Conformal field theory correlators from classical field theory on anti–de Sitter space: Vector and spinor fields

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(Received 3 June 1998; published 8 October 1998)

We use the AdS-CFT correspondence to calculate CFT correlation functions of vector and spinor fields. The connection between the AdS and boundary fields is properly treated via a Dirichlet boundary value problem. [S0556-2821(98)08620-2]

PACS number(s): 11.25.Hf, 11.10.Kk

I. INTRODUCTION

The study of conformal field theories (CFT's) in dimensions larger than 2 [1,2] has recently been boosted by Maldacena's conjecture that the large N limit of certain conformal field theories in d dimensions can be described by supergravity and string theory on $(d+1)$-dimensional anti–de Sitter (AdS) space [3]. Subsequently, this conjecture has been given a more precise formulation [4,5] and it has been shown that, in fact, any field theory on AdS_{d+1} is linked to a conformal field theory on the AdS boundary [5]. This observation is entirely due to the fact that one obtains a metric on the AdS boundary by multiplying the AdS metric with a function, which has a single zero on the boundary in order to counteract the divergence of the AdS metric. However, this function is otherwise arbitrary, which imposes the symmetries of the conformal group on the boundary metric. All one needs then is a suitable connection between the fields on AdS_{d+1} and its boundary. Schematically, this connection is given by

$$Z_{AdS}[\phi_0]=\int_{\phi_0}\mathcal{D}\phi\ \exp(-I[\phi])$$
$$\equiv Z_{CFT}[\phi_0]=\left\langle \exp\left(\int d^dx\mathcal{O}\phi_0\right)\right\rangle, \qquad (1)$$

where ϕ_0 is a suitably defined boundary value of the AdS field ϕ and couples as a current to the boundary conformal field theory operator $\mathcal{O}$. In the classical approximation the path integral on the left-hand side (lhs) is, of course, redundant.

Field theories on AdS spaces have been the subject of research in the past [6–14]. More recently, the AdS-CFT correspondence has been investigated for scalar fields [15–17], gauge fields [17], spinors [18], classical gravity [19] and type IIB string theory [20,21]. For a comprehensive list of recent references see [17].

Using as representation of AdS_{d+1} the upper half space $x_0>0$, $x_i\in\mathbb{R}$, with the metric

$$ds^2=\frac{1}{x_0^2}dx^\mu dx^\mu \qquad (2)$$

$(\mu=0,1,\ldots,d)$, its boundary is compactified $\mathbb{R}^d$ (the points with $x_0=0$ and the single point $x_0=\infty$). We will frequently denote AdS vectors by $(x_0,\mathbf{x})$ and use x_i to specify the components of $\mathbf{x}$.

The fact that the AdS metric diverges on the boundary presents a difficulty in the AdS-CFT correspondence, which is to be met with care. The natural solution is to calculate the AdS action on a surface, $x_0=\epsilon$, and then take the limit $\epsilon\to0$. However, the exact connection between the AdS fields ϕ and the boundary fields ϕ_0 is subtle. Whereas Witten [5] stated that ϕ should approach ϕ_0 times a certain power of x_0 as $x_0\to0$, it was soon realized [17] that in certain cases, in order to satisfy Ward identities, one must formulate a proper Dirichlet boundary value problem on the surface $x_0=\epsilon$ and take the limit $\epsilon\to0$ at the very end. A detailed investigation taking into account this subtlety has so far been done only for scalar fields [16,17]. We find it therefore necessary to extend our previous investigation of the scalar field [16] to the vector and Dirac fields on AdS_{d+1}. To be general, we shall include a mass term in the vector field action, which is considered in Sec. II. In Sec. III we will give account of the Dirac field. The minimal coupling of the Dirac and gauge fields is considered in Sec. IV, and Sec. V contains the conclusions.

II. VECTOR FIELD

The starting point is the action

$$I=\int d^{d+1}x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}m^2A_\mu A^\mu\right) \qquad (3)$$

with the usual relation $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. The equation of motion derived from Eq. (3) is

$$\nabla_\mu F^{\mu\nu}-m^2A^\nu=0, \qquad (4)$$

which implies the subsidiary condition

$$\nabla_\mu A^\mu=0. \qquad (5)$$

Within our representation of anti–de Sitter space (2) one can use Eqs. (4) and (5) to obtain an equation for A_0:

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$$[x_0^2\partial_\mu\partial_\mu+(1-d)x_0\partial_0-(m^2-d+1)]A_0=0. \tag{6}$$

