Perturbative stability of the Padé-summed Higgs-boson coupling in the standard electroweak model

Loyal Durand* and Gregory Jaczko†
Department of Physics, University of Wisconsin-Madison, 1150 University Avenue, Madison, Wisconsin 53706
(Received 21 April 1998; published 28 October 1998)

We apply the Padé summation to the $\beta(\lambda)$ function for the quartic Higgs coupling λ in the standard electroweak model. We use the β function calculated to five loops in the minimal subtraction scheme to demonstrate the improvement resulting from the summation, and then apply the method to the more physical on-mass-shell renormalization scheme where β is known to three loops. We find that the apparent convergence of the perturbation series for β is significantly improved, and apply the result to the study of the running coupling $\lambda(s)$. [S0556-2821(98)00921-7]

PACS number(s): 12.15.−y, 11.10.Hi, 14.80.Bn

I. INTRODUCTION

In the standard model of elementary particle physics, the $SU(2)\times U(1)$ symmetry is spontaneously broken to a residual $U(1)_{EM}$, generating mass for the $W^{\pm}$ and Z gauge bosons and the matter fields. A possible cause for the symmetry breaking is the presence of an additional scalar field, the Higgs field. Although there is as yet no experimental evidence for the expected Higgs boson, we can still explore the implications of this symmetry-breaking mechanism using radiative corrections to standard-model processes. For example, the condition that perturbative calculations be reliable provides a theoretical upper bound on the mass M_H of a weakly interacting Higgs boson [1–11].

In a recent paper [11], Nierste and Riesselmann analyzed one- and two-scale processes involving the Higgs field with a particular emphasis on the running of the quartic Higgs coupling $\lambda(\mu)$. They assessed the reliability of perturbation theory using two criteria: the relative difference of physical quantities calculated in different renormalization schemes; and the dependence of λ on the renormalization scale μ. If perturbation theory is to be reliable, the choices of the renormalization scheme and scale should not be important for physical quantities. To determine $\lambda(\mu)$ in their analysis, Nierste and Riesselmann integrated the renormalization-group equation using the three-loop β function, and solved the resulting equation for λ iteratively using four different approximation schemes. The solutions differed significantly for large values of the coupling or mass scale, and determined one constraint on M_H in a perturbative theory. This uncertainty in $\lambda(\mu)$ carries over to physical quantities such as scattering amplitudes and again affects the ranges of M_H and μ over which perturbative calculations are reliable.

We show here that is is possible to explore the regime of large coupling without the ambiguities that arise from the direct iterative solution for the coupling. We approach the problem by emphasizing the β function $\beta(\lambda)$, and show that it can apparently be determined reliably to rather large values of λ by using Padé approximates [12,13] to sum the perturbation series for $\beta(\lambda)$ [14]. Integration of the renormalization-group equation then gives an implicit equation for $\lambda(\mu)$ that can be inverted numerically. The results can be used to study the validity of perturbation theory for scattering amplitudes in the region of large Higgs-boson masses and high energies where the running coupling $\lambda(\mu)$ is the natural renormalization-group expansion parameter. We will not pursue those applications here as a number of authors [4–11,15] have considered them in detail.

We first investigate the Padé approach in Sec. II using the the results for $\beta(\lambda)$ in the minimal subtraction (MS) renormalization scheme for which the perturbation series for β is known to five loops [16,17]. After establishing the effectiveness of the Padé approach, we apply it in Sec. III to the more physical on-mass-shell (OMS) renormalization scheme where β is only known to three loops [11,18]. We find that Padé summation of the series apparently gives a reliable result for $\beta(\lambda)$ for quite large values of the coupling, $\lambda\leqslant 10$, and conclude, after inversion of the renormalization-group expression, that $\lambda(\mu)$ is known reliably in the OMS scheme for $\mu\leqslant 4$ TeV for $M_H\leqslant 800$ GeV. The region in which $\lambda(\mu)$ is known well extends to very large mass scales if M_H is sufficiently small, for example, to 10^{17} GeV for $M_H \leqslant 155$ GeV.

II. PADÉ SUMMATION OF THE β FUNCTION

A. Preliminary considerations

In the following, we deal with the quartic Higgs-boson coupling λ defined at tree level in terms of M_H and the electroweak vacuum expectation value $v=246$ GeV or the Fermi coupling G_F by $\lambda=M_H{}^2/2v^2=G_F M_H{}^2/\sqrt{2}$. We will work in the interesting limit of large Higgs-boson masses, corresponding to the limit of large quartic couplings, and neglect the effects of couplings with fermions.

The running coupling $\lambda(\mu)$ is defined as the solution of the renormalization-group equation

$$\mu\frac{d\lambda(\mu)}{d\mu}=\beta(\lambda) \tag{1}$$

*Electronic address: ldurand@theory2.physics.wisc.edu
†Electronic address: jaczko@theory1.physics.wisc.edu

at the energy scale μ. The function $\beta(\lambda)$ is given in perturbation theory as a power series in λ:

$$\beta(\lambda)=\frac{\lambda^2}{16\pi^2}\sum_{n=0}\beta_n\left(\frac{\lambda}{16\pi^2}\right)^n \tag{2}$$

$$=\beta_0\frac{\lambda^2}{16\pi^2}\left(1+\sum_{n=1}B_n\lambda^n\right). \tag{3}$$

