Gauge symmetry in phase space with spin, a basis for conformal symmetry and duality among many interactions

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We show that a simple OSp(1/2) world line gauge theory in 0-brane phase space $X^M(\tau),P^M(\tau)$ with spin degrees of freedom $\psi^M(\tau)$, formulated for a $(d+2)$-dimensional spacetime with two times $X^0(\tau),X^{0'}(\tau)$, unifies many physical systems which ordinarily are described by a one-time formulation. Different systems of one-time physics emerge by choosing gauges that embed ordinary time in $d+2$ dimensions in different ways. The embeddings have different topology and geometry for the choice of time among the $d+2$ dimensions. Thus, two-time physics unifies an infinite number of one-time physical interacting systems, and establishes a kind of duality among them. One manifestation of the two times is that all of these physical systems have the same quantum Hilbert space in the form of a unique representation of SO$(d,2)$ with the same Casimir eigenvalues. By changing the number of spinning degrees of freedom $\psi_a^M(\tau)$, $a=1,2,\ldots,n$ (including no spin $n=0$), the gauge group changes to OSp$(n/2)$. Then the eigenvalue of the Casimir operators of SO$(d,2)$ depend on n and the content of the one-time physical systems that are unified in the same representation depend on n. The models we study raise new questions about the nature of spacetime. [S0556-2821(98)05620-3]

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I. INTRODUCTION

In two recent papers [1,2] we showed that various physical systems, which normally are considered unrelated, are actually unified by the same theory that establishes a kind of duality among them. Examples of such systems included the free relativistic particle in d spacetime dimensions, the H atom in $d-1$ space dimensions, and the harmonic oscillator in $d-2$ space dimensions with its mass identified with a momentum in an extra dimension. Related ideas were considered in Refs. [3–6]. Our aim in this paper is twofold. First, to generalize the theory to describe spinning systems, and second to present an infinite array of interacting models (relativistic, nonrelativistic, arbitrary potentials, curved backgrounds, etc.) with spin, that are unified by gauge transformations (dualities) in the same theory. Our understanding of the quantum version of the theory is solidified by working out many examples and gauge choices in detail.

Through all these examples we emphasize that one-time physics with various interactions are unified in some geometrical sense as two-time physics. The theory in Refs. [1,2] is a simple Sp(2) gauge theory on the world line. Sp(2) is the global isometry group of the quantum relations $[x,p]=i$ and it transforms (x,p) as a doublet. The idea was to turn this group into a local symmetry of some theory. This was loosely motivated by the fact that all known dualities involve a transformation of canonically conjugate phase space variables. The Sp(2) gauge theory on the world line achieved this, but it required that the world line vectors $(X^M(\tau),P^M(\tau))$ be in a spacetime with two timelike coordinates $X^0(\tau),X^{0'}(\tau)$. This turned out to be a boon rather than a drawback. The presence of two times together with the larger gauge symmetry allowed the possibility of gauge choices that are ghost free and physical (unitary). The gauge fixed theory has a single time. The ability to choose time in various ways turned out to be equivalent to different choices of Hamiltonians that describe ordinary one-time physics. In this way, different looking physics corresponds to gauge choices within the same theory. The gauge transformations that map them into each other may be interpreted as dualities (in a universe of two times).

In this paper the theory is generalized by including anticommuting phase space variables for world line fermions $\psi^M(\tau)$. The gauge group becomes OSp(1/2), and (ψ^M,X^M,P^M) form a triplet. At the end of the paper we further generalize this to n world line fermions $\psi_a^M(\tau)$, $a=1,2,\ldots,n$, and gauge group OSp$(n/2)$. The requirement that this be a two-time theory remains the same for any n. The content of the one-time physical dual sectors changes as a function of n. However, for a fixed n all dual physics is described within the same quantum Hilbert space that corresponds to a unique unitary representation of SO$(d,2)$ with fixed Casimir eigenvalues.

The paper is organized as follows. After formulating the OSp(1/2) gauge theory, we quantize it covariantly, and show that the gauge invariant states must be described by a unique representation of SO$(d,2)$ with fixed Casimir eigenvalues. Next we choose specific gauges, which we call ''particle gauge,'' ''light-cone gauge,'' ''H-atom gauge,'' ''anti–de Sitter (AdS) gauge,'' ''conformal gauge,'' and study the quantum theory in each of those gauges. We show that the physics looks different according to the gauge choice of time, but that the Hilbert space is the same in each case, and that it has the same eigenvalues of the Casimir operators of SO$(d,2)$. In a semiclassical approach in the H-atom gauge we show that any Hamiltonian of the form $H=\mathbf{p}^2/2+V(\mathbf{r},\mathbf{p},\mathbf{S})$ with any potential energy function V, emerges as a gauge choice. At the end of the paper we argue that when n changes, the spin content changes. For example, in the particle gauge the relativistic particle that is described corresponds to the antisymmetric form $A_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ that couples to p-branes, with $p=n/2-1$ for even n, and similar fermionic counterparts for odd n.

The message of our work is that two-time physics is not only possible, but also is a basis for unifying many features of one-time physics in some geometrical manner. This raises new questions about the nature of time and space. Our gauge symmetry approach in 0-brane phase space connects together dualities and two-time physics inextricably from each other, and gives a new rich area to explore further and generalize to higher p-branes. Our work supports the idea that the fundamental theory of our universe may be better understood in a two-time formulation, as various hints and theories have suggested from different directions [7–18].

II. GAUGING OSp(1/2)

OSp(1/2) has two local fermionic $s^i(\tau)$ and three local bosonic $\omega^{ij}(\tau)$ parameters. Under the subgroup Sp(2,R) the s^i with $i=1,2$ form a doublet, while the symmetric $\omega^{ij}=\omega^{ji}$ form a triplet. Consider the OSp(1/2) triplets $\Phi_a^M(\tau)=(\psi^M,X_1^M,X_2^M)$ (one for each M) which transform similar to the fundamental representation of OSp(1/2):

$$\delta\psi^M=s^iX_i^M,\quad \delta X_i^M=\varepsilon_{ik}(\omega^{kl}X_l^M-is^k\psi^M). \tag{1}$$

The complex number i is introduced in δX_i^M to insure that the product of fermions $is^k\psi^M$ is Hermitian, assuming that each fermion is Hermitian individually. For each M, ψ^M is a singlet of Sp(2,R) while X_i^M is a doublet of Sp(2). Two such triplets Φ_a^M,Φ_b^N form an OSp(1/2) invariant I^{MN} under the dot product with the metric g^{ab} given by

$$I^{MN}=\Phi_a^M g^{ab}\Phi_b^N=X_i^M\varepsilon^{ij}X_j^N-i\psi^M\psi^N. \tag{2}$$

Fermionic and bosonic gauge potentials (F^i,A^{ij}) are introduced in one to one correspondence with the parameters. There are two fermions $F^i(\tau)$ and three bosons $A^{ij}(\tau)=A^{ji}(\tau)$. They transform as

$$\delta F^i=\partial_\tau s^i+\omega^{ik}\varepsilon_{kl}F^l-A^{ik}\varepsilon_{kl}s^l, \tag{3}$$

$$\delta A^{ij}=\partial_\tau\omega^{ij}+\omega^{ik}\varepsilon_{kl}A^{lj}+\omega^{jk}\varepsilon_{kl}A^{il}-is^iF^j-is^jF^i. \tag{4}$$

The following covariant derivatives $D_\tau\Phi_a^M=(D_\tau\psi^M,D_\tau X_i^M)$:

$$D_\tau\psi^M=\partial_\tau\psi^M-F^iX_i^M, \tag{5}$$

$$D_\tau X_i^M=\partial_\tau X_i^M-\varepsilon_{ik}(A^{kl}X_l^M-iF^k\psi^M), \tag{6}$$

transform similar to OSp(1/2) triplets

$$\delta(D_\tau\psi^M)=s^iD_\tau X_i^M, \tag{7}$$

$$\delta(D_\tau X_i^M)=\varepsilon_{ik}(\omega^{kl}D_\tau X_l^M-is^kD_\tau\psi^M). \tag{8}$$

We can then construct more OSp(1/2) invariants by using the covariant derivatives and the metric defined in Eq. (2) $(D_\tau\Phi_a^M)g^{ab}\Phi_b^N$. In particular we construct a gauge invariant action

$$S_0=\frac{1}{2}\int_0^T d\tau(D_\tau\Phi_a^M)g^{ab}\Phi_b^N\eta_{MN} \tag{9}$$

$$=\frac{1}{2}\int_0^T d\tau[D_\tau X_i^M\varepsilon^{ij}X_j^N-iD_\tau\psi^M\psi^N]\eta_{MN} \tag{10}$$

$$=\int_0^T d\tau\left[X_2\cdot\partial_\tau X_1+\frac{i}{2}\psi\cdot\partial_\tau\psi-\frac{1}{2}A^{ij}X_i\cdot X_j+iF^iX_i\cdot\psi\right]. \tag{11}$$