Introducing $\tilde{m}^2=m^2-d+1$ we know from the consideration of the scalar field that the solution of Eq. (6), which does not diverge for $x_0\to\infty$, is given by

$$A_0(x)=\int \frac{d^dk}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\mathbf{x}}x_0^{d/2}a_0(\mathbf{k})K_{\tilde{\alpha}}(kx_0), \tag{7}$$

with

$$\tilde{\alpha}=\sqrt{\frac{d^2}{4}+\tilde{m}^2}=\sqrt{\frac{(d-2)^2}{4}+m^2}. \tag{8}$$

It is useful to introduce fields with Lorentz indices by

$$\tilde{A}_a=e_a^\mu A_\mu=x_0A_a, \tag{9}$$

where e_a^μ denotes the vielbein $(a=0,1,\ldots,d)$. The virtue of this is seen when considering the components $\tilde{A}_i$ $(i=1,2,\ldots,d)$, whose equation of motion is again obtained from Eqs. (4) and (5) and is given by

$$[x_0^2\partial_\mu\partial_\mu+(1-d)x_0\partial_0-\tilde{m}^2]\tilde{A}_i=2x_0\partial_i\tilde{A}_0. \tag{10}$$

The solution of the homogeneous part of Eq. (10) can be taken over from A_0 and the inhomogeneous equation is solved by making a good guess as to which form the solution should have. One obtains

$$\tilde{A}_i(x)=\int \frac{d^dk}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\mathbf{x}}x_0^{d/2}\bigg(a_i(\mathbf{k})K_{\tilde{\alpha}}(kx_0)+ia_0(\mathbf{k})\frac{k_i}{k}x_0K_{\tilde{\alpha}+1}(kx_0)\bigg). \tag{11}$$

We have now to impose the subsidiary condition (5), which in terms of the Lorentz index fields reads

$$x_0\partial_\mu\tilde{A}_\mu-d\tilde{A}_0=0. \tag{12}$$

Inserting Eqs. (7) and (11) into Eq. (12) yields

$$a_0\bigg(\tilde{\alpha}-\frac{d}{2}+1\bigg)=ia_ik_i, \tag{13}$$

which determines a_0 in the generic case of massive vector fields, but leaves it undetermined in the massless case. In order to find a prescription which is valid for both cases, let us first impose the boundary conditions on the fields $\tilde{A}_i$. It is useful to write

$$a_i=b_i+bk_i. \tag{14}$$

Setting $x_0=\epsilon$ in Eq. (11) we then find

$$b_iK_{\tilde{\alpha}}+k_i\bigg[bK_{\tilde{\alpha}}+ia_0\frac{\epsilon}{k}K_{\tilde{\alpha}+1}\bigg]=\epsilon^{-d/2}\tilde{A}_{\epsilon,i}(\mathbf{k}), \tag{15}$$

where the argument $k\epsilon$ of the modified Bessel functions has been omitted and $\tilde{A}_{\epsilon,i}(\mathbf{k})$ denotes the Fourier transform of the Dirichlet boundary value of the field $\tilde{A}_i$. We can determine b_i and a_0 from Eq. (15) by identifying the first term on the LHS with the RHS and demanding that the second term on the LHS be zero. This yields

$$b_i=\epsilon^{-d/2}\frac{\tilde{A}_{\epsilon,i}(\mathbf{k})}{K_{\tilde{\alpha}}}, \tag{16}$$

$$a_0=i\,\frac{kbK_{\tilde{\alpha}}}{\epsilon K_{\tilde{\alpha}+1}}. \tag{17}$$

Substituting Eqs. (14) and (17) into Eq. (13) we find the missing coefficient

$$b=\frac{b_ik_i}{k^2}\,\frac{k\epsilon K_{\tilde{\alpha}+1}}{(1-\tilde{\Delta})K_{\tilde{\alpha}}-k\epsilon K_{\tilde{\alpha}-1}}, \tag{18}$$

where a functional relation of the modified Bessel functions has been used to rearrange the denominator and we have defined $\tilde{\Delta}=\tilde{\alpha}+d/2$.

Let us use the AdS-CFT correspondence to calculate the two-point functions of currents J_i, which couple to the massive vector fields $A_{0,i}$. After integration by parts and using Eq. (4) the action (3) takes the value

$$I=-\frac{1}{2}\int d^dx\,\epsilon^{-d}\tilde{A}_{\epsilon,i}[-\tilde{A}_{\epsilon,i}+\epsilon\tilde{F}_{\epsilon,0i}], \tag{19}$$

where $\tilde{F}_{0i}=\partial_0\tilde{A}_i-\partial_i\tilde{A}_0$ contains the interesting part. Using the solutions (7) and (11) with the coefficients obtained in Eqs. (14), (16), (17) and (18) one finds

$$\tilde{F}_{\epsilon,0i}=\bigg(\frac{d}{2}-\tilde{\alpha}\bigg)\frac{1}{\epsilon}\tilde{A}_{\epsilon,i}+\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\tilde{A}_{\epsilon,j}(\mathbf{k})k\frac{K_{\tilde{\alpha}-1}}{K_{\tilde{\alpha}}}\times\bigg[-\delta_{ij}+\frac{k_ik_j}{k^2}\,\frac{k\epsilon K_{\tilde{\alpha}+1}}{(\tilde{\Delta}-1)K_{\tilde{\alpha}}+k\epsilon K_{\tilde{\alpha}-1}}\bigg]. \tag{20}$$