The coefficients β_n are renormalization-scheme dependent beyond two loops. They are known through three loops in the on-mass-shell renormalization scheme [11,18],

$$\text{OMS:}\quad \beta_0=24,\quad \beta_1=-312,\quad \beta_2=4238.23, \tag{4}$$

and to five loops in the minimal subtraction scheme [16,17],

$$\text{MS:}\quad \beta_0=24,\quad \beta_1=-312,\quad \beta_2=12\,022.7,$$

$$\beta_3=-690\,759,\quad \beta_4=4.91261\times10^7. \tag{5}$$

Alternatively, the coefficients B_n are given by

$$\text{OMS:}\quad B_0=1,\quad B_1=-0.082\,323,\quad B_2=0.007\,081\,6, \tag{6}$$

$$\text{MS:}\quad B_0=1,\quad B_1=-0.082\,323,\quad B_2=0.020\,089,$$

$$B_3=-0.007\,309\,0,\quad B_4=0.003\,291\,7. \tag{7}$$

To determine the running coupling, one must integrate the renormalization group equation, Eq. (1), and solve the implicit equation

$$\ln\frac{\mu}{\mu_0}=\int_{\lambda(\mu_0)}^{\lambda(\mu)}\frac{dx}{\beta(x)}. \tag{8}$$

This equation determines $\lambda(\mu)$ in terms of the initial and final mass scales μ_0 and μ and the initial value of the coupling at the scale μ_0, defined as $\lambda_0=\lambda(\mu_0)$. Different, typically iterative, methods of solution lead to different results for $\lambda(\mu)$, with the differences increasing for large vales of M_H or λ_0 and for $\mu\gg\mu_0$ [11]. Since the β function is only known to finite order, the only constraint on this standard approach is that the different solutions satisfy the renormalization-group equation, Eq. (8), to that order. However, the resulting ambiguities for large values of M_H can compromise tests of the reliability of perturbation theory, and the determination of limits on M_H in a weakly interacting theory. It is therefore useful to approach the problem differently, and concentrate on the β function itself. If $\beta(\lambda)$ is known accurately for some range of λ, the integral in Eq. (8) will also be accurately determined, and the equation can be inverted numerically to find $\lambda(\mu)$ in that region.

B. Padé summation and $\beta(\lambda)$

Padé approximates [12,13] give a very useful way of summing or extrapolating a series for which only a finite number of terms are known. The $[N,\ M]$ Padé approximate for a function $f(z)$ defined by a truncated power series

$$f(z)=\sum_{j=0}^{m}c_jz^j+O(z^{m+1}) \tag{9}$$

is a ratio of two polynomials,

$$P[N,\ M](z)\equiv\frac{\sum_{n=0}^{N}a_nz^n}{\sum_{n=0}^{M}b_nz^n},\quad b_0=1,\quad N+M=m. \tag{10}$$

The coefficients a_n, b_n are determined uniquely by the requirement that the series expansion of $P[N,\ M](z)$ agree term-by-term with the series for $f(z)$ through terms of order z^m.

The sequence of Padé approximates $P[N,\ M]$ is known to converge to $f(z)$ as $N,\ M\to\infty$ with $N-M$ fixed for large classes of functions [12,13], but the approximates can also give useful and rapidly convergent asymptotic approximations for finite N and M even if the sequence and the original series for $f(z)$ do not converge [13].

In the present case, the function in question is $\beta(\lambda)$, known perturbatively to orders λ^4 and λ^6, that is, to three and five loops, in the OMS and MS renormalization schemes, respectively. The perturbation series for β is not expected to converge, but a Padé summation of the series may still be useful for λ not too large. Because the perturbative expansion of $\beta(\lambda)$ starts at order λ^2, we will extract the leading power explicitly, redefine the Padé coefficients, and define the $[N,\ M]$ approximate for the n-loop β function as[1]

$$\beta[N,\ M]=\beta_0\frac{\lambda^2}{16\pi^2}\frac{1+a_1\lambda+a_2\lambda^2+\cdots+a_N\lambda^N}{1+b_1\lambda+b_2\lambda^2+\cdots+b_M\lambda^M},$$

$$N+M=n-1. \tag{11}$$

Note that the approximates $\beta[n-1,0]$ are just the perturbation series for β carried to n loops.

The series for $\beta(\lambda)$ defined by Eq. (3) are alternating series in which the ratios of coefficients B_{n+1}/B_n change only slowly in either OMS or MS renormalization in the range in which the B's are known. This suggests that the diagonal approximates $\beta[N,N]$ with $M=N$ or the subdiagonal approximates with $M=N+1$ may be particularly effective in estimating the series. In the case of OMS renormalization, the β's are known only to three loops, so $M+N\leqslant2$. The possible choices are then $\beta[1,1]$ or $\beta[0,2]$ if we use all the three-loop information, or $\beta[0,1]$ if the per-

[1]Padé summation of β was considered by Yang and Ni [14], but without applications to the present problem. Those authors did not extract the overall factor λ^2, so use a different labeling of the approximates, and miss the diagonal approximates used here.

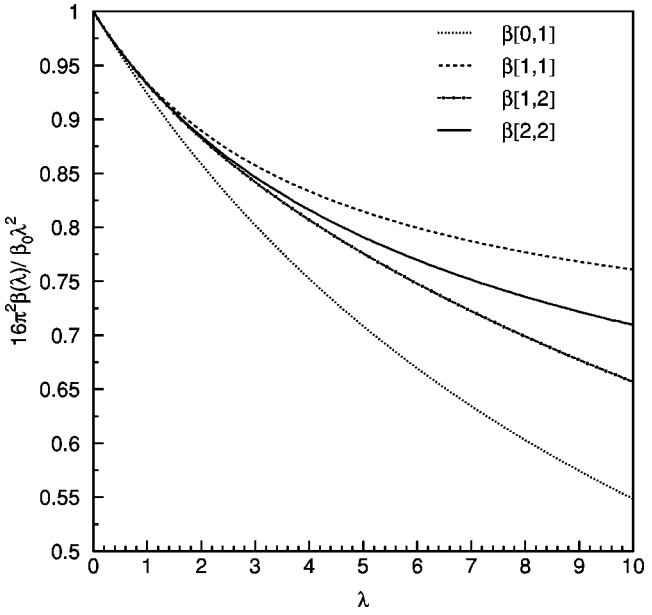

FIG. 1. A comparison of the sequence of five-loop diagonal and subdiagonal Padé approximates for $\beta(\lambda)$ in the MS renormalization scheme. Note that alternate approximates are too large or too small, and that the sequence converges rapidly with the final result presumably in the band between $\beta[1,2]$ and $\beta[2,2]$.

