldsymbol{p}_{2} + \boldsymbol{\epsilon}_{v}^{(m)}\boldsymbol{v}(p_{2} \cdot v) - \boldsymbol{\varepsilon}_{v}^{(m)} \cdot p_{2} \right],$$
(26)

where  $\varepsilon_v^{(m)}$  is the axial meson polarization in the heavy quark limit. Using Eqs. (16) in the limit (19), we obtain, finally,

$$\langle 0|A^{\mu}|1_{3/2}^{+},m\rangle = \frac{\sqrt{N_{c}}}{8\sqrt{3}} \frac{1}{\sqrt{2v^{0}}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \frac{1}{p_{2}^{0}} \frac{\sqrt{p_{2} \cdot v}}{\sqrt{p_{2} \cdot v + m_{2}}} \varphi_{3/2}(\mathbf{k}_{2}) \times \operatorname{Tr}\{\gamma^{\mu}(m_{2} + p_{2})(1 + p)[-\boldsymbol{\epsilon}_{v}^{(m)}\boldsymbol{p}_{2} + \boldsymbol{\epsilon}_{v}^{(m)}\boldsymbol{\psi}(p_{2} \cdot v) - \boldsymbol{\epsilon}_{v}^{(m)} \cdot p_{2}](1 + p)\}.$$

$$(27)$$

This expression is covariant and satisfies heavy quark scaling. In the rest frame, particularizing to m=0, one gets, because of rotational invariance of  $\varphi_{3/2}(\mathbf{k}_2)$ ,

$$\langle 0|A^{z}|1_{3/2}^{+}, \mathbf{P}=\mathbf{0}, m=0 \rangle = \frac{\sqrt{N_{c}}}{\sqrt{3}} \frac{1}{\sqrt{2v^{0}}} \int \frac{d\mathbf{k}_{2}}{(2\pi)^{3}} \frac{1}{k_{2}^{0}} \frac{\sqrt{k_{2}^{0}}}{\sqrt{k_{2}^{0}+m_{2}}} \varphi_{3/2}(\mathbf{k}_{2}) \{-[(k_{2}^{x})^{2}+(k_{2}^{y})^{2}]+2(k_{2}^{z})^{2}\}=0.$$
(28)

Therefore the second relation of Eqs. (4),  $g^{(3/2)} = 0$ , follows. The vanishing of  $g^{(3/2)}$  can be seen also from the following covariant argument. Contracting Eq. (27) with the four-vector  $\varepsilon_v^{(m)}$ , we see that  $g^{(3/2)}$  is proportional to the integral

$$g^{(3/2)} \sim \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{p_2 \cdot v}}{\sqrt{p_2 \cdot v + m_2}} \varphi_{3/2}(\mathbf{k}_2) \\ \times [3(\varepsilon_v^{(m)} \cdot p_2)^2 - (v \cdot p_2)^2 + m_2^2].$$
(29)

Two types of integrals appear:

$$I_{\mu\nu} = \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{p_2 \cdot v}}{\sqrt{p_2 \cdot v + m_2}} \varphi_{3/2}(\mathbf{k}_2) p_{2\mu} p_{2\nu}, \quad (30)$$

$$I = \int \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{p_2 \cdot v}}{\sqrt{p_2 \cdot v + m_2}} \varphi_{3/2}(\mathbf{k}_2).$$
(31)

From covariance it follows that

$$I_{\mu\nu} = A v_{\mu} v_{\nu} + B g_{\mu\nu},$$
  
$$I = C,$$
 (32)

where A,B,C are constants. Contracting  $I_{\mu\nu}$  with  $g_{\mu\nu}$  it follows that

$$4B + A = m_2^2 C \tag{33}$$

and therefore

$$g^{(3/2)} \sim 4B + A - m_2^2 C = 0. \tag{34}$$

### **III. SUM RULES**

Let us now show that the QCD heavy quark limit sum rules [10]

$$X(w) \equiv \sum_{n} \frac{f^{(n)}}{f^{(0)}} \xi^{(n)}(w) = 1,$$
(35)

DECAY CONSTANTS OF HEAVY-LIGHT MESONS IN ...

$$T_{1/2}(w) \equiv \sum_{n} \frac{f_{1/2}^{(n)}}{f^{(0)}} \tau_{1/2}^{(n)}(w) = \frac{1}{2}$$
(36)

are satisfied within the present BT scheme.

In order to prove Bjorken sum rule, in Ref. [5] we used the completeness relation at *fixed*  $\mathbf{P}$  that holds in the BT formalism for the internal wave functions. We proceed here in the same way. We need to compute expressions of the form

$$\sum_{n} \langle 0|\tilde{O}|\mathbf{P}',n\rangle\langle\mathbf{P}',n|O|\mathbf{P},0\rangle, \qquad (37)$$

where  $\langle 0|\tilde{O}|\mathbf{P}',n\rangle$  and  $\langle \mathbf{P}',n|O|\mathbf{P},0\rangle$  will be related, respectively, to the decay constants and to the Isgur-Wise func-

tions. We need  $\mathbf{P}'$  different from  $\mathbf{P}$  in order to demonstrate the sum rules (35) and (36) for any value of the scaling variable *w*.

