Understanding the success of nonrelativistic potential models for relativistic quark-antiquark bound states

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We investigate the connection between relativistic potential models for quark-antiquark bound states and the nonrelativistic models that have been used successfully to fit and predict the spectra of relativistic systems, as in the work of Martin. We use Martin's operator inequality $\sqrt{p^2+m^2} \leqslant (p^2+M^2+m^2)/2M$ to motivate the approximation of the relativistic kinetic energy terms in the spinless Salpeter equation by expressions of the nonrelativistic form $M+\epsilon+p^2/2M$ for each quark. To investigate the validity of the resulting approximation numerically, we generate energy spectra for $q\bar{q}$ mesons composed of two light or two heavy quarks using the spinless Salpeter equation with the linear-plus-Coulomb potential typical of phenomenological fits to $q\bar{q}$ data, and then fit the lowest few states of each type using the effective Schrödinger description with the same potential. We find good fits to the lowest four calculated $c\bar{c}$ and the lowest three $s\bar{s}$ states either taking M fixed at the value $M_q=\sqrt{\langle p^2\rangle+m_q^2}$ that minimizes the Martin bound, or allowing M_q to vary in the fit. The energies of the lowest few $c\bar{s}$ states are then predicted with similar accuracy. The reasons for the success of the nonrelativistic approximation are identified, and explain the success of Martin's nonrelativistic predictions for the spectra of relativistic light-heavy mesons. However, we note that the agreement between the nonrelativistic and relativistic wave functions is not good, a point of potential concern for the calculation of transition matrix elements. [S0556-2821(98)00823-6]

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I. INTRODUCTION

The development of potential models to describe the energy spectra of mesonic and baryonic systems has proved extremely successful. Phenomenological models that use a simple relativistic kinetic energy term and a scalar potential that incorporates the linear confinement and the short-distance color-Coulomb interaction suggested by QCD give good descriptions of the observed spectra of both heavy- and light-quark mesons and baryons [1–5]. Moreover, Duncan, Eichten, and Thacker [6] have demonstrated a nontrivial connection between the relativistic potential models and rigorous numerical results from lattice QCD, showing that both the spectrum and the lattice wave functions for light-quark mesons are reproduced very well when the lattice potential is used in the relativistic wave equation

$$[\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}+V(r)]\psi(\mathbf{r})=E\psi(\mathbf{r}). \qquad (1)$$

This equation, the spinless Salpeter equation, can be derived as a limit of the full Salpeter equation in which the ''small-small'' components of the Salpeter wave function are neglected and spin effects are averaged out as discussed, for example, in [3]. In the expression above p is the momentum of either quark in the center-of-momentum frame, m_1 and m_2 are the quark masses, and $V(r)$ is the effective potential between the quarks.

We will be concerned here with the description of mesonic systems described as quark-antiquark bound states $q\bar{Q}$, where the quarks q and Q may be the same or different. We assume that these systems can be described by the spinless Salpeter equation as demonstrated in [6], and will take $V(r)$ as the linear-plus-Coulomb potential used in much phenomenological work. This also gives a good approximation to the lattice potential. For heavy quarks, the kinetic terms in Eq. (1) can be expanded in inverse powers of the quark mass to obtain the usual nonrelativistic Schrödinger Hamiltonian. This gives successful descriptions of the $b\bar{b}$ and $c\bar{c}$ states [7], even though the latter are close to being relativistic. More surprisingly, Martin [8,9] showed that a nonrelativistic model based on a power-law potential could be extended to include the clearly relativistic $s\bar{s}$ states, and was able using that model to predict successfully the masses of a number of then unmeasured light-heavy states [10].

Although numerical methods have been developed which allow one to treat a relativistic kinetic term as easily as a nonrelativistic term [3,11–13], it is important to understand why an ostensibly nonrelativistic treatment works and allows useful predictions to be made for relativistic systems as in the work of Martin and others. In this paper, we explore this problem theoretically, and develop a nonrelativistic approximation to the Hamiltonian in Eq. (1) based on an effective-mass expansion of the kinetic energy terms. We then study the accuracy of this approximation in reproducing the energy spectra and wave functions of relativistic $q\bar{Q}$ bound states by

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using the corresponding Schrödinger equation to fit ''data'' obtained by solving the spinless Salpeter equation. We find that it is possible to fit the energy spectra for the low-lying energy levels to within a few MeV for both heavy-heavy ''$c\bar{c}$'' and light-light ''$s\bar{s}$'' states. We are then able, using the nonrelativistic description, to predict the energies of the low-lying $c\bar{s}$ states to within 11 MeV. The effective quark mass M found in the fits is considerably larger than either the input quark mass m or the natural effective mass $\sqrt{\langle p^2\rangle + m^2}$ expected from various arguments [14,10]. However, we can also obtain quite good fits to the relativistic $c\bar{c}$ and $s\bar{s}$ spectra, and good absolute predictions for the $c\bar{s}$ energies, using $M=\sqrt{\langle p^2\rangle + m^2}$.

We study the $q\bar{q}$ wave functions in detail, and find qualitative agreement between the relativistic and nonrelativistic functions in the regions in which both are large provided the effective quark mass is used as a parameter in the fitting procedure. However, systematic differences are evident, and the nonrelativistic wave functions can be seriously in error locally, a problem that can limit the usefulness of the approximate wave functions in calculations of such quantities as transition matrix elements.

In the next section, we develop the theory of the nonrelativistic approximation. We then outline the numerical techniques used to determine the energy spectra and wave functions, and discuss the results of the heavy and light fits and the light-heavy predictions in Sec. III, and summarize our conclusions in Sec. IV.