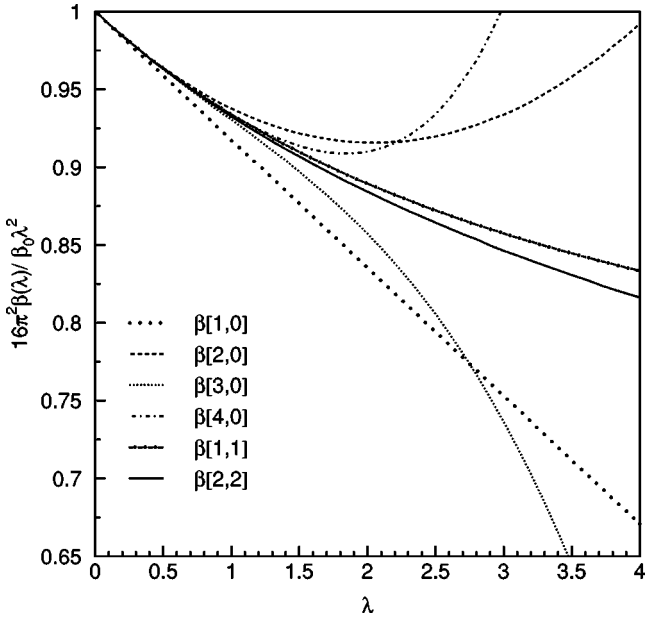

FIG. 2. Demonstration of the slow convergence of successive perturbative approximations to the β function toward the diagonal Padé approximate $\beta[2,2]$ for MS renormalization.

turbation series is truncated at two loops. $\beta[2,0]$ and $\beta[1,0]$ are just the three- and two-loop perturbation series. In the case of MS renormalization, β is known to five loops, $M+N\leqslant 4$, and we will consider the approximates $\beta[1,2]$ at the four-loop level, and $\beta[2,2]$ at five loops, keeping $M=N$ or $M=N+1$. The additional five-loop approximates $\beta[1,3]$, $\beta[3,1]$, and $\beta[0,4]$ are members of sequences two or more steps off the diagonal. These are not expected to converge as rapidly as the sequences we consider. The coefficients a_j, b_j for these approximates are given in Appendix A.

C. Tests of Padé summation using MS renormalization

The fact that the perturbation series for β is known to five loops gives us the opportunity to test the Padé summation procedure using known results. Having established its reliability, we will then apply the method in Sec. III to the more physical OMS renormalization scheme in which the connection between λ and M_H is known.

1. Convergence of the Padé sequence

Based upon the general convergence properties of Padé approximates and the alternating character of the series at hand, we expect the sequence $\beta[1,1]$, $\beta[1,2]$, and $\beta[2,2]$ to converge as we progress from three to five loops. We plot these approximates in Fig. 1 to demonstrate that convergence. The convergence of the Padé sequence is, in fact, relatively fast. For low values of λ there is excellent agreement. Even for $\lambda=10$, $\beta[1,1]$ and $\beta[1,2]$ differ by $<10\%$ with the diagonal five-loop approximate $\beta[2,2]$ lying roughly halfway between the other two. We interpret the agreement and the pattern of convergence as strong evidence for the effectiveness of the $\beta[N,N]$ sequence in summing the series for $\beta(\lambda)$, and conclude that it is unlikely that β would be found to differ significantly from $\beta[2,2]$ in the region shown if higher-loop contributions were calculated.

In Fig. 2, we look at the problem from the point of view of the purely perturbative approach, and show the sequence of the N-loop perturbation series $\beta[N-1,0]$ for β. This is not a sequence in which N and M increase together with the difference $N-M$ fixed, so the standard results on Padé convergence do not apply. The convergence of the sequence is very slow as shown in the figure, with large differences between successive terms already present for $\lambda\simeq 3$. For comparison, we also show the three- and five-loop diagonal approximates $\beta[1,1]$ and $\beta[2,2]$. These forms interpolate the perturbative sequence very well, eliminating the dominance of the last term in the series for λ large. Since $\beta[1,1]$, $\beta[2,2]$, and the four-loop approximate $\beta[1,2]$ differ from each other by less than 5% for $\lambda<10$, all are effective in extrapolating the perturbation series. We conclude, in particular, that the three-loop approximate $\beta[1,1]$ already gives a reasonable extrapolation for $\beta(\lambda)$.