The equation of motion of the gauge fields give the following constraints:

$$X_i\cdot X_j=0,\quad X_i\cdot\psi=0. \tag{12}$$

As in the purely bosonic case the signature of the metric η^{MN} must be $(d,2)$ including two timelike dimensions otherwise the constraints have no nontrivial solutions. The action is manifestly invariant under SO$(d,2)$ transformations since the metric η^{MN} is invariant. The conserved generators of the symmetry have an orbital part L^{MN} and spin part S^{MN}:

$$J^{MN}=L^{MN}+S^{MN}, \tag{13}$$

$$L^{MN}=X_1^MX_2^N-X_1^NX_2^M, \tag{14}$$

$$S^{MN}=\frac{1}{2i}(\psi^M\psi^N-\psi^N\psi^M). \tag{15}$$

The total generators J^{MN} are OSp(1/2) gauge invariant according to Eq. (2) (take the antisymmetric part of I^{MN}).

From the action we obtain the canonical conjugate pairs $X_1^M=X^M$ and $X_2^M=P^M$. Furthermore, the canonical conjugate to ψ^M is naively $i\psi^M/2$, however this is also a second class constraint. Once the second class constraint is taken into account, the commutation rules for quantizing the system covariantly are

$$[X^M,P^N]=i\eta^{MN},\quad \{\psi^M,\psi^N\}=\eta^{MN}. \tag{16}$$

The ψ^M form a Clifford algebra which is represented by gamma matrices $\psi^M=\gamma^M/\sqrt{2}$, where the gamma matrices are normalized in the standard way $\{\gamma^M,\gamma^N\}=2\eta^{MN}$. The quantum system is subject to first class constraints (12)

$$X\cdot X=P\cdot P=X\cdot P=X\cdot\psi=P\cdot\psi=0, \tag{17}$$

which will be imposed on the Hilbert space. These constraints form the OSp(1/2) superalgebra defined by three bosonic and two fermionic generators

$$J_3=\frac{1}{4}(X^2+P^2),\quad J_1=\frac{1}{4}(X\cdot P+P\cdot X), \tag{18}$$

$$J_2=\frac{1}{4}(X^2-P^2),\quad S_\pm=\frac{1}{2\sqrt{2}}(P\pm iX)\cdot\psi, \tag{19}$$

$$J_\pm=\pm\frac{1}{4i}(P\pm iX)^2=J_1\pm iJ_2. \tag{20}$$

The OSp(1/2) superalgebra among these first class constraints is given by

$$[J_3,J_1]=iJ_2,\quad [J_3,J_2]=-iJ_1,\quad [J_1,J_2]=-iJ_3, \tag{21}$$

$$[J_3,S_\pm]=\pm\frac{1}{2}S_\pm,\quad [J_1,S_\pm]=\frac{i}{2}S_\mp,$$

$$[J_2,S_\pm]=\mp\frac{1}{2}S_\mp, \tag{22}$$

$$\{S_\pm,S_\pm\}=\pm\frac{i}{2}(J_1\pm iJ_2),\quad \{S_+,S_-\}=\frac{J_3}{2}, \tag{23}$$

where $J_\pm=(J_1\pm iJ_2)$. Thus (J_1,J_2,J_3) are represented on the row (S_+,S_-) by the Pauli matrices $(\Sigma^i)^\beta{}_\alpha=(i\sigma_1/2,i\sigma_2/2,\sigma_3/2)$, respectively, i.e., $[J_i,S_\alpha]=iS_\beta(\Sigma_i)^\beta{}_\alpha$ and the last line may be written as $\{S_\alpha,S_\beta\}=\Sigma^i_{\alpha\beta}J_i$.

The quadratic and cubic Casimirs of the superalgebra OSp(1/2) are

$$C_2(\mathrm{OSp}(1/2))=J_3^2-J_1^2-J_2^2-S_+S_-+S_-S_+, \tag{24}$$

$$C_3(\mathrm{OSp}(1/2))=S_+J_3S_-+S_-J_3S_+ +iS_+(J_1-iJ_2)S_+-iS_-(J_1+iJ_2)S_-.$$

These commute with all the generators J_i,S_α. In terms of the canonical operators, with the orders of operators taken into account, the quadratic Casimir becomes

$$C_2(\mathrm{OSp}(1/2))=\frac{1}{4}(X^MP^2X_M-X\cdot PP\cdot X)+\frac{1}{16}(d^2-4) +\frac{1}{4}\left(\frac{1}{2i}[\psi_M,\psi_N](X^MP^N-X^NP^M)\right) +\frac{1}{4}\psi\cdot\psi. \tag{25}$$

Similarly, the cubic Casimir operator is computed in terms of the canonical operators.

Next, consider the quadratic Casimir operator of SO$(d,2)$ given by

$$\begin{aligned}C_2(\mathrm{SO}(d,2))&=\frac{1}{2}J^{MN}J_{MN}=\frac{1}{2}L^{MN}L_{MN}+\frac{1}{2}S^{MN}S_{MN}+L^{MN}S_{MN}\\&=(X^MP^2X_M-X\cdot PP\cdot X)+\frac{1}{4}[2(\psi\cdot\psi)^2-\psi\cdot\psi]\\&\quad+\frac{1}{2i}[\psi_M,\psi_N](X^MP^N-X^NP^M).\end{aligned} \tag{26}$$

Therefore, we find the following relation between the quadratic Casimir operators of SO$(d,2)$ and OSp(1/2):

$$C_2(\mathrm{SO}(d,2))=4C_2(\mathrm{OSp}(1/2))-\frac{1}{8}(d+2)(d-1), \tag{27}$$

where we have used $\psi\cdot\psi=\frac{1}{2}(d+2)$. Similarly, the higher order Casimir operators of the conformal group $C_n=(1/n!)\mathrm{Tr}(iJ)^n$ are obtained in terms of the Casimir operators of OSp(1/2).

In the gauge invariant sector the quadratic Casimir operator of the gauge group must vanish $C_2(1/2)=0$. Therefore, the physical sector is characterized by

$$C_2(\mathrm{OSp}(1/2))=0,\quad C_2(\mathrm{SO}(d,2))=-\frac{1}{8}(d+2)(d-1). \tag{28}$$

Similarly, the eigenvalues $C_n(\mathrm{SO}(d,2))$ are completely fixed after setting all OSp(1/2) Casimir operators equal to zero. We will not use the higher Casimir operators in this paper. We will verify the result for $C_2(\mathrm{SO}(d,2))$ in noncovariant quantization in several gauges.

III. PARTICLE GAUGE AND DIRAC EQUATION

Consider the basis $X^M=[X^{+'},X^{-'},X^\mu]$ with non-zero metric components $\eta^{+'-'}=\eta^{-'+'}=-1$ and $\eta^{\mu\nu}=\mathrm{diag}(-1,+1,\cdots,+1)$ Minkowski metric. Choose two bosonic gauges $X^{+'}=1$, $P^{+'}=0$, and one fermionic gauge $\psi^{+'}=0$, and solve explicitly two bosonic and one fermionic constraints $X^2=X\cdot P=X\cdot\psi=0$. We will call this the relativistic particle gauge. The remaining degrees of freedom x^μ,p^μ,ψ^μ are in Minkowski spacetime and they parametrize X^M,P^M,ψ^M as follows:

$$M=[+',-',\mu],$$

$$X^M=[1,x^2/2,x^\mu],$$

$$P^M=[0,x\cdot p,p^\mu],\quad p^2=0,$$

$$\psi^M=[0,x\cdot\psi,\psi^\mu],\quad \psi^\mu p_\mu=0. \tag{29}$$

There is manifest SO($d-1$,1) Lorentz symmetry. There remains one bosonic and one fermionic gauge degrees of freedom and the corresponding constraints $p^2=0$, $\psi^\mu p_\mu=0$. The quantum rules are $[x^\mu,p^\nu]=i\eta^{\mu\nu}$ and $\{\psi^\mu,\psi^\nu\}=\eta^{\mu\nu}$. The quantum states are labeled by $|\alpha,p\rangle$ or $|\alpha,x\rangle$ with $p^2=0$ and $\psi^\mu p_\mu=0$ to be satisfied on states. The index α is a spinor index in d dimensions, and ψ^μ acts like the Dirac gamma matrix $\psi^\mu\to\gamma^\mu/\sqrt{2}$ on these states. Note that $(\sqrt{2}\psi^\mu p_\mu)^2=p^2$, so that the constraint $p^2=0$ need not be considered separately.

