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String baryonic model ''triangle'': Hypocycloidal solutions and the Regge trajectories

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The considered model of baryon consists of three pointlike masses (quarks) bounded pairwise by relativistic strings forming a curvilinear triangle. Classic analytic solutions for this model corresponding to a planar uniform rotation about the system center of mass are found and investigated. These solutions describe a rotating curve composed of segments of a hypocycloid. The curve is a curvilinear triangle or a more complicated configuration with a set of internal massless points moving at the speed of light. Different topological types of these motions are classified in connection with different forms of hypocycloids in zero quark mass limit. An application of these solutions to the description of baryon states on Regge trajectories is considered. [S0556-2821(98)01521-5]

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INTRODUCTION

The string baryon model ''triangle'' is genetically connected with the meson model of relativistic string with massive ends [1,2]. The latter model including two pointlike massive quarks bounded by a relativistic string gives the possibility of describing the meson orbitally excited states on the Regge trajectories [3].

On the basis of this meson model, string models of baryons, were suggested in some variants [4–7]. These variants differ from each other in the type of spatial junction of three pointlike quarks by relativistic strings: (a) the first quark is bounded with the second and the second quark with the third, (b) the ''three-string'' model or Y configuration with three strings from three quarks joined in the fourth massless point, (c) the quark-diquark model, and (d) the ''triangle'' model. The first variant was investigated qualitatively [4], the ''three-string'' [4–6] and mesonlike quark-diquark models [7,8] in a more detailed way.

In the present paper the ''triangle'' model of the baryon [9] is under consideration. In this model three material points (quarks) are pairwise connected by three relativistic strings forming a curvilinear triangle in space at each instant of time. If tensions of these three strings are equal, such an object could be regarded as a closed string carrying three pointlike masses. From the point of view of describing quark strong interaction in the orbitally excited baryon this model looks rather natural in comparison with three others. Some arguments in favor of the ''triangle'' baryon model in comparison with the Y configuration are given by Cornwall [10] in the QSD Wilson loop operator approach.

The transformation of the ''meson'' string with massive ends or the three-string model of baryon to the model ''triangle'' results in some additional difficulties. In particular, a string world surface in this model has discontinuities of derivatives on quark trajectories; a parametrization with these trajectories as coordinate curves does not exist in general and spacelike coordinate lines are not closed in general.

In the present paper these difficulties are overcome and classic analytic solutions are found for a set of motions—uniform planar rotations of the system. This kind of motion is an analogue and generalization of well known rotations of a straight relativistic string with massive ends [1–3]. The latter class of motions was a base of applying this model [3,8] and the relativistic tube model [11] to the description of meson Regge trajectories.

In this paper rotational motions in the baryonic model ''triangle'' and their applications are investigated. In Sec. I equations of evolution and conditions on the quark trajectories are deduced from the action of the system. In Sec. II solutions of these equations corresponding to rotational motions of the system (quarks and the string of hypocycloidal form) are described and classified. In Sec. III the possibility of a description of the baryon states on Regge trajectories by these solutions is discussed.

The string solutions obtained here in Sec. II are applicable not only to the particle physics, but to various branches of string or M-brane theory. In particular, the massive points placed on the string (the number of these points is arbitrary) could be regarded as 0-branes.

Note that the rotational solutions of the considered type also take place for a closed massless string. Such a string has a form of a rotating hypocycloid with singular points moving at the speed of light.

I. MODEL AND EQUATIONS

Let us consider the baryon model ''triangle'' as a closed relativistic string with tension γ carrying three pointlike masses m_1,m_2,m_3. The action of this system is [9]

$$S=-\int_{\tau_1}^{\tau_2}\left\{\gamma\int_{\sigma_0(\tau)}^{\sigma_3(\tau)}\sqrt{-g}\,d\sigma+\sum_{i=1}^{3}m_i\sqrt{V_i^2(\tau)}\right\}d\tau. \quad (1)$$

Here $g=\dot{X}^2X'^2-(\dot{X}X')^2$ is a determinant of induced metric on a string world surface $x^\mu=X^\mu(\tau,\sigma)$, $\mu=0,1,\ldots,$ in d-dimensional Minkowski space with signature $+,-,-,\ldots$; $\dot{X}^\mu=\partial_\tau X^\mu$, $X'^\mu=\partial_\sigma X^\mu$, the speed of light in these units $c=1$, $(\tau,\sigma)\in\Delta=\Delta_1\cup\Delta_2\cup\Delta_3$, $V_i^\mu=(d/d\tau)X^\mu[\tau,\sigma_i(\tau)]$ is a tangent vector to the ith quark trajectory with an inner equation $\sigma=\sigma_i(\tau)$, $i=0,1,2,3$ (Fig. 1). The equations $\sigma=\sigma_0(\tau)$ and $\sigma=\sigma_3(\tau)$ define the trajectory of the same third

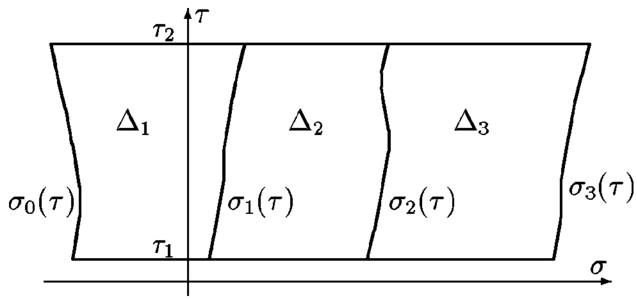

FIG. 1. Domain of integration in Eq. (1).

quark. It is connected with the fact that the string is closed and may be rewritten in the common form

$$X^\mu[\tau,\sigma_0(\tau)]=X^\mu[\tau^*,\sigma_3(\tau^*)]. \tag{2}$$

Note that the parameters τ and τ^* in these two parametrizations of one curve (2) are not equal in general. This means that coordinate curves $\tau=\text{const}$ on the world surface are not closed—the beginning of this curve at $\sigma=\sigma_0$ does not coincide spatially with its end at $\sigma=\sigma_3$. The equality $\tau=\tau^*$ may be obtained only by a special choice of τ and σ, for example, $\tau=t\equiv X^0$.

The parametrization of the world surface $X^\mu(\tau,\sigma)$ is continuous in Δ, but on the lines $\sigma_i(\tau)$ its derivatives [except for tangential V_i^μ and $(d/d\tau)V_i^\mu$] have discontinuities in general. Nevertheless, by a local choice of parameters τ and σ we can obtain the induced metric $ds^2=\dot{X}^2d\tau^2+2(\dot{X}X')d\tau d\sigma+X'^2d\sigma^2$ continuous on these lines. The action (1) is invariant with respect to an arbitrary nondegenerate reparametrization $\tau=\tau(\tilde{\tau},\tilde{\sigma})$, $\sigma=\sigma(\tilde{\tau},\tilde{\sigma})$.

The equations of motion and the boundary conditions on the quark trajectories in this model are deduced by variation and minimization of action (1). This procedure is partially similar to that for the model of relativistic string with massive ends [2] and results in the same equations of motion

$$\frac{\partial}{\partial\tau}\frac{\partial L}{\partial\dot{X}^\mu}+\frac{\partial}{\partial\sigma}\frac{\partial L}{\partial X'^\mu}=0,\quad (\tau,\sigma)\in\Delta,\quad L\equiv\sqrt{-g}. \tag{3}$$

But to derive boundary conditions in the model ''triangle'' we are to take into account the discontinuities of $\dot{X}^\mu,X'^\mu$ on the lines $\sigma=\sigma_i(\tau)$. Thereby the term

$$\int\int_\Delta\left[\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot{X}^\mu}\delta X^\mu\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial L}{\partial X'^\mu}\delta X^\mu\right)\right]d\tau\, d\sigma$$

in $\delta S[X^\mu]$ transformed using the Green's formula equals the sum of three curvilinear integrals of internal boundary values along the borders of the domains Δ_1, Δ_2, Δ_3 and, therefore, in the following boundary conditions:

$$m_i\frac{d}{d\tau}\frac{V_{i\mu}}{|V_i|}-\gamma\left[\frac{\partial L}{\partial X'^\mu}-\frac{\partial L}{\partial\dot{X}^\mu}\sigma_i'(\tau)\right]\Bigg|_{\sigma=\sigma_i+0}+\gamma\left[\frac{\partial L}{\partial X'^\mu}-\frac{\partial L}{\partial\dot{X}^\mu}\sigma_i'\right]\Bigg|_{\sigma=\sigma_i-0}=0,\quad i=1,2,3. \tag{4}$$

For the third quark $(i=3)$ in the first two summands we are to put $\sigma=\sigma_0(\tau)$, and in the third we put $\sigma=\sigma_3(\tau^*)$ in accordance with the closure condition (2). From the physical point of view Eqs. (4) are the second Newtonian law for the material points m_i, moduli of the applied tension forces equal to γ.