2. Estimates of unknown coefficients

Padé approximates often converge to the limit function faster than the power series used to construct them. In that case, the terms in the expansion of a Padé approximate beyond the matched order may give reasonable estimates for the unknown higher-order coefficients in the power series. As a simple test of this expectation in the present case, we can expand the three- and four-loop approximates $\beta[1,1]$ and $\beta[1,2]$ to one order higher in λ than the finite power

series used to construct them, and compare the new coefficient with the known four- and five-loop results. Thus, the expansion

$$\beta[1,1]=\frac{\lambda^2}{16\pi^2}\beta_0[1+B_1\lambda+B_2\lambda^2+(B_2^2/B_1)\lambda^3 + (B_2^3/B_1^2)\lambda^4+\cdots] \quad (12)$$

gives the estimates

$$B_3'\equiv B_2^2/B_1, \quad B_4'\equiv B_2^3/B_1^2, \quad (13)$$

for the four- and five-loop coefficients B_3 and B_4, results equivalent to

$$\beta_3'\equiv\beta_2^2/\beta_1=-463\,286, \quad \beta_4'\equiv\beta_2^3/\beta_1^2=1.785\times10^7. \quad (14)$$

The actual four- and five-loop results are

$$\beta_3=-690\,759, \quad \beta_4=4.913\times10^7. \quad (15)$$

The estimates of β_3 and β_4 from the three-loop are therefore about 0.67 and 0.36 of the actual coefficients and have the correct signs.

In the case of $\beta[1,2]$, we can estimate only β_4, with the result

$$B_4'=-(B_3^2-2B_1B_2B_3+B_2^3)/(B_1^2-B_2). \quad (16)$$

This estimate gives $\beta_4'=3.48\times10^7$, and a ratio $\beta_4'/\beta_4=0.71$.

The estimates for the first missing terms in the perturbation series are too small in both of the cases considered. We can understand this result qualitatively as resulting from the averaging of an alternating series by the approximates, with the corresponding tendency to avoid large higher coefficients in the expansion. We will use this observation below.

The effects of incorrect estimates of B_3 on the approximate $\beta[1,2]$ are shown in Fig. 3. In these calculations, we have taken B_3 as five- and ten times the estimated value, and calculated $\beta[1,2]$ using the new value as input. The result is a $<10\%$ change in β for $\lambda<10$ despite the very large values of the new coefficient.

D. The running coupling $\lambda(\mu)$ in MS renormalization

The effect of the uncertainty in $\beta(\lambda)$ on the running of $\lambda(\mu)$ can be studied by integrating the renormalization-group equation, Eq. (1), and solving numerically for λ as a function of its initial value λ_0 and the ratio of energy scales μ/μ_0.[2] We have done this calculation using the approximates $\beta[1,1]$ and $\beta[1,2]$, choosing initial values $\lambda_0=1,\ 3,\ 5$. The results are shown in Fig. 4. The result for the optimum five-loop approximate, $\beta[2,2]$, lies near the center of the shaded regions in that figure, as would be expected from the comparison of the approximates in Fig. 1. We believe the estimated range of uncertainty is quite generous given the rapid convergence of the sequence shown there toward $\beta[2,2]$.

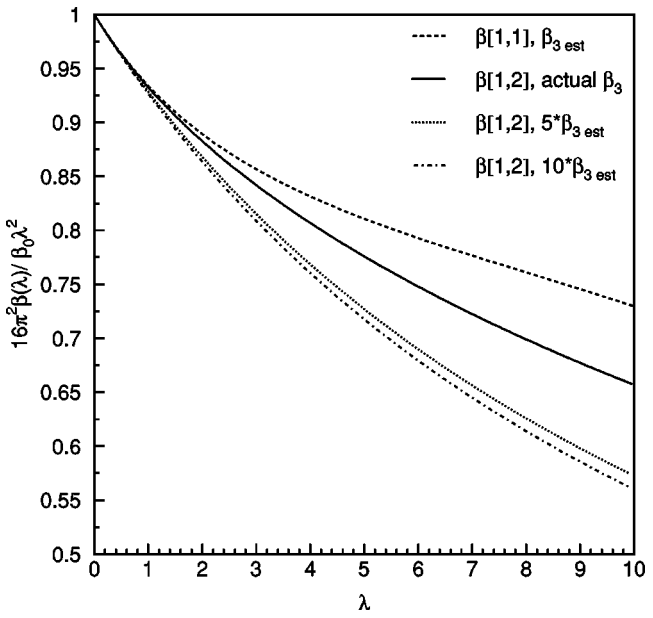

FIG. 3. Plots of the Padé approximate $\beta[1,2]$ to the MS β function using the actual value of the four-loop coefficient β_3 and values five and ten times the estimate obtained from the three-loop approximate $\beta[1,1]$.

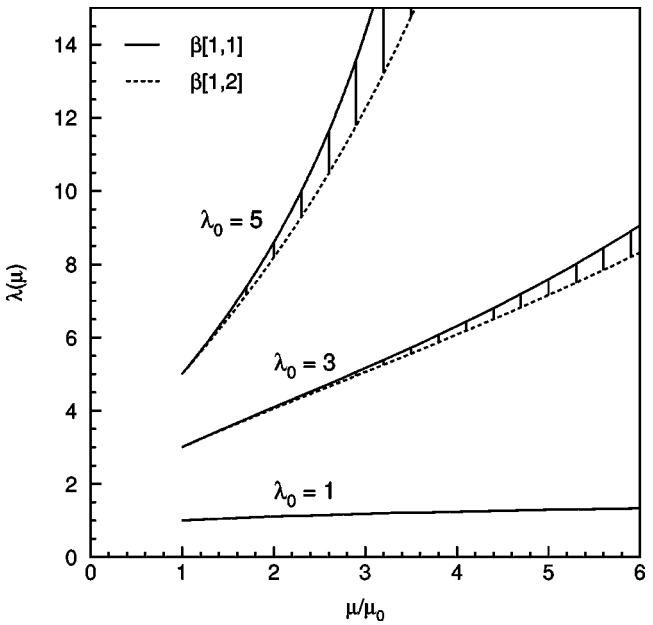

FIG. 4. Plots showing the running of $\lambda(\mu)$ as a function of the ratio of scales μ/μ_0 for different initial choices of λ_0 in the MS renormalization scheme. The differences between the curves obtained using the Padé approximates $\beta[1,1]$ and $\beta[1,2]$ corresponding to three- and four-loop summations of β indicates the range of uncertainty in the result. The curves for the five-loop β function lie near the center of the band of uncertainty.