For the general physical state $|\Psi\rangle$ the constraint $\sqrt{2}\psi^\mu p_\mu|\Psi\rangle=0$ becomes the Dirac equation for a massless particle. When expressed in x-space $\langle x,\alpha|\Psi\rangle=\Psi_\alpha(x)$ the physical state constraint takes the form of the Dirac equation

$$\langle x,\alpha|(\sqrt{2}\psi^\mu p_\mu)|\Psi\rangle=-i(\gamma^\mu\partial_\mu\Psi)_\alpha=0. \tag{30}$$

The effective field theory is therefore given by the action for the free Dirac field in d dimensions

$$S_{\rm eff}=\int d^dx\,\bar{\Psi}i\gamma\cdot\partial\Psi. \tag{31}$$

The conformal generators (13) in this gauge take the form

$$J^{+'-'}=\frac{1}{2}(x\cdot p+p\cdot x)+is_0, \tag{32}$$

$$J^{+'\mu}=p^\mu,\quad J^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu+s^{\mu\nu}, \tag{33}$$

$$J^{-'\mu}=\frac{1}{2}x_\lambda p^\mu x^\lambda-\frac{1}{2}x^\mu p\cdot x-\frac{1}{2}x\cdot px^\mu-is_0x^\mu-s^{\mu\nu}x_\nu, \tag{34}$$

where $s^{\mu\nu}=(i/2)(\psi^\mu\psi^\nu-\psi^\nu\psi^\mu)$. The operators x,p,ψ are quantum ordered so that the J^{MN} satisfy the correct algebra for any complex s_0. The parameter s_0 is an operator ordering constant which is fixed by hermiticity according to the Lorenz invariant dot product for states $\langle\Psi|\Psi\rangle=\int d^{d-1}x\bar{\Psi}\gamma^0\Psi$. Hermiticity $\langle J^{MN}\Psi|\Psi\rangle=\langle\Psi|J^{MN}\Psi\rangle$ fixes $s_0=1/2$. In contrast, in the purely bosonic case we had $s_0=1$. Thus the presence of the complex $is_0=i/2$ is required for Hermitian $J^{+'-'},J^{-'\mu}$. Furthermore s_0 should be consistent with the correct dimension of the Dirac field in d dimensions. When the dimension operator $iJ^{+'-'}$ is applied on the Dirac field $\langle x,\alpha|iJ^{+'-'}|\Psi\rangle=(x\cdot\partial+\frac{1}{2}d-s_0)\Psi_\alpha(x)$ we must obtain $(\frac{1}{2}d-s_0)=(d-1)/2$. Thus we find again $s_0=1/2$. The quadratic Casimir operator becomes (orbital parts x,p drop out)

$$\begin{aligned}C_2(\mathrm{SO}(d,2))&=-\frac{d^2}{4}+s_0^2+\frac{1}{2}s^{\mu\nu}s_{\mu\nu}\\&=-\frac{d^2}{4}+\frac{1}{4}+\frac{1}{8}(d^2-d)\\&=-\frac{1}{8}(d+2)(d-1),\end{aligned} \tag{35}$$

where we have used $\frac{1}{2}s^{\mu\nu}s_{\mu\nu}=\frac{1}{2}(\psi\cdot\psi)^2-\frac{1}{4}\psi\cdot\psi$ and $\psi\cdot\psi=\gamma\cdot\gamma/2=d/2$. This agrees precisely with the OSp(1/2) gauge invariance requirements (28) obtained in covariant quantization in the previous section.

Hence all of the Dirac particle's states correspond to a single and very special representation of SO(d,2). This feature is a reflection of the two-time nature of the spacetime that underlies the Dirac particle, as is clear in our formulation. As we will see, the same quantum representation describes many other physical systems by simply choosing other gauges in the two-time spacetime. In this sense, in the two-time spacetime, the Dirac particle is dual to all the other physical systems that we will describe below.

IV. LIGHT CONE GAUGE AND HARMONIC OSCILLATOR

A. Free particle in light cone gauge

Consider the basis $X^M=(X^{+'},X^{-'},X^+,X^-,X^i)$ with the metric η^{MN} taking the values $\eta^{+'-'}=\eta^{+-}=-1$ in the light cone type dimensions, while $\eta^{ij}=\delta^{ij}$ for the remaining $d-2$ space dimensions. Thus one time $X^{0'}$ is a linear combination of $X^{\pm'}$, and the other X^0 is a linear combination of $X^\pm$. The gauge group OSp(1/2) has three bosonic and two fermionic gauge parameters, hence we can make three bosonic and two fermionic gauge choices. We define the light cone gauge as $X^{+'}=1$, $P^{+'}=0$, $X^+=\tau$, and $\psi^{+'}=\psi^0=0$. There is no more gauge freedom left over, so all remaining degrees of freedom are physical. Inserting this gauge into the constraints (17), and solving them, one finds the following components expressed in terms of the remaining independent degrees of freedom $(x^-,p^+,\vec{x}^i,\vec{p}^i,\vec{\psi}^i)$:

$$M=[+',-',+,-,i],$$

$$X^M=[1,(\vec{x}^2/2-\tau x^-),\tau,x^-,\vec{x}^i], \tag{36}$$

$$P^M=\left[0,\left(\vec{x}\cdot\vec{p}-x^-p^+-\frac{\tau\vec{p}^2}{2p^+}\right),p^+,\frac{\vec{p}^2}{2p^+},\vec{p}^i\right], \tag{37}$$

$$\psi^M=\left[0,\vec{x}\cdot\vec{\psi}-\tau\frac{\vec{p}\cdot\vec{\psi}}{p^+},0,\frac{\vec{p}\cdot\vec{\psi}}{p^+},\vec{\psi}^i\right]. \tag{38}$$

One can verify that this gauge corresponds to the free relativistic massless particle, by inserting the gauge fixed form (36) into the action (11). Since all constraints have been solved, the A^{ij},F^i terms drop out, and we get

$$\begin{aligned}S_0&=\int_0^T d\tau\left(\partial_\tau X^MP^N\eta_{MN}+\frac{i}{2}\psi\cdot\partial_\tau\psi+0+0\right)\\&=\int_0^T d\tau\left(\partial_\tau\vec{x}\cdot\vec{p}-\partial_\tau x^-p^+-\frac{\vec{p}^2}{2p^+}+\frac{i}{2}\psi^i\partial_\tau\psi^i\right).\end{aligned} \tag{39}$$

This is the action of the free massless spinning relativistic particle in the light cone gauge, in the first order formalism, with the correct Hamiltonian $p^-=\vec{p}^2/2p^+$. Note that both time coordinates have been gauge fixed, $X^{+'}=1$ and $X^+=\tau$, to describe the free particle. This is the light cone ''time.''

The quantization rules are $[x^-,p^+]=i\eta^{+-}=-i$, $[\vec{x}^i,\vec{p}^j]=i\delta^{ij}$, and $\{\vec{\psi}^i,\vec{\psi}^j\}=\delta^{ij}$. The physical quantum states for $\vec{\psi}$ correspond to the basis for the Clifford algebra (with $d-2$ transverse $\vec{\psi}'s$). These consist of left spinors of dimension $2_L^{(d-2)/2-1}$ and right spinors of dimension $2_R^{(d-2)/2-1}$ in even dimensions

$$\begin{aligned} d&=12{:}16_L\oplus 16_R,\\ d&=10{:}8_L\oplus 8_R,\\ d&=8{:}4_L\oplus 4_R,\\ d&=6{:}2_L\oplus 2_R,\\ d&=4{:}1_L\oplus 1_R. \end{aligned}\tag{40}$$

For odd dimensions one gets the sum of the L and R spinors of the lower even dimension. These are the helicity states for massless fermions from the light cone point of view. For example in four dimensions there is one degree of freedom for a massless left handed ''neutrino'' and one degree of freedom for a massless right handed ''neutrino.''