Let the induced metric on the world surface be conformally flat, i.e., conditions of orthonormality be tied:

$$\dot{X}^2+X'^2=0,\quad (\dot{X}X')=0. \tag{5}$$

These equalities in Δ may always be obtained by the reparametrization $\tau=\tau(\tilde{\tau},\tilde{\sigma})$, $\sigma=\sigma(\tilde{\tau},\tilde{\sigma})$ (new coordinate lines $\tilde{\tau}\pm\tilde{\sigma}=\text{const}$ on the world sheet are integral curves of equations $\dot{X}^2d\tau+[(\dot{X}X')\pm L]d\sigma=0$). We use the same notation for τ and σ below and suppose that equalities (5) are satisfied. Under conditions (5) the equations of motion (3) become linear

$$\ddot{X}^\mu-X''^\mu=0, \tag{6}$$

Equations (5) and (6) are invariant with respect to reparametrizations $\tau\pm\sigma=f_\pm(\tilde{\tau}\pm\tilde{\sigma})$ [2]. Choosing these two arbitrary functions $f_\pm$ one can fix two (of four) functions $\sigma_i(\tau)$, for example, in the form

$$\sigma_1(\tau)=0,\quad \sigma_2(\tau)=\pi. \tag{7}$$

The boundary equations (4) on these lines under conditions (5) and (7) take the form

$$m_i\frac{d}{d\tau}\frac{\dot{X}^\mu(\tau,\sigma_i)}{[\dot{X}^2(\tau,\sigma_i)]^{1/2}}-\gamma X'^\mu(\tau,\sigma_i+0)+\gamma X'^\mu(\tau,\sigma_i-0)=0,\quad i=1,2. \tag{8}$$

Reparametrizations of the mentioned type with $f_+(\eta)=f_-(\eta)=\eta+\phi(\eta)$, $\phi(\eta+2\pi)=\phi(\eta)$, $|\phi'(\eta)|<1$ [12] preserving the form of equations (7) do not permit us to fix $\sigma_3=\text{const}$ (or $\sigma_0=\text{const}$) for all τ in general.

Thus choosing τ and σ one can not fix three functions $\sigma_0(\tau)$, $\sigma_3(\tau)$, and $\tau^*(\tau)$ for an arbitrary motion in a convenient form. The necessity of determinating these functions from initial data essentially sophisticates the initial-boundary-value problem for the model ''triangle'' in comparison with the string model of a meson [13]. In the present paper the functions $\sigma_0(\tau)$, $\sigma_3(\tau)$, and $\tau^*(\tau)$ are defined from properties of symmetry for a class of uniform planar rotations of the system.

II. ROTATIONAL MOTIONS

Let the closed string with three material points uniformly rotate (preserving its form in time) in a plane xy around the origin of coordinates. The quark trajectories in $(2+1)$-dimensional Minkowski space are the screw lines. For this motion one can choose on the world surface a parametrization with screw lines $\sigma=\text{const}$ and with an uniform growth of τ along these lines. In these coordinates on the quark trajectories

$$\sigma_i(\tau)=\text{const},\quad i=0,1,2,3,\qquad |V_i|=\sqrt{\dot{X}^2}|_{\sigma=\sigma_i}=C_i=\text{const},\quad i=1,2,3. \tag{9}$$

All four functions $\sigma_i(\tau)$ are fixed simultaneously, σ_1 and σ_2 are in the form (7) nonlimiting a generality.

Let coordinate curves $\tau=\text{const}$ be orthogonal trajectories to the specified lines $\sigma=\text{const}$ and conditions (5) be satisfied. These lines $\tau=\text{const}$ (do not coincide with sections $t=\text{const}$) are not closed, but a connection between τ and τ^* in the closure condition (2) is very simple: $\tau^*=\tau+\text{const}$. It is a consequence of the symmetry of this motion that the world surface in Minkowski space coincide with itself after a rotation about the $t=x^0$ axis with simultaneous translation along this axis.

Under these circumstances and conditions (5), (9) the third Eq. (4) takes the form

$$m_3C_3^{-1}\ddot{X}^\mu(\tau,\sigma_0)-\gamma X'^\mu(\tau,\sigma_0+0)+\gamma X'^\mu(\tau^*,\sigma_3-0)=0,\quad \tau^*=\tau+\text{const}. \tag{10}$$

A solution of the string oscillatory equation (6) satisfying the conditions (5), (7)–(9) may be found by the Fourier method: $X^\mu=\sum_k e_k^\mu u_k(\sigma)T_k(\tau)$. The functions $u_k(\sigma)$ and $T_k(\tau)$ with the same k as a consequence of Eq. (6) are linear functions or harmonics with the same frequency ω. Taking into account the above described properties of the rotational motion and its parametrization one can find the Fourier series for X^μ in $(2+1)$ Minkowski space (with the unique frequency ω) in the form

$$X^\mu=\{t_0+a\tau+b\sigma;u(\sigma)\cos\omega\tau-\tilde{u}(\sigma)\sin\omega\tau;u(\sigma)\sin\omega\tau+\tilde{u}(\sigma)\cos\omega\tau\}. \tag{11}$$

The functions $u(\sigma)$ and $\tilde{u}(\sigma)$ are continuous in $[\sigma_0,\sigma_3]$, may have discontinuities of derivatives at $\sigma=0$, $\sigma=\pi$ and in the segments $[\sigma_{i-1},\sigma_i]$ are

$$u(\sigma)=\begin{cases}A_0\cos\omega\sigma+B_0\sin\omega\sigma, & \sigma\in[\sigma_0,0],\\ A_1\cos\omega\sigma+B_1\sin\omega\sigma, & \sigma\in[0,\pi],\\ A_2\cos\omega\sigma+B_2\sin\omega\sigma, & \sigma\in[\pi,\sigma_3];\end{cases}$$

$$\tilde{u}(\sigma)=\tilde{A}_i\cos\omega\sigma+\tilde{B}_i\sin\omega\sigma,\quad \sigma\in[\sigma_i,\sigma_{i+1}]. \tag{12}$$

Let the functions $e^\mu u(\sigma)T(\tau)$ and $e^\mu\tilde{u}(\sigma)T(\tau)$ (with $T=\cos\omega\tau$ or $T=\sin\omega\tau$) satisfy the boundary conditions (8) independently. With the continuity conditions at $\sigma=0$ and $\sigma=\pi$ this results in four equations both for u and $\tilde{u}$ which may be presented in the form solved with respect to $A_1\equiv A$ and $B_1\equiv B$ (the same formulas express $\tilde{A}_i,\tilde{B}_i$ by $\tilde{A}_1\equiv\tilde{A}$ and $\tilde{B}_1\equiv\tilde{B}$):

$$A_0=A,\quad B_0=h_1A+B,\qquad A_2=(1+h_2c_1s_1)A+h_2s_1^2B,\qquad B_2=-h_2c_1^2A+(1-h_2c_1s_1)B. \tag{13}$$

Here $c_1=\cos\pi\omega$, $s_1=\sin\pi\omega$, $h_i=\omega m_i/(\gamma C_i)$.

Under relations (13) solution (11)–(12) satisfies conditions (8). Substitution Eqs. (11), (12) into the second of the orthonormality conditions (5) results in three equations:

$$A_i\tilde{B}_i-\tilde{A}_iB_i=ab/\omega^2,\quad i=0,1,2. \tag{14}$$

But among Eqs. (14) only one is independent, for example, with $i=1$. If it is satisfied and the relations (13) take place then the two other conditions (14) are satisfied too. Also substitution (11), (12) into the first condition (5) results in $A_i^2+B_i^2+\tilde{A}_i^2+\tilde{B}_i^2=(a^2+b^2)/\omega^2$—three independent equations. Transform this system taking into account Eqs. (13) in the following equivalent form:

$$A^2+B^2+\tilde{A}^2+\tilde{B}^2=(a^2+b^2)/\omega^2, \tag{15}$$

$$h_1(A^2+\tilde{A}^2)+2(AB+\tilde{A}\tilde{B})=0, \tag{16}$$

$$\lambda_1(A^2+\tilde{A}^2)=\lambda_2(B^2+\tilde{B}^2). \tag{17}$$

Here $\lambda_1=(h_1h_2-2)c_1s_1+h_1(1-2c_1)-h_2c_1^2$ and $\lambda_2=h_2s_1^2-2c_1s_1$.