To obtain the sum rule (35) for *S* waves, the simplest way is to take  $\mathbf{P}'=0$  for the intermediate states and choose the currents  $O = \overline{Q}' \gamma^{\nu} Q$  and  $\widetilde{O} = \overline{q} \gamma^{\mu} \gamma_5 Q'$  with  $\mu = 0$ , where *Q* and *Q'* are heavy quarks (e.g., *b* and *c*) and *q* is a light quark. Then only the 0<sup>-</sup> intermediate states contribute, because the 1<sup>+</sup> states do not at  $\mathbf{P}'=0$ , since then the time component of the polarization  $\varepsilon_0^{(m)}=0$ . Alternatively, to obtain the sum rule (36) for *P* waves, we will take  $\mathbf{P}'=0$  and O $= \overline{Q}' \gamma^{\nu} \gamma_5 Q$ , and  $\widetilde{O} = \overline{q} \gamma^{\mu} Q'$  with  $\mu = 0$ . Then only the 0<sup>+</sup> intermediate states contribute, since the 1<sup>-</sup> do not at  $\mathbf{P}' = 0$ .

We start from the completeness relation at fixed  $\mathbf{P}'$ :

$$\sum_{n} \Psi_{s_{1}'',s_{2}''}^{(n)}(\mathbf{P}'-\mathbf{p}_{2}'',\mathbf{p}_{2}'')\Psi_{s_{1}'s_{2}'}^{(n)*}(\mathbf{P}'-\mathbf{p}_{2}',\mathbf{p}_{2}') = \delta_{s_{1}'s_{1}''}\delta_{s_{2}'s_{2}''}(2\pi)^{3}\delta(\mathbf{p}_{2}'-\mathbf{p}_{2}'').$$
(38)

We need the matrix elements

$$\langle \mathbf{P}', n | O | \mathbf{P}, 0 \rangle = \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_2'}{(2\pi)^3} \sum_{\{s_i\}} \sum_{\{s_i'\}} \Psi_{s_1's_2'}^{n(\mathbf{P}')*}(\mathbf{p}_1', \mathbf{p}_2') O_{s_1's_1}(\mathbf{p}_1', \mathbf{p}_1) \Psi_{s_1s_2}^{0(\mathbf{P})}(\mathbf{p}_1, \mathbf{p}_2) (2\pi)^3 \delta(\mathbf{p}_2 - \mathbf{p}_2') \delta_{s_2's_2}, \quad (39)$$

$$\langle 0|\tilde{O}|\mathbf{P},n\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sum_{\{s_i\}} \tilde{O}_{s_1s_2}(\mathbf{p}_1,\mathbf{p}_2) \Psi_{s_2s_1}^{n(\mathbf{P})}(\mathbf{p}_1,\mathbf{p}_2), \tag{40}$$

related, respectively, to the IW functions and decay constants. In these expressions,

$$O(\mathbf{p}_{1}',\mathbf{p}_{1}) = \frac{\sqrt{m_{1}m_{1}'}}{\sqrt{p_{1}^{0}p_{1}'^{0}}} \frac{1+\gamma^{0}}{2} \mathbf{B}_{p_{1}'}^{-1} O \mathbf{B}_{p_{1}} \frac{1+\gamma^{0}}{2},$$
(41)

$$\tilde{O}(\mathbf{P}' - \mathbf{p}_2, \mathbf{p}_2) = \sqrt{\frac{m_1}{p_1'^0}} \sqrt{\frac{m_2}{p_2^0}} \frac{1 + \gamma^0}{2} \sigma_2 \mathbf{B}_{p_1'}^{-1} i \gamma^0 \gamma_5 \tilde{O} \mathbf{B}_{p_2} \frac{1 + \gamma^0}{2},$$
(42)

and here O and  $\tilde{O}$  are the Dirac matrices of the corresponding currents. After some algebra we obtain, using the completeness relation (38),

$$\sum_{n} \langle 0|\tilde{O}|\mathbf{P}',n\rangle\langle\mathbf{P}',n|O|\mathbf{P},0\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sum_{\{s_i\}} \sum_{s_1'} \tilde{O}_{s_1's_2}(\mathbf{P}'-\mathbf{p}_2,\mathbf{p}_2)O_{s_1's_1}(\mathbf{P}'-\mathbf{p}_2,\mathbf{P}-\mathbf{p}_2)\Psi_{s_1s_2}^{(0)}(\mathbf{P}-\mathbf{p}_2,\mathbf{p}_2), \quad (43)$$

where the ground state wave function is given by [1]