II. THEORETICAL BACKGROUND

Our objective is to approximate the relativistic potential model defined by the spinless Salpeter equation using a nonrelativistic Schrödinger description. Since only a small number of low-lying heavy- and light-quark bound states are actually known experimentally, an approximation will be successful for practical purposes if it reproduces the wave functions and the energy spectra for those limited sets of states. We will suppose that the potential $V(r)$ is known and is kept fixed.[1] This requires that our approximation for the kinetic energy terms in Eq. (1) be accurate in some average sense for the low-lying states in both the heavy- and light-quark systems.

Our nonrelativistic approximation to the kinetic terms is suggested by Martin's [9] operator bound

$$\sqrt{p^2+m^2} \leqslant \frac{M}{2} + \frac{p^2}{2M} + \frac{m^2}{2M}, \tag{2}$$

valid for an arbitrary mass M. The right hand side of this equation has the form of a nonrelativistic kinetic energy operator with an effective mass M, plus an additive constant that shifts the total energy. Given M, the equality in Eq. (2) holds in momentum space at the momentum $p_0 = \sqrt{M^2-m^2}$. The curves defined by the two sides of the inequality are tangent at that point, suggesting that M be defined in terms of m and the point of tangency p_0 by

$$M^2 = m^2 + p_0^2. \tag{3}$$

Because the Martin bound is an operator relation, the inequality in Eq. (2) holds for expectation values in single states, and for averages of expectation values over sets of states for arbitrary values of M or p_0. The special choice $p_0^2 = \langle p^2\rangle$ puts the point of equality in Eq. (2) at the average value of p^2 for the state or set of states under consideration. We would expect this choice for p_0^2 to yield a reasonably accurate nonrelativistic approximation for the relativistic kinetic energy, a point noted in different contexts by other authors [7,14,15,10]. More important theoretically, the corresponding effective mass $M=\sqrt{\langle p^2\rangle + m^2}$ minimizes the average value of the right hand side of Eq. (2), so gives a least upper bound for the average of the relativistic kinetic energies when the average is calculated using the actual eigenfunctions for the relativistic problem. Using this value for M we obtain the relation

$$\sqrt{p^2+m^2} \leqslant M + \frac{p^2}{2M} - \frac{\langle p^2\rangle}{2M}, \quad M=\sqrt{\langle p^2\rangle + m^2}. \tag{4}$$

The physical content of this result can be illustrated through a direct expansion of the square root operator. The standard expansion

$$\sqrt{p^2+m^2} = m + \frac{p^2}{2m} - \frac{p^4}{8m^3} + \cdots \tag{5}$$

in powers of p^2/m^2 may be reliable for heavy-quark systems, but fails for light-quark systems. A possible solution to this problem is to consider an expansion about a fixed momentum p_0^2,

$$\sqrt{p^2+m^2} = \sqrt{p^2 - p_0^2 + M^2} = M + \frac{p^2-p_0^2}{2M} - \frac{(p^2-p_0^2)^2}{8M^3} + \cdots \tag{6}$$

[1]Possible differences between the effective potentials $V(r)$ for the heavy- and light-quark systems are outside our concern. However, we note that the potential is often varied in making phenomenological fits to data on different systems. This eases the problem of fitting the heavy- and light-quark systems together.

where $M=\sqrt{m^2+p_0^2}$. The expansion will give a good average approximation to the relativistic kinetic energy provided the relevant values of p^2 are concentrated near p_0^2 with $\langle(p^2-p_0^2)^2\rangle \ll M^4$.[2] The numerator in this ratio has its minimum value for $p_0^2=\langle p^2\rangle$. A comparison of Eqs. (2) and (6) shows that the net effect of all the terms in Eq. (6) beyond the simple nonrelativistic result $M+p^2/2M$ is to decrease the kinetic energy. Note that the ''relativistic correction'' $-(p^2-p_0^2)^2/8M^3$ to the kinetic energy operator in Eq. (6) does not have the standard form $-p^4/8M^3$, and would be expected to be much smaller in magnitude for p_0^2 close to $\langle p^2\rangle$.

To remove the strict inequality in Eq. (2) in the following discussion, we will allow for an energy shift ϵ' that includes the average contribution of the ''relativistic corrections'' in Eq. (6), taken as constant, and will use a nonrelativistic approximation to the relativistic kinetic energy operator of the form

$$\sqrt{p^2+m^2}\approx M+\frac{p^2}{2M}+\frac{1}{2}\epsilon, \tag{7}$$

where

$$\epsilon=-\frac{\langle p^2\rangle}{M}+\epsilon'\approx-\frac{\langle p^2\rangle}{M}-\frac{\langle(p^2-\langle p^2\rangle)^2\rangle}{4M^3}+\cdots. \tag{8}$$

The content of this approximation is best illustrated in momentum space. In Fig. 1, we compare a model relativistic operator with $m=0.5$ GeV, $\langle p^2\rangle=3.75$ GeV^2, and $M=2$ GeV with the nonrelativistic approximation in Eq. (4). The curves corresponding to the relativistic and nonrelativistic expressions are tangent at $p^2=\langle p^2\rangle$.

