Chiral solitons in a current coupled Schrödinger equation with self-interaction

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Recently nontopological chiral soliton solutions were obtained in a derivatively coupled nonlinear Schrödinger model in 1+1 dimensions. We extend the analysis to include a more general self-coupling potential (which includes the previous cases) and find chiral soliton solutions. Interestingly, even the magnitude of the velocity is found to be fixed. The energy and $U(1)$ charge associated with this nontopological chiral soliton are also obtained. [S0556-2821(98)00322-1]

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The study of solitons, localized travelling solutions with a finite energy density in nonlinear scalar field theories, has a long history [1]. In recent times (2+1) dimensional theory with matter fields coupled to gauge fields governed by Chern-Simons action has also been found to have soliton solutions, which is of relevance to the quantum Hall effect [2].

In a recent work, a derivatively coupled scale invariant nonlinear Schrödinger equation in 1+1 dimensions, obtained partially by the dimensional reduction of a (2+1) dimensional theory, was shown to have a novel, soliton solution: the soliton exists only for a fixed sign of velocity, but for a range of magnitude [3]. Such solitons are likely to have applications in one-dimensional systems such as quantum wires and in the description of chiral waves which are travelling edge states, in the quantum Hall effect [4]. Apart from its potential application, it is of intrinsic interest to see if the known nonlinear equation can be modified to admit solitons, which travel only unidirectionally. Such studies have been carried out recently for generalized Korteweg–de Vries (KdV) and other nonlinear equations [5] and also for the multicomponent nonlinear Schrödinger equation [6]. In this letter, we extend the analysis of Ref. [3], which introduced the current coupled nonlinear Schrödinger equation, to include a more general self-coupling and study the soliton solution of the model.

The nonlinear Schrödinger equation in 1+1 dimension, with cubic nonlinearity, given by

$$i\hbar\partial_t\psi=-\frac{\hbar^2}{2m}\partial_x^2\psi-g(\psi^*\psi)\psi \tag{1}$$

is well studied and soliton solutions are constructed. This model is found to be completely integrable. Recently a new nonlinear Schrödinger equation was constructed, which has nonlinearity due to current coupling, rather than charge density coupling in the equation:

$$i\hbar\partial_t\psi=-\frac{\hbar^2}{2m}\partial_x^2\psi-\lambda j(x,t)\psi(x,t), \tag{2}$$

where $j=\hbar/m\,\mathrm{Im}(\psi^*\partial_x\psi)$is the current density. This derivative nonlinear Schrödinger (DNLS) equation was found to have soliton solutions which (i) are chiral, i.e., having only a fixed sign of velocity and (ii) have a velocity which cannot be arbitrarily reduced, i.e., the absence of Galilean invariance. Note that the Eq. (2) belongs to the same class of the Derivative Nonlinear Schrödinger equation of Kaup and Newell [7] which reads

$$i\partial_t\psi=-\partial_x^2\psi\pm i\partial_x(\psi^*\psi^2). \tag{3}$$

Equation (2) cannot be directly obtained from a local Lagrangian. Instead, as shown in Ref. [3], the following Lagrangian density provides an equation of motion equivalent to that of Eq. (2), by a suitable redefinition of the field, as indicated below:

$$\mathcal{L}=i\hbar\Phi^*\partial_t\Phi-\frac{\hbar^2}{2m}\left|\left(\partial_x+i\frac{\lambda}{2}\rho\right)\Phi\right|^2 \tag{4}$$

where $\rho=\Phi^*\Phi$.

The equation of motion following from the above Lagrangian is

$$i\hbar\partial_t\Phi=-\frac{\hbar^2}{2m}\left(\partial_x+i\frac{\lambda}{2}\rho\right)^2\Phi+\frac{\hbar}{2}\lambda J\Phi, \tag{5}$$

where

$$J=\frac{\hbar}{m}\mathrm{Im}\left(\Phi^*\left(\partial_x+i\frac{\lambda}{2}\rho\right)\Phi\right), \tag{6}$$

obeys the continuity equation

$$\partial_t(\phi^*\phi)+\partial_xJ(x)=0. \tag{7}$$

By redefining the field Φ by

$$\Phi=\exp\left(i\frac{\lambda}{2}\int dy\,\rho(y,t)\right)\psi, \tag{8}$$

Eq. (2) is obtained after using continuity Eq. (5) for J.