The SO$(d,2)$ generators of Eq. (13) now take the form (at $\tau=0$)

$$J^{ij}=\vec{x}^i\vec{p}^j-\vec{x}^j\vec{p}^i+S^{ij},\quad \mathrm{SO}(d-2),\tag{41}$$

$$\left.\begin{aligned} &J^{+'-}=\frac{\vec{p}^2}{2p^+},\quad J^{+-}=-\frac{1}{2}(x^-p^++p^+x^-),\\ &J^{-'+}=\frac{1}{2}\vec{x}^2p^+,\quad J^{+'+}=p^+,\\ &J^{+'-'}=\frac{1}{2}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x}-x^-p^+-p^+x^-),\\ &J^{-'-}=\begin{bmatrix} \frac{1}{8p^+}(\vec{x}^2\vec{p}^2+\vec{p}^2\vec{x}^2-2\alpha)\\ -\frac{x^-}{2}(\vec{x}\cdot\vec{p}+\vec{p}\cdot\vec{x})+x^-p^+x^-\\ +\frac{1}{2p^+}(\vec{x}^i\vec{p}^j-\vec{x}^j\vec{p}^i)S^{ij}\end{bmatrix}\end{aligned}\right\}\mathrm{SO}(2,2),\tag{42}$$

$$J^{+'i}=\vec{p}^i,\quad J^{+i}=-\vec{x}^ip^+,\tag{43}$$

$$J^{-i}=x^-\vec{p}^i-\frac{1}{2p^+}\vec{p}^j\vec{x}^i\vec{p}^j-\frac{1}{p^+}S^{ij}\vec{p}_j,\tag{44}$$

$$J^{-'i}=\begin{bmatrix}\frac{1}{2}\vec{x}^j\vec{p}^i\vec{x}^j-\frac{1}{2}\vec{x}\cdot\vec{p}\vec{x}^i-S^{ij}\vec{x}^j\\ -\frac{1}{2}\vec{x}^i\vec{p}\cdot\vec{x}+\frac{1}{2}\vec{x}^i(x^-p^++p^+x^-)\end{bmatrix},\tag{45}$$

where

$$S^{ij}=\frac{1}{2i}[\vec{\psi}^i,\vec{\psi}^j]\tag{46}$$

is a spin operator in the transverse dimensions. The quantum operators are ordered so that all generators J^{MN} are Hermitian. The constant α that appears in $J^{-'-}$ arises due to quantum operator ordering ambiguities. It is fixed to $\alpha=2-d$ by demanding the correct closure for the commutator

$$[L^{-'i},L^{-j}]=i\delta^{ij}L^{-'-},\quad\rightarrow\alpha=2-d.\tag{47}$$

In contrast, in the purely bosonic case we had a similar $\alpha_{\mathrm{bose}}=-1$ [1,2].

The Hilbert space may be labeled by the commuting momentum operators of the free particle as well as its spin in the form of helicity states as given above in Eq. (40)

$$|\vec{p},p^+,p^-=\vec{p}^2/2p^+;\text{helicities}\rangle.\tag{48}$$

This is the free particle Hilbert space, which is complete. It is unitary and has the usual delta function normalization. Wave packets with finite positive norm are constructed as usual, and they correspond to the solutions of the Dirac equation of the previous section written in light cone coordinates. The operators J^{MN} given above act on the states in a natural way and these states form a basis for SO$(d,2)$. The Casimir eigenvalues are easily computed directly by squaring the operators. By using our previous calculation of the purely bosonic case [1,2], and the property $\vec{\psi}\cdot\vec{\psi}=(d-2)/2$, we find (see also below)

$$\frac{1}{2}J^{MN}J_{MN}=-\frac{1}{8}(d+2)(d-1),\tag{49}$$

in agreement with fully covariant quantization and Lorentz covariant quantizations given in the previous sections.

The interpretation of the physics in this basis of SO$(d,2)$ is, of course, the same as the previous section. Next we show that the same construction of SO$(d,2)$ has a different physical interpretation.

B. Harmonic oscillator with spin

The same realization of SO$(d,2)$, with the same eigenvalues of the Casimir operators, is also related to the Harmonic oscillator. In Eq. (42) the SO(2,2) subgroup which is equivalent to $\mathrm{SL}(2,R)_L\otimes\mathrm{SL}(2,R)_R$ has the following generators $G_{0,1,2}^{L,R}$ with the standard algebra $[G_0,G_1]=iG_2$: $[G_0,G_2]=-iG_1$, $[G_1,G_2]=-iG_0$,

$$\mathrm{SL}(2,R)_R:G_2^R=\frac{1}{2}(J^{+'-'}-J^{+-}),\quad G_0^R\pm G_1^R=J^{\pm'\mp},\tag{50}$$

$$\mathrm{SL}(2,R)_L:G_2^L=\frac{1}{2}(J^{+'-'}+J^{+-}),\quad G_0^L\pm G_1^L=J^{\pm'\pm}.\tag{51}$$

Thus the compact generator G_0^R of SL$(2,R)_R$ is given by the harmonic oscillator Hamiltonian

$$G_0^R=\frac{1}{2}(J^{+'-}+J^{-'+})=\frac{\vec{p}^2}{4p^+}+\frac{1}{4}\vec{x}^2p^+. \tag{52}$$

The mass of the harmonic oscillator is $M=2p^+$ and the frequency is $\omega=1/2$. The mass is given by the generator $J^{+'+}=p^+=G_0^L+G_1^L$ of SL$(2,R)^L$.

Even though the particle has spin degrees of freedom, the harmonic oscillator Hamiltonian is independent of spin. For a fixed mass $M=2p^+$, its quantum eigenstates $|p^+,E_n,l,s,j\rangle$ are labeled with the eigenvalues of energy $E=G_0^R$, orbital and spin angular momentum l,s and/or j for total SO$(d-2)$ spin J^{ij}. Of course, from the solution of the harmonic oscillator quantum mechanics in $d-2$ space dimensions, we already know that the energy quantum numbers should be $E=\omega[n+\frac{1}{2}(d-2)]=n/2+\frac{1}{4}(d-2)$, with the angular momentum also determined:

$$n=0,1,2,\ldots, \tag{53}$$

$$l=n,(n-2),(n-4),\ldots,(0 \;\text{ or }\; 1). \tag{54}$$

The degeneracy of the state at level n corresponds to an SU$(d-2)$ multiplet described by a single row Young tableau with n boxes, times the degeneracy of the spin states which is the same at every level. This Young tableau decomposed under SO$(d-2)$ gives completely symmetric traceless tensors with l indices $T_{i_1i_2\cdots i_l}(\vec{x})$, with the values of l indicated above. The total J^{ij} SO$(d-2)$ spin j is obtained by combining the orbital and spin parts for SO$(d-2)$. To make a connection to the group theory below it is useful to rewrite $n=l+2n_r$ where both l,n_r are positive integers, and n_r has the meaning of radial quantum number. So we may write the energy eigenvalue in the form

$$E=G_0^R=\frac{1}{4}(d-2)+\frac{l}{2}+n_r. \tag{55}$$

Now we explain how these harmonic oscillator quantum numbers fully label the same unique representation of SO$(d,2)$, and how the full set of harmonic oscillator states at all energy levels provide a *single* irreducible representation. The key here is that the mass $2p^+$ as well as the spin are labels of the representation and they must transform under SO$(d,2)$. In this sense the mass is the analogue of a modulus parameter that transforms under duality. Furthermore, the choice of G_0^R as Hamiltonian implies a different choice of time as embedded in $d+2$ dimensions, as compared to the free particle time.

A basis for the group theory representation space is labeled by the SO$(d,2)$ Casimir eigenvalues, and the SO$(d-2)\otimes$SL$(2,R)_L\otimes$SL$(2,R)_R$ subgroups

$$|\text{Casimir eigenvalues;SO}(d-2);\text{SL}(2,R)_L;\text{SL}(2,R)_R\rangle$$
$$=|\text{Casimir eigenvalues};l,s;j_Lp^+;j_Rm_R\rangle. \tag{56}$$

The SL$(2,R)_L\otimes SL(2,R)_R$ subspace is labeled by $|j_Lp^+;j_Rm\rangle$, where m is the eigenvalue of the compact generator of SL$(2,R)_R$ that coincides with the Hamiltonian $G_0^R=m=E$, and p^+ is the eigenvalue of the SL$(2,R)_L$ generator $J^{+'+}=G_0^L+G_1^L=p^+$. We will compare these quantum numbers to those of the harmonic oscillator given above.

First we compare m to the energy eigenvalue E. The quantum number m is determined from representation theory of SL$(2,R)$. Since G_0^R is a positive operator, the only possible representation is the positive discrete series, for which $E=G_0^R=m=j_R+1+n_r$ with $n_r=0,1,2,\ldots$. There remains to show that j_R+1 is the remaining part of Eq. (55), which we will do below.