For all other momenta, the nonrelativistic approximation lies above the actual relativistic kinetic energy, as expected from the Martin bound. To improve the agreement between the operators for momenta away from the point of tangency,

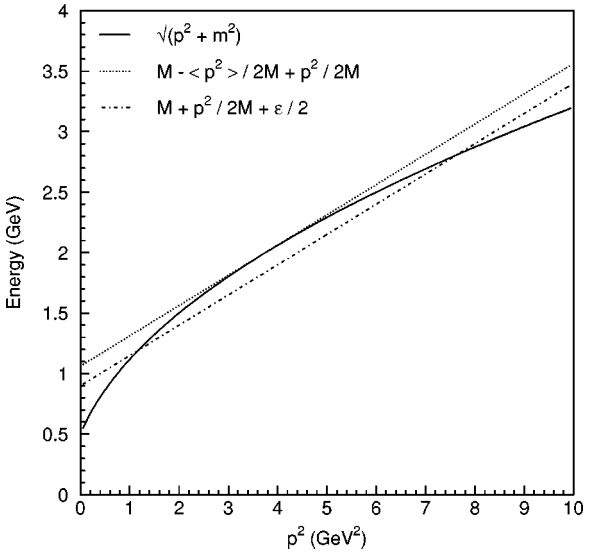

FIG. 1. We show the relation between the relativistic kinetic energy operator and the nonrelativistic approximation given by the Martin bound, Eq. (2). The effective mass M and the quark mass m are related by $M^2=m^2+\langle p^2\rangle$. The values used are $M=2$ GeV and $m=0.5$ GeV which give the local equality at $p^2=3.75$ GeV^2. We also plot the nonrelativistic approximation in Eq. (7) with an energy shift $\epsilon/2=-1.1$ GeV instead of the shift $-\langle p^2\rangle/2M=-0.94$ GeV given by the optimum Martin bound. The agreement between the two expressions is improved at low and high momenta by the extra shift $\epsilon'=-0.16$ GeV defined in Eq. (8).

we can add a negative shift ϵ' to the nonrelativistic approximation as suggested above and shown in Fig. 1. Because of the negative curvature of the relativistic kinetic energy, it is also advantageous to increase the value of M relative to $\sqrt{\langle p^2\rangle+m^2}$ to move the point of tangency outward and reduce the slope of the nonrelativistic curve. This will be seen in our numerical results. The quality of the resulting approximation is evident in Fig. 2, in which we compare the exact and approximate kinetic energies for the $c\bar{c}$ system over the region in which the wave function for the second excited $c\bar{c}$ state is large. The products of the kinetic energy operators with the squares of the momentum-space wave functions for the Salpeter and Schrödinger equations are compared in Fig. 3. The details and interpretation of the fit are discussed in Sec. III B.

III. NUMERICAL INVESTIGATION OF THE NONRELATIVISTIC APPROXIMATION

In this section, we will explore the accuracy of the nonrelativistic approximations derived above in the case of the $c\bar{c}$, $s\bar{s}$, and $c\bar{s}$ systems by comparing the results for the energy spectra and wave functions obtained by solving the corresponding Salpeter and Schrödinger equations. The relativistic kinetic energy operators will be approximated as in Eq. (7), so that, for example,

$$H_c=2\sqrt{p^2+m_c^2}+V(r)\approx 2M_c+\epsilon_c+\frac{p^2}{M_c}+V(r) \tag{9}$$

[2]Basdevant and Boukraa [14] consider the approximation obtained by including only the linear term in $p^2-\langle p^2\rangle$ in the expansion. A very different approximation which leads to a smaller effective mass $M'=M/2$ was proposed in [7] and [16] and studied in more detail by Lucha, Schöberl, and Moser [17]. This approximation was obtained from the inequality $\langle\sqrt{(p^2+m^2)}\rangle\leqslant M\equiv\sqrt{\langle p^2\rangle+m^2}$ for matrix elements by treating the right hand side as the expectation value of the operator $(p^2+m^2)/M$, thus introducing a kinetic term of nonrelativistic type. However, this construction does not give an operator inequality. The result and the expression in Eq. (4) agree at the point $p^2=\langle p^2\rangle$, but Martin's approximation is tangent to the curve $E=\sqrt{p^2+m^2}$ while the foregoing approximation has twice the slope as a function of p^2. The construction is also not unique. For example, the optimum Martin bound is obtained by rewriting the right hand side of the inequality as $M/2+M/2$ and replacing only the second M by $(p^2+m^2)/M$. The key point is that the Martin bound holds for all p as an operator relation, and can be minimized in expectation values by the choice of M above. The effective mass M' obtained in [7,16,17] is substantially too small as will be seen in Sec. III B.

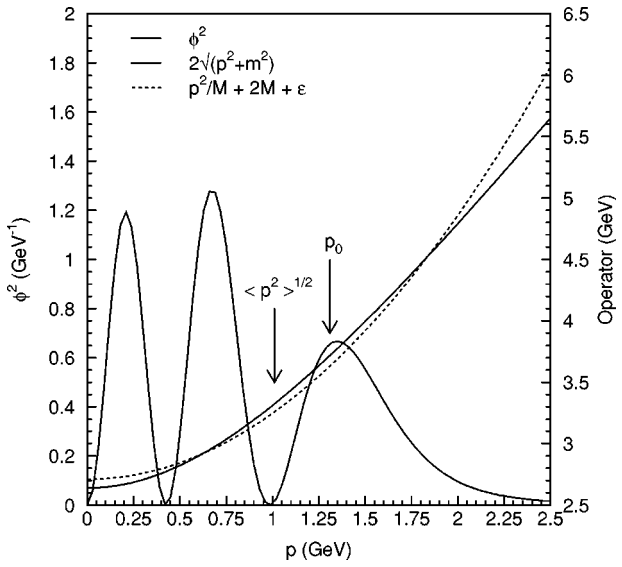

FIG. 2. We plot the relativistic and the approximate nonrelativistic kinetic energy operators with the square of the second excited state wave function for the $c\bar{c}$ system of Sec. III B superposed to show the approximate agreement of those operators in the region in which the wave function is large. $M=1861$ MeV, $\epsilon=-1009$ MeV, and $m_c=1320$ MeV.

for charmonium. We will take a standard linear-plus-Coulomb form for $V(r)$,

$$V(r)=Ar-\frac{B}{r}, \tag{10}$$

with $A=0.203$ GeV^2 and $B=0.437$. These values correspond to the potential parameters used by Fulcher for fits to the charmonium system [4]. We will concentrate on the $L=0$ states, and will consider the possibility of varying M as well as that of keeping M fixed at the value $M=\sqrt{\langle p^2\rangle+m^2}$ determined by a relativistic calculation of $\langle p^2\rangle$. The best values of M and ϵ in the Schrödinger equation, or of ϵ alone, will be determined by making a least squares fit to the relativistic ''data'' calculated using the Salpeter equation.