$$\Psi_{s_{1}s_{2}}^{(0)}(\mathbf{P}-\mathbf{p}_{2},\mathbf{p}_{2}) = \sqrt{\frac{\Sigma p_{j}^{0}}{M_{0}}} \prod_{i=1,2} \frac{\sqrt{k_{i}^{0}}}{\sqrt{p_{i}^{0}}} \sum_{\{s_{i}'\}} [D(\mathbf{R}_{1})]_{s_{1}s_{1}'} [D(\mathbf{R}_{2})]_{s_{2}s_{2}'} \varphi_{s_{1}'s_{2}'}^{(0)}(\mathbf{k}_{2})$$

$$= \sqrt{\frac{\Sigma p_{j}^{0}}{M_{0}}} \prod_{i=1,2} \frac{\sqrt{k_{i}^{0}}}{\sqrt{p_{i}^{0}}} [D(\mathbf{R}_{1})\varphi^{(0)}(k_{2})D^{t}(\mathbf{R}_{2})]_{s_{1}s_{2}}, \qquad (44)$$

and using Eqs. (6), in an obvious notation,

$$\sum_{n} \langle 0|\tilde{O}|\mathbf{P}',n\rangle\langle\mathbf{P}',n|O|\mathbf{P},0\rangle = \sqrt{N_c} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \sqrt{\frac{\Sigma p_j^0}{M_0}} \prod_{i=1,2} \frac{\sqrt{k_i^0}}{\sqrt{p_i^0}} \frac{i}{\sqrt{2}} \varphi^{(0)}(\mathbf{k}_2) \\ \times \operatorname{Tr}[\tilde{O}^t(\mathbf{P}'-\mathbf{p}_2,\mathbf{p}_2)O(\mathbf{P}'-\mathbf{p}_2,\mathbf{P}-\mathbf{p}_2)D(\mathbf{R}_1)\boldsymbol{\sigma}_2 D^t(\mathbf{R}_2)],$$
(45)

where the matrices  $\mathbf{R}_i$  are given by Eq. (11).

To isolate the  $0^-$  states, we take, as argued above:

$$\sum_{n} \langle 0|A^{0}|\mathbf{0},n\rangle\langle\mathbf{0},n|V^{\nu}|\mathbf{P},0\rangle = \sqrt{N_{c}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \sqrt{\frac{\Sigma p_{j}^{0}}{M_{0}}} \prod_{i=1,2} \frac{\sqrt{k_{i}^{0}}}{\sqrt{p_{i}^{0}}} \sqrt{\frac{m_{1}m_{1}'}{p_{1}^{0}p_{1}'^{0}}} \sqrt{\frac{m_{2}}{p_{2}^{0}}} \frac{i}{\sqrt{2}} \varphi^{(0)}(\mathbf{k}_{2}) \\ \times \operatorname{Tr}\left\{ \left[\frac{1+\gamma^{0}}{2} \sigma_{2} \mathbf{B}_{p_{1}'}^{-1} i \gamma^{0} \mathbf{B}_{p_{2}} \frac{1+\gamma^{0}}{2}\right]^{t} \left[\frac{1+\gamma^{0}}{2} \mathbf{B}_{p_{1}'}^{-1} \gamma^{0} \gamma^{\nu} \mathbf{B}_{p_{1}} \frac{1+\gamma^{0}}{2}\right] D(\mathbf{R}_{1}) \sigma_{2} D^{t}(\mathbf{R}_{2}) \right\}.$$
(46)

Using  $\sigma_2(\mathbf{B}_p)^t \sigma_2 = \mathbf{B}_p^{-1}$ ,  $\sigma_2(\gamma^{\mu})^t \sigma_2 = \gamma^{\mu}$ ,  $\mathbf{B}_p[(1+\gamma^0)/2]\mathbf{B}_p^{-1} = \frac{1}{2}(1+p/m)$ , the fact that  $\frac{1}{2}(1+\gamma^0)$  commutes with rotations, and the relations

$$\mathbf{B}_{v}\mathbf{B}_{k_{1}}\mathbf{B}_{v}^{-1} = \frac{m_{1} + \not p_{1}\not v}{\sqrt{2m_{1}(k_{1}^{0} + m_{1})}}, \quad \mathbf{B}_{v}\mathbf{B}_{k_{2}}^{-1}\mathbf{B}_{v}^{-1} = \frac{m_{2} + \not v \not p_{2}}{\sqrt{2m_{2}(k_{2}^{0} + m_{2})}}, \tag{47}$$

one obtains, after some algebra, in the heavy quark limit  $(p_1 \rightarrow m_1 v, k_1^0 \rightarrow m_1)$ , for the right-hand side (RHS) of Eq. (46),

