No-boundary wave function of the anti–de Sitter space-time and the quantization of Λ

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The application of the ''no-boundary'' condition to space-times with initially open spatial sections may lead to extra conditions on physical quantities of these space-times. In the present Brief Report, we verify the above statement in a model where we compute the semi-classical approximation to the ''no-boundary'' wave function of the anti–de Sitter space-time. For this model the ''no-boundary'' conditions impose a well defined, discrete spectrum for the cosmological constant. As a by-product of our investigations we also find that among the space-times contributing to the above wave function there are two complex conjugate ones that show a new type of signature change. [S0556-2821(98)03820-X]

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The ''no-boundary'' condition is a very successful set of boundary conditions [1]. Since its introduction it has been shown to fix uniquely the wave function of several space-times of cosmological as well as astrophysical interest. In addition to that, it also produced some important predictions on the physical behavior of these space-times.

For a given model, the ''no-boundary'' conditions specify the set of four-geometries which must be summed over, so that a wave function may be derived through the path integral formalism. These four-geometries should be compact, regular, with a single three-dimensional, spacelike boundary hypersurface. Therefore, if one foliates these four-geometries in three-dimensional, spacelike hypersurfaces plus a timelike line, the hypersurfaces must necessarily be closed.

Most of the cases studied so far dealt with space-times whose spacelike hypersurfaces are naturally closed [1–3], and very few considered the possibility of initially open, spacelike hypersurfaces [4]. Of course, in the last case a mathematical technique of compactification had to be used [5].

In Ref. [4], the space-times contributing, in the semiclassical approximation, to a specific ''no-boundary'' wave function, were foliated by flat, compactified, spacelike hypersurfaces. There, it was shown that the requirement that these space-times were regular resulted in extra conditions upon them. In particular, their actions were labeled by ordered pairs of coprime integers.

In this Brief Report we would like to show another example of the application of the regularity and compactness conditions from the ''no-boundary'' proposal, upon space-times whose spacelike hypersurfaces are initially open. We shall see that it will lead to extra conditions upon physical quantities.

We shall compute the semiclassical approximation to the ''no-boundary'' wave function (Ψ_{nb}), Ref. [2], of the anti–de Sitter space-time. As a matter of simplicity we shall restrict our attention to $2+1$ dimensions. As a by-product, we shall see that among the space-times contributing to Ψ_{nb}, there are two complex conjugate ones that show a new type of signature change [6,7].

The anti–de Sitter space-time represents an homogeneous, isotropic, constant negatively curved, universe, which only source of stress energy is a negative cosmological constant (Λ). In addition to this, it is also foliated by open, constant negatively curved, spacelike hypersurfaces.

We start by writing down the appropriate Euclidean metric ansatz in its Arnowitt-Deser-Misner (ADM) form [8]

$$ds^2 = +N^2(t)dt^2 + a^2(t)[d\chi^2 + \sinh^2\chi d\theta^2], \qquad (1)$$

where $N(t)$ is the lapse function, $a(t)$ is the scale factor, and the coordinates vary over their usual domain [8].

From the line element Eq. (1), we see that the spatial sections are pseudo-spheres which are open. As we have mentioned above, the ''no-boundary'' conditions restrict these spatial sections to be closed therefore we must compactify the pseudospheres.

There are several works in the literature where the explicit process of compactification of the pseudosphere or H^2, is performed. Here, we shall follow Ref. [9], where the H^2 is transformed into a double torus. Then, the spatial sections of the line element Eq. (1), will be double tori. Because of the compactification process the coordinate χ will vary, now, over a finite domain.

In order to obtain Ψ_{nb}, we must solve the Einstein's equations [8], for Eq. (1), subjected to the ''no-boundary'' conditions. For our case these conditions are given in Ref. [10], and result in the following choices.

We shall choose the point where the scale factor vanishes to be the zero value of the time scale. With this choice we shall be able to picture our universe starting at $t=0$, from this surface of zero volume and evolving until $t=t_1$, where we shall furnish the other required value of the scale factor, say, $a(t_1)=a_1$. Introducing those quantities in the Einstein equations, and rescaling the time in order that $t_1=1$, we get the four solutions given below.

The Lorentzian solutions

$$N = iN_I, \qquad (2)$$

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where

$$N_I=\alpha\left[\frac{\pi}{2}\pm\arccos(a_1/\alpha)\right]$$

and

$$a(t)=\alpha\sin(N_I t/\alpha),\qquad \alpha^2=\frac{1}{|\Lambda|} \tag{3}$$

valid for $a_1<\alpha$. These space-times, Eqs. (2) and (3), are known in the literature as anti–de Sitter space-times [11].