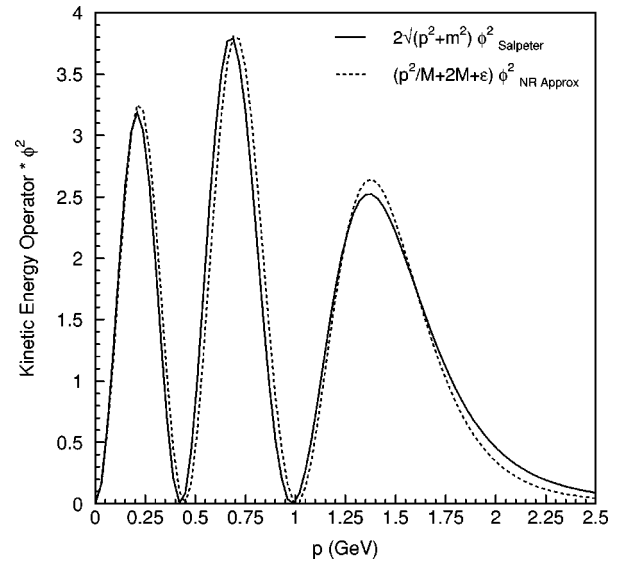

FIG. 3. We compare the relativistic momentum-space kinetic energy density for the second excited state of the $c\bar{c}$ system of Sec. III B with the density obtained using the nonrelativistic approximation in Eq. (7) with $M=1861$ MeV and $\epsilon=-1009$ MeV.

A. Numerical methods

We have calculated the relativistic energy spectra and wave functions using now-standard numerical methods developed elsewhere [11–13]. We first construct matrix representations for the potential $V(r)$ and the positive operators $E_i^2=p^2+m_i^2=-\nabla_i^2+m_i^2$ in a suitable orthonormal basis of angular momentum eigenstates. The matrix E_i^2 can be diagonalized by an orthogonal transformation U, $E_i^2=U\Lambda_i U^{-1}$. The eigenvalues are necessarily positive. The square-root operator $E_i=\sqrt{p^2+m_i^2}$ is then defined as $U\Lambda_i^{1/2}U^{-1}$ where $\Lambda_i^{1/2}$ is the diagonal matrix of the square roots of the eigenvalues [11,12]. With a finite basis, this construction reduces the solution of the Salpeter equation to the matrix eigenvalue problem

$$(E_1+E_2+V-E)R_l=0, \tag{11}$$

where R_l is the column-vector representation of the radial wave functions in the given basis for orbital angular momentum l. This equation can be solved by standard methods.

As shown by Fulcher [13], the matrix elements needed in this construction can be calculated analytically using basis wave functions

$$\psi_{l,m}^n(\vec{r})=X_{n,l}(r)Y_{l,m}(\hat{r}) \tag{12}$$

with the angular dependence given by the spherical harmonics $Y_{l,m}$ and the radial wave function $X_{n,l}(r)$ given by

$$X_{n,l}(r)=\beta^{3/2}(2\beta r)^l e^{-2\beta r}L_n^{2l+2}(2\beta r). \tag{13}$$

Here β is a length scale parameter and L_n^{2l+2} is the associated Laguerre polynomial [18]. This set has been investigated by several authors [2,4,11–13,19]. We find that a matrix size of 20×20 is sufficient to produce stable eigenvalues and wave functions for the low-lying energy eigenstates. The same basis functions can be used to solve the Schrödinger equation as a matrix problem.

In various figures which appear later we will use the radial wave functions $u_{n,l}(r)=rR_{n,l}(r)$, where $\psi_{n,l,m}(\mathbf{r})=R_{n,l}(r)Y_l^m(\hat{r})$ with Y_l^m a spherical harmonic. The radial probability density for the quarks is just $|u_{n,l}(r)|^2$. We will also use the corresponding momentum-space wave functions $\phi_{n,l}(p)=p\Phi_{n,l}(p)$ with $\tilde{\psi}_{n,l,m}(\mathbf{p})=(-i)^l\Phi_{n,l}(p)Y_l^m(\hat{p})$. The radial probability density in momentum space is then $|\phi_{n,l}(p)|^2$. The functions $u_{n,l}(r)$ and $\phi_{n,l}(p)$ are related by the Fourier-Bessel transform

$$\phi_{n,l}(p)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty dr\, prj_l(pr)u_{n,l}(r), \tag{14}$$

TABLE I. Comparison of the exact Salpeter energy spectrum for the ''$c\bar{c}$'' system of two heavy quarks with the spectra obtained in various nonrelativistic approximations. The Schrödinger approximation involves the kinetic energy p^2/m_c. The nonrelativistic (NR) approximation is defined in Eq. (7), and is considered both with the effective mass M_c fixed at the value $\sqrt{\langle p^2\rangle + m_c^2}$, and with M_c allowed to vary. ϵ_c is the energy shift defined in Eq. (8).

Model	M_c (MeV)	ϵ_c (MeV)	E_1 (MeV)	E_2 (MeV)	E_3 (MeV)	E_4 (MeV)	ΔE rms (MeV)
Salpeter	1320	—	3067	3668	4112	4486	—
Schrödinger	1320	0	3114	3755	4241	4659	119
NR, M fixed	1662	−669	3049	3660	4116	4507	14.5
NR, M free	1861	−1009	3069	3667	4109	4488	2.12

where j_l is the standard spherical Bessel function.