$$\sqrt{N_{c}} \frac{1}{\sqrt{v^{0}v'^{0}}} \frac{1}{\sqrt{v'^{0}}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \frac{1}{p_{2}^{0}} \sqrt{m_{2}} \sqrt{k_{2}^{0}} \frac{1}{\sqrt{2}} \varphi^{(0)}(\mathbf{k}_{2}) \operatorname{Tr} \left\{ \frac{1}{2} (1+\gamma^{0}) \gamma^{\nu} \frac{1}{2} (1+\psi) \mathbf{B}_{v} \mathbf{B}_{k_{1}} \mathbf{B}_{k_{2}}^{-1} \mathbf{B}_{v}^{-1} \right\}$$

$$= \sqrt{N_{c}} \frac{1}{\sqrt{4v^{0}v'^{0}}} \frac{1}{\sqrt{2v'^{0}}} \int \frac{d\mathbf{p}_{2}}{(2\pi)^{3}} \frac{1}{p_{2}^{0}} \frac{\sqrt{p_{2} \cdot v}}{\sqrt{p_{2} \cdot v + m_{2}}} \frac{1}{\sqrt{2}} \varphi^{(0)}(\mathbf{k}_{2}) \frac{1}{2}$$

$$\times \operatorname{Tr} \{ (1+\gamma^{0}) \gamma^{\nu} (1+\psi) (m_{2}+\psi_{2}) \}$$

$$= \frac{1}{\sqrt{4v^{0}v'^{0}}} \frac{1}{\sqrt{2v'^{0}}} \sqrt{M} f^{(0)} (g^{0\nu} + v^{\nu}).$$
(48)

On the other hand, the LHS of Eq. (46) reads

$$\sum_{n} \langle 0|A^{0}|\mathbf{0},n\rangle\langle\mathbf{0},n|V^{\nu}|\mathbf{P},0\rangle = \frac{1}{\sqrt{4v^{0}v'^{0}}} \frac{1}{\sqrt{2v'^{0}}} \sum_{n} \sqrt{M} f_{P}^{(n)} \xi^{(n)}(w) (g^{0\nu} + v^{\nu}),$$
(49)

and therefore the sum rule (35) follows.

The sum rule for the *P* states, Eq. (36), follows straightforwardly in similar manner by taking  $\mathbf{P}'=0$  and  $O=\bar{Q}'\gamma^{\nu}\gamma_{5}Q$ ,  $\tilde{O}=\bar{q}\gamma^{\mu}Q'$  with  $\mu=0$ , since then only the 0<sup>+</sup> intermediate states contribute, as pointed out above. However, to illustrate the methods of calculation in the BT formalism, we will now verify the sum rule (36) from the general expressions for  $f_{1/2}^{(n)}$  and  $\tau_{1/2}^{(n)}(w)$ , without appealing to any particular frame.

We want to evaluate the expression

$$T_{1/2}(w)f^{(0)} = \sum_{n} f^{(n)}_{1/2} \tau^{(n)}_{1/2}(w).$$
(50)

From the explicit expressions (7) for  $f_{1/2}^{(n)}$  and  $\tau_{1/2}^{(n)}(w)$  from Ref. [6],

$$\tau_{1/2}^{(n)}(w) = \frac{1}{2\sqrt{3}} \int \frac{d\mathbf{p}_2'}{(2\pi)^3} \frac{1}{p_2'^0} \frac{\sqrt{(p_2' \cdot v')(p_2' \cdot v)}}{\sqrt{(p_2' \cdot v + m_2)(p_2' \cdot v' + m_2)}} \varphi_{1/2}^{(n)}(\mathbf{k}_2')^* \varphi^{(0)}(\mathbf{k}_2'') \times \frac{(p_2' \cdot v')(p_2' \cdot v + m_2) - (p_2' \cdot v)(p_2' \cdot v + wm_2) + (1 - w)m_2^2}{1 - w},$$
(51)

with

$$k_2' = \mathbf{B}_v^{-1} p_2', \quad k_2'' = \mathbf{B}_{v'}^{-1} p_2'.$$
 (52)

The computation of the sum (50) leads to the expression

$$\sum_{n} \varphi_{1/2}^{(n)}(\mathbf{k}_{2}')^{*} \varphi_{1/2}^{(n)}(\mathbf{p}_{2}) = 6 \pi^{2} \frac{1}{\mathbf{p}_{2}^{2} \mathbf{k}_{2}'^{2}} \delta(|\mathbf{p}_{2}| - |\mathbf{k}_{2}'|),$$
(53)

where the RHS follows from Ref. [6]. One can then perform the integration over  $\mathbf{p}_2$  that amounts to replacing all  $p_2^0$  by  $(p'_2 \cdot v)$ . Realizing then that

$$\frac{(p_2' \cdot v')(p_2' \cdot v + m_2) - (p_2' \cdot v)(p_2' \cdot v + wm_2) + (1 - w)m_2^2}{1 - w} = \mathbf{k}_2'^2 \left( -1 + \frac{p_2' \cdot v' - w(p_2' \cdot v)}{(p_2' \cdot v - m_2)(1 - w)} \right),$$
(54)

one gets

$$2T_{1/2}(w)f^{(0)} = 2\sum_{n} f_{1/2}^{(n)}\tau_{1/2}^{(n)}(w) = f^{(0)} - \sqrt{\frac{2N_c}{\pi}} \int \frac{d\mathbf{p}_2'}{(2\pi)^3} \frac{1}{p_2'^0} \frac{\sqrt{(p_2' \cdot v')}}{\sqrt{(p_2' \cdot v' + m_2)}} \varphi^{(0)}(\mathbf{k}_2'') \frac{p_2' \cdot v - w(p_2' \cdot v')}{1 - w}.$$
 (55)