We take the limit $\epsilon\to 0$ by substituting the first terms of the series expansion of the modified Bessel functions in Eq. (20). The series expansion is given by

$$K_\nu(z)=z^{-\nu}2^{\nu-1}\Gamma(\nu)\bigg[1-\bigg(\frac{z}{2}\bigg)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}+\cdots\bigg], \tag{21}$$

where the ellipsis indicates terms of order z^{2n} and $z^{2\nu+2n}$ $(n=1,2,\ldots)$. Our experience from the scalar field [16] tells us that the relevant terms are proportional to $k^{2\tilde{\alpha}}\delta_{ij}$ and $k^{2\tilde{\alpha}-2}k_ik_j$. We obtain these by keeping only the leading order terms for the denominators in Eq. (20) and using the appropriate terms for the numerators. In particular, the term $k^{2\tilde{\alpha}}$ from Eq. (21) is needed only for $K_{\tilde{\alpha}-1}$ in the numerator of Eq. (20). One obtains

$$\tilde{F}_{\epsilon,0i}=\left(\frac{d}{2}-\tilde{\alpha}\right)\frac{1}{\epsilon}\tilde{A}_{\epsilon,i}+\left(\frac{\epsilon}{2}\right)^{2\tilde{\alpha}-1}\frac{\Gamma(1-\tilde{\alpha})}{\Gamma(\tilde{\alpha})}\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\tilde{A}_{\epsilon,j}(\mathbf{k})\times\left(-k^{2\tilde{\alpha}}\delta_{ij}+\frac{2\tilde{\alpha}}{\tilde{\Delta}-1}k^{2\tilde{\alpha}-2}k_ik_j+\cdots\right), \tag{22}$$

where the ellipsis denotes all other terms representing either contact terms in the two-point function or terms of higher order in ϵ. Performing the integrals in Eq. (22) and inserting the result into Eq. (19) yields

$$I=\frac{1}{2}\left(\tilde{\alpha}+1-\frac{d}{2}\right)\int d^dx\,\epsilon^{-d}\tilde{A}_{\epsilon,i}(\mathbf{x})\tilde{A}_{\epsilon,i}(\mathbf{x})-\frac{1}{2}\frac{2\tilde{c}\tilde{\alpha}\tilde{\Delta}}{\tilde{\Delta}-1}\int d^dx\,d^dy\,\tilde{A}_{\epsilon,i}(\mathbf{x})\tilde{A}_{\epsilon,i}(\mathbf{y})\frac{\epsilon^{2(\tilde{\Delta}-d)}}{|\mathbf{x}-\mathbf{y}|^{2\tilde{\Delta}}}\times\left(\delta_{ij}-2\frac{(x-y)_i(x-y)_j}{|\mathbf{x}-\mathbf{y}|^2}\right)+\cdots, \tag{23}$$

with

$$\tilde{c}=\frac{\Gamma(\tilde{\Delta})}{\pi^{d/2}\Gamma(\tilde{\alpha})}.$$

Identifying

$$A_{0,i}(\mathbf{x})=\lim_{\epsilon\to 0}\epsilon^{\tilde{\Delta}-d}\tilde{A}_{\epsilon,i}(\mathbf{x}) \tag{24}$$

and using the AdS-CFT correspondence of the form

$$\exp(-I_{AdS})\equiv\left\langle\exp\left(\int d^dx\,J_j(\mathbf{x})A_{0,j}(\mathbf{x})\right)\right\rangle, \tag{25}$$

we can read off from Eq. (23) the finite distance two-point function as

$$\langle J_i(\mathbf{x})J_j(\mathbf{x})\rangle=\frac{2\tilde{c}\tilde{\alpha}\tilde{\Delta}}{\tilde{\Delta}-1}\left(\delta_{ij}-2\frac{(x-y)_i(x-y)_j}{|\mathbf{x}-\mathbf{y}|^2}\right)\times|\mathbf{x}-\mathbf{y}|^{-2\tilde{\Delta}}, \tag{26}$$

which is of the form dictated by conformal invariance. It shows in particular that J_i has the conformal dimension $\tilde{\Delta}$. This is of course as expected, but in view of the fact that the integrals in Eq. (22) have to combine to give exactly the terms in parentheses in Eq. (26) it is a non-trivial check of our derivation. Moreover, our result coincides for the massless case with the one obtained in [17].