[2]In the case of MS renormalization, λ is connected only indirectly to the physical pole mass of the Higgs boson, so we cannot state the results in terms of μ and M_H without using a separate calculation of the self-energy function.

The range of uncertainty in $\lambda(\mu)$ at fixed μ/μ_0 is quite small for $\lambda_0=1$, 3 over the entire range shown, $\mu/\mu_0\leqslant 6$. The uncertainty is larger for $\lambda_0=5$, roughly 16%, at $\mu/\mu_0=3$, but even then the boundary curves differ from the curve for $\beta[2,2]$ by $<8\%$.

The rather small effect of uncertainties in β on $\lambda(\mu)$ can be understood rather simply. The renormalization-group equation involves $1/\beta$ rather than β. The prefactor λ^2 in the Padé expression in Eq. (11) leads to a rapid decrease in the integrand, and the value of the integral is determined mainly by the region near λ_0, the lower endpoint of the integration. For λ_0 small, β is well determined in the most important region, and the uncertainty in the integral is small. The uncertainty in the integral, hence the uncertainty in $\lambda(\mu)$, becomes large only for renormalization-group evolution away from a large starting value for λ_0.

III. APPLICATIONS: RANGES OF RELIABILITY OF $\beta(\lambda)$ AND $\lambda(\mu)$ IN OMS RENORMALIZATION

A. Padé approximates for $\beta(\lambda)_{\rm OMS}$

Having tested the use of Padé approximates in the MS scheme, we consider the implications of Padé summation for the OMS scheme. The most significant difference is the limited order, three loops, to which the perturbation series for β is known. We are therefore restricted to two approximates that use the full information available, the diagonal approximate $\beta[1,1]$ and the subdiagonal approximate $\beta[0,2]$. We can also use $\beta[0,1]$ at the two-loop level. Based upon the convergence of the Padé sequence demonstrated for the MS scheme, and the apparent reduction in the size of the coefficients in the OMS scheme,[3] we will assume that these approximates again provide an accurate estimate for the β function, with the diagonal approximate probably the most reliable.

To determine the range of λ for which the β function is reliable, we first considered the differences among the three-loop functions $\beta[1,1]$ and $\beta[0,2]$, and the two-loop function $\beta[0,1]$. These approximates can barely be distinguished over the range of λ shown in Fig. 5 with the scale used there, so only $\beta[1,1]$ is shown. This agreement is the result of the nearly geometric growth of the first coefficients in the perturbation series. The three-loop approximates $\beta[1,1]$ and $\beta[0,2]$ continue to agree well to much larger values of λ. While one is tempted on this basis to conclude that the OMS β function is reliably known for $\lambda\leqslant 10$, the range of current interest, the geometric character of the low-order perturbation series may well be accidental. We have therefore attempted to estimate a wider range of uncertainty in the β function in a different way by supposing, in agreement with the results of the MS analysis, that the coefficient B_3' estimated by expanding $\beta[1,1]$ is too small, and constructing a new ''four-loop'' approximate $\beta[1,2]$ using a greatly increased value of B_3. The result obtained using $B_3=5B_3'$ is shown in Fig. 5. The change in the extrapolation of the perturbation series is quite small, with a difference of less than 2% between $\beta[1,1]$ and $\beta[1,1]$ for $\lambda<10$. We also show the perturbation series for the β function, $\beta[2,0]$, in Fig. 5 for comparison.

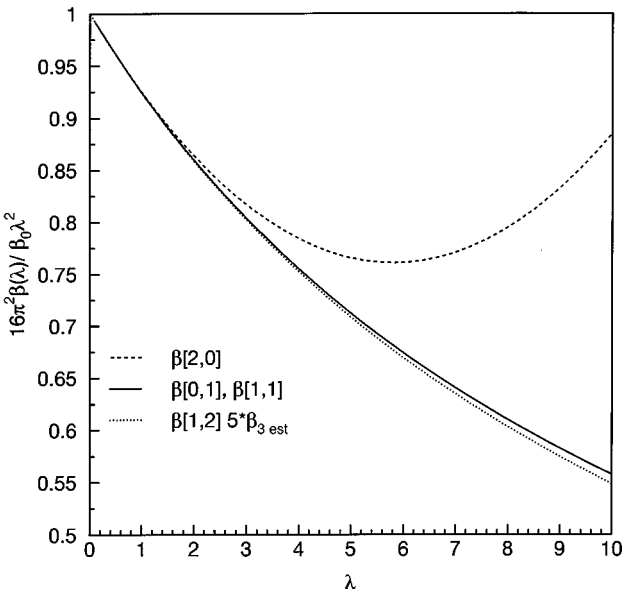

FIG. 5. Plots of the two- and three-loop Padé approximates $\beta[0,1]$ and $\beta[1,1]$ for β in the OMS scheme. The functions are identical on the scale of the figure. The function $\beta[1,2]$ obtained using a coefficient β_3 five times as large as that estimated from $\beta[1,1]$ is shown to indicate a range of uncertainty. The three-loop perturbation series $\beta[2,0]$ is shown for comparison.