Expression (11) is a solution of the given problem if the last necessary conditions (2) and (10) are satisfied. Denote $-\theta/\omega$ as the constant in Eq. (10):

$$\tau^*=\tau-\theta/\omega,\quad \theta=(\sigma_3-\sigma_0)\omega b/a=D\omega b/a. \tag{18}$$

The expression for θ results from the substitution of $X^0=t_0+a\tau+b\sigma$ into the closure condition (2). The angle θ has the following geometrical sense: θ is the phase shift on a screw trajectory of the third quark between the beginning (at $\sigma=\sigma_0$) and the end (at $\sigma=\sigma_3$) of an unclosed coordinate line $\tau=\text{const}$.

Substitute Eqs. (11)–(13), (18) into the closure (2) and boundary conditions (10) with $\mu=1,2$. Values of $u,\tilde{u}$ and their derivatives at $\sigma=\sigma_0$ and σ_3 express through $A,B,\tilde{A},\tilde{B}$ by Eqs. (13), for example, $u(\sigma_3)=[\cos\omega\sigma_3-h_2c\sin\omega(\sigma_3-\pi)]A+[\sin\omega\sigma_3-h_2s\sin\omega(\sigma_3-\pi)]B$.

Equating similar terms with $\cos\omega\tau$ and $\sin\omega\tau$ in the four Eqs. (2), (10) with $\mu=1,2$ we obtain eight homogeneous equations with respect to $A,B,\tilde{A},\tilde{B}$ which reduce to four pairs of coinciding ones. For the sake of simplicity and explication of its intrinsic structure we write this homogeneous system with the matrix notation

$$M_1\alpha=M_2\beta,\quad M_3\alpha=M_4\beta. \tag{19}$$

Here

$$\alpha=\begin{pmatrix}A\\ \tilde{A}\end{pmatrix},\quad \beta=\begin{pmatrix}B\\ \tilde{B}\end{pmatrix}$$

and matrices

$$M_1=(h_1s_0-c_0)I+(c-h_2c_1s_2)U,$$

$$M_2=-s_0I-(s-h_2s_1s_2)U,$$

$$M_3=[(1-h_1h_3)s_0+(h_1+h_3)c_0]I +(s+h_2c_1c_2)U,$$

$$M_4=(h_3s_0-c_0)I+(c-h_2s_1c_2)U$$

are linear combinations of the identity matrix I and the matrix

$$U=U(\theta)=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

The coefficients are

$$c_i=\cos\omega d_i,\quad s_i=\sin\omega d_i;\quad c=\cos\omega\sigma_3,\quad s=\sin\omega\sigma_3;$$

$$d_i=\sigma_{i+1}-\sigma_i:\quad d_0=-\sigma_0,\quad d_1=\pi,$$

$$d_2=\sigma_3-\pi;\quad D=\sigma_3-\sigma_0=d_0+d_1+d_2.$$

Taking into account the mutual commutability of M_k one can exclude α or β from the system (19)

$$M\alpha=0,\quad M\beta=0, \tag{20}$$

$M=M_1M_4-M_2M_3=I+U^2-FU=(2\cos\theta-F)U$ [an equality $I+U^2(\theta)=2\cos\theta\cdot U$ is used]. The parameter F may be transformed to the simple form

$$F=2\cos\omega D-\sum_i h_i\sin\omega D+\sum_{i<j} h_ih_js_i\sin\omega(d_{i-1}+d_j) -h_1h_2h_3s_1s_2s_0=G_1+G_2+G_3-G_1G_2G_3$$

through the following notation:

$$G_i=\frac{h_is_{i-1}s_i-\sin\omega(d_{i-1}+d_i)}{s_{i+1}}. \tag{21}$$

The notation here is cyclically equivalent: $d_{i+3}\equiv d_i$, $s_{i+3}\equiv s_i$, $G_{i+3}\equiv G_i$, for example, $d_3\equiv d_0$, $s_4\equiv s_1$.

Homogeneous systems (20) have a desirable nontrivial solution if and only if $\det M=(2\cos\theta-F)^2=0$, i.e.,

$$2\cos\theta=G_1+G_2+G_3-G_1G_2G_3. \tag{22}$$

Under condition (22) the matrix $M=0$ and an arbitrary nonzero column α or β is its eigenvector. It is connected with the rotational symmetry of the problem. So one can choose an optional pair A and $\tilde{A}$, B and $\tilde{B}$, or A and B and determine two other constants from Eq. (19) [under condition (22) two systems (19) are equivalent], in particular,

$$\tilde{A}=-K(h_1A+2B),\quad \tilde{B}=K(2HA+h_1B), \tag{23}$$

where

$$K=\frac{s_0s_1(G_2G_3-1)}{2s_2\sin\theta},\quad H=\frac{1+h_1^2K^2}{4K^2}. \tag{24}$$

Values (23) must obey conditions (14)–(17) descending from the orthonormality conditions (5). Substitution of Eq. (23) in Eqs. (16) and (17) after transformations results in relations

$$\frac{G_{i+1}-G_i}{G_iG_{i+1}-1}=\frac{\sin\omega(d_{i-1}-d_{i+1})}{s_i},\quad i=1,2,3. \tag{25}$$

One of these equations $(i=2)$ is a consequence of Eq. (16), the second of (17), and the third of the previous two.

Substitution of Eq. (23) in Eqs. (14) and (15) after transformations taking into account Eqs. (18), (21)–(25) results in two equations which may be written in the form

$$a^2=2KD\omega^3\theta^{-1}(HA^2+h_1AB+B^2), \tag{26}$$

$$\frac{D\omega\theta}{D^2\omega^2+\theta^2}=\frac{2K}{1+(4+h_1^2)K^2} \tag{27}$$

with K from Eq. (24).

The latter equation determines a set of acceptable frequencies ω if the parameters G_i, d_i, and θ are given. All these parameters defining a rotational motion of the model (except for translations and a scale factor) are related by the system of nonlinear equations (21), (22), (24), (25), and (27).

The simplest way to construct solutions of the considered problem is to start with fixing three parameters G_1,G_2,G_3 as initial data. In the next step we determine the angle θ by Eq. (22). The result of this procedure is not unique—for every triplet G_i one can find a countable set of values $\theta=\theta_{j_1}$. Further, the lengths d_i are defined from Eqs. (25) by the following two steps (the value $d_1=\pi$ was already chosen):

$$\delta=d_0-d_2=\frac{1}{\omega}\left[(-1)^{j_2}\arcsin s\,\frac{G_2-G_1}{G_1G_2-1}+\pi j_2\right],$$

$$d_0=\frac{1}{\omega}\left[\arctan\frac{\sin\omega(\delta+\pi)}{\cos\omega(\delta+\pi)-(G_1-G_3)/(G_1G_3-1)} +\pi j_3\right],$$

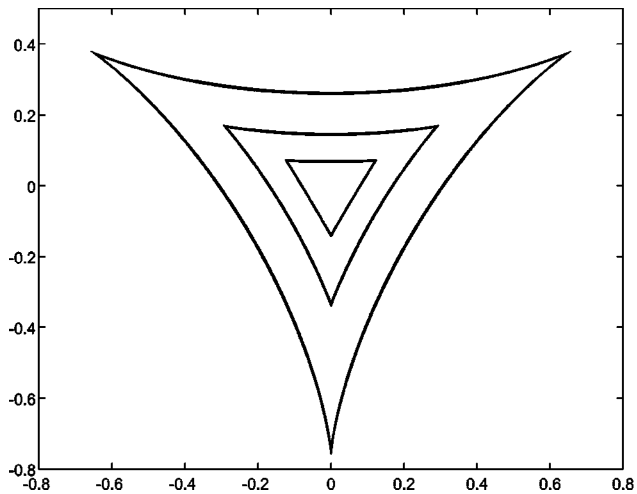

FIG. 2. The simple states with various rotational rates for the system with equal masses $m_1 = m_2 = m_3 = 1$.

and $d_2 = d_0 - \delta$ with arbitrary integer j_2, j_3. Substitution of G_i, d_i, θ, and K into Eq. (27) results in a countable set of frequencies ω. The latter equation is solved numerically by the secant method [14]. After a choice of the amplitudes A and B one can determine the values $\tilde{A}, \tilde{B}, a, b$, correspondingly, by Eqs. (23), (26), (18) and through Eqs. (12)–(13)—the world surface (11).