In contrast to the two-point function, which is determined by a boundary integral, interactions involve integrals over the volume of AdS_{d+1}. Hence, higher correlation functions are not sensitive to the order of taking the $\epsilon\to 0$ limit and we shall take it for the fields A_μ. Substituting Eqs. (14), (16), (17) and (18) into Eq. (11) and replacing $K_\nu(k\epsilon)$ by the leading order term of its asymptotic expansion (21) one finds

$$A_i^{bulk}(x)=\frac{\tilde{c}\tilde{\Delta}}{\tilde{\Delta}-1}\int d^dy\,A_{0,j}(\mathbf{y})\frac{x_0^{\tilde{\Delta}-1}}{(x_0^2+|\mathbf{x}-\mathbf{y}|^2)^{\tilde{\Delta}}}\times\left(\delta_{ij}-2\frac{(x-y)_i(x-y)_j}{x_0^2+|\mathbf{x}-\mathbf{y}|^2}\right). \tag{27}$$

Similarly, taking the limit in Eq. (7) yields

$$A_0^{bulk}(x)=-\frac{2\tilde{c}\tilde{\Delta}}{\tilde{\Delta}-1}\int d^dy\,A_{0,j}(\mathbf{y})\frac{x_0^{\tilde{\Delta}}(x-y)_j}{(x_0^2+|\mathbf{x}-\mathbf{y}|^2)^{\tilde{\Delta}+1}}. \tag{28}$$

III. FREE DIRAC FIELD

Let us start with the action

$$I[\bar{\psi},\psi]=\int d^{d+1}x\sqrt{g}\,\bar{\psi}(x)(\not{D}-m)\psi(x)+G\int d^dx\sqrt{h}\,\bar{\psi}(\mathbf{x})\psi(\mathbf{x}), \tag{29}$$

where we supplemented the dynamical bulk action with a surface term [18] with an undetermined coefficient G. The surface term is necessary in order to obtain a two-point function of spinors in the boundary conformal field theory. The equation of motion for ψ derived from the action (29) is the Dirac equation

$$(\not{D}-m)\psi(x)=\left(x_0\gamma_\mu\partial_\mu-\frac{d}{2}\gamma_0-m\right)\psi(x)=0, \tag{30}$$

where the matrices γ_μ are the Dirac matrices of $(d+1)$-dimensional Euclidian space, i.e., $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}$. Acting with $\gamma_\mu\partial_\mu$ on Eq. (30) one obtains the second order differential equation

$$\left[\partial_\mu\partial_\mu-\frac{d}{x_0}\partial_0-\frac{1}{x_0^2}\left(m^2-\frac{d^2}{4}-\frac{d}{2}-\gamma_0 m\right)\right]\psi(x)=0. \tag{31}$$

The solution of Eq. (31), which does not diverge for $x_0\to\infty$, is obtained in a similar fashion as in the scalar and vector cases and is given by

$$\psi(x)=\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot\mathbf{x}}x_0^{(d+1)/2}\,[a^+(\mathbf{k})K_{m-1/2}(kx_0)+a^-(\mathbf{k})K_{m+1/2}(kx_0)], \tag{32}$$

where the spinors $a^\pm$ satisfy $\gamma_0 a^\pm=\pm a^\pm$. The expression (32) is in general not a solution of the Dirac equation (30). In fact, substituting Eq. (32) into Eq. (30) we find that the spinors a^+ and a^- must be related by

$$a^{-}=\frac{i}{k}k_i\gamma_i a^{+}. \tag{33}$$

Our next task is to impose boundary conditions on the solution (32). However, there is a major difference to the scalar and vector cases. The origin of this difference lies in the nature of the differential equations, which serve as the equations of motion for the fields. In the scalar case [16] and vector case (cf. Sec. II) we have second order differential equations. Hence, we could impose two sets of boundary data, namely the field and its derivative. Instead of the latter we demand that the field be well behaved in the volume of AdS_{d+1}, i.e. for $x_0\to\infty$, which yields a unique solution to the Dirichlet problem. On the other hand, the Dirac equation (30) is a first order differential equation. The $x_0\to\infty$ behavior of the solutions of the Dirac equation is crucial from the AdS field theory point of view and cannot be abandoned. Hence, only half of the general solutions are available for fitting the boundary data, which means that only half the components of the spinor ψ can be prescribed on the boundary, the other half being fixed by a relation which will be determined in a moment. This result is important also from a CFT point of view. Considering the boundary term of the action (29) we realize that, if one could prescribe the entire boundary spinor, then there would be only a contact term in the CFT two-point function. The trade-off is that we can obtain only correlators for spinors, which have half the number of components as the field ψ. This means that the boundary spinors are Weyl or Dirac spinors for d even or odd, respectively [18].