Finally, to determine the best values of M and ϵ, we minimize the function

$$\sum_{k=1}^{N} (E_{R,k} - E_{NR,k})^2 \tag{15}$$

for the N lowest energy levels, varying M and ϵ in the nonrelativistic Schrödinger equation with the calculated relativistic energies $E_{R,k}$ held fixed.

B. Results for heavy-quark systems

We will use the $c\bar{c}$ system for our study of bound states of two heavy quarks. We use the quark mass $m_c = 1.320$ GeV and the linear-plus-Coulomb potential determined by Fulcher [13] in his Salpeter-equation fit to the charmonium spectrum. After calculating the exact Salpeter energy spectrum for those parameters to obtain our ''data,'' we fit the four lowest energy levels using a sequence of nonrelativistic approximations. Since it is frequently argued that the $c\bar{c}$ is almost nonrelativistic, we consider the standard Schrödinger kinetic energy $2m_c + p^2/m_c$ as well as the effective-mass approximation discussed above. In the latter case, we take M_c either as fixed at the value $\sqrt{\langle p^2\rangle + m_c^2}$ obtained using the Salpeter value of $\langle p^2\rangle$ averaged over the states in question, or allow M_c to vary along with the energy shift ϵ_c. Our results are given in Table I.

We see from Table I that the Schrödinger approximation is rather poor, with deviations of the fitted energies from the exact values ranging from 47 MeV in the ground state to 173 MeV in the third excited state. The Schrödinger energies are all too high, and increase much too rapidly for the excited states, with a total change in the deviation of +119 MeV over the states considered. The failure of the Schrödinger approximation is not surprising given the rather large mean momentum in the Salpeter $c\bar{c}$ states, $\langle p^2\rangle = 1.021$ GeV^2, where

$$\langle p^2\rangle = \frac{1}{4}\sum_{n=1}^{4} \langle n|p^2|n\rangle. \tag{16}$$

This corresponds to a root-mean-square relativistic velocity $\langle v^2\rangle^{1/2} = 0.61$ for the quarks, and the system is semirelativistic.

The energies obtained using the approximation in Eq. (7) with $M_c = \sqrt{\langle p^2\rangle + m_c^2}$ are substantially better, with deviations ranging from −18 MeV for the ground state to +21 MeV in the third excited state. Moreover, the approximate energies increase less rapidly than those for the Schrödinger approximation, with an excess increase of only 39 MeV relative to the Salpeter energies over the four states shown. The overall fit is good. The improvement in the mean energy is the result of including the energy shift ϵ_c. The flattening of the deviations is the result of the larger value of the effective mass, with $M_c = 1.662$ GeV rather than the input mass $m_c = 1.320$ GeV. The fitted value of the energy shift, $\epsilon = -669$ MeV, is close to, and larger in magnitude than the average kinetic term $-\langle p^2\rangle/M_c = -614$ MeV as expected from Eq. (8). The extra shift is associated with the terms omitted in Eq. (4).

Finally, if we allow M_c to vary along with ϵ in the fitting procedure, we obtain an excellent fit to the relativistic spectrum, with errors less than 3 MeV and a root-mean-squared (rms) deviation of 2.12 MeV as shown in Table I. However, M_c is now quite large, $M_c = 1.861$ GeV, while $\epsilon_c = -1.009$ GeV. The large value of M_c is needed to slow the growth of the nonrelativistic kinetic energy with increasing p, and improve its agreement with the relativistic kinetic energy as remarked earlier. However, the resulting effective mass is not directly related to the charm-quark mass m_c.

As shown in Fig. 3, the variable-M nonrelativistic approximation leads to a seemingly excellent result for the kinetic-energy density. However, the relativistic and nonrelativistic wave functions do not agree precisely even for this fit as seen either in momentum space in Fig. 4, or in position space in Fig. 5. Some quantities of interest such as leptonic [20] and electromagnetic transition rates are sensitive to these differences, and the nonrelativistic model must therefore be used with care.

The increase in the heights of successive peaks in the nonrelativistic position-space wave function relative to the relativistic wave function, can be understood on the basis of the relativistic WKB approximation [21].[3] In particular, the velocity of a nonrelativistic particle is larger semiclassically than that of a relativistic particle in the region near the origin

[3]Numerical calculations show that the approximation is rather good in this case.

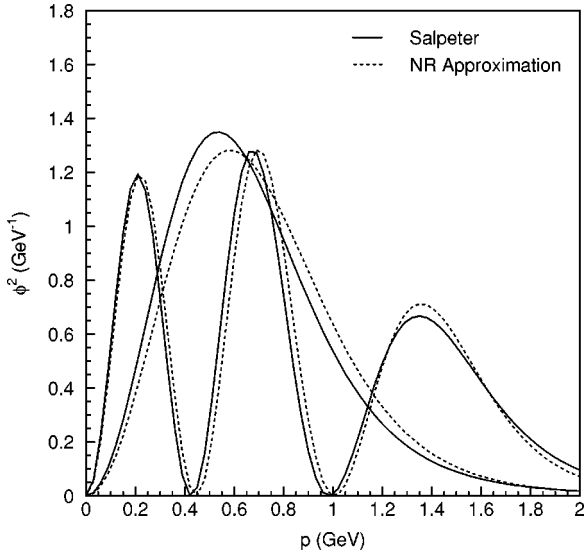

FIG. 4. We compare the Salpeter momentum-space probability densities $|\phi(p)|^2$ for the ground state and second excited state of the $c\bar{c}$ system of Sec. III B with the densities obtained using the nonrelativistic approximation for the kinetic energy given in Eq. (7) with $M=1861$ MeV and $\epsilon=-1009$ MeV.

where the color-Coulomb potential is large, so the particle spends less time in that region and its wave function is consequently smaller. Correspondingly, its wave function is larger near the outer turning point.