To investigate the constructed world surface one can consider its section $t = t_0 = \text{const}$ as a ‘‘photograph’’ of the string position at time moment t_0. These sections (curvilinear triangles) are shown in Fig. 2–5. There are some different curves placed in each figure by a choice of the amplitude factor B. Without limiting generality $A = 0$ is supposed in these examples—a transition to another ‘‘gauge’’ with $A \neq 0$ does not change the form of such a curve, but only rotates it.

A parametrization of these curves is

$$x = u(\sigma)\cos\frac{\theta}{D}\sigma + \tilde{u}(\sigma)\sin\frac{\theta}{D}\sigma,$$

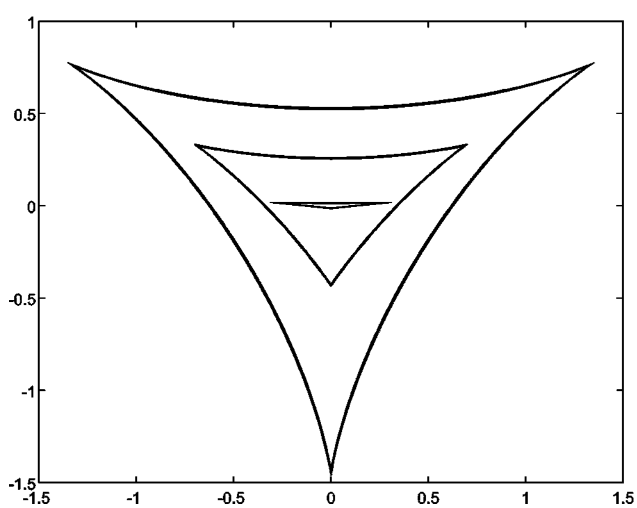

FIG. 3. The simple states for the system with $m_1 = 3$, $m_2 = m_3 = 1$.

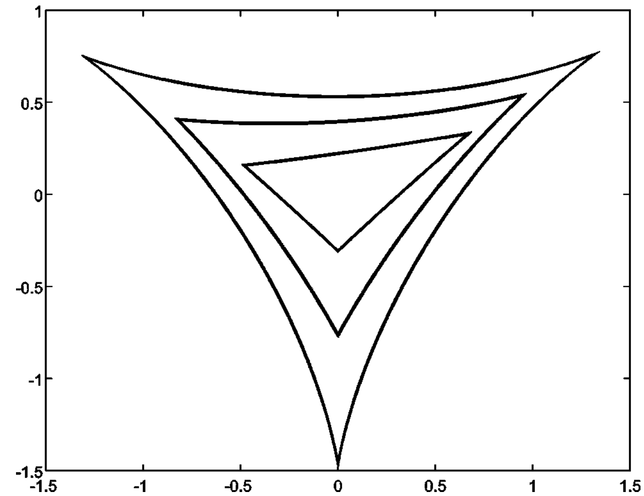

FIG. 4. The simple states for the system with $m_1 = 4$, $m_2 = 2$, $m_3 = 3$.

$$y = -u(\sigma)\sin\frac{\theta}{D}\sigma + \tilde{u}(\sigma)\cos\frac{\theta}{D}\sigma, \tag{28}$$

in particular, for two sides of the ‘‘triangle’’ ($A = 0$)

$$u = B\ \sin\ \omega\sigma, \quad \tilde{u} = BK(h_1\ \sin\ \omega|\sigma| - 2\ \cos\ \omega\sigma),$$
$$\sigma \in [\sigma_0, \pi].$$

The curve (28) is composed of three segments of a hypocycloid joined at nonzero angles in three points (the quark positions). Hypocycloid is the curve drawing by a point of a circle (with radius r) that is rolling in another fixed circle with larger radius R [15]. In the case (28) a relation of these radii

$$R/r = 2/(1 - |b|/a) = 2/(1 - |\theta/(D\omega)|) \tag{29}$$

is irrational in general.

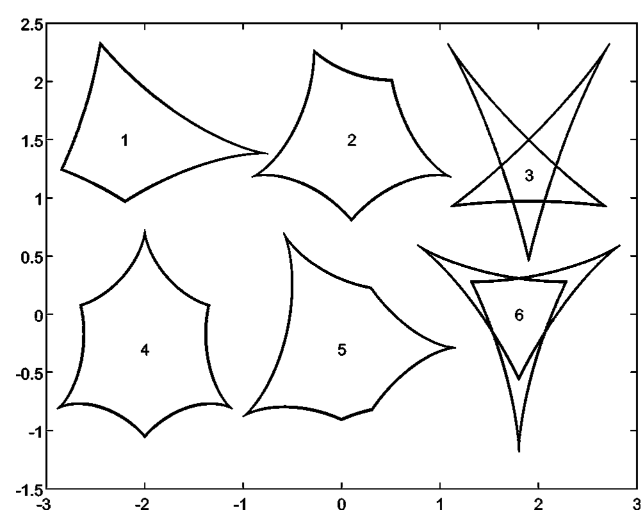

FIG. 5. The exotic states with various configurations.

Differentiating Eqs. (28) results in the following fact: the curve (28) (for its smooth segments) is the hypocycloid if and only if the parameters of the curve are bounded by Eq. (27).

The curves in Figs. 2–5 rotate in the xy plane at the angular velocity $\Omega=\omega/a$ where a is determined by Eq. (26); three quarks move at speeds

$$v_i=\sqrt{\frac{\theta s_{i-1}s_i(G_{i-1}G_{i+1}-1)}{\omega D s_{i+1}\sin\,\theta}},\quad i=1,2,3,\qquad(30)$$

along circles with radii $R_i=v_i/\Omega=av_i/\omega$.

A free choice of the mentioned integer parameters j_1,j_2,j_3,l results in a very large number of different motions of the system distinguishing from each other by their topological structure. A motion or state of the system we will denominate ''simple'' if the position of the string (section $t=\mathrm{const}$) is a curvilinear triangle with smooth sides (Figs. 2–4). In the opposite case if there are some singular massless points on the sides of the ''triangle'' we will denominate the state ''exotic'' (Fig. 5). These singular points move at the speed of light.

The motion of the system is simple if its parameters satisfy the following conditions:

$$|\theta|<\pi,\quad |\omega|d_i<\pi,\quad G_i>1,\quad i=1,2,3,$$

$$G_1+G_2+G_3-G_1G_2G_3>-2.\qquad(31)$$

In particular, if two quark masses are equal, for example, $m_2=m_3$ (Fig. 3), the conditions (31) for G_i take the form $G_1>1$, $1<G_2=G_3<1+2/G_1$ and in a symmetric case $m_1=m_2=m_3$ (Fig. 2) the limitation (31) for the simple states is $1<G_1=G_2=G_3<2$.

The dependence of a form of the curvilinear triangle on its rotational speed is shown in Figs. 2–4 for the case of simple motion of the system with fixed quark masses m_i. In each figure the ''photographs'' of the same system in various rotational states are placed, the higher the quark speeds, the larger the size of the curvilinear triangle. The dependence $R_i(v_i)$ for $m_i/\gamma=\mathrm{const}$ is too sharp so the smallest (inner) triangles in Fig. 2–4 are magnified, and the largest are diminished in comparison with the natural size (natural is in this case $m_i/\gamma=\mathrm{const}$) through homothetic multiplication by a scale factor B/B_n. For the middle curves $\gamma=1$ is taken.

The speed of rotation could be measured by anyone of the parameters v_i, ω, Ω, θ, G_i, the energy E, the angular momentum J (Sec. III), etc. In the symmetric case $m_1=m_2=m_3$ (Fig. 2) which is considered in Ref. [9] for the simple states the following parameters are equal: $G_1=G_2=G_3=G$, $d_1=d_2=d_3=\pi$, $v_1=v_2=v_3=v$. Some rounded values of these and other parameters (in particular, the minimal v_1 and maximal v_2 quark speeds in the case of different m_i, the scale factors B/B_n) for the simple motions in Fig. 2–4 are presented in Tables I and II.

In the symmetric case $m_1=m_2=m_3$ (Fig. 2) a value of the parameter G in the interval (1,2) is taken as the measure of rotational rate, the corresponding values of θ and ω are determined from Eqs. (22) and (27). The above described procedure of determinating all other parameters in the solution (11) is simplified as a consequence of the symmetry $d_1=d_2=d_3=\pi$ [9].

TABLE I. Some values of parameters for Fig. 2.