Letting $x_0=\epsilon$ in (32) we find

$$\psi_\epsilon(\mathbf{k})=\epsilon^{(d+1)/2}\left(K_{m-1/2}+i\frac{k_i\gamma_i}{k}K_{m+1/2}\right)a^{+}(\mathbf{k}), \tag{34}$$

where $\psi_\epsilon(\mathbf{k})$ is the Fourier transform of the boundary spinor and we have omitted the argument $k\epsilon$ of the modified Bessel functions. We can determine a^{+} from Eq. (34) in two ways, namely by

$$a^{+}(\mathbf{k})=\epsilon^{-(d+1)/2}\frac{\psi_\epsilon^{+}(\mathbf{k})}{K_{m-1/2}} \tag{35}$$

or

$$a^{+}(\mathbf{k})=\epsilon^{-(d+1)/2}\frac{k_i\gamma_i}{ik}\frac{\psi_\epsilon^{-}(\mathbf{k})}{K_{m+1/2}}, \tag{36}$$

where $\psi_\epsilon^{\pm}=\frac{1}{2}(1\pm\gamma_0)\psi_\epsilon$. Substituting Eq. (36) into Eq. (35) we find that ψ_ϵ^{+} and ψ_ϵ^{-} are related by

$$\psi_\epsilon^{+}(\mathbf{k})=-i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}}\psi_\epsilon^{-}(\mathbf{k}). \tag{37}$$

The question as to which of the functions $\psi_\epsilon^{\pm}$ should be used as boundary data is, in general, not a matter of choice, but is dictated by the $\epsilon\to0$ limit. Here we have to distinguish three cases. If $m>0$, $K_{m-1/2}$ diverges slower than $K_{m+1/2}$ for $\epsilon\to0$ and thus we find that $\psi_\epsilon^{+}\to0$, if we fix ψ_ϵ^{-}. This is in agreement with the condition found in [18]. On the other hand, we cannot prescribe ψ_ϵ^{+} for $m>0$, as ψ_ϵ^{-} would then diverge. The case $m<0$ is just the opposite. For $m=0$ we have $K_{-1/2}=K_{1/2}$ and hence one can prescribe either of the functions $\psi_\epsilon^{\pm}$.

We shall in the following consider the case $m\geqslant0$. Inserting Eqs. (36) and (33) into Eq. (32) we finally find

$$\begin{aligned}\psi(x)=\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\frac{x_0}{\epsilon}\right)^{(d+1)/2}\\ \times\left(-i\frac{k_i\gamma_i}{k}K_{m-1/2}(kx_0)+K_{m+1/2}(kx_0)\right)\\ \times\frac{\psi_\epsilon^{-}(\mathbf{k})}{K_{m+1/2}(k\epsilon)}.\end{aligned} \tag{38}$$

In a similar fashion one can solve the equation of motion for the conjugate spinor:

$$\bar\psi(x)(\overleftarrow{\not{D}}+m)=\bar\psi(x)\left(\overleftarrow{\partial}_\mu\gamma_\mu x_0-\frac{d}{2}\gamma_0+m\right)=0. \tag{39}$$

The solution in the case $m\geqslant0$ is

$$\begin{aligned}\bar\psi(x)=\int\frac{d^dk}{(2\pi)^d}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\left(\frac{x_0}{\epsilon}\right)^{(d+1)/2}\frac{\bar\psi_\epsilon^{+}(\mathbf{k})}{K_{m+1/2}(k\epsilon)}\\ \times\left(i\frac{k_i\gamma_i}{k}K_{m-1/2}(kx_0)+K_{m+1/2}(kx_0)\right),\end{aligned} \tag{40}$$

where $\bar\psi_\epsilon^{\pm}=\bar\psi_\epsilon\frac{1}{2}(1\pm\gamma_0)$. Again we find a relation between the components of the boundary spinor, which is given by

$$\bar\psi_\epsilon^{-}(\mathbf{k})=\bar\psi_\epsilon^{+}(\mathbf{k})\,i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}}. \tag{41}$$

Let us turn now to the two-point function for the boundary spinors χ^{+} and $\bar\chi^{-}$, which couple to $\bar\psi_0^{+}$ and ψ_0^{-}, respectively. Inserting the solutions of the equations of motion into the action (29), the bulk term vanishes and the surface term can be written as

$$I=G\epsilon^{-d}\int\frac{d^dk}{(2\pi)^d}\,[\bar\psi^{+}(\mathbf{k})\psi^{+}(-\mathbf{k})+\bar\psi^{-}(\mathbf{k})\psi^{-}(-\mathbf{k})]. \tag{42}$$