The SO$(d-2)$ quantum numbers (l,s) are determined by orbital l and spin quantum numbers s. In the construction of SO$(d,2)$ given in Eq. (42) orbital angular momentum L^{ij} can only have representations labeled by integers l that corresponds to the completely symmetric traceless tensor $T_{i_1i_2\cdots i_l}(\vec{x})$ with l indices in $(d-2)$ dimensions. Similarly s is limited to the spinor representations listed in Eq. (40). The direct product of these representations is what is symbolized by the quantum numbers (l,s). So, these are the same angular momentum labels as the harmonic oscillator.

There remains to specify the values of j_L,j_R. They are computed through the Casimir operators

$$j_{L,R}(j_{L,R}+1)=(G_0^{L,R})^2-(G_1^{L,R})^2-(G_2^{L,R})^2. \tag{57}$$

Using the $\vec{x},\vec{p},\vec{\psi}$ representation for $G_{0,1,2}^{L,R}$ given in Eqs. (42), (50) we find that they $j_{L,R}$ are not independent of the orbital and spin angular momenta

$$j_R(j_R+1)=\frac{1}{8}L_{ij}L^{ij}+\frac{1}{16}(d-2)(d-6), \tag{58}$$

$$j_L(j_L+1)=\frac{1}{8}L_{ij}L^{ij}+\frac{1}{16}(d-2)(d-6)+\frac{1}{2}L_{ij}S^{ij}$$
$$=\frac{1}{4}J_{ij}J^{ij}-\frac{1}{8}L_{ij}L^{ij}-\frac{3}{16}(d-2). \tag{59}$$

The allowed eigenvalues for SO$(d-2)$ orbital angular momentum are $\frac{1}{2}L_{ij}L^{ij}=l(l+d-4)$ (completely symmetric traceless tensor with l indices in $d-2$ dimensions), and the allowed values of SO$(d-2)$ spin are $\frac{1}{2}S_{ij}S^{ij}=\frac{1}{8}(d-2)(d-3)$ [from $\vec{\psi}\cdot\vec{\psi}=(d-2)/2$]. From these we deduce the allowed values of j_R,j_L and SO$(d-2)$ total angular momentum j,

$$j_R(j_R+1)=\frac{1}{4}l(l+d-4)+\frac{1}{16}(d-2)(d-6) \tag{60}$$

gives

$$j_R=\frac{1}{2}l+\frac{1}{4}d-\frac{3}{2},\quad l=0,1,2,\ldots. \tag{61}$$

Similarly, we obtain j_L, for $d=5$,

$$j_{L_{d=5}}0=\begin{cases} d=5:\mathrm{SO}(3), s=\pm 1/2,\\ \frac{1}{2}J_{ij}J^{ij}=j(j+1),\\ j=l\pm 1/2,\quad j=\frac{1}{2}^{+},\frac{3}{2}^{+},\frac{3}{2}^{-},\frac{5}{2}^{+},\frac{5}{2}^{-},\ldots,\\ j_L(j_L+1)=\frac{1}{2}(l+s)(l+s+1)-\frac{1}{4}l(l+1)-\frac{9}{16},\\ j_L=-\frac{1}{2}+\frac{1}{2}\sqrt{\left(j+\frac{1}{2}\right)\left(j+\frac{1}{2}\pm 1\right)-2}\end{cases} \tag{62}$$

and $d=6$

$$j_{L_{d=6}}=\begin{cases} d=6:\mathrm{SO}(4)=\mathrm{SU}(2)_L\otimes \mathrm{SU}(2)_R, s_{L,R}=\pm 1/2,\\ \frac{1}{2}J_{ij}J^{ij}=2j_1(j_1+1)+2j_2(j_2+1),\\ (j_1,j_2)=\left(\frac{l}{2}\pm\frac{1}{2},\frac{l}{2}\right)\oplus\left(\frac{l}{2},\frac{l}{2}\pm\frac{1}{2}\right)\\ =\left(j,j\mp\frac{1}{2}\right)\oplus\left(j\mp\frac{1}{2},j\right) j=\frac{1}{2},1,\frac{3}{2},2,\ldots,\\ j_L(j_L+1)=\left(\frac{l}{2}+s\right)\left(\frac{l}{2}+s+1\right)-\frac{3}{4},\\ j_L(j_L+1)=j(j+1)-\frac{3}{4}\\ j_L=-\frac{1}{2}+\frac{1}{2}\sqrt{(2j+1)^2-3}.\end{cases} \tag{63}$$

For other values of d the computation of j_L is a technical matter. This verifies that the energy eigenvalue and other quantum numbers coincide with the group theoretical representation labels. The only independent labels are those of the harmonic oscillators, including mass and spin, while the remaining group theory labels are determined by them. Among the group theory labels we must include the mass $M=2p^+$.

The realization of SO(d,2) on this harmonic oscillator system is quite nontrivial. As already verified, the Casimir eigenvalues for SO(d,2) are the ones determined by OSp(1/2) gauge invariance in Eq. (28). The choice of time as embedded in $d+2$ dimensions has a different topology than the free particle. The quantum space is dual to the free particle, while both systems represent the same two-time quantum theory in unitarily equivalent bases.

V. ''H ATOM'' WITH SPIN

The free Dirac particle may be described in the $x^0=\tau$ gauge (see next section) instead of the $x^+=\tau$ gauge of the previous section. To describe the H atom we take a gauge that is dual to the free Dirac particle. The duality relation to the free particle is obtained by flipping the roles of $\mathbf{r},\mathbf{p}$ followed by a discrete Sp(2) transformation. The resulting gauge is (at fixed time $\tau=0$) $X^{+'}=0$, $P^{+'}=1$, $P^0=0$, $\psi^{+'}=0$:

$$M=(+',-',0,i),$$

$$X^M=(0,\mathbf{r}\cdot\mathbf{p},r,\mathbf{r}^i), \tag{64}$$

$$P^M=\left(1,\frac{\mathbf{p}^2}{2},0,\mathbf{p}^i\right), \tag{65}$$

$$\psi=\left(0,\psi\cdot\mathbf{p},\frac{1}{r}\psi\cdot\mathbf{r},\psi^i\right), \tag{66}$$

where $i=1,2,\ldots,(d-1)$. All constraints, $X^2=P^2=X\cdot P=X\cdot\psi=\psi\cdot P=0$, are explicitly solved. The generators of the conformal group (13) take the form (recall $\eta^{+'-'}=\eta^{00}=-1$)

$$J^{-'+'}=\frac{1}{2}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}), \tag{67}$$

$$J^{0+'}=r,\quad J^{i+'}=\mathbf{r}^i, \tag{68}$$

$$J^{0-'}=\frac{1}{2}\mathbf{p}^i r\mathbf{p}^i+\frac{a}{r}+\frac{1}{2r}\mathbf{S}_{ij}\mathbf{L}^{ij}, \tag{69}$$

$$J^{i-'}=-\frac{1}{2}\mathbf{p}\cdot\mathbf{r}\mathbf{p}^i-\frac{1}{2}\mathbf{p}^i\mathbf{r}\cdot\mathbf{p}+\frac{1}{2}\mathbf{p}^j\mathbf{r}^i\mathbf{p}^j+b\frac{\mathbf{r}^i}{r^2}+\mathbf{S}^{ij}\mathbf{p}_j, \tag{70}$$

$$J^{i0}=-\frac{1}{2}(r\mathbf{p}^i+\mathbf{p}^i r)+\frac{1}{r}\mathbf{S}^{ij}\mathbf{r}_j, \tag{71}$$

$$J^{ij}=\mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i+\mathbf{S}^{ij}, \tag{72}$$

where

$$\mathbf{S}^{ij}=\frac{1}{2i}(\psi^i\psi^j-\psi^j\psi^i). \tag{73}$$

As in the purely bosonic case [2] there are ordering ambiguities represented by the constants a,b that appear in $J_{0+'}$ and $J_{+'i}$ respectively. By using the basic commutation relations among $(\mathbf{r},\mathbf{p},\psi)$ one can check that the SO(d,2) commutation relations are indeed satisfied for any a, while b is fixed to $b=-a$ by demanding correct closure for the commutator

$$[J^{0-'},J^{i0}]=-iJ^{i-'}\rightarrow b=-a. \tag{74}$$

In contrast, in the purely bosonic case we had $b_{\mathrm{bose}}=-a_{\mathrm{bose}}-(d-2)/4$. The remaining parameter a will be fixed by the OSp(1/2) gauge invariance, not by the SO(d,2) algebra, as will be discussed below.