Fig. 2		inner	middle	outer
	G	1.05	1.6	1.95
	ω	0.102	0.451	0.791
$m_1=1$	θ	0.087	1.211	2.47
$m_2=1$	v	0.183	0.689	0.953
$m_3=1$	E	3.157	6.904	46.50
	J	0.011	1.679	125.6
	B/B_n	10	1	0.18

In the case with different masses m_i given as initial data (Figs. 3, 4) the mentioned procedure needs some complement. The given quark masses are connected with the other parameters of the system by the expressions

$$m_i=\frac{\gamma}{\omega}C_ih_i=\frac{\gamma}{\omega}h_ia\sqrt{1-v_i^2}.\qquad(32)$$

The values of the parameters for the rotational states in Figs. 3 and 4 were calculated as follows: a value of G_1 was chosen as a measure of rotational rate, G_2 and G_3 were taken as tentative at the first step of the iteration. After realization of the mentioned procedure of determination of θ, d_i, ω, etc., the masses (32) [or relations $m_i/m_1=(h_i/h_1)\sqrt{(1-v_i^2)/(1-v_1^2)}$, $i=2,3$] were found and compared with the given values. The two-dimensional secant method [14] was applied in this iterative process.

The simple states in Figs. 2–4 demonstrate the following asymptotics in nonrelativistic and ultrarelativistic limits. If the quark velocities v_i, the system energy E, the momentum J, and the values ω and θ decrease, the curvilinear triangle tends to a rectilinear triangle. A form of the latter depends on the answer to the question: is the triangle inequality for the quark masses m_1,m_2,m_3 satisfied?

If this inequality is satisfied, i.e., each of the quark masses m_i is less then a sum of two others (Figs. 2 and 4) in the nonrelativistic limiting case the parameters ω, θ, v_i, R_i tend to 0, $G_i\to 1+0$ for all $i=1,2,3$; the triangle tends to the rectilinear one, and lengths of its sides l_{ij} (between the ith and jth quark) in this limit are proportional to the associated d_i and opposite quark masses:

$$\frac{l_{12}}{d_1}=\frac{l_{23}}{d_2}=\frac{l_{31}}{d_0},\quad \frac{d_1}{m_3}=\frac{d_2}{m_1}=\frac{d_0}{m_2},\quad v_i\to 0.\qquad(33)$$

If one of these masses is larger then a sum of two others, for example, $m_1>m_2+m_3$ (Fig. 3), in the low energy limit $\theta\to 0$ the obtuse angle at the corner with the largest mass m_1 tends to π and the triangle tends to a rectilinear segment. This limit is attained with $\theta=0$, $R_1=0$, $v_1=0$, $G_2=G_3=1+0$, and nonzero values of ω, v_2, v_3, $G_1>1$. These limiting values are connected by the equations resulting from Eqs. (21), (24)–(27), (30), (32):

TABLE II. Some values of parameters for Fig. 3 and Fig. 4.

	Fig. 3: $m_i=3,1,1$			Fig. 4: $m_i=4,2,3,$		
	inner	middle	outer	inner	middle	outer
G_1	1.61	2.3	2.7	1.15	1.7	2.2
G_2	1.002	1.31	1.68	1.041	1.312	1.69
ω	0.168	0.387	0.77	0.143	0.386	0.752
θ	0.053	1.063	2.46	0.139	0.923	2.328
d_2	6.25	4.38	3.392	3.955	3.493	3.222
v_1	0.025	0.433	0.908	0.145	0.497	0.909
v_2	0.503	0.768	0.969	0.354	0.711	0.954
E	6.025	10.28	70.77	9.832	16.38	95.61
J	0.218	3.13	289.0	0.231	7.005	520.0
B/B_n	1.7	1	1/5	5	1	1/7

$$v_2=s_1,\quad d_2=d_0+d_1,\quad G_1=1+2d_2\omega s_0 s_1 s_2^{-1},$$

$$v_3=s_0,\quad \frac{m_2}{m_3}=\frac{c_1^2 s_0}{c_0^2 s_1},\quad \frac{m_1}{m_3}=d_2\omega s_0 s_1+s_2. \tag{34}$$

If v_2 and v_3 become less then the limiting values (34), the heaviest quark occupies a position at the rotational center and the string rotates as the rectilinear segment. It looks similar to the string mesonic model with two light quarks, bounded by two relativistic strings (details in Sec. III) with a supplement—the heavy quark at rest.

In the ultrarelativistic limit $v_i\to 1$ for the simple states the values d_0 and d_2 tend to $d_1=\pi$, $|\omega|\to 1-0$, $|\theta|\to\pi-0$, and the curvilinear triangle tends to a hypocycloid with three arcs (deltoid)

$$x=B\left(2\sin\frac{2}{3}\sigma-\sin\frac{4}{3}\sigma\right),$$

$$y=-B\left(2\cos\frac{2}{3}\sigma+\cos\frac{4}{3}\sigma\right),$$

$$\sigma\in[-\pi,2\pi]. \tag{35}$$

The form of the limiting curve (35) does not depend on the (fixed) values m_1,m_2,m_3. So one can deduce Eqs. (35) by the simplest way in the symmetric case $m_1=m_2=m_3$. In this case the ultrarelativistic limit $v_i\to 1$ corresponds to a limit $G_i=G\to 2-0$. Substitution of expressions $\omega=1-\delta$, $G=2-g^2$ with infinitesimals δ, g into Eqs. (24), (27) results in the limiting relation $\frac{3}{10}=\lim_{g\to 0}\pi\delta g(g^2+\pi^2\delta^2)^{-1}$. The root $\lim_{g\to 0}\pi\delta/g=3$ of this square equation corresponds to the desirable physical case $m_i>0$. The following terms of expansion ω and θ in Eq. (27) are

$$\omega\simeq 1-\frac{3}{\pi}g+\frac{15}{8\pi}g^3,\quad \theta\simeq\pi-3g,\quad g\to +0. \tag{36}$$

Substitution of Eq. (36) in Eqs. (22) and (13) results in limiting expressions at $g\to +0$ (in the case $A=0$) $u(\sigma)=B\sin\sigma$, $\tilde{u}(\sigma)=-3B\cos\sigma$, $\sigma\in[-\pi,2\pi]$, and the world surface (11):

$$X^\mu=B\{t_0+3\tau+\sigma;\sin\sigma\cos\tau+3\cos\sigma\sin\tau;\sin\sigma\sin\tau-3\cos\sigma\cos\tau\}. \tag{37}$$

A section of this surface $t=\text{const}$ is the hypocycloid (35).

Let us consider a situation where the condition of ''simplicity'' (31) are not satisfied. Such a motion was denominated as exotic. Its world surface has peculiarities $\dot{X}^2=X'^2=0$ on the world lines of singular points (cusps) of the hypocycloid (28) which move at the speed of light.

There are many types of exotic motion differing from each other by the number and positions of these peculiarities. Some examples are shown in Fig. 5. One must differ the peculiar points (cusps) on these curves from the quark positions. At the point of quark position two segments of the string are joined at a nonzero angle. For each curve 1–5 in Fig. 5 the first quark is situated at the lowest point, two others are along the string in the counterclockwise direction.

The number of the curve is in the center of rotation. Curve 1 in Fig. 5 represents the simplest exotic state with one peculiar point (in this example between the first and the second quark). The chosen values of parameters for this state are $G_1=0.2$, $G_2=-0.4$, $G_3=-0.2$, $\omega\simeq 1.23$, $\theta\simeq 1.78$ [θ and ω were determined by Eqs. (22), (27) after a choice of the discrete parameters j_i and l]. The quark masses for the curve 1: $m_1\simeq 3.04$, $m_2\simeq 2.59$, $m_3\simeq 3.36$, if $\gamma=1$.

Other values of G_i, j_i, l result in other types of curvilinear ''triangles.'' The pentagonal line 2 ($G_1=0.2$, $G_2=0.1$, $G_3=0.5$, $\theta\simeq 5.12$) and the starlike curve 3 ($G_1=1.9$, $G_2=G_3=1.7$, $\omega\simeq 1.37$, $\theta\simeq 1.67$) both contain two singularities and represent two different topological configurations of the string. In points of self-intersection different parts of the string with the action (1) do not interact.