The complex solutions

$$N=N_R+iN_I, \tag{4}$$

where

$$N_I=\alpha\frac{\pi}{2},\quad N_R=\pm\alpha\,\mathrm{arcsinh}\,\beta,\quad \beta=\sqrt{\left(\frac{a_1}{\alpha}\right)^2-1} \tag{5}$$

and

$$a(t)=a_R(t)+ia_I(t), \tag{6}$$

where

$$a_R(t)=\alpha\sin(N_I t/\alpha)\cosh(N_R t/\alpha), \tag{7}$$

$$a_I(t)=-\alpha\cos(N_I t/\alpha)\sinh(N_R t/\alpha), \tag{8}$$

and this solution holds for $a_1>\alpha$.

In order to identify the signature change nature of these complex solutions Eqs. (4) and (6), we follow closely Ref. [6].

First, we rewrite the general expression of the Euclidean action [12] for the metric ansatz Eq. (1):

$$I[N,a_1]=-\mathcal{A}\int[\dot{a}^2(t)-N^2+a^2(t)N^2|\Lambda|]\frac{1}{N}dt, \tag{9}$$

where $\mathcal{A}$ is a finite, defined number proportional to the volume of the compact, spacelike hypersurfaces.

Then, we introduce a complex variable $\overline{T}$ defined as the product

$$\overline{T}=Nt, \tag{10}$$

and rewrite the action Eq. (9) in terms of $\overline{T}$.

The contour C along which we must evaluate the action in the complex $\overline{T}$ plane is a straight line. Its initial and final points are given with the aid of Eqs. (4), (5), and (10), respectively, by

$$\overline{T}_i=0\quad\text{and}\quad \overline{T}_f=N_R+iN_I. \tag{11}$$

Next, for the present case of a negative Λ, we deform the above contour C producing a new contour C'. C' starts running along the imaginary $\overline{T}$ axis from 0 to $\overline{T}=iN_I$ where it turns abruptly to run parallel to the real $\overline{T}$ axis up to $\overline{T}_f$ Eq. (11).

If we rewrite the complex scale factor Eqs. (6)–(8) in terms of $\overline{T}$ Eq. (10), we obtain the following expressions for $a_R(\overline{T})$ and $a_I(\overline{T})$:

$$a_R(\overline{T})=\alpha\sin(\overline{T}_I/\alpha)\cosh(\overline{T}_R/\alpha), \tag{12}$$

$$a_I(\overline{T})=-\alpha\cos(\overline{T}_I/\alpha)\sinh(\overline{T}_R/\alpha), \tag{13}$$

where $\overline{T}_R$ and $\overline{T}_I$ are, respectively, the real and imaginary parts of $\overline{T}$. In terms of $a_R(\overline{T})$, $a_I(\overline{T})$, and the contour C', it is easy to identify the signature change nature of the complex solution.

Along C', $a_I(\overline{T})$ Eq. (13) is nil. This happens because on the first part of the circuit we have $\overline{T}_R=0$, and on the second part $\overline{T}_I=\alpha\pi/2$. $a_R(\overline{T})$ is initially nil, as demanded by the "no-boundary" proposal for minisuperspace models [10], for $\overline{T}_I=0$. Then, it behaves as

$$a_R(\overline{T}_I)=\alpha\sin(\overline{T}_I/\alpha), \tag{14}$$

until reaching the value α, for $\overline{T}_I=\alpha\pi/2$. From there onward it goes as

$$a_R(\overline{T}_R)=\alpha\cosh(\overline{T}_R/\alpha), \tag{15}$$

up to the final surface where it is a_1.

Observing the transformation Eq. (10), one can see that on the first part of C', N is imaginary. Therefore, the metric Eq. (1) has a Lorentzian signature there. On the other hand, on the second part of C', it has an Euclidean signature ($L\to E$).

One may verify the correctness of the above contour choice by proving that the extrinsic curvature at the boundary surface where $\overline{T}=i\alpha\pi/2\,(\Sigma)$ between the Lorentzian and Euclidean sectors is nil [6]. For our model, this junction condition translates to the condition $da(\overline{T})/dT=0$ at Σ.

Along the first part of C', $a(\overline{T})$ is given by Eq. (14), therefore,

$$\frac{da(\overline{T})}{d\overline{T}}=\cos(\overline{T}_I/\alpha). \tag{16}$$

The junction condition is then easily seen to be satisfied at Σ. As a by-product of the above analysis we obtain that next to $t=0$, the complex solutions are anti–de Sitter space-times.

The regularity of the Lorentzian and complex solutions, the final requirement of the "no-boundary" conditions, may be studied together because from Eqs. (1), (3), (6)–(8), we notice that all of them have singularities only at $t=0$. As a matter of fact this regularity investigation has already been done in Ref. [9], where a systematic procedure to identify the presence of conical singularities, called the holonomy method [13], was used in the above space-times.