In Fig. 6 we show the effect of varying the mass M_c on the wave function for the second excited state of the $c\bar{c}$ system. The lower masses shown bracket the input value of m_c, while the highest mass is close to that obtained in the variable-mass fit, $M_c=1.86$ GeV. It is clear from the figure

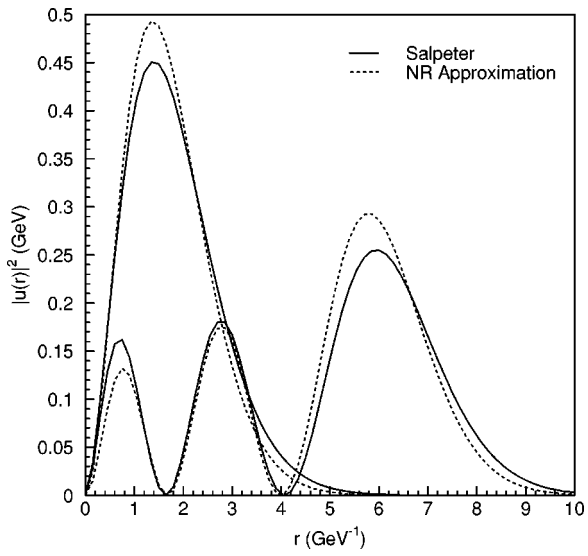

FIG. 5. We compare the Salpeter position-space probability densities $|u(r)|^2$ for the ground state and second excited state of the $c\bar{c}$ system of Sec. III B with the densities obtained using nonrelativistic approximation for the kinetic energy given in Eq. (7) with $M=1861$ MeV and $\epsilon=-1009$ MeV.

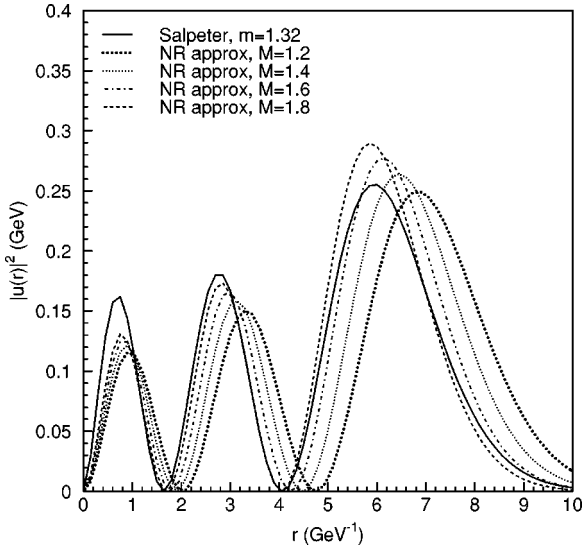

FIG. 6. We show the effect of varying the mass M in the nonrelativistic approximation for the kinetic energy operator, Eq. (7), on the quark radial probability density $|u(r)|^2$ for the second excited state of the $c\bar{c}$ system of Sec. III B with a fixed energy shift $\epsilon=-1010$ MeV. The best agreement of the wave functions is achieved for the large effective mass $M\approx 1.8$ needed in fitting the energy spectrum.

that the wave functions are quite inaccurate for the lower masses and are not especially good even for the large effective mass $\sqrt{\langle p^2\rangle+m_c^2}=1.66$ GeV or the mass 1.86 GeV obtained in the variable-mass fit. The trends in the wave functions discussed above are also clearly evident.

Finally, in Fig. 7 we compare the total energy densities u^*Hu for the second excited states for the Salpeter equation and the optimal nonrelativistic approximation with $M_c=1.861$ GeV. We note that the relativistic kinetic energy operator in position space is nonlocal. Its action on the radial wave function $u_{n,l}(r)$ can be defined following Nickisch *et al.* [11] in terms of the integral

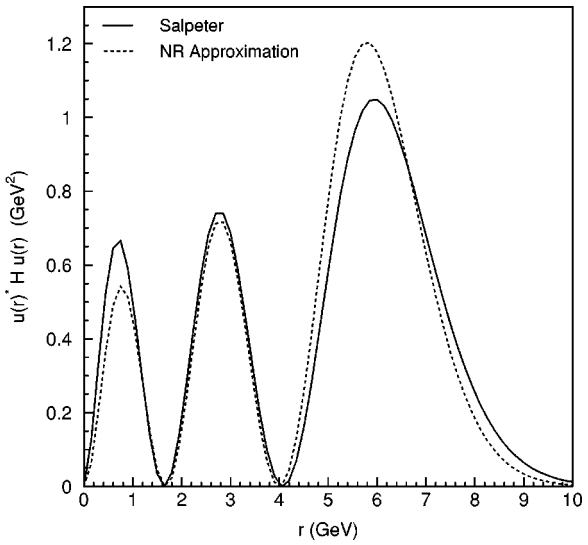

FIG. 7. Plot of the radial energy density u^*Hu for the second excited state of the $c\bar{c}$ system.

TABLE II. Comparison of the exact Salpeter energy spectrum for the ''$s\bar{s}$'' system of two light quarks with the spectra obtained using the nonrelativistic (NR) approximation defined in Eq. (7), taken either with the effective mass M_s fixed at the value $\sqrt{\langle p^2\rangle + m_s^2}$ or allowed to vary. ϵ_s is the energy shift defined in Eq. (8).