The second term involves the integral

$$\int \frac{d\mathbf{p}_{2}'}{(2\pi)^{3}} \frac{1}{p_{2}'^{0}} \frac{\sqrt{(p_{2}' \cdot v')}}{\sqrt{(p_{2}' \cdot v' + m_{2})}} \varphi^{(0)}(\mathbf{k}_{2}'') p_{\mu}' = C v_{\mu}',$$
(56)

where *C* is some constant, because of covariance. Inserting the RHS of Eq. (56) into Eq. (55), we see that the second term of the RHS of Eq. (55) vanishes for  $w \neq 1$ , and then the sum rule (36) follows. For w=1, one can choose  $v = (v^0, \mathbf{v}), v' = (1, \mathbf{0})$  and make an expansion of  $(w-1)^{-1}$ for  $\mathbf{v} \rightarrow \mathbf{0}$ . Then the second term on the RHS of Eq. (55) vanishes from rotational invariance. The same method allows one also to obtain the sum rule (35) along similar lines.

## **IV. NUMERICAL RESULTS**

As explained at length in Ref. [1], the dynamics in the BT formalism depends on the form of the mass operator M at P=0. This mass operator can be of any form

$$M = K(\{\mathbf{k}_i\}) + V(\{\mathbf{r}_i, \mathbf{p}_i\}), \qquad (57)$$

and one obtains covariance of the form factors and IW scaling with only the very general assumption of the rotational invariance of *M*. We are going now to give the results for the decay constants for various *Ansätze* of the operator *M*, not only various forms for the potential  $V({\bf r}_i, {\bf p}_i)$ , but also for the kinetic energy  $K({\bf k}_i)$ , which can be taken to be of the nonrelativistic  ${\bf k}_i^2/2m_i$  or relativistic  $\sqrt{{\bf k}_i^2 + m_i^2}$  forms. We choose such models in order to emphasize the physics involved in the decay constants, sensitive to the short-distance part of the potential and to the scaling behavior of the kinetic energy (quadratic or linear in *k*). Of course, any scheme that we adopt should give a reasonable fit of the whole meson spectrum. The success of quark models in the description of the spectrum with either nonrelativistic or relativistic kinetic

energies shows that the spectrum by itself does not constrain the form of this kinetic energy. Interestingly, we have shown in Ref. [7] that quark models of form factors in the BT formalism show a clear preference for the relativistic form of the kinetic energy, since they give a slope of the elastic IW function  $\rho^2 \cong 1$ , while models with a nonrelativistic kinetic energy give a larger, phenomenologically unacceptable value.

We list here a number of phenomenological quark models of the hadron spectrum that we will use, specifying their interesting features.

(1) Isgur-Scora-Grinstein-Wise (ISGW) spectroscopic model [11] (to be distinguished from the ISGW nonrelativistic model of form factors): nonrelativistic kinetic energy  $\mathbf{k}_i^2/2m_i$  with linear plus Coulomb potential.

(2) Veseli-Dunietz (VD) model [12]: relativistic kinetic energy  $\sqrt{\mathbf{k}_i^2 + m_i^2}$  with linear plus Coulomb potential. The Coulomb part is not regularized.

(3) Godfrey-Isgur (GI) model [13]: relativistic kinetic energy  $\sqrt{\mathbf{k}_i^2 + m_i^2}$  with linear plus a regularized short-distance part. This scheme also incorporates the fine structure of the potential.

(4) Richardson potential with relativistic kinetic energy, as used by Colangelo, Nardulli, and Pietroni (CNP) [14]. This model exhibits the asymptotic freedom behavior at short distances, but the Coulomb singularity (logarithmically corrected) is regularized by a cut at small r.

These models are solved numerically using a harmonic oscillator basis. In Table I we give the masses and decay constants of the states n=0, l=0 or 1 as a function of  $N_{\text{max}}$  in the different models.  $N_{\text{max}}$  means the number of radial excitations included in the truncated basis, the ground state Gaussian plus up to  $N_{\text{max}}$  radially excitated harmonic oscillator wave functions:  $N_{\text{max}}+1$  is then the dimension of the truncated Hilbert space. In Table II we give the decay constants of a number of radial excitations in the GI model.