B. The running coupling $\lambda(\mu)$

In the OMS renormalization scheme, the parameter λ is defined by the relation $\lambda=G_F M_H{}^2/\sqrt{2}$ to all orders in perturbation theory [9,20]. We will choose the starting value λ_0 of the running coupling $\lambda(\mu)$ to have this value. What remains to be decided is the energy scale μ_0 at which this relation should be taken to hold. The natural energy scale would appear to be $\mu_0=M_H$. However, other choices have been made. Thus, in an early investigation, Sirlin and Zucchini [19] calculated the one-loop corrections to the four-point Higgs-boson scattering amplitude and defined the parameters in the theory so that large electromagnetic effects appear only in such standard relations as that between G_F and the muon decay rate. With this definition, the high mass limit of the four-point function gives [19]

$$h(\mu)=\lambda_0\left[1+\frac{\lambda_0}{16\pi^2}\left(24\ln\frac{\mu}{M_H}+25-3\sqrt{3}\pi\right)\right]. \quad (17)$$

The logarithm in the expression above is just that which appears in the expansion of the one-loop expression for $\lambda(\mu)$,

[3]The known value of β_2 in the OMS scheme is smaller than that in the MS scheme by roughly a factor of 3 [11,18]. Nierste and Riesselmann [11] have found similar reductions in the coefficients in the expansion of physical amplitudes. We assume that the reductions in the size of the coefficients persist at higher orders.

$$\lambda(\mu)=\lambda_0\left(1-\beta_0\frac{\lambda_0}{16\pi^2}\ln\frac{\mu}{\mu_0}\right)^{-1} \tag{18}$$

for $\beta_0=24$ and $\mu_0=M_H$. The ambiguity in the choice of μ_0 is in the treatment of the remaining constants in Eq. (17). These have been incorporated in the running coupling by some authors [19,8,9] by redefining μ_0 as $\mu_0=M_H\exp[(-25+3\sqrt{3}\pi)/24]$. However, the constants do not appear naturally in the expression for the four-point function at two loops [9]. It is probably most reasonable, therefore, to treat them as separate ''radiative corrections'' and write $h(\mu)$ to one loop as $h(\mu)=\lambda(\mu)[1+\delta]$, with $\lambda(\mu)$ the one-loop running coupling defined above, and δ incorporating the remaining scale-independent corrections.

This question has been studied in more detail by Nierste and Riesselmann [11], who showed that the convergence of the perturbation series was improved for several physical amplitudes by adopting the natural scale $\mu_0=M_H$ instead of the choice noted above. They note, furthermore, that in order to cancel large logarithmic terms in the perturbative result when one considers two-scale physical processes such as scattering, the scale μ must be related to the energy scale of the interaction by $\mu=\sqrt{s}$ [11]. We will follow Nierste and Riesselmann and make the definite, physically motivated choices $\mu_0=M_H$ and $\mu=\sqrt{s}$ in the following analysis. This specification amounts as already noted to a definite specification of the ''radiative corrections'' in perturbatively calculated amplitudes once the couplings are expressed in terms of $\lambda(\mu)$.

With λ_0 and μ_0 specified, and the range $\lambda_0\leqslant\lambda\leqslant\lambda_{\max}$ of stability and presumed reliability of the β function established, it is straightforward to integrate the renormalization-group equation and invert the result numerically to obtain $\lambda(\mu)$. However, the results will only be reliable for $\mu_0\leqslant\mu\leqslant\mu_{\max}$, where $\mu_{\max}$ is determined implicitly by the condition $\lambda(\mu_{\max})=\lambda_{\max}$. The behavior of $\lambda(\mu)$ for $\mu\gg\mu_{\max}$ is not determined. The uncertainty in $\lambda(\mu)$ can be specified in terms of that in β. With this procedure, it is not necessary to obtain the solution of the renormalization-group equation as a series in λ_0, or in a different approximate form. We note in this connection that the ''naive'' and ''consistent'' forms for $\lambda(\mu)$ in the nomenclature in [11] correspond, respectively, to the approximates $\beta[N,0]$, the perturbation series for β, and $\beta[0,N]$, the series obtained by expanding β. Neither sequence is expected to converge well with increasing N.

Our results for $\lambda(\mu)$ are shown in Fig. 6 for $M_H=500$ and 800 GeV and $\mu_0\leqslant\mu\leqslant4$ TeV. We find for $M_H=500$ GeV that all Padé approximates, including the perturbation series, agree very well for $\mu<5$ TeV, a region in which $\lambda_0<5$. The residual uncertainty in $\lambda(\mu)$ is small enough not to affect perturbative results for physical processes.

Different Padé approximates also give very similar extrapolations for $\lambda(\mu)$ for $M_H=800$ GeV, even when the predicted value of β_3 is changed by a large factor. The only significant deviation involves the perturbation series $\beta[2,0]$ which we do not believe is reliable on the basis of our earlier investigation. Even if we restrict the range of λ in which we take β as reliable to $\lambda<10$ as in Fig. 5, the result for $\lambda(\mu)$ remains reliable for $\mu<2$ TeV, a value well into the energy region of interest for experiments at the Large Hadron Collider at CERN.