It is evident that the operators $\mathbf{J}^{ij}$ form the algebra of the rotation subgroup SO$(d-1)$. Its quadratic Casimir operator is given by

$$\frac{1}{2}\mathbf{J}_{ij}\mathbf{J}^{ij}=\frac{1}{2}\mathbf{L}_{ij}\mathbf{L}^{ij}+\mathbf{L}_{ij}\mathbf{S}^{ij}+\frac{1}{2}\mathbf{S}_{ij}\mathbf{S}^{ij}$$
$$\equiv(\mathbf{r}^j\mathbf{p}^2\mathbf{r}^j-\mathbf{r}\cdot\mathbf{p}\mathbf{p}\cdot\mathbf{r})+\mathbf{L}_{ij}\mathbf{S}^{ij}+\frac{1}{4}[2(\psi\cdot\psi)^2-\psi\cdot\psi]. \tag{75}$$

Similarly, the following three operators form a SO(1,2) subalgebra:

$$J^{-'+'}\equiv J_2,\quad J^{0-'}\equiv\frac{1}{2}(J_0+J_1),\quad J^{0+'}\equiv J_0-J_1,$$
$$J_2=\frac{1}{2}(\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}),\quad (J_0+J_1)=\mathbf{p}^i r\mathbf{p}^i+\frac{2a}{r}+\frac{1}{r}\mathbf{S}_{ij}\mathbf{L}^{ij},$$
$$J_0-J_1=r. \tag{76}$$

For any a they close correctly

$$[J_0,J_1]=iJ_2,\quad [J_0,J_2]=-iJ_1,\quad [J_1,J_2]=-iJ_0. \tag{77}$$

The compact generator J_0 is given in terms of the canonical operators as

$$J_0=J^{0-'}+\frac{1}{2}J^{0+'}=\frac{1}{2}\mathbf{p}^i r\mathbf{p}^i+\frac{a}{r}+\frac{r}{2}+\frac{1}{2r}\mathbf{S}_{ij}\mathbf{L}^{ij}. \tag{78}$$

The quadratic Casimir operator for this subalgebra takes the form

$$J(J+1)=J_0^2-J_1^2-J_2^2=J^{0-'}J^{0+'}+J^{0+'}J^{0-'}-(J^{-'+'})^2$$
$$=\frac{1}{2}\mathbf{L}_{ij}\mathbf{L}^{ij}+\mathbf{S}_{ij}\mathbf{L}^{ij}+\frac{1}{4}(d-2)^2-\frac{1}{4}+2a$$
$$=\frac{1}{2}\mathbf{J}_{ij}\mathbf{J}^{ij}-\frac{1}{2}\mathbf{S}^{ij}\mathbf{S}_{ij}+\frac{1}{4}(d-2)^2-\frac{1}{4}+2a$$
$$=\frac{1}{2}\mathbf{J}_{ij}\mathbf{J}^{ij}+\frac{1}{8}(d-1)(d-4)+2a, \tag{79}$$

where we have used $\frac{1}{2}\mathbf{S}^{ij}\mathbf{S}_{ij}=\frac{1}{2}(\psi\cdot\psi)^2-\frac{1}{4}\psi\cdot\psi$ and $\psi\cdot\psi=(d-1)/2$. We see that the quadratic Casimir operators of the SO(1,2) subalgebra and that of the rotation subgroup SO$(d-1)$ are related to each other in this representation of SO$(d,2)$. The overall quadratic Casimir operator for SO$(d,2)$ may now be evaluated. All orbital parts $\mathbf{r},\mathbf{p}$ drop out, and the result is

$$C_2=\frac{1}{2}J_{MN}J^{MN}$$
$$=-(J^{-'+'})^2+J^{0-'}J^{0+'}+J^{0+'}J^{0-'}-J^{i-'}J^{i+'}$$
$$-J^{i+'}J^{i-'}-J^{i0}J^{i0}+\frac{1}{2}\mathbf{J}_{ij}\mathbf{J}^{ij}$$
$$=-\frac{1}{8}d^2-\frac{1}{8}d-\frac{3}{4}+4a$$
$$=-\frac{1}{8}(d+2)(d-1)\rightarrow a=\frac{1}{4}. \tag{80}$$

We see that $a=\frac{1}{4}$ is fixed by the requirement of OSp(1/2) gauge invariance (28) that was obtained in covariant quantization. Therefore the last step fixes the values of a and b uniquely in the gauge invariant sector

$$a=-b=\frac{1}{4}. \tag{81}$$

These values correspond to the following quantum ordering of the operators in $J^{0-'}$:

$$J^{0-'}=\frac{1}{2}\mathbf{p}^i r\mathbf{p}^i+\frac{1}{4r}+\frac{1}{2r}\mathbf{S}_{ij}\mathbf{L}^{ij}$$
$$=r^{1/2}\left[\frac{1}{2}\mathbf{p}^2\right]r^{1/2}-\frac{1}{8r}(3-2d)+\frac{1}{2r}\mathbf{S}_{ij}\mathbf{L}^{ij}. \tag{82}$$

We now proceed to solve the system algebraically, and show its relation to the H-atom. A basis for the quantum theory is chosen to diagonalize the Hamiltonian. In our case we will show that this corresponds to the SO$(d,2)$ representation basis labeled by the subgroups

$$|\text{Casimir eigenvalues;SO}(d-1);\text{SO}(1,2)\rangle, \tag{83}$$

and that all the states of the ''H atom'' with spin correspond to a single irreducible representation of SO$(d,2)$, with the Casimir eigenvalues given before by the covariant quantization [i.e., OSp(1/2) gauge invariance].

As explained earlier, since we have two timelike dimensions, the choice of ''time'' corresponds to a choice of Hamiltonian as a combination of the generators of SO$(d,2)$. One such choice is dual to another via OSp(1/2) gauge transformations. We now make the following choice for ''Hamiltonian'' $h=J^{0'0}=J_0$ which is the compact generator of the SO(1,2) subgroup. One way of justifying this gauge choice is the algebraic demonstration below that it corresponds to the $1/r$ potential. Another way is to choose a (canonically related) gauge in which the original action reduces to the interacting system with $1/r$ potential, and then show that $J^{0'0}$ is related to the Hamiltonian. This was done explicitly in [2] for the spinless case. See also the last paragraph of section (VI B). Thus, consider $h=J^{0'0}=J_0$ in the form

$$h=J_0=J^{0-\prime}+\frac{1}{2}J^{0+\prime}$$

$$=r^{1/2}\left[\frac{1}{2}\mathbf{p}^2+\frac{1}{2}-\frac{(3-2d)}{8r^2}+\frac{1}{2r^2}\mathbf{S}_{ij}\mathbf{L}^{ij}\right]r^{1/2}$$

$$=r^{1/2}\left[\begin{array}{c}\frac{1}{2}\left(p_r^2+\frac{1}{2r^2}\mathbf{L}^{ij}\mathbf{L}_{ij}+\frac{1}{4r^2}(d-2)(d-4)\right)\\ +\frac{1}{2}-\frac{(3-2d)}{8r^2}+\frac{1}{2r^2}\mathbf{S}_{ij}\mathbf{L}^{ij}\end{array}\right]r^{1/2}$$

$$=r^{1/2}\left\{\frac{1}{2}\left[p_r^2+\frac{1}{r^2}\left(\frac{1}{2}\mathbf{J}^{ij}\mathbf{J}_{ij}-\frac{1}{2}\mathbf{S}^{ij}\mathbf{S}_{ij}\right.\right.\right.$$

$$\left.\left.\left.+\frac{1}{4}d^2-\frac{4}{4}d+\frac{5}{4}\right)\right]+\frac{1}{2}\right\}r^{1/2}$$

$$=r^{1/2}\left\{\frac{1}{2}\left[p_r^2+\frac{1}{r^2}\left(\frac{1}{2}\mathbf{J}^{ij}\mathbf{J}_{ij}+\frac{1}{8}(d^2-5d+8)\right)\right]+\frac{1}{2}\right\}r^{1/2}$$

$$=r^{1/2}\left[\frac{1}{2}\left(p_r^2+\frac{1}{r^2}J(J+1)\right)+\frac{1}{2}\right]r^{1/2}. \tag{84}$$

Here we have used

$$\mathbf{p}^2=p_r^2+\frac{1}{2r^2}\mathbf{L}^{ij}\mathbf{L}_{ij}+\frac{1}{4r^2}(d-2)(d-4),$$

where

$$p_r=\frac{1}{2r}\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}\,\frac{1}{2r}$$

is the Hermitian radial momentum canonically conjugate to r, and have shown that $J(J+1)$, which is the quadratic Casimir of SO(1,2) as given in Eq. (79), emerges in the radial equation. From here one may proceed in two ways. Either one may solve the radial equation given below directly, or use an algebraic approach. The agreement between the two is a check of our calculation.