TABLE I. Decay constants of n=0, l=0 and n=0, l=1,  $j=\frac{1}{2}$  mesons for the various spectroscopic models. The ISGW is nonrelativistic. In the VD, GI, and CNP models, the kinetic energy is relativistic. In the VD model the *S*-wave decay constants diverge due to the Coulomb singularity. In the GI and CNP models the Coulomb singularity is regularized and the decay constants are finite.  $N_{\text{max}}$  stands for the number of radial excitations included in the truncated variational harmonic oscillator basis.

Model	$M_Q$ (GeV)	N <sub>max</sub>	$\begin{array}{c} M^{(0)} - M_Q \\ (\text{GeV}) \end{array}$	$M_{1/2}^{(0)} - M_Q$ (GeV)	$\frac{\sqrt{M}f^{(0)}}{(\text{GeV}^{3/2})}$	$\sqrt{M} f_{1/2}^{(0)}$ (GeV <sup>3/2</sup> )
ISGW [11]	$10^{4}$	10	0.0438	0.5467	0.422	0.235
	$10^{4}$	15	0.0436	0.5467	0.428	0.235
	$10^{4}$	20	0.0435	0.5467	0.431	0.236
		infinite			finite	finite
VD [12]	$10^{4}$	10	0.119	0.620	1.36	0.603
	$10^{4}$	15	0.108	0.620	1.58	0.617
	$10^{4}$	20	0.108	0.620	1.76	0.631
		infinite			infinite	finite
GI [13]	$10^{4}$	10	0.386	0.792	0.649	0.620
	$10^{4}$	15	0.386	0.792	0.662	0.632
	$10^{4}$	20	0.386	0.792	0.667	0.640
	$10^{4}$	infinite			finite	finite
CNP [14]	$10^{4}$	10	0.389	0.859	0.747	0.669
	$10^{4}$	15	0.387	0.858	0.798	0.691
	$10^{4}$	20	0.386	0.858	0.828	0.704
		infinite			finite	finite

Let us begin our discussion with the models with relativistic kinetic energy. A first important remark to make is that dimensional analysis implies that a Hamiltonian (or mass operator in the BT formalism) with kinetic energy of the relativistic form  $\sqrt{\mathbf{k}_i^2 + m_i^2}$  and a Coulomb potential implies divergent wave functions at the origin  $\Psi^{(n)}(0)$  (or decay constants  $f^{(n)}$ ) for S waves. The reason for this behavior is that the kinetic and potential energies exhibit the same scaling properties at large momentum or small distances. The

TABLE II. Decay constants  $f^{(n)}$  and  $f^{(n)}_{1/2}$  for the first radial excitations of l=0 and l=1,  $j=\frac{1}{2}$  mesons in the GI spectroscopic model. The error in parentheses is estimated by comparing the number of radial excitations included in the truncated variational harmonic oscillator basis,  $N_{\text{max}}=20$  and 10. For  $n \ge 5$  the error becomes larger than 20%.

Radial excitation	$\frac{\sqrt{M}f^{(n)}}{(\text{GeV}^{3/2})}$	$\sqrt{M}f_{1/2}^{(n)}$ (GeV <sup>3/2</sup> )
n = 0	0.67(2)	0.64(2)
n = 1	0.73(4)	0.66(4)
n=2	0.76(5)	0.71(5)
n=3	0.78(9)	0.73(8)
n=4	0.80(10)	0.76(11)
n = 5	0.81(17)	0.77(17)
n = 6	0.82(15)	0.78(15)
n = 7	0.82(28)	0.78(27)
n=8	0.83(25)	0.79(25)
n=9	0.80(40)	0.76(40)
n = 10	0.83(42)	0.79(40)

relativistic kinetic energy is not efficient enough in smoothing the *r*-space wave function. This divergence of the wave function at the origin is the cause of the existence of a critical coupling  $\alpha^{\text{crit}}$  in this class of models: for  $\alpha > \alpha^{\text{crit}}$  arises the so-called phenomenon of a fall in the center. For a discussion, see, for example, the paper by Hardekopf and Sucher [15]. On the contrary, the decay constants for *P*-wave mesons remain finite even in this case. A short-distance Coulomb part corrected by asymptotic freedom exhibits the same phenomenon, although  $f^{(n)}$  diverges only logarithmically in this latter case, instead of as a power in the former. These general features are exhibited by the model of Veseli and Dunietz [12], as shown in Table I, and have been underlined by these authors. In our Table I, the finite values obtained for  $\sqrt{M}f^{(0)}$  using the singular model for  $N_{\text{max}}=10, 15, \text{ or } 20$  are just an artifact of the truncation method.

The GI model [13], a model of the meson spectrum for all  $q\bar{q}$ ,  $Q\bar{q}$ , and  $Q\bar{Q}$  systems, chooses a short-distance part with a fully regularized Coulomb singularity. Within this model it is possible to compute both types of decay constants  $f^{(n)}$  and  $f_{1/2}^{(n)}$ . In Table I we give the scaling-invariant quantities  $\sqrt{M}f^{(0)}$  and  $\sqrt{M}f_{1/2}^{(0)}$  for S- and P-wave n=0 mesons, computed in the heavy quark limit.