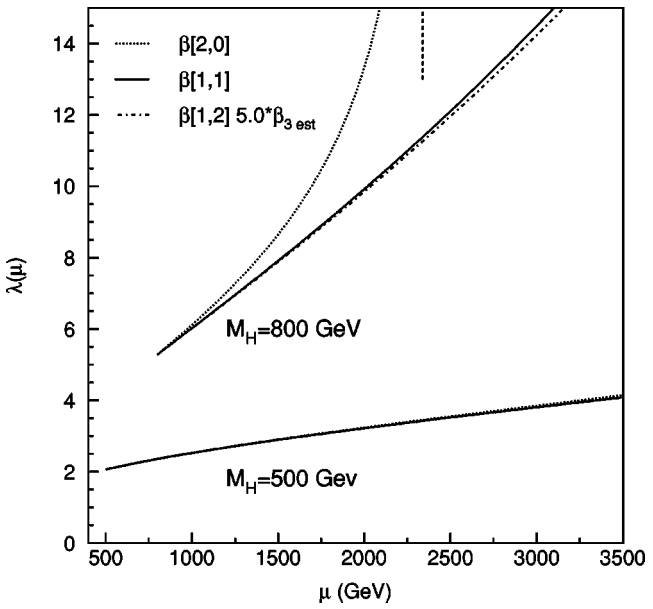

FIG. 6. Plots showing the running of $\lambda(\mu)$ in the OMS renormalization scheme as a function of the mass scale μ for different initial choices of the Higgs-boson mass M_H. The differences between the curves obtained using the three-loop Padé approximate $\beta[1,1]$ and the function $\beta[1,2]$ obtained using a coefficient β_3 five times as large as that estimated from $\beta[1,1]$ is shown to indicate a range of uncertainty. The perturbative result for β given by $\beta[2,0]$ has a Landau pole at $\mu=2.3$ TeV for $M_H=800$ GeV, but is not reliable and is included only to illustrate the effects of a nearby pole.

The rapid growth of $\lambda(\mu)$ for the perturbative approximate $\beta[2,0]$ in Fig. 6 is the result of a Landau pole at $\mu=2339$ GeV. A pole can appear in $\lambda(\mu)$ if the integral of $1/\beta$ converges for $\lambda\to\infty$, with the position of the pole in μ determined by the condition

$$\ln\frac{\mu}{\mu_0}=\lim_{\lambda\to\infty}\int_{\lambda_0}^{\lambda}\frac{d\lambda}{\beta}. \tag{19}$$

No pole can actually appear when the integration is restricted to the finite range of λ in which β is known reliably, but the likely presence of a pole would be indicated by very rapid growth of $\lambda(\mu)$ with increasing μ in that region. In the present case, there is no reason to expect the perturbation series $\beta[2,0]$ to be accurate for λ large. The results in Fig. 5 indicate, in fact, that the perturbative approximation begins to fail badly for $\lambda\approx5$, while the starting point for the evolution of $\lambda(\mu)$ shown in Fig. 6 is at $\lambda=5.3$ for $M_H=800$ GeV. The remaining approximates do not lead to poles in the region shown.

C. Conclusions

We have shown that Padé summation of the β function apparently improves the perturbative stability of β in an in-

terval $\lambda_0 \leqslant \lambda \leqslant \lambda_{\max}$ in which the Padé estimates are stable. The method gives a best estimate for β, and removes much of the uncertainty associated with different determinations of the running quartic Higgs coupling $\lambda(\mu)$ at the three-loop level [11] in a corresponding interval $\mu_0 \leqslant \mu \leqslant \mu_{\max}$, but the behavior of $\lambda(\mu)$ for $\mu \gg \mu_{\max}$ is not determined.

We have tested the Padé method using the β function in the MS renormalization scheme, where β is known to five loops in the perturbation expansion. The test results suggest rapid convergence of the diagonal and subdiagonal Padé sequences. Our applications are to the more physical OMS renormalization scheme, where the first scheme-dependent coefficient in the OMS expansion is significantly smaller than in the MS expansion. This more rapid apparent convergence is reflected in the excellent agreement among the leading Padé approximates for $\beta_{\rm OMS}$ in the diagonal-subdiagonal sequence even for rather large values of the first unknown coefficient, β_3. Calculation of β_3 in the OMS scheme would give an important test of whether this method captures the behavior of β_3 at large λ as well as it appears to do.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-95ER40896. L.D. would like to thank the Aspen Center for Physics for its hospitality while parts of this work were done.

APPENDIX A: PADÉ APPROXIMATES

The coefficients of the Padé forms used in our analysis are given below. We will state the results in terms of the coefficients $B_n = (\beta_n/\beta_0)/(16\pi^2)^n$.

At two loops, $N+M=1$ and we have only the truncated perturbation series $\beta[1,0]$ and the approximate $\beta[0,1]$ with

$$\beta[0,1]: \quad b_1 = -B_1. \tag{A1}$$

At three loops, $N+M=2$ and we have the new approximants $\beta[1,1]$ and $\beta[0,2]$. The coefficients are given by

$$\beta[1,1]: \quad a_1 = (B_1^2 - B_2)/B_1, \tag{A2}$$

$$b_1 = -B_2/B_1,$$

$$\beta[0,2]: \quad b_1 = -B_1, \tag{A3}$$

$$b_2 = B_1^2 - B_2.$$

At five loops, $N+M=4$ and we will consider the new approximants $\beta[3,1]$, $\beta[2,2]$, $\beta[1,3]$, and $\beta[0,4]$. The coefficients are given by

$$\beta[3,1]: \quad a_1 = (B_1B_3 - B_4)/B_3, \tag{A4}$$

$$a_2 = (B_2B_3 - B_1B_4)/B_3,$$

$$a_3 = (B_3^2 - B_2B_4)/B_3,$$

$$b_1 = -B_4/B_3,$$

$$\beta[2,2]: \quad a_1 = (B_1B_2^2 - B_1^2B_3 + B_1B_4 - B_2B_3)/A_{22}, \tag{A5}$$