Incidentally, the Lagrangian density (4) is related to (2+1) dimensional nonrelativistic field coupled to an $U(1)$ gauge field, whose kinetic term is the Chern-Simon term, by dimensional reduction. In the ensuing (1+1) dimensional theories, for $A_3=B(x,t)$ field, which is nonpropagating, a kinetic term of the form $(dB/dt)(dB/dx)$ is added by hand.

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By using Hamiltonian reduction (and by a phase redefinition of ψ) $A_\mu(x,t)$ and $B(x,t)$ can be eliminated resulting in Eq. (4).

In Ref. [3], an interesting nontopological soliton solution was discovered for this novel, derivative coupled nonlinear Schrödinger equation, having chiral motion in a global rest frame, when $V(\phi^*\phi)$ is absent and also when $V(\phi^*\phi)$ is repulsive cubic. It should be of interest to examine whether these type of solutions exist for other interactions. In this note, we extend the analysis of Ref. [3] to include a more general nature of anharmonicities (potential) and discuss the nature of the soliton solutions in this model. The potential we add to the Lagrangian is of the form

$$V(\rho^2)=a\rho^{2n+2}+b\rho^{n+2}+c\rho^4, \tag{9}$$

where $\rho^2=(\phi^*\phi)$ and a, b, and c are dimensionful coupling constants. The $n=1$ and $n=2$ cases are included in Ref. [3].

Such anharmonic potential terms has been considered earlier in the context of models describing both first-and second-order transitions, and topological, nontopological and periodic solutions were obtained [8].

Interestingly we find, as shown below, except for $n=1$ and 2, soliton solutions are of a fixed velocity (with fixed magnitude and sign). This has to be contrasted with the result of Ref. [3], where only the sign of the velocity has fixed. Such velocity selection is also present in a modified sine-Gordon theory [9]. We also calculate the $U(1)$ charge and energy of these soliton solutions.

The equations of motion following from the Lagrangian (4), with the potential (9) added is,

$$i\hbar\partial_t\phi=-\frac{\hbar^2}{2m}\left(\partial_x+\frac{i\lambda}{2}\rho^2\right)^2\phi+\frac{\lambda\hbar}{2}J\phi+V'\phi, \tag{10}$$

where $V'=dV/d\rho^2$. In order to construct the soliton solution make the ansatz,

$$\phi=\rho(x-vt)e^{i\theta(x,t)} \tag{11}$$

where v is the velocity. Using Eq. (11) in (7), one gets

$$\theta'=\frac{mv}{\hbar}-\frac{\lambda\rho^2}{2}. \tag{12}$$

Using Eq. (12) and Eq. (11), the equation of motion (10) becomes

$$\rho''=\left[\left(\frac{mv}{\hbar}\right)^2+\frac{2m\omega_0}{\hbar}\right]\rho+\frac{mv\lambda}{\hbar}\rho^3+\frac{2m}{\hbar}\frac{dV}{d\rho}, \tag{13}$$

where $\omega_0=\partial_t\theta$ and

$$\frac{dV}{d\rho}=(a(2n+2)\rho^{2n+1}+b(n+2)\rho^{n+1}+4c\rho^3).$$

Note for $n=1$ and $n=2$, Eq. (13) reduces to that considered in Ref. [3]. This nonlinear equation admits a localized solution for $n>2$,

$$\rho(x-vt)=\left(\frac{2mb}{\hbar}M^{-2}\sqrt{1-\frac{a}{mb^2}M^2}\right)^{-1/n}\times\left(\frac{1}{\sqrt{1-(a/mb^2)M^2}}+\cosh(Mnx)\right)^{-1/n} \tag{14}$$

only when the velocity

$$v=-\left(\frac{8c}{\lambda}\right), \tag{15}$$

where

$$M=\left[\left(\frac{mv}{\hbar}\right)^2+\frac{2m\omega_0}{\hbar}\right]^{1/2}.$$

Note that sign of λ determines the direction of velocity, implying the chiral nature of the solitons. Chirality is due to the constraint on v rather than on v^2. Note that the magnitude of the velocity is also fixed.

In the case, when $n=1$ and 2, the solution $\rho(x,t)$ is valid without any restriction on the magnitude of the velocity. Chirality nature of the solution still exists for these cases except for $n=2$ and $a<0$ as shown in Ref. [3].