We proceed with the algebraic approach. Since $h=J_0$ is a generator of the SO(1,2) algebra it is diagonalized on the usual SO(1,2) basis $|Jm\rangle$ where m is the quantized eigenvalue of the compact generator J_0. Evidently the operator h is positive, therefore m can only be positive. This is possible only in the positive unitary discrete series representation of SO(1,2), and therefore the spectrum of m must be

$$m=J+1+n_r,\quad n_r=0,1,2,\dots\ , \tag{85}$$

where, as we will see shortly, the integer n_r will play the role of the radial quantum number.

Let us now show the relation to the hydrogen atom Hamiltonian. Applying h on these states we have

$$r^{1/2}\left[\frac{1}{2}\mathbf{p}^2+\frac{1}{2}+\cdots\right]r^{1/2}|Jm\rangle=m|Jm\rangle. \tag{86}$$

Multiplying it with the operator $r^{-1/2}$ from the left, this equation is rewritten as

$$\left[\frac{1}{2}\mathbf{p}^2+\frac{1}{2}-\frac{m}{r}+\cdots\right](r^{1/2}|Jm\rangle)=0. \tag{87}$$

We now recognize that the states $|\psi_m\rangle=(r^{1/2}|jm\rangle)$ are eigenstates of the hydrogen atom Hamiltonian. Actually this is a rescaled form of the standard Hamiltonian equation written in terms of dimensionful coordinates and momenta $\tilde{\mathbf{r}},\tilde{\mathbf{p}}$

$$\left[\frac{\tilde{\mathbf{p}}^2}{2M}-\frac{\alpha}{\tilde{r}}+\cdots\right]|\psi_m\rangle=E_m|\psi_m\rangle. \tag{88}$$

The quantized energy is

$$E_n=-\frac{\alpha^2}{m^2}=-\frac{\alpha^2}{(J+1+n_r)^2}. \tag{89}$$

We still need to figure out the values of J as a function of the quantized integers l and n_r. As an example consider $d=4$, $\mathrm{SO}(d-1)=\mathrm{SO}(3)$, for which we are familiar with the use of addition of angular momentum. Adding spin 1/2 to orbital quantum number l, gives $j=l\pm 1/2$ with $l=0,1,2,\dots$. From these we compute $J(j)$ by inserting $j(j+1)=\frac{1}{2}J_{ij}J^{ij}$ in Eq. (79):

$$J(j)=-\frac{1}{2}+\frac{1}{2}\sqrt{(3+4j^2+4j)}\quad j=\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2},\dots$$
$$=-\frac{1}{2}+\frac{1}{2}\sqrt{(6+4l^2+8l)_{j=l+1/2}\ \ or\ \ (2+4l^2)_{j=l-1/2}}. \tag{90}$$

The energy depends on both on j and n_r and therefore there is no accidental degeneracies, such as the SO(4) in 4D. For other dimensions d we compute J by using a similar procedure.

Although we have named this gauge the ''H-atom gauge'' (with quotes) because of the $1/r$ potential, evidently the system does not describe the usual H atom with the usual spin correction, since the spin dependence has a different r dependence than the usual $\mathbf{L}\cdot\mathbf{S}$ correction (here $1/r^2$, whereas the usual correction is $1/r^3$). Nevertheless, as seen below, it is possible to find a gauge that gives any interacting nonrelativistic Hamiltonian, including the correct spin correction for the H atom. However, the SO(d,2) representation becomes considerably more complicated and the quantum ordering issues for all the generators become technically difficult to resolve. Our aim here was to show that for the cases for which we could resolve the quantum ordering, the model does correspond to the same representation of SO(d,2) in all gauges, and hence the corresponding physical systems are dual in this sense in the quantum theory.

VI. ARBITRARY INTERACTIONS AS GAUGE CHOICES

The action in a general gauge is given in Eq. (11). We will show that we can construct any interacting nonrelativistic system in $d-1$ space dimensions, with Hamiltonian of the form

$$H=\frac{\mathbf{p}^2}{2}+V(\mathbf{r},\mathbf{p},S_{ij}), \tag{91}$$

with any potential function V, and SO$(d-1)$ spin $\frac{1}{2}S_{ij}S_{ij}=\frac{1}{8}(d-1)(d-2)$, simply by taking appropriate gauge choices for time. We must emphasize that we expect that there are more gauge choices that would yield other forms of Hamiltonians.

A. Free spinning particle in timelike gauge

First let us remind ourselves of the method by reconsidering the free massless particle of Eq. (29) in the timelike gauge $[x^0(\tau)=\tau]$. The same method will be applied to the more general case. The following parametrization solves all of the constraints (17):

$$M=(+',-',0,i),$$

$$X^M=\left(1,\frac{1}{2}(\mathbf{r}^2-\tau^2),\tau,\mathbf{r}^i\right), \tag{92}$$

$$P^M=(0,\mathbf{r}\cdot\mathbf{p}-|\mathbf{p}|\tau,|\mathbf{p}|,\mathbf{p}^i), \tag{93}$$

$$\psi=\left[0,\left(\mathbf{r}\cdot\psi-\frac{\mathbf{r}\cdot\mathbf{p}}{\mathbf{p}^2}\mathbf{p}\cdot\psi\right),0,\psi^i-\frac{\mathbf{p}^i}{\mathbf{p}^2}\mathbf{p}\cdot\psi\right)+\chi X^M+\xi P^M, \tag{94}$$

where χ,ξ represents fermionic gauge freedom, and they can be chosen at convenience so as to obtain the simplest possible action or SO$(d,2)$ generators. This gauge is OSp(1/2) dual to the H-atom gauge (64) used in the previous section. At $\tau=0$, the duality transformation consists of choosing $\chi=0$ and $\xi=-(1/\mathbf{p}^2)\mathbf{p}\cdot\psi$, plus a discrete Sp(2) transformation that interchanges X^M,P^M, and then renaming $\mathbf{r}\leftrightarrow\mathbf{p}$. By inserting this gauge choice, with $\xi=\chi=0$, into the action (11) we obtain

$$S_0=\int_0^T d\tau\left[P\cdot\partial_\tau X+\frac{i}{2}\psi\cdot\partial_\tau\psi+0+0\right]$$
$$=\int_0^T d\tau\left[\dot{\mathbf{r}}\cdot\mathbf{p}-|\mathbf{p}|+\frac{i}{2}\tilde{\psi}^i\partial_\tau\tilde{\psi}^j\right], \tag{95}$$

where

$$\tilde{\psi}^i=\left(\delta_{ij}-\frac{\mathbf{p}_i\mathbf{p}_j}{\mathbf{p}^2}\right)\psi^j. \tag{96}$$

This action describes the free massless relativistic spinning particle, with Hamiltonian $H=|\mathbf{p}|$. Using Noether's theorem, we see that the generator of rotations is $J^{ij}=L^{ij}+\tilde{S}^{ij}$, where $\tilde{S}^{ij}=1/2i(\tilde{\psi}^i\tilde{\psi}^j-\tilde{\psi}^j\tilde{\psi}^i)$. Only the spin components perpendicular to momentum can appear since $\tilde{\psi}\cdot\mathbf{p}=0$. This is as it should be for a massless particle that has only helicity components. We kept $d-1$ components in ψ^i instead of $d-2$ components that would have been possible by taking the gauge $\psi\cdot\mathbf{p}=0$. The reason is to maintain manifest rotation symmetry SO$(d-1)$, and for this we paid the price of having the projector $\delta_{ij}-\mathbf{p}_i\mathbf{p}_j/\mathbf{p}^2$. This projector appears in the anticommutation relations $\{\tilde{\psi}^i,\tilde{\psi}^j\}=\delta_{ij}-\mathbf{p}_i\mathbf{p}_j/\mathbf{p}^2$. The generators of SO$(d,2)$ are obtained by inserting the gauge choice above into the general expression; obviously the fermions appear only in the form $\tilde{S}^{ij}$.