Model	M_s (MeV)	ϵ_s (MeV)	E_1 (MeV)	E_2 (MeV)	E_3 (MeV)	ΔE rms (MeV)
Salpeter	364	—	1531	2222	2744	—
NR, M fixed	828	−795	1503	2219	2775	24.2
NR, M free	989	−1022	1533	2218	2746	2.83

$$\sqrt{p^2+m^2}\,u_{n,l}(r) = \frac{m^2}{\pi}\int_0^\infty dr' G_l(mr,mr') \times\left[-\frac{d^2}{dr'^2}+\frac{l(l+1)}{r'^2}+m^2\right]u_{n,l}(r'). \tag{17}$$

The kernels G_l are defined in that reference. For $l=0$,

$$G_0(x,x') = K_0(|x-x'|) - K_0(x+x'), \tag{18}$$

where $K_0(z)$ is a hyperbolic Bessel function, and decreases exponentially for $z>1$. This form makes it clear that the relativistic kinetic term samples the wave function over a range of r of order $1/m$. As a result, it is less sensitive to local changes in $u_{n,l}$ than the approximate nonrelativistic kinetic energy. The latter depends on the local curvature of $u_{n,l}$ through the term $p^2/2M \to -[d^2/dr^2 + l(l+1)/r^2]/2M$. The effect in the present case is a reduction in the relativistic kinetic contribution to u^*Hu at short distances where the potential is strongly attractive and u varies rapidly, and an increase at large distances. The result is a shift of the probability and Hamiltonian densities u^*u and u^*Hu toward smaller radii, and a flattening of the distributions relative to the nonrelativistic distribution. This is evident in Fig. 7.

C. Results for light-quark systems

Our results for $s\bar{s}$ system of two light quarks are given in Table II. We have used a strange-quark mass $m_s = 364$ MeV in these calculations following Fulcher [4], but have not introduced an extra overall shift in the potential for the $s\bar{s}$ system as he did, preferring to use the same potential for the light- and heavy-quark systems so as to be able to treat both systems simultaneously and predict the $c\bar{s}$ spectrum. The results are actually rather insensitive to m_s because $\langle p^2\rangle^{1/2} \approx 744$ MeV $\gg m_s$. The system is clearly relativistic, with an rms velocity $\langle v^2\rangle^{1/2} = 0.90$ for the quarks.

The energies obtained with the effective mass $M_s = \sqrt{\langle p^2\rangle + m_s^2}$ are reasonably good on the average, but the approximate energies again increase too fast relative to the Salpeter spectrum. The fit obtained when M_s is allowed to vary is excellent, with the energies differing from the Salpeter energies by less than 4 MeV for the three lowest states considered. The fitted value of M_s has essentially no relation to the input mass m_s.

Unfortunately, the wave functions obtained in this case are poor even for the best fit to the spectrum. We compare the Salpeter and approximate energy densities in Fig. 8. The differences are due mainly to differences in the wave functions. Even the kinetic energy densities shows significant pointwise disagreement in this case. The very significant flattening of the relativistic energy density u^*Hu relative to the nonrelativistic approximation seen in Fig. 7 is again the result of the nonlocality of the kinetic energy operator $\sqrt{p^2+m^2}$ in position space. The nonlocality is now over a distance $\sim 1/m_s \approx 2.7$ GeV^{-1} which is quite important on the radial scale of the wave function. As implied by the striking success of the nonrelativistic approximation in reproducing the relativistic energy spectrum, the unwanted tilt in the nonrelativistic energy density averages essentially to zero in the expectation value $\langle H\rangle$. The accuracy of the approximation for the energies does not imply corresponding accuracy for the wave functions or expectation values other than $\langle H\rangle$.

D. Predictions for the light-heavy system

We consider finally the light-heavy system corresponding to the relativistic Hamiltonian of Eq. (1) with $m_1 = m_c$ and

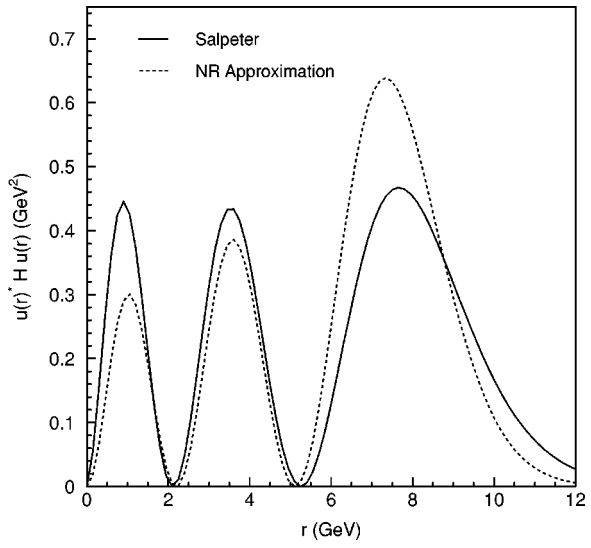

FIG. 8. Plot of the radial energy density u^*Hu for the second excited state of the $s\bar{s}$ system. The differences between the relativistic and nonrelativistic cases result mostly from differences between the wave functions.

TABLE III. Comparison of the exact Salpeter energy spectrum of the heavy-light ''$c\bar{s}$'' system with the spectrum obtained using the nonrelativistic approximation for the kinetic energy given in Eq. (19) taken either with the effective masses M_i fixed at the values $\sqrt{\langle p^2\rangle_i + m_i^2}$ or allowed to vary. The normalized nonrelativistic spectrum with the masses M_i fixed is obtained by adjusting the energy shifts to match the ground states of the $c\bar{c}$ and $s\bar{s}$ systems exactly.