The solutions of the following equation, derived in Ref. [9], guarantee the absence of conical singularities in the event $t=0$ of the space-times above with compactified, spacelike hypersurfaces:

$$Np=2\pi n, \tag{17}$$

where p is the period of a compactified direction and n is a nonzero, positive integer.

For the complex solutions we learned that near $t=0$, N is given by N_I, Eq. (5), and for the Lorentzian solution by Eq. (2). Therefore, if we demand that these space-times be regular by solving Eq. (17) for the appropriate N and impose by simplicity that the resulting spectra of $|\Lambda|$ be identical, we find

$$|\Lambda|_{n,m}=[n+(-1)^m a_1]^2\frac{m^2\pi^2}{n^2}. \tag{18}$$

This equation, gives $|\Lambda|$ as a function of a_1, n, and m, which is a nonzero, positive integer.

We are now in position to write down the Ψ_{nb} Ref. [2], for the present model. As a matter of simplicity we shall not introduce any extra condition but convergence in order to fix the integration contour for Ψ_{nb} [6].

For the case where $a_1\sqrt{|\Lambda|_{n,m}}<1$, the Lorentzian solutions Eqs. (2) and (3), will give the following Ψ_{nb} (up to renormalization), with the aid of the expression for I Eq. (9), for a fixed pair (n,m):

$$\Psi^L_{nb(n,m)}=2\exp\left(\frac{-\imath\mathcal{A}}{\sqrt{|\Lambda|_{n,m}}}\frac{\pi}{2}\right)\times\cos\left\{\frac{\mathcal{A}}{\sqrt{|\Lambda|_{n,m}}}[x\sqrt{1-x^2}-\arccos(x)]\right\}, \tag{19}$$

where we introduced the new variable $x\equiv a_1\sqrt{|\Lambda|_{n,m}}$. For the case where $a_1\sqrt{|\Lambda|_{n,m}}>1$, the complex solutions Eqs. (4) and (6), give for a fixed pair (n,m) (up to renormalization),

$$\Psi^C_{nb(n,m)}=\exp\left(\frac{-\imath\mathcal{A}}{\sqrt{|\Lambda|_{n,m}}}\frac{\pi}{2}\right)\exp\left\{\frac{\mathcal{A}}{\sqrt{|\Lambda|_{n,m}}}[-x\sqrt{x^2-1}+\operatorname{arcsinh}\sqrt{x^2-1}]\right\}. \tag{20}$$

It is clear then, that for a given value of a_1 greater (smaller) than $1/\sqrt{|\Lambda|_{n,m}}$, we shall have an infinite number of $|\Lambda|_{n,m}$, Eq. (18), and of associated $\Psi^C_{nb(n,m)}$, Eq. (20) [$\Psi^L_{nb(n,m)}$ Eq. (19)], one for each pair (n,m). So, for a given a_1 the state of the universe in the present model is specified by the pair (n,m).

We may try to obtain some information about the model universe by studying the wave-functions, Eqs. (19) and (20) and their suitability to describe the classical anti–de Sitter space-time at the semi-classical level. Let us describe the behavior of the two wave functions above in terms of the variable x. Note that for fixed values of the cosmological constant this variable gives a direct measure of the scale factor for final hypersurfaces.

We notice that each of the wave functions above correspond to distinct regions, whether x is smaller or greater than 1. $\Psi^L_{nb(n,m)}$, is an oscillatory function of x, and is defined in the region where $x<1$. So, this is the classically allowed region. $\Psi^C_{nb(n,m)}$, is a decreasing exponential function of x, and is defined in the region where $x>1$. So, this is the classically forbidden region.

The universe has a probability proportional to $\Psi^{L*}_{nb(n,m)}\Psi^L_{nb(n,m)}$ to be in the classically allowed region. This probability varies with $\cos^2$, from Eq. (19). On the other hand, the universe has an exponentially decreasing probability, proportional to $\Psi^{C*}_{nb(n,m)}\Psi^C_{nb(n,m)}$, Eq. (20), to be found in the classically forbidden region.

The properties of the quantum density probabilities mentioned above derived from the wave functions Eqs. (19) and (20), are those one would expect to be semiclassically associated with the anti–de Sitter space-time. From Refs. [8] and [11], we know that the scale factor $a(t)$ for the anti–de Sitter space-time is a sine or cosine function of the time coordinate t. It varies from zero up to a maximum value $1/\sqrt{|\Lambda|}$, and then decreases back to zero. Therefore, classically, $a(t)$ cannot be greater than $1/\sqrt{|\Lambda|}$.

Finally, it is important to mention that the papers in Ref. [14] were the first ones to compute the ‘‘no-boundary’’ wave-function for a negatively curved space-time. One may not compare the results derived here with the ones derived there because first, and most importantly, they have not considered contributions to the wave function from space-times foliated by spacelike hypersurfaces. Secondly, they have not restricted their attention to the anti–de Sitter space-time.

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