Curves 4–6 in Fig. 5 describe a system with equal masses $m_1=m_2=m_3$ and equal G_i. These curves contain three singular points with various arrangements—in symmetric lines 4 and 6 these cusps alternate with quark positions. The symmetric curvilinear hexagon 4 corresponds to $G_i=G=0$, $\theta=5\pi/2$, $\omega\simeq 1.35$; line 6 with self-intersections—$G=-1$, $\theta=\pi$, $\omega\simeq 1.27$. Curve 5 has 1 cusp between the second and the third, and two cusps between the third and the first quark. This state corresponds to $\theta\simeq 5.31$, $\omega\simeq 0.12$; equal values

$G_i=1.5$, $m_i\simeq 0.4$, but different $d_0\simeq 17.7\pi$ and $d_2\simeq 9.4\pi$.

These topological configurations of the exotic states may be classified by investigation of the massless $m_i\to 0$ or ultrarelativistic $v_i\to 1$ limit. In this limit for the exotic states Eqs. (22)–(27), Eq. (30) results in the expressions

$$\lim_{m_i\to 0}\frac{|\omega|d_i}{\pi}=1+n_i,\quad \lim_{m_i\to 0} h_i=0,\quad \lim_{m_i\to 0} 2K=\frac{n}{k},$$

where

$$n=\lim_{m_i\to 0}\frac{|\omega|D}{\pi}=n_1+n_2+n_3+3,\quad k=\lim_{m_i\to 0}\frac{\theta}{\pi}. \quad (38)$$

Here n_1 is the number of singular points between the first and second quark, n_2 between the second and third and n_3 between the third and first. k is an integer.

Substitution of these expressions into Eq. (23) with $A=0$ results in the following limiting form of the world surface for all parts of the string as a generalization of Eq. (37):

$$X^\mu=B\{n\tau+k\sigma;k\sin\sigma\cos\tau+n\cos\sigma\sin\tau;k\sin\sigma\sin\tau-n\cos\sigma\cos\tau\}. \quad (39)$$

Here $\sigma\in[0,\pi n]$, the integer parameters (38) n and k are restricted by the conditions

$$n\geqslant 2,\quad |k|\leqslant n-2,\quad n-k \quad \text{is even}. \quad (40)$$

For the simple states (31) $n=3$, $|k|=1$.

Note that world surfaces (39) describe motions of a closed massless relativistic string. Expression (39) is a solution of Eq. (6) and satisfies the orthonormality conditions (5) and the closure condition $X^\mu(\tau,0)=X^\mu(\tau-\pi k,\pi n)$ with $n\geqslant 2$ and k restricted by Eq. (40).

A section $t=\text{const}$ of world surface (39) is a closed hypocycloid with rational relation of the two radii

$$R/r=2n/(n-|k|)$$

[compare with Eq. (29)]. If $|k|=n-2$, this relation equals n and the curve has no self-intersections. If $|k|\leqslant n-4$, the hypocycloid is starlike. The singular points of these hypocycloids move at the speed of light. Topological types of rotational motions of the considered system may be exhaustively classified by pointing out a set of the mentioned integer parameters $(n,k;n_1,n_2,n_3)$ which are connected by Eq. (38) and satisfy the inequalities (40).

The states of the system differing from each other only by changing k to $-k$ should be interpreted as the same topological type. This results from the fact that replacement of θ by $-\theta$ in Eqs. (18)–(28) changes only the bypass direction of the curvilinear ''triangle.''

In these terms the classification of rotational states in Figs. 2–5 looks as follows: simple motions in Figs. 2–4 have the type (3,1;0,0,0); exotic states in Fig. 5, curve 1 (4,2;1,0,0), curve 2 (5,3;1,0,1), curve 3 (5,1;1,0,1) (for the case $n=5$ both possible values $k=1$ and $k=3$ are shown), curve 4 (6,4;1,1,1), curve 5 (6,4;0,1,2), and curve 6 (6,2;1,1,1).

In the case $k=0$ (it is possible for even n) the exotic state has the form of a uniformly rotating rectilinear string that is folded. The simplest of these states $n=2$, $k=0$ is the case of the coincidence of two quarks (one of d_i equals 0). In this state the model ''triangle'' practically reduces to the quark-diquark one with the quark and diquark connected by a double string with tension 2γ. This rectilinear segment is the particular case of the hypocycloid with $R/r=2$.

If $n\geqslant 4$, $k=0$ the quarks and the massless peculiarities $\dot X^2=0$ are situated at the fold points. In this case in Eq. (11) $b=\theta=0$, $\tilde u(\sigma)=u(\sigma)\text{const}$. These states have analogues in the meson string model with massive ends. A solution [16]

$$X^\mu=\{\alpha\tau;Bu_n(\sigma)\cos\omega_n\tau;Bu_n(\sigma)\cos\omega_n\tau\} \quad (41)$$

describes a rotation of an $n-1$ times folded rectilinear open string. Here $u_n(\sigma)=\cos\omega_n\sigma-\omega_n Q_1^{-1}\sin\omega_n\sigma$, $\sigma\in[0,\pi]$, $Q_i=\gamma m_i^{-1}\sqrt{\dot X^2}|_{\sigma=\sigma_i}=\text{const}$ and ω_n is the nth positive root of the transcendental equation $\tan\pi\omega=(Q_1+Q_2)\omega/(\omega^2-Q_1Q_2)$.

III. ENERGY AND ANGULAR MOMENTUM OF ROTATIONAL STATES

In this section the possibility of the application of the considered solutions for a description of baryon states on Regge trajectories is briefly discussed. The Regge trajectory includes states of baryons with the same quark composition and almost the same set of quantum numbers. This trajectory is linear dependent (without a satisfactory theoretical explanation) between the square of mass or rest energy of the particle $M^2=E^2$ and its spin or angular momentum J: $J=\alpha' E^2+\alpha_0$.

Let us find a connection between the energy E and angular momentum J of the rotational state (11) of the baryonic model ''triangle'' on the classic level. The same problem for the string model of the meson is solved in Refs. [3,8].

In accordance with Refs. [2,3] consider new parameters t,σ on the world surface, where $t=X^0$ is time, and σ is the former parameter. The Lagrangian in action (1) is $\Lambda=-\gamma\int_{\sigma_0}^{\sigma_3}L(\overline{X}_t,\overline{X}_\sigma),d\sigma-\Sigma_{i=1}^3 m_i\sqrt{1-\overline{X}_t^2(t,\sigma_i)}$, where $L=[(\overline{X}_t\overline{X}_\sigma)^2+\overline{X}_\sigma^2(1-\overline{X}_t^2)]^{1/2}$, $\overline{X}_t=\partial_t\overline{X}$, $\overline{X}_\sigma=\partial_\sigma\overline{X}$, and $\overline{X}=\{X(t,\sigma),Y(t,\sigma)\}$ is a two-dimensional ($2D$) vector; the scalar product is Euclidean.

In coordinates t,σ the orthonormality conditions (5) are not satisfied, so the canonical momentum $\overline{P}(t,\sigma)=\delta\Lambda/\delta\overline{X}_t=-\gamma[(\overline{X}_t\overline{X}_\sigma)\overline{X}_\sigma-\overline{X}_\sigma^2\overline{X}_t]/L+\Sigma_{i=1}^3 m_i\overline{X}_t(1-\overline{X}_t^2)^{-1/2}\ \delta(\sigma-\sigma_i)$ is nonlinear with respect to $\overline{X}_t$.

The energy of the system $E=\int_{\sigma_0}^{\sigma_3}(\overline{X}_t\overline{P})d\sigma-\Lambda=\gamma\int_{\sigma_0}^{\sigma_3}L^{-1}\overline{X}_\sigma^2 d\sigma+\Sigma_{i=1}^3 m_i/\sqrt{1-v_i^2}$ has the form

$$E=\gamma D\frac{a^2-b^2}{a}+\sum_{i=1}^3\frac{m_i}{\sqrt{1-v_i^2}}, \quad (42)$$

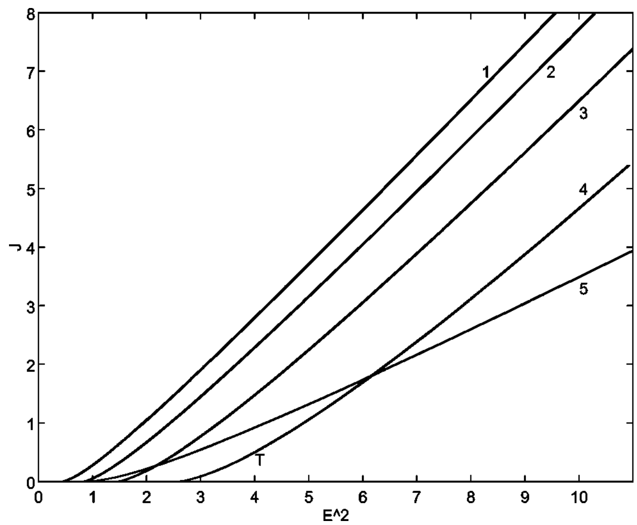

FIG. 6. Dependence $J(E^2)$ for the simple motions with various m_i (1–4) and for the exotic state (5).

where $v_i^2=\overline{X}_t^2(t,\sigma_i)$. The following expressions resulting from Eqs. (11)–(17) were used in the calculations: $\overline{X}_\sigma^2=(a^2-b^2)(1-\overline{X}_t^2)=(a^2-b^2)b^{-1}(\overline{X}_t\overline{X}_\sigma)=(a^2-b^2)a^{-1}L$. The parameters $D,a,b=a\theta/(D\omega)$, and v_i are defined by Eqs. (18), (26), and (30).