Model	E_1	E_2	E_3	ΔE rms (MeV)
Salpeter	2319	2957	3438	—
NR, M, ϵ fixed	2296	2963	3474	24.9
Normalized	2319	2986	3497	38.0
NR, M free	2319	2961	3449	6.8

$m_2 = m_s$ corresponding to the masses used in the discussion above. We use the nonrelativistic Hamiltonian

$$H_{c\bar{s}} = M_c + M_s + \frac{p^2}{2M_c} + \frac{p^2}{2M_s} + \frac{1}{2}(\epsilon_c + \epsilon_s) + V(r) \quad (19)$$

obtained by replacing the square-root operators in Eq. (1) by the approximation in Eq. (7). The kinetic term is of the standard Schrödinger form with a reduced mass $M = M_s M_c/(M_s + M_c)$ given in terms of the effective masses rather than the quark masses. For the purpose of making predictions, we will keep the energy shifts ϵ_i and the effective masses M_i fixed at the values determined separately for the heavy- and light-quark systems. These quantities would all be expected to change somewhat in the light-heavy system. For example, the masses $M_i = \sqrt{\langle p^2\rangle + m_i^2}$ that minimize the Martin bound on the total kinetic energy change because of the different value of $\langle p^2\rangle$ in the light-heavy system. The value of this quantity averaged over the three lowest states is $\langle p^2\rangle_{c\bar{s}} = 0.835$ GeV^2, a value intermediate between the values $\langle p^2\rangle_{c\bar{c}} = 1.021$ GeV^2 and $\langle p^2\rangle_{s\bar{s}} = 0.744$ GeV^2 obtained for the heavy- and light-quark systems. The energy shifts are given to leading approximation by $\epsilon_i \approx -\langle p^2\rangle/M_i$, so also change. However, the conditions for minimizing the bound make the kinetic energy stationary with respect to the masses M_c and M_s. As a result, by the Feynman-Hellman theorem [22], there is no first-order change in the energies for small changes in $\langle p^2\rangle$. More physically, the original nonrelativistic approximations for the kinetic energy operators are already good over a wide range of momenta as shown in Fig. 4, so the effect of the changes on the spectrum is not expected to be large.

Our predictions for the Salpeter energy spectrum for the light-heavy system are shown in Table III. If we use the masses $M_i = \sqrt{\langle p^2\rangle_i + m_i^2}$ with the values of $\langle p^2\rangle_i$ determined for the $c\bar{c}$ and $s\bar{s}$ systems, the energies of the four lowest $c\bar{s}$ states are predicted to within 36 MeV as shown in the table. We note that the ground state is predicted to lie at too low an energy as a result of the large negative value of the energy shift defined above. However, an examination of Tables I and II shows that the predicted ground-state energies of the $c\bar{c}$ and $s\bar{s}$ systems are also too small. The usual fitting procedure adjusts the energy shift to minimize the deviations between the theory and the input data over the set of states considered. If we consider instead adjusting the energy shifts ϵ_c and ϵ_s to fit the $c\bar{c}$ and $s\bar{s}$ ground-state energies exactly, a reasonable procedure phenomenologically, we predict the normalized energies given in the third row in Table III. The ground state is now predicted correctly. However, the energies of the excited states $c\bar{s}$ increase too rapidly. This too rapid increase was also present for the $c\bar{c}$ and $s\bar{s}$ states. We note in this connection that the energies of the $c\bar{s}$ states are very close to the average of the energies of the corresponding $c\bar{c}$ and $s\bar{s}$ states.

If we use instead the fitted values of the masses and energy shifts for the heavy- and light-quark systems, we predict the energies of the lowest three $c\bar{s}$ states to within 11 MeV as shown in Table III. The largest difference occurs for the second excited state. The fits to the $c\bar{c}$ and $s\bar{s}$ energies are already excellent, and there is no reason in this case to renormalize the energy shifts. The closeness of the predictions to the actual energies would be expected given the results obtained for the $c\bar{c}$ and $s\bar{s}$ systems. In particular, the nonrelativistic approximations to the kinetic energy operators are good in the regions in which the momentum-space wave functions are large. However, the final position-space $c\bar{s}$ wave functions are again not accurate.

IV. CONCLUSIONS

We find that the apparent success of nonrelativistic models for relativistic systems can be understood in terms of an approximation to the relativistic kinetic energy operator motivated by the Martin bound [9] in Eq. (2). Although the physical content of the approximation can be understood in terms of an expansion of the relativistic operator about a mean momentum squared p_0^2, given optimally from the bound as $p_0^2 = \langle p^2\rangle$, the series expansion is not necessary. What is important is to obtain a good average representation of the kinetic energy operator of Schrödinger form. We observe in this connection that the approximation can be improved significantly by allowing an extra energy shift to eliminate the inequality, and, if desired, also allowing the effective mass M to vary.

We have investigated the effectiveness of this procedure in detail by using the nonrelativistic approximation to fit ''data'' obtained by solving the relativistic Salpeter equation for the linear-plus-Coulomb potential used by Fulcher [13] in fits to the charmonium spectrum. We find that the nonrelativistic approximation for the kinetic energy operator in Eq. (7) gives generally good descriptions of the Salpeter energy spectra for the $c\bar{c}$ and $s\bar{s}$ systems, taken as examples of bound states of heavy and light quark pairs. The results obtained with the effective masses fixed at the values $\sqrt{\langle p^2\rangle + m^2}$ suggested by minimizing the Martin bound over a set of states are good, but the excited state energies gener-

ally increase too rapidly if the potential is kept fixed. The results obtained when M is allowed to vary in the fitting procedure are accurate to a few MeV is all cases, a striking result.

We believe that the theoretical understanding of the success of the nonrelativistic effective-mass approximation developed here provides a justification for Martin's nonrelativistic treatment of heavy- and light-quark systems, and explains the unexpected success of his predictions for the masses of light-heavy systems [9,8,10].

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