The conserved $U(1)$ charge associated with the soliton solution is

$$\begin{aligned}N&=\int dx(\phi^*\phi)\\&=\frac{1}{Mn(\alpha\beta)^{2/n}}\int\frac{dy}{\left(1/\beta+\cosh y\right)^{2/n}}\\&=\frac{1}{Mn(\alpha\beta)^{2/n}}\frac{\beta}{\sqrt{1-\beta^2}}Q_{(2-n)/n}\left(\frac{1}{\sqrt{1-\beta^2}}\right),\end{aligned} \tag{16}$$

where

$$\alpha=\frac{2mb}{\hbar}M^{-2},$$

$$\beta=\sqrt{1-\frac{aM^2}{mb^2}}$$

and $Q_\nu(k)$ is Legendre function of second kind [10].

Energy associated with this soliton configuration is

$$\mathcal{E}=\int dx\mathcal{H}, \tag{17}$$

where

$$\mathcal{H}=\frac{\hbar^2}{2m}|D_x\phi|^2+V(\rho^2). \tag{18}$$

is the Hamiltonian density. Substituting the ansatz (12), and using Eq. (13) in Eq. (11), we get

$$\mathcal{E}=\int dx\,\frac{\hbar^2}{2m}\left[(\rho')^2+\left(\frac{mv}{\hbar}\right)^2\rho^2+\frac{2m}{\hbar^2}V(\rho^2)\right]. \tag{19}$$

Now substituting the solution (13),

$$\mathcal{E}=\left[M^2+\left(\frac{mv}{\hbar}\right)^2\right]N-\frac{1}{n(\alpha\beta)^{(n+2)/n}}\left(2M\alpha-\frac{2mb}{\hbar^2 M}\right)\times\left(\frac{\beta^2}{1+\beta^2}\right)^{-(n+2)/2n}Q_{(2/n)}\left(\frac{1}{\sqrt{1+\beta^2}}\right)+\frac{1}{n(\alpha\beta)^{(2n+2)/n}}\left[M\alpha^2(1-\beta^2)+\frac{2ma}{\hbar^2 M}\right]\times\left(\frac{\beta^2}{1+\beta^2}\right)^{-(2n+2)/2n}Q_{(n+2)/n}\left(\frac{1}{\sqrt{1+\beta^2}}\right)\times\frac{2mc}{\hbar^2 Mn(\alpha\beta)^{4/n}}\left[\frac{\beta^2}{1+\beta^2}\right]^{-2/n}Q_{(4-n)/n}\left(\frac{1}{\sqrt{1+\beta^2}}\right). \tag{20}$$

Here Q_ν^μ are the associated Legendre functions of second kind and $Q_\nu^0=Q_\nu$. Interestingly, for both attractive and repulsive cases of highest anharmonicity ($a>0$ and $a<0$), finite energy chiral soliton solution exists. The study of interesting cases of topological soliton and soliton with nontrivial boundary conditions (i.e., with nonzero density at spatial infinity) is in progress.

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[1] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[2] R. Jackiw and S. Y. Pi, Phys. Rev. D **42**, 3500 (1990); G. Dunne, *Self-Dual Chern-Simon Theories* (Springer, Verlag, Berlin, 1995), Vol. 36.
[3] U. Aglietti, L. Griguolo, R. Jackiw, S.-Y. Pi, and D. Seminara, Phys. Rev. Lett. **77**, 4406 (1996); L. Griguolo and D. Seminara, Nucl. Phys. **B516**, 467 (1998).
[4] X.-G. Wen, Phys. Rev. B **41**, 12 838 (1990); X.-G. Wen, Int. J. Mod. Phys. B **6**, 1711 (1992).
[5] D. Bazeia and F. Moraes, solv-int/9802002; D. Bazeia, solv-int/9802007.
[6] M. Hisakado, ''Chiral solitons from dimensional reduction of Chern-Simons gauged coupled nonlinear Schrödinger model,'' hep-th/9712255.
[7] D. J. Kaup and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[8] S. N. Behera and A. Khare, Pramana **14**, 327 (1980); E. Magyari, Z. Phys. B **43**, 345 (1981).
[9] M. Peyrard, B. Piette, and W. J. Zakrzewski, Physica D **64**, 355 (1993).
[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).