The angular momentum $J=\int_{\sigma_0}^{\sigma_3}(XP_y-YP_x)d\sigma$ of the state (11) is calculated in a similar way:

$$J=\frac{a}{2\omega}\left(E-\sum_{i=1}^{3} m_i\sqrt{1-v_i^2}\right). \tag{43}$$

The latter relation between E, J, and the angular frequency $\Omega=\omega/a$ has almost the same form as in the string model of a meson [3].

Expressions (42) and (43) set an implicit nonlinear connection between E and J of the considered system. A form of this connection depends on the topological type $(n,k;n_1,n_2,n_3)$ of the state of the system.

In Fig. 6 the results of the numerical calculation the dependence J on E^2 are shown for various states of the systems with fixed $m_2=m_3=0.3$, $\gamma=3/(16\pi)$. Such a choice of γ approximately corresponds to the experimental value $\alpha'\simeq 1$ GeV^{-2}. One can suppose conditionally that E^2 in Fig. 6 is measured in GeV^2 and J in units $\hbar$.

Curves 1, 2, 3, and 4 describe the simple motions of the system correspondingly with $m_1=0.05$, $m_1=0.3$, $m_1=0.6$, and $m_1=1$. Curve 5 is the exotic state of the symmetric system with equal masses $m_1=m_2=m_3=0.3$ and with topological type of this state (6,2;1,1,1) (curve 6 in Fig. 5).

The symbol T on curve 4 shows the point of transformation of the triangular configuration of this system with $m_1=1>m_2+m_3=0.6$ to the rectilinear configuration. This point of transformation corresponds to the satisfying of Eqs. (34). A form of the curvilinear triangle in the vicinity of such a point is shown in Fig. 3.

The analysis shows that in the nonrelativistic limit the asymptotic behavior of the function $J(E)$ depends on satisfaction of the triangle inequality between three masses m_i. If each mass is less than the sum of two others, the limiting relations (33) for the simple state take place, and energy (42) of this state has the form $E=\Sigma_{i=1}^3 m_i+\frac{3}{2}m_1m_2m_3^{-1}\pi^2\omega^2+o(\omega^2)$, where ω is an infinitesimal. The considered asymptotic relation in this case is

$$J\simeq\left(\frac{2}{3}\right)^{3/2}\frac{\sqrt{m_1m_2m_3}}{\gamma(m_1+m_2+m_3)}\left(E-\sum_{i=1}^{3} m_i\right)^{3/2}, \quad v_i\ll 1.$$

If one of the masses, for example, m_1, is larger then the sum of two others (see Fig. 3 and curve 4 in Fig. 6), then the nonrelativistic asymptotic case describes a rotation of a rectilinear double string with two masses m_2 and m_3 at the ends and a mass m_1 at the rotational center. In this limit for the simple state the following expression takes place:

$$J\simeq\left(\frac{2}{3}\right)^{3/2}\frac{1}{2\gamma}\sqrt{\frac{m_2m_3}{m_2+m_3}}\left(E-\sum_{i=1}^{3} m_i\right)^{3/2}, \quad v_i\ll 1.$$

It looks similar to the formula in Ref. [3] for the string model of a meson and may be deduced from solution (41), but the tension of the string equals 2γ.

The exponent 3/2 is the same for both cases considered. So graphs 1–4 in Fig. 6 have similar forms and curve 4 in the vicinity of the transformation point is rather smooth.

In the opposite ultrarelativistic limit $v_i\to 1$, $E\to\infty$, $J\to\infty$ the analysis of dependence $J(E)$ includes substituting limiting formulas (38) and expressions $\omega d_i=\pi(n_i+1)-\delta_i$, $\theta=\pi k(1-\delta_\theta)$, $\sqrt{1-v_i^2}=\varepsilon_i$ with infinitesimals δ_i, δ_θ, ε_i [generalization of Eq. (36)] into Eqs. (22)–(30), (42) and (43). Expansion in series in Eqs. (25), (27), and (30) results in the following relations between the infinitesimals:

$$h_i\simeq 2\frac{\sqrt{n^2-k^2}}{n}\varepsilon_i\left(1+\frac{n^2-2k^2}{2(n^2-k^2)}\varepsilon_i^2\right),$$

$$\frac{\delta_i}{\sqrt{m_i}+\sqrt{m_{i+1}}}\simeq\frac{\delta_j}{\sqrt{m_j}+\sqrt{m_{j+1}}}\simeq\frac{n}{\sqrt{n^2-k^2}}\frac{\varepsilon_i}{\sqrt{m_i}},$$

$$\sum_{i=1}^{3}\delta_i\simeq\frac{2nm_1^{-1/2}}{\sqrt{n^2-k^2}}\left(\sum_{i=1}^{3} m_i^{1/2}\right)\varepsilon_1 + \frac{nm_1^{-3/2}}{(n^2-k^2)^{3/2}}\left(\frac{n^2-2k^2}{2}m_1\sum_{i=1}^{3} m_i^{1/2} - \frac{n^2-6k^2}{6}\sum_{i=1}^{3} m_i^{3/2}\right)\varepsilon_1^3.$$

By substitution of these and analogous relations into Eqs. (32), (42), and (43) we obtain the ultrarelativistic asymptotic dependence for a state with an arbitrary type $(n,k;n_1,n_2,n_3)$

$$J\simeq\alpha' E^2+\alpha_1E^{1/2}, \quad v_i\to 1, \tag{44}$$

where

$$\alpha' = \frac{1}{2\pi\gamma}\frac{n}{n^2-k^2}, \quad \alpha_1 = -\frac{\sqrt{2}n(n^2-k^2)^{-3/4}}{3\sqrt{\pi}\gamma}\sum_{i=1}^{3} m_i^{3/2}.$$

This is close to the standard linear form $J=\alpha' E^2+\alpha_0$.

The slope coefficient in Eq. (44) differs from the Nambu value for the mesonic model $\alpha'=1/(2\pi\gamma)$ by the factor $n/(n^2-k^2)$. This factor equals 3/8 for simple motions and attains the maximal value 1/2 (under admissible n and k) for ''quark-diquark'' motions with $n=2$, $k=0$. The latter case differs from the quark-diquark baryon model only by the substitution $\gamma\rightarrow 2\gamma$.

The ''quark-diquark'' state is preferable if we assume the principle of minimal energy: the string system with given J chooses the configuration with the minimal energy [7,8].

The first summand in Eq. (42) that could be interpreted as the ''string energy'' or ''gluon energy'' in the limit $v_i\rightarrow 1$ or $\varepsilon_i\rightarrow 0$ grows as ε_i^{-2}, but the last summands, ''quark kinetic energy'' $\Sigma m_i/\varepsilon_i$, grow as ε_i^{-1}. So in the ultrarelativistic limit the ''string energy'' dominates, and the slope coefficient α' in Eq. (44) does not depend on quark masses m_i.

The coefficient α_1, otherwise, is determined by the combination $\Sigma m_i^{3/2}$. This fact gives the possibility of estimating (in the model frameworks) the mentioned sum and quark masses m_i. This estimation will be accurate only for baryons which satisfy two conditions: the quark motion is to be relativistic and close to classic (the model is classic with spinless quarks). The latter condition is equivalent to the standard inequality $J/\hbar \gg 1$ and in particular, results from the comparison of typical sizes of the ''triangle'' system in the relativistic case [if $E\gg m_i\sqrt{1-v_i^2}$ in Eq. (43)]

$$R_i=\frac{a}{\omega}v_i\simeq\frac{2J}{E}v_i\simeq 3.95\times 10^{-14}\frac{J}{\hbar}\frac{1\ \ \text{GeV}}{E}\frac{v_i}{c}\ \text{cm}$$

with the corresponding length $\hbar/p\simeq\hbar/E$. Furthermore, the motion is relativistic if the quarks are not very heavy: $m_i \ll M=E$.