Using the relations (37) and (41) one finds

$$\begin{aligned}I=G\epsilon^{-d}\int d^dx\,d^dy\int\frac{d^dk}{(2\pi)^d}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\bar\psi_\epsilon^{+}(\mathbf{x})\\ \times\left(2i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}}\right)\psi_\epsilon^{-}(\mathbf{y}).\end{aligned} \tag{43}$$

We use the expansion (21) for the modified Bessel functions in the numerator and the leading order term in the denominator. Hence, we find, after integration,

$$I=-2\hat{c}G\int d^dx d^dy\,\overline{\psi}_0^+(\mathbf{x})\,\frac{\gamma_i(x_i-y_i)}{|\mathbf{x}-\mathbf{y}|^{d+2m+1}}\psi_0^-(\mathbf{y}), \quad (44)$$

where we defined

$$\psi_0^-=\lim_{\epsilon\to 0}\epsilon^{m-d/2}\psi_\epsilon^- \quad \text{and} \quad \overline{\psi}_0^+=\lim_{\epsilon\to 0}\epsilon^{m-d/2}\overline{\psi}_\epsilon^+ \quad (45)$$

for the $\epsilon\to 0$ limit and

$$\hat{c}=\frac{\Gamma\left(\frac{d+1}{2}+m\right)}{\pi^{d/2}\Gamma\left(m+\frac{1}{2}\right)}.$$

In the case $m=0$ the k integration in Eq. (43) can be done without the asymptotic expansion and leads to the same result. Using the AdS-CFT correspondence

$$\exp(-I_{AdS})\equiv\left\langle \exp\left(\int d^dx(\overline{\chi}^-\psi_0^-+\overline{\psi}_0^+\chi^+)\right)\right\rangle, \quad (46)$$

the two-point function reads

$$\langle\chi^+(\mathbf{x})\overline{\chi}^-(\mathbf{y})\rangle=2\hat{c}G\,\frac{\gamma_i(x_i-y_i)}{|\mathbf{x}-\mathbf{y}|^{d+2m+1}}. \quad (47)$$

Hence, the spinors χ and $\overline{\chi}$ have the conformal dimension $m+d/2$. Our result agrees up to the appropriate normalization with the one found in [18].

For calculating interactions we are interested in the bulk behavior of the spinors ψ and $\overline{\psi}$. It is obtained by replacing $K_{m+1/2}(k\epsilon)$ by the leading order term of its asymptotic expansion in Eqs. (38) and (40). One finds the expressions

$$\psi^{bulk}(x)=\hat{c}\int d^dy[x_0-\gamma_i(x_i-y_i)]\times(x_0^2+|\mathbf{x}-\mathbf{y}|^2)^{-(d+1)/2-m}x_0^{d/2+m}\psi_0^-(\mathbf{y}) \quad (48)$$

and

$$\overline{\psi}^{bulk}(x)=\hat{c}\int d^dy\,\overline{\psi}_0^+(\mathbf{y})[x_0+\gamma_i(x_i-y_i)]\times(x_0^2+|\mathbf{x}-\mathbf{y}|^2)^{-(d+1)/2-m}x_0^{d/2+m}, \quad (49)$$

which coincide with those in [18] up to normalization. A good check of the derivation of these expressions is provided by the case $m=0$. Since $K_{\pm 1/2}(z)=(\sqrt{\pi/2z})e^{-z}$, it is possible to carry out the integration in Eq. (38) with the result

$$\psi(x)=\int d^dy\,\frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{d/2}\Gamma\left(\frac{1}{2}\right)}x_0^{d/2}[(x_0-\epsilon)^2+|\mathbf{x}-\mathbf{y}|^2]^{-(d+1)/2}\times[x_0-\epsilon-\gamma_i(x_i-y_i)]\psi_0^-(\mathbf{y}). \quad (50)$$

IV. INTERACTION BETWEEN SPINOR AND GAUGE FIELDS

Calculating the first order interaction between the spinor and massless vector fields serves two purposes. First, it provides another detail of the AdS-CFT correspondence in the form of the vector-spinor-spinor three-point function. In contrast to the scalar three-point function, conformal symmetry does not fix, but only restricts the form of this particular three-point function [1]. Hence, the calculation will yield more than just a coefficient in front of a universal function. Second, a check of the Ward identity corresponding to gauge invariance will reveal that no supplementary surface term of the order of the gauge coupling is needed.