$$a_2 = (B_2^3 - 2B_1B_2B_3 + B_1^2B_4 + B_3^2 - B_2B_4)/A_{22},$$

$$b_1 = (B_1B_4 - B_2B_3)/A_{22},$$

$$b_2 = (B_3^2 - B_2B_4)/A_{22},$$

$$A_{22} = B_2^2 - B_1B_3,$$

$$\beta[1,3]: \quad a_1 = (B_1^4 - 3B_1^2B_2 + 2B_1B_3 + B_2^2 - B_4)/A_{13}, \tag{A6}$$

$$b_1 = (-B_1^2B_2 + B_2^2 + B_1B_3 - B_4)/A_{13},$$

$$b_2 = (B_1B_2^2 - B_2B_3 - B_1^2B_3 + B_1B_4)/A_{13},$$

$$b_3 = (2B_1B_2B_3 - B_2^3 - B_3^2 + B_2B_4 - B_1^2B_4)/A_{13},$$

$$A_{13} = B_1^3 - 2B_1B_2 + B_3.$$

We will also consider the approximates $\beta[1,2]$, the subdiagonal approximate for the four-loop expansion. The coefficients in this case are

$$\beta[1,2]: \quad a_1 = (B_1^3 - 2B_1B_2 + B_3)/(B_1^2 - B_2), \tag{A7}$$

$$b_1 = (B_3 - B_1B_2)/(B_1^2 - B_2),$$

$$b_2 = (B_2^2 - B_1B_3)/(B_1^2 - B_2).$$

APPENDIX B: ANALYTIC RESULTS

The Padé approximates we have used are all integrable analytically. We will give only the results needed in our investigation of the OMS renormalization scheme:

$$\beta[0,1]: \quad \frac{\beta_0}{16\pi^2}\int^{\lambda}\frac{d\lambda}{\beta[0,1]} = -\frac{1}{\lambda} - B_1 \ln\lambda, \tag{B1}$$

$$\beta[2,0]: \quad \frac{\beta_0}{16\pi^2}\int^{\lambda}\frac{d\lambda}{\beta[2,0]} = -\frac{1}{\lambda} - B_1\ln\lambda + \frac{1}{2}B_1\ln(1+B_1\lambda+B_2\lambda^2) + \frac{B_1^2-2B_2}{\sqrt{4B_2-B_1^2}}\arctan\frac{B_1+2B_2\lambda}{\sqrt{4B_2-B_1^2}}, \tag{B2}$$

$$\beta[1,1]: \quad \frac{\beta_0}{16\pi^2}\int^{\lambda}\frac{d\lambda}{\beta[1,1]} = -\frac{1}{\lambda} - B_1\ln\lambda + B_1\ln\left(1+\frac{B_1^2-B_2}{B_1}\lambda\right), \tag{B3}$$

$$\beta[0,2]: \quad \frac{\beta_0}{16\pi^2}\int^{\lambda}\frac{d\lambda}{\beta[0,2]} = -\frac{1}{\lambda} - B_1\ln\lambda + (B_1^2-B_2)\lambda. \tag{B4}$$

These expressions are to be equated to $(\beta_0/16\pi^2)\ln(\mu/\mu_0)$.

[1] D.A. Dicus and V.S. Mathur, Phys. Rev. D **7**, 3111 (1973).
[2] B.W. Lee, C. Quigg, and H.B. Thacker, Phys. Rev. Lett. **38**, 883 (1977); Phys. Rev. D **16**, 1519 (1977).
[3] M.S. Chanowitz and M.K. Gaillard, Nucl. Phys. **B261**, 379 (1985).
[4] W. Marciano, G. Valencia, and S. Willenbrock, Phys. Rev. D **40**, 1725 (1989).
[5] S. Dawson and S. Willenbrock, Phys. Rev. Lett. **62**, 1232 (1989); Phys. Rev. D **40**, 2880 (1989).
[6] G. Passarino, Phys. Lett. **156B**, 231 (1985); Nucl. Phys. **B343**, 31 (1990).
[7] C.E. Vayonakis, Europhys. Lett. **12**, 29 (1990).
[8] L. Durand, J.M. Johnson, and J.L. Lopez, Phys. Rev. Lett. **64**, 1215 (1990); Phys. Rev. D **45**, 3112 (1992).
[9] L. Durand, P. Maher, and K. Riesselmann, Phys. Rev. D **48**, 1084 (1993).
[10] L. Durand, B.A. Kniehl, and K. Riesselmann, Phys. Rev. Lett. **72**, 2534 (1994); **74**, 1699(E) (1995); Phys. Rev. D **51**, 5007 (1995).
[11] U. Nierste and K. Riesselmann, Phys. Rev. D **53**, 6638 (1996).
[12] G. Baker, Jr., *Essentials of Padé Approximants* (Academic, New York, 1975).
[13] C. Bender and S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw Hill, New York, 1978).
[14] J.-F. Yang and G.-J. Ni, Commun. Math. Phys. **22**, 207 (1994).
[15] A. Ghinculov, Phys. Lett. B **337**, 137 (1994); **346**, 426(E) (1994).
[16] A.A. Vladimirov, D.I. Kazakov, and O.V. Tarasov, Sov. Phys. JETP **50**, 521 (1979).
[17] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin, and S.A. Larin, Phys. Lett. B **272**, 39 (1991); **319**, 545(E) (1993).
[18] M. Lüscher and P. Weisz, Phys. Lett. B **212**, 472 (1988).
[19] A. Sirlin and R. Zucchini, Nucl. Phys. **B266**, 389 (1986).
[20] L. Durand and K. Riesselmann, Phys. Rev. D **55**, 1533 (1997).