Express the combination $\Sigma m_i^{3/2}$ from Eq. (44)

$$\sum_{i=1}^{3} m_i^{3/2}\simeq\frac{3(n^2-k^2)^{-1/4}}{2^{3/2}\sqrt{\pi}}\left(E^{3/2}-\frac{J}{\alpha'\sqrt{E}}\right). \tag{45}$$

It is natural to suppose that the states of the model are simple ($n=3$, $k=1$) or ''quark-diquark'' ($n=2$, $k=0$). For these two cases the mass estimations differ from each other by the small factor $\simeq 2^{1/6}$.

The expression in the parentheses on the right-hand side (RHS) of Eq. (45) is a small difference of two large values. So it is very sensitive to errors in J and E. The simplest quantum correction to these values due to quark spins implies an addition $S=\Sigma_{i=1}^{3}s_i$ (quark spin projection) to the classic angular momentum (43) and $\Delta E=\Delta E_{\text{SS}}+\Delta E_{\text{SO}}$ to the energy (42). The latter correction results from spin-spin (ΔE_{SS}) and spin-orbit (ΔE_{SO}) interaction of quarks. The value ΔE_{SO} is supposed to be due to pure Thomas precession of quark spins [8,17], but there are some doubts as to the form and the sign of this correction. A precise form of ΔE is to be found only from a consecutive quantum theory of this baryon model that has not yet been constructed.

TABLE III. Effective quark mass predictions.

J	1/2	5/2	9/2
Particle	n,p	$N(1680)$	$N(2220)$
m_{ud}	0.138 ± 0.015	0.105 ± 0.03	0.11 ± 0.02
Particle	Λ	$\Lambda(1815)$	$\Lambda(2350)$
m_s	0.41 ± 0.03	0.345 ± 0.07	0.35 ± 0.055
m_s^*	0.34 ± 0.035	0.26 ± 0.07	0.27 ± 0.06

In the examples below the spin correction was not made. The results of using Eq. (45) for estimating quark masses on examples of two Regge trajectories (nucleonic and for strange Λ particles) are shown in Table III. Masses of u and d quarks are assumed to be equal. Here m_{ud} and m_s are effective quark masses measured in GeV; J is in units of $\hbar$. The values m_s were calculated under the assumption that $m_{ud}\ll m_s$, and m_s^* under the assumption $m_{ud}\simeq 0.1$ GeV.

The error ranges in determinating m_i are due to error ranges in particle masses M which influence the value α'. The small difference between the results for the simple and ''quark-diquark'' configurations is also included in the error ranges. Note that the considered model (and other mentioned string models) is applicable only to the orbitally excited baryon states (resonances) with $J\geq 5/2$ and is not adequate for p, n, and Λ particles.

We may conclude that quark masses calculated by Eq. (45) are steady with growing J. But the found values $m_{ud} \simeq 100$ MeV and $m_s\simeq 250$ MeV (larger than other data for free quark masses [18] and less then the constituent masses [8]) are preliminary and depend on the spin correction.

The necessity of the spin correction is demonstrated by the following fact. For the Regge trajectory with Δ resonances ($S=3/2$) Eq. (45) results in small negative values of $\Sigma m_i^{3/2}$ (error boxes include some positive range). But with substituting $J-1/2$ instead of J the formula (45) gives the steady value $m_{ud}\simeq 0.1$ GeV for heavy Δ resonances.

With growing E and J the influence of the unknown ΔE on the values m_i in Eq. (45) diminishes, but too slowly—as $E^{-1/2}$ or $J^{-1/4}$. So for the available baryon mass range 1–3 GeV the spin correction in Eq. (45) is required for a valid estimation of the quark masses in the frameworks of the considered model.

CONCLUSION

In the present paper a set of rotational motions of the baryon model ''triangle,'' interesting from a geometric point of view, were investigated on the classic level. The quantization in this model as in the string model of meson with massive ends [2,12] encountered some problems connected with the nonlinear form of the boundary conditions (8). Progress in this direction, for example, a description of quark spins, will give the possibility of a precise model prediction of the effective quark masses through comparison of calculated dependence $J(E^2)$ with the experimental Regge trajectories. But the problem of quantization needs special research which is beyond the present paper.

On the other hand, the slope α' was finally determined by Eq. (44) on the classic level. It was mentioned that this coefficient in the baryonic model ''triangle'' differs from the mesonic slope $\alpha' = 1/(2\pi\gamma)$ by a factor 1/2 for the ''quark-diquark'' states and by a factor 3/8 for the simple states. The experimental value $\alpha' \simeq 1\ \mathrm{GeV}^{-2}$ is approximately equal for mesons and baryons. So an effective value of string tension γ in the model ''triangle'' is to be about 1/2 or 3/8 of the tension in the model of a meson. This is probably connected with different energies of QCD interaction in the pairs: quark-quark and quark-antiquark. For the sake of comparison note that in the three-string model [4–7] this factor equals 2/3, i.e., the Regge slope in the ultrarelativistic limit is $\alpha' = \frac{2}{3}(2\pi\gamma)^{-1}$, and the effective string tension is to differ by the same factor from the mesonic one.

In the quark-diquark model and in the linear configuration the Regge slope $\alpha' = 1/(2\pi\gamma)$ equals the mesonic one. So these models explain the equality of values α' for mesons and baryons in a natural way (the rotational motions of these models are mesonlike). But this advantage is balanced on an explicit dissymetry of the quarks in both models. Furthermore, the ''triangle'' and Y configurations unlike the two others string baryon models are QSD motivated in the Wilson loop operator approach [10].

For a description of baryons on the Regge trajectories the ''quark-diquark'' states and the simple states (Fig. 2–4) of the model ''triangle'' were used. Under the assumption that the energy of the orbitally excited string state for the given angular momentum J is minimal [7,8] these configurations are preferable, and among them is the ''quark-diquark'' one.

The exotic states (Fig. 5) naturally emerging in this model are probably too exotic for physical applications. Perhaps, they have some connection with such undetected particles as hybrids.

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[1] A. Chodos and C. B. Thorn, Nucl. Phys. **B72**, 509 (1974).
[2] B. M. Barbashov and V. V. Nesterenko, *Introduction to the Relativistic String Theory* (World Scientific, Singapore, 1990).
[3] B. M. Barbashov, in *Strong Interactions at Long Distances* (Hadronic, Palm Harbor, FL, 1995), pp. 257–275.
[4] X. Artru, Nucl. Phys. **B85**, 442 (1975); Phys. Rep. **97**, 147 (1983).
[5] P. A. Collins and J. F. L. Hopkinson, Nucl. Phys. **B100**, 157 (1975); K. Sundermeyer and A. de la Torre, Phys. Rev. D **15**, 1745 (1977).
[6] M. S. Plyuschai, G. P. Pron'ko, and A. V. Razumov, Theor. Math. Phys. **63**, 389 (1985); S. V. Klimenko *et al.*, *ibid.* **64**, 810 (1985).
[7] K. Johnson and C. B. Thorn, Phys. Rev. D **13**, 1934 (1976).
[8] Yu. I. Kobzarev, L. A. Kondratyuk, B. V. Martemyanov, and M. G. Schepkin, Yad. Fiz. **45**, 526 (1987).
[9] G. S. Sharov, Theor. Math. Phys. **113**, 1263 (1997).
[10] John M. Cornwall, Phys. Rev. D **54**, 6527 (1996).
[11] Dan LaCourse and M. C. Olsson, Phys. Rev. D **39**, 2751 (1989); M. C. Olsson and Ken Williams, *ibid.* **48**, 417 (1993).
[12] G. S. Sharov, Theor. Math. Phys. **107**, 487 (1996).
[13] B. M. Barbashov and G. S. Sharov, Theor. Math. Phys. **101**, 1332 (1994).
[14] J. F. Traub, *Iterative Methods for the Solutions of Equations* (Chelsea, New York, 1982).
[15] E. H. Lockwood, *A Book of Curves* (Cambridge University Press, Cambridge, England, 1961).
[16] V. P. Petrov and G. S. Sharov, Theor. Math. Phys. **109**, 1388 (1996).
[17] T. J. Allen, M. C. Olsson, S. Veseli and Ken Williams, Phys. Rev. D **55**, 5408 (1997).
[18] Particle Data Group, R. M. Barnett *et al.*, Phys. Rev. D **54**, 1 (1996).