We shall use the action for minimally coupled spinor and gauge fields, together with the spinor surface term,

$$I=\int d^{d+1}x\sqrt{g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\overline{\psi}(\not{D}-iq\not{A}-m)\psi\right]+G\int d^dx\sqrt{h}\overline{\psi}\psi. \quad (51)$$

The equations of motion derived from Eq. (51) are

$$\nabla_\mu F^{\mu\nu}=-iq\mathrm{e}_a^\nu\overline{\psi}\gamma_a\psi, \quad (52)$$

$$(\not{D}-m)\psi=iq\not{A}\psi \quad (53)$$

and its conjugate

$$\overline{\psi}(\overleftarrow{\not{D}}+m)=-iq\overline{\psi}\not{A}. \quad (54)$$

We split the gauge field into its free part $A^{(0)}$ and the remainder $A^{(1)}$. Substituting Eq. (53) into Eq. (51) and using the equation of motion for $F^{(0)}$, we find

$$I=\int d^dx\,\epsilon^{-d}\left(-\frac{1}{2\epsilon}A_i^{(0)}F^{(0),0i}-\frac{1}{\epsilon}A_i^{(1)}F^{(0),0i}+G\overline{\psi}\psi\right)+\mathcal{O}(q^2). \quad (55)$$

Most importantly, the bulk terms vanish. Moreover, using the appropriate Green's function to calculate $A^{(1)}$ (cf. [16] for the scalar field analogue), we realize that also the second term in Eq. (55) is zero. The first term only yields the two-point function for the conserved currents J. However, the last term will give the two-point function for the spinors and the three-point function coupling J and the spinors. This surprising fact comes about as follows. Going back to the derivation of the spinor two-point function, we realize that it was generated by the relations (37) and (41) between the + and − components of the spinors on the boundary. These relations will be altered by the presence of the interaction. Writing

$$\psi(x)=\psi^{(0)}(x)+\psi^{(1)}(x)+\mathcal{O}(q^2), \quad (56)$$

$$\psi^{(1)}(x)=iq\int d^{d+1}y\sqrt{g}S(x,y)\not{A}(y)\psi^{(0)}(y), \quad (57)$$

where $S(x,y)$ is the spinor Green's function defined by

$$(\not{D}_x - m)S(x,y) = \frac{\delta(x-y)}{\sqrt{g(x)}}, \tag{58}$$

we find, using Eq. (37),

$$\psi^+(\mathbf{k}) = -i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}}\psi^-(\mathbf{k}) + \frac{1+\gamma_0}{2}\left(1+i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}}\right)\psi^{(1)}(\mathbf{k}) + \mathcal{O}(q^2), \tag{59}$$

where we omitted the argument $k\epsilon$ of the modified Bessel functions. Similarly, one finds, for the conjugate field,

$$\bar{\psi}(x) = \bar{\psi}^{(0)}(x) + \bar{\psi}^{(1)}(x) + \mathcal{O}(q^2), \tag{60}$$

$$\bar{\psi}^{(1)}(x) = iq\int d^{d+1}y\sqrt{g}\,\bar{\psi}^{(0)}(y)\not{A}(y)\bar{S}(y,x), \tag{61}$$

with[1]

$$\bar{S}(y,x)(\overleftarrow{\not{D}}_x + m) = -\frac{\delta(x-y)}{\sqrt{g(x)}} \tag{62}$$

and, using Eq. (41),

$$\bar{\psi}^-(\mathbf{k}) = \bar{\psi}^+(\mathbf{k})\,i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}} + \bar{\psi}^{(1)}(\mathbf{k})\times\left(1-i\frac{k_i\gamma_i}{k}\frac{K_{m-1/2}}{K_{m+1/2}}\right)\frac{1-\gamma_0}{2} + \mathcal{O}(q^2). \tag{63}$$

Substituting Eqs. (59) and (63) into the spinor surface term in the form (42), one finds that the contribution to the action of first order in q is

$$I^{(1)} = G\epsilon^{-d}\int d^dx\{[\bar{\psi}^{(0)+}(\mathbf{x}) - \bar{\psi}^{(0)-}(\mathbf{x})]\psi^{(1)}(\mathbf{x}) - \bar{\psi}^{(1)}(\mathbf{x})[\psi^{(0)+}(\mathbf{x}) - \psi^{(0)-}(\mathbf{x})]\} + \mathcal{O}(q^2). \tag{64}$$

On the other hand, from Eqs. (62) and (58) one can obtain

$$\psi^{(0)}(x) = \epsilon^{-d}\int d^dy\,\bar{S}(x,\mathbf{y})[\psi^{(0)+}(\mathbf{y}) - \psi^{(0)-}(\mathbf{y})] \tag{65}$$

and

$$\bar{\psi}^{(0)}(x) = -\epsilon^{-d}\int d^dy[\bar{\psi}^{(0)+}(\mathbf{y}) - \bar{\psi}^{(0)-}(\mathbf{y})]S(\mathbf{y},x), \tag{66}$$

[1]The relation between $\bar{S}$ and S is of no importance here.

respectively. Inserting Eqs. (57), (66), (61) and (65) into Eq. (64) one then finds

$$I^{(1)} = -2Giq\int d^{d+1}x\sqrt{g}\,\bar{\psi}^{(0)}\not{A}\psi^{(0)}. \tag{67}$$