We have also made the calculation for the CNP model (Richardson potential plus relativistic kinetic energy) [14].

In the case of a kinetic energy of the nonrelativistic form  $\mathbf{k}_i^2/2m_i$ , the *S*-wave decay constants are finite even in the presence of a Coulomb singularity. This feature is exemplified by the ISGW [11] model in Table I. Moreover, the decay constants are smaller in this case than for the models of relativistic kinetic energy, another manifestation of the singular behavior of the latter.



FIG. 1. Convergence of the heavy quark limit S-wave sum rule (35) in the GI model [13] for different values of the scaling variable w as one increases  $N_{\text{max}}$ , the number of radial levels included in the truncated variational harmonic oscillator basis. The Ox axis represents  $(N_{\text{max}}+1)^{-1}$  and the Oy axis  $\sum_{n=0}^{N_{\text{max}}} f^{(n)}/f^{(0)})\xi^{(n)}(w)$ , which the sum rule predicts to be equal to 1 for  $N_{\text{max}} \rightarrow \infty$ . The different lines correspond to w = 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, from up to down.

It is interesting to notice that in the heavy quark limit and for  $N_{\text{max}}$ =20 one gets, for the models with relativistic kinetic energy and regularized Coulomb singularity,

$$\sqrt{M}f^{(0)} = 0.67 \text{ GeV}^{3/2}$$
 (GI model),  
0.83 GeV<sup>3/2</sup> (CNP model),  
(58)

$$\sqrt{M}f_{1/2}^{(0)} = 0.64 \text{ GeV}^{3/2}$$
 (GI model),  
0.70 GeV<sup>3/2</sup> (CNP model).

Applying this *asymptotic* result to the *B* meson, one obtains

$$f_B \cong 300 - 350 \text{ MeV.}$$
 (59)

This value is not far away from, although slightly larger than, the values obtained in *lattice QCD in the static limit*, to which it should naturally be compared, which ranges between 220 and 290 MeV [16]. However, one should keep in mind that the lattice QCD result includes logarithmic corrections absent in our phenomenological scheme.

It is worth noticing that we obtain the same order of magnitude for the decay constant  $f_{1/2}^{(0)}$ . This is phenomenologically important because, reasoning within the factorization assumption, this means that the *emission* of  $D^{**}(0^+)$  and  $D^{**}(1^+, j = \frac{1}{2})$  is expected to be important in *B* decays. On the contrary, the emission of  $D^{**}(j = \frac{3}{2})$  will be suppressed. Table II shows that the decay constants of radial excitations are of the same order of magnitude as in the ground state. However, the error due to the truncation is larger as n increases.

Finally, let us study the convergence of the sum rules (35) and (36), which we have shown formally to hold, in models that give finite results (GI, CNP, and ISGW) for which we can compute the decay constants  $f^{(n)}$ ,  $f_{1/2}^{(n)}$  and the IW functions  $\xi^{(n)}(w)$ ,  $\tau_{1/2}^{(n)}(w)$  [8]. The convergence of the Bjorken-Isgur-Wise sum rule [4] has been studied in Ref. [8].

Let us define

$$X^{(N)}(w) = \sum_{n=0}^{N} \frac{f^{(n)}}{f^{(0)}} \xi^{(n)}(w),$$

$$T^{(N)}_{1/2}(w) = \sum_{n=0}^{N} \frac{f^{(n)}_{1/2}}{f^{(0)}} \tau^{(n)}_{1/2}(w).$$
(60)

We compute the sums for  $N=N_{\text{max}}$  in the different models for various values of w and see how they compare to the RHS of the sum rules (35) and (36). Let us recall that  $N_{\text{max}}$  is the maximal number of radial excitations included in the truncation method (the dimension of the variational base is  $N_{\text{max}}+1$ ). We show the results for the Godfrey-Isgur model in Figs. 1 and 2. The Ox axis represents  $1/(N_{\text{max}}+1)$ . We observe that these sums converge fairly well towards the RHS of the sum rules (respectively, 1 and  $\frac{1}{2}$ ) as we increase  $N_{\text{max}}$ . For fixed  $N_{\text{max}}$  we can ask how the partial sums  $X^{(N)}(w)$  and  $T_{1/2}^{(N)}(w)$  ( $N \le N_{\text{max}}$ ) behave as a function of N, i.e., how fast  $X^{(N)}(w)$ ,  $T_{1/2}^{(N)}(w)$  approach  $X^{(N_{\text{max}})}(w)$ ,  $T_{1/2}^{(N_{\text{max}})}(w)$  when N increases ( $N \ge 0$ ). Let us give the example  $N_{\text{max}} = 20$ . The convergence is rather fast, but it degrades as w increases. Concerning the X sum rule, for w= 1 one has trivially  $X^{(0)}(1) = X^{(N_{\text{max}})}(1)$ , because the ground