Equation (67) has the same form as the minimal coupling term, but is multiplied by $2G$. It is determined by a bulk integral, which means that the bulk behavior for the fields can be used. Substituting Eqs. (27), (28) (with $\tilde{\Delta} = d-1$), (48) and (49) into Eq. (67), the following tedious calculation involves Feynman parametrization of the denominator and heavy numerator algebra. The result is

$$\langle J_j(\mathbf{x}_2)\chi^+(\mathbf{x}_1)\bar{\chi}^-(\mathbf{x}_3)\rangle = \frac{-iGq\hat{c}\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}(d-1+2m)} \times\left[(d-2)\gamma_i\gamma_j\gamma_k\frac{x_{12i}x_{23k}}{x_{12}^d x_{23}^d x_{13}^{2m+1}} + (2m+1)\gamma_i x_{13i}\times\frac{x_{12}^2 x_{23j} + x_{23}^2 x_{12j}}{x_{12}^d x_{23}^d x_{13}^{2m+3}}\right], \tag{68}$$

where $\mathbf{x}_{ab} = \mathbf{x}_a - \mathbf{x}_b$. After further algebra one finds that Eq. (68) can be written in the form

$$\langle J_j(\mathbf{x}_2)\chi^+(\mathbf{x}_1)\bar{\chi}^-(\mathbf{x}_3)\rangle = \frac{-iGq\hat{c}\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}(d-1+2m)}\frac{1}{x_{12}^d x_{23}^{d-2} x_{13}^{2m}} \times\frac{\gamma_i x_{13i}}{x_{13}}\left(\delta_{jk} - 2\frac{x_{23j}x_{23k}}{x_{23}^2}\right) \times\left(\frac{x_{13l}}{x_{13}^2} - \frac{x_{23l}}{x_{23}^2}\right) \times[(d-2)\gamma_l\gamma_k + (2m+1)\delta_{kl}], \tag{69}$$

which is a specific case of the general expression dictated by conformal invariance [1].

Finally, let us confirm the Ward identity [22]

$$\frac{\partial}{\partial x_2^j}\langle J_j(\mathbf{x}_2)\chi^+(\mathbf{x}_1)\bar{\chi}^-(\mathbf{x}_3)\rangle = -iq\langle\chi^+(\mathbf{x}_1)\bar{\chi}^-(\mathbf{x}_3)\rangle \times[\delta(\mathbf{x}_{23}) - \delta(\mathbf{x}_{12})]. \tag{70}$$

From Eq. (68) one finds

$$\frac{\partial}{\partial x_2^j}\langle J_j(\mathbf{x}_2)\chi^+(\mathbf{x}_1)\bar{\chi}^-(\mathbf{x}_3)\rangle = -iGq2\hat{c}\,\frac{\gamma_i x_{13i}}{x_{13}^{d+2m+1}} \times[\delta(\mathbf{x}_{23}) - \delta(\mathbf{x}_{12})]. \tag{71}$$

Comparing Eqs. (71) and (47) with Eq. (70) we see that the Ward identity is satisfied. This result is significant, since it tells us that, to first order in q, no supplementary surface term except the one used already for the free Dirac field is required in the action for interacting fields.

V. CONCLUSIONS

In the present paper we used the AdS-CFT correspondence to calculate CFT correlators from the classical AdS theories of vector and Dirac fields. We took care to address the proper treatment of the $\epsilon \to 0$ limit when calculating the two-point functions. As for the scalar field [16,17], this was particularly important for the vector field with nonzero mass.

Our calculation for the free Dirac field revealed the full details as to why only half the number of spinor components can be given as boundary data. For odd d this is exactly what one wants, because the boundary spinor representation has only half the number of components as the bulk spinors. For even d the dimensions of the spinor representations are the same and γ_0 acts as the chirality operator on the boundary spinors. This means that for even d we calculated only the correlation functions for chiral spinors. However, the formalism can be extended to Dirac spinors by coupling χ^- to an AdS spinor ψ_1 with positive mass m and χ^+ to a field ψ_2 with mass $-m$.

Minimally coupling the Dirac and massless vector field, we calculated the CFT vector-spinor-spinor three-point function. The result should be interesting from a CFT point of view, as the form of this correlator is not totally fixed by conformal invariance. Thus, our result could indicate which CFT is obtained by the AdS-CFT correspondence. Finally, we confirmed the validity of the Ward identity and found that no interaction surface terms are required in the action.

ACKNOWLEDGMENTS

We would like to thank D. Freedman and K. Sfetsos for commenting on the first version of this paper, which led to a better understanding of the interaction considered in Sec. IV. This work was supported in part by an operating grant from NSERC. W.M. gratefully acknowledges support from Simon Fraser University.

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