FIG. 2. Convergence of the heavy quark limit S-wave sum rule (36) in the GI model [13] for different values of the scaling variable w as one increases  $N_{\text{max}}$ , the number of radial levels included in the truncated variational harmonic oscillator basis. The Ox axis represents  $(N_{\text{max}}+1)^{-1}$  and the Oy axis  $\sum_{n=0}^{N_{\text{max}}} (f_{1/2}^{(n)}/f^{(0)}) \tau_{1/2}^{(n)}(w)$ , which the sum rule predicts to be equal to  $\frac{1}{2}$  for  $N_{\text{max}} \rightarrow \infty$ . The different lines correspond to w = 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, from up to down.

state saturates the sum, since  $\xi^{(n)}(1)=0$  (n>0). As one increases *w*, one one needs to sum up to N=3, 4, 5, 6, or 7, respectively, for w=1.1, 1.2, 1.3, 1.4, or 1.5 to approach  $X^{(N_{\text{max}})}(w)$  at the 5% level. Concerning the  $T_{1/2}$  sum rule, one needs N=3, 4, 5, 6, 7, or 8, respectively, for w=1.1, 1.2, 1.3, 1.4, or 1.5 to approach  $T_{1/2}^{(N_{\text{max}})}(w)$  at the 10% level.

For the CNP model and the nonrelativistic ISGW model, the convergences both in  $N_{\text{max}}$  and in N for fixed  $N_{\text{max}}$  are not as good. For the case of the VD model, one gets finite  $f_{1/2}^{(n)}$  and  $\tau_{1/2}^{(n)}(w)$ , but the convergence toward the RHS of the sum rule (30) does not improve as one increases  $N_{\text{max}}$ . This results from the divergence of the denominator  $f^{(0)}$  in Eqs. (35), (36) when  $N_{\text{max}} \rightarrow \infty$ .

In conclusion, we have studied the decay constants of heavy-light mesons in the heavy mass limit in a class of models of Bakamjian and Thomas, with different *Ansätze* for the dynamics at rest, with nonrelativistic or relativistic kinetic energies. Each particular model gives an acceptable phenomenological description of the spectrum. The models with relativistic kinetic energy, which yield a slope of the elastic IW function  $\rho^2 \cong 1$  (as shown in Ref. [7]), give finite decay constants if the Coulomb singularity of the potential is regularized, as in the GI model. At the *B* mass, one finds  $f_B$ slightly larger than in the static limit of lattice QCD. The decay constants of  $D^{**}$  with  $j = \frac{1}{2}$  are of the same order of magnitude. Moreover, we have shown that heavy quark limit sum rules involving decay constants [10] are satisfied by these class of quark models of the Bakamjian-Thomas type, and in the case of the Godfrey-Isgur model the convergence of the sum rules is quite fast.

### ACKNOWLEDGMENTS

LPTHE is Laboratoire associé au CNRS URA D0063.

- A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Lett. B 365, 319 (1996).
- [2] B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953).
- [3] N. Isgur and M. B. Wise, Phys. Lett. B 237, 527 (1990).
- [4] J. D. Bjorken, presented at Les Rencontres de Physique de la Vallée d'Aoste, La Thuile, SLAC Report No. SLAC-PUB-5278, 1990 (unpublished); N. Isgur and M. B. Wise, Phys. Rev. D 43, 819 (1991); J. D. Bjorken, I. Dunietz, and J. Taron, Nucl. Phys. B371, 111 (1992).
- [5] A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Lett. B 386, 304 (1996).

- [6] V. Morénas, A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Lett. B 386, 315 (1996).
- [7] V. Morénas, A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Lett. B 408, 357 (1997).
- [8] V. Morénas, A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Rev. D 56, 5668 (1997).
- [9] A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, *Hadron Transitions in the Quark Model* (Gordon and Breach, New York, 1988), Sec. 2.1.3.
- [10] A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Lett. B 387, 582 (1996).

- [11] N. Isgur, D. Scora, B. Grinstein, and M. B. Wise, Phys. Rev. D 39, 799 (1989).
- [12] S. Veseli and I. Dunietz, Phys. Rev. D 54, 6803 (1996).
- [13] S. Godfrey and N. Isgur, Phys. Rev. D 32, 189 (1985).
- [14] J. L. Richardson, Phys. Lett. 82B, 272 (1979); P. Colangelo,

G. Nardulli, and M. Pietroni, Phys. Rev. D 43, 3002 (1991).

- [15] G. Hardekopf and J. Sucher, Phys. Rev. A 31, 2020 (1985).
- [16] S. Aoki *et al.*, Nucl. Phys. B (Proc. Suppl.) 53, 356 (1997); C. Bernard *et al.*, *ibid.* 53, 358 (1997); G. M. de Divitiis, *ibid.* 53, 362 (1997).