B. Arbitrary potential

We now show that any interacting system corresponds to another gauge choice, with a rather different topology for embedding time in $(d+2)$ spacetime. Consider the basis $X^M=(X^{0'},X^0,X^I)$ and $P^M=(P^{0'},P^0,P^I)$ with metric $\eta^{0'0'}=\eta^{00}=-1$ and $\eta^{IJ}=\delta^{IJ}$. Choose one gauge such that the four functions $X^{0'},X^0,P^{0'},P^0$ are expressed in terms of three functions F,G,u

$$X^{0'}=F\cos u,\quad X^0=F\sin u, \tag{97}$$

$$P^{0'}=-G\sin u,\quad P^0=G\cos u. \tag{98}$$

Inserting this form in the constraints (17) gives

$$X^M=F[\cos u,\sin u,n^I],$$
$$P^M=G[-\sin u,\cos u,m^I], \tag{99}$$
$$\psi^M=[\psi^{0'},\psi^0,\psi^I],$$

where

$$\psi^{0'}=\cos u\, n\cdot\psi-\sin u\, m\cdot\psi, \tag{100}$$

$$\psi^0=\sin u\, n\cdot\psi+\cos u\, m\cdot\psi, \tag{101}$$

and n^I,m^I are *Euclidean* unit vectors that are orthogonal. We choose the following parametrization for these unit vectors in the basis $I=[1',i]$ where $I=1'$ denotes the extra space dimension and $i=1,2,\ldots,(d-1)$ labels ordinary space:

$$n^I=\left[\frac{1}{rV}\sqrt{-2H}\mathbf{r}\cdot\mathbf{p},\quad\left(\frac{1}{r}\mathbf{r}^i+\frac{\mathbf{r}\cdot\mathbf{p}}{rV}\mathbf{p}^i\right)\right], \tag{102}$$

$$m^I=\left[\left(1+\frac{\mathbf{p}^2}{V}\right),\quad-\sqrt{-2H}\frac{1}{V}\mathbf{p}^i\right],$$

where

$$H=\frac{\mathbf{p}^2}{2}+V, \tag{103}$$

and $V(\mathbf{r},\mathbf{p},\psi)$ is any potential energy function, giving any Hamiltonian. We emphasize that this is the most general solution of the constraints (17) that had taken the form $n^In^I=m^Im^I=1$, and $m^In^I=0$. Even though the solution is expressed with a particular choice of coordinates, and an arbitrary function V, this does not involve a gauge choice. We still have the freedom of choosing two bosonic gauge functions and two fermionic gauge functions. These gauge choices will be made as needed in the discussion below.

Since all the constraints are explicitly solved, the A^{ij},F^i terms drop out in the action (11) and we get

$$S_0=\int_0^T d\tau\left(\partial_\tau X^M P^N+\frac{i}{2}\psi^M\partial_\tau\psi^N+0+0\right)\eta_{MN}$$

$$=\int_0^T d\tau\left[\begin{array}{c} GF(-\partial_\tau u+m^I\partial_\tau n^I)+\frac{i}{2}\psi^I\partial_\tau\psi^I \\ -\frac{i}{2}(m\cdot\psi n\cdot\psi-n\cdot\psi m\cdot\psi)\partial_\tau u \\ -\frac{i}{2}(n\cdot\psi\partial_\tau(n\cdot\psi)+m\cdot\psi\partial_\tau(m\cdot\psi))\end{array}\right] \quad (104)$$

$$=\int_0^T d\tau\left(\mathbf{p}^i\partial_\tau\mathbf{r}^i+\frac{i}{2}\psi^i\partial_\tau\psi^i-H\right).$$

A total derivative $\partial_\tau(-\mathbf{r}\cdot\mathbf{p})$ has been dropped in the last line. To derive the last line we have used

$$m^I\partial_\tau n^I=-\frac{\sqrt{-2H}}{rV}\times[\mathbf{r}\cdot\mathbf{p}\partial_\tau(\ln\sqrt{-2H})-\partial_\tau(\mathbf{r}\cdot\mathbf{p})+\mathbf{p}\cdot\partial_\tau\mathbf{r}], \quad (105)$$

which follows from the m^I,n^I given above, and we have made the following choices of gauges: one bosonic gauge choice

$$GF=-\frac{rV}{\sqrt{-2H}}, \quad (106)$$

to insure that the $\mathbf{p}\cdot\partial_\tau\mathbf{r}$ term is correctly normalized so that the momentum $\mathbf{p}$ is indeed the canonical conjugate to the coordinate $\mathbf{r}$, and another bosonic gauge choice for ''time'' τ

$$u(\tau)=-\int^\tau d\tau'[H+Q]\frac{\sqrt{-2H}}{rV}, \quad (107)$$

where Q is given below, so that the only term containing $\partial_\tau\mathbf{r}$ in the Lagrangian is of the form $\mathbf{p}\cdot\partial_\tau\mathbf{r}$, and the only term containing $\partial_\tau\psi^i$ is of the form $(i/2)\psi^i\partial_\tau\psi^i$. With these gauge choices we insure that the remaining term in the action gives just the Hamiltonian H. We find H in the last line of Eq. (104) provided Q is chosen (i.e., u is chosen) as follows:

$$\left[1-\frac{i}{2}(m\cdot\psi n\cdot\psi-n\cdot\psi m\cdot\psi)\frac{\sqrt{-2H}}{rV}\right]Q=\mathbf{r}\cdot\mathbf{p}\partial_\tau(\ln\sqrt{-2H})+\frac{i}{2}(m\cdot\psi n\cdot\psi-n\cdot\psi m\cdot\psi)\frac{\sqrt{-2H}}{rV}H+\frac{i}{2}\psi^{1'}\partial_\tau\psi^{1'}$$
$$-\frac{i}{2}[n\cdot\psi\partial_\tau(n\cdot\psi)+m\cdot\psi\partial_\tau(m\cdot\psi)]. \quad (108)$$

There still remains freedom to choose fermionic gauges to simplify this expression. For example, one may take

$$\psi^{1'}=\alpha V\psi\cdot\mathbf{r}+\beta\mathbf{p}\cdot\psi, \quad (109)$$

where α,β may be chosen as arbitrary functions. For example, taking $\alpha=0$ and $\beta=-1/\sqrt{-2H}$ gives

$$\psi^{1'}=-m\cdot\psi=-\frac{\mathbf{p}\cdot\psi}{\sqrt{-2H}},\quad n\cdot\psi=\psi\cdot\mathbf{r}/r, \quad (110)$$

and this simplifies the SO(d,2) generators (see below). Another choice that simplifies generators is $\alpha V=\sqrt{-2H}/\mathbf{r}\cdot\mathbf{p}$ and $\beta=\sqrt{-2H}/V$. This simplifies also Q by giving $(m\cdot\psi n\cdot\psi-n\cdot\psi m\cdot\psi)=0$.

The last form of the action (104) is the first order formalism, with the Hamiltonian given in Eq. (103). This form shows that the unconstrained variables $(\mathbf{r}^i,\mathbf{p}^i)$ and ψ^i are the standard canonical variables. The middle line of Eq. (104) shows that the system in $(d-1)$ space dimensions has SO(d) dynamical symmetry. The first line shows that the **H** atom has a dynamical symmetry SO(d,2) which mixes the two timelike coordinates with the d space coordinates.

Note that time τ is embedded in the $(d+2)$-dimensional spacetime in a rather complicated way as given through Eqs. (97), (106)–(108). In this section the $(d+2)$-dimensional X^M space has the topology of $S^2\oplus S^d$. This and other topologies (e.g., as discussed in other sections) are permitted as solutions of the same set of constraints that followed from the action (11). The detailed parametrization of S^2 and S^d involves the potential V as well as phase space $(\mathbf{r},\mathbf{p})$, and such details affect the choice of time through equations (107), (108). Conversely, one may view the presence of the potential V as a result of the gauge choice for time. Thus, the topology of the $(d+2)$-dimensional space, as well as the geometry of its phase space in $d+1,d,d-1$ dimensions are equivalent to the presence of forces that are represented by the potential V. In some sense, embedding time as a curve in $d+2$ dimensions, and then arranging the evolution of the system as a function of this curve, corresponds to a Hamiltonian with a potential V. Thus the choice of the time curve is equivalent to the choice of the Hamiltonian. So a specific V corresponds to a specific time curve. Changing the time curve changes the interaction.

The generators of SO$(d,2)$ may now be constructed for any interacting system with spin. All we need to do is to insert the gauge fixed form for X^M, P^M, ψ^M at $\tau=0$ (or $u=0$) into 13. In the classical version, in which operator ordering is not taken into consideration, we obtain for any choice of $\psi^{1'}$

$$J^{0'0}=\frac{-rV}{\sqrt{-2H}}+S^{IJ}n_I m_J,$$

$$J^{IJ}=\frac{-rV}{\sqrt{-2H}}(n^I m^J-n^J m^I)+S^{IJ}, \quad (111)$$

$$J^{0'I}=\frac{-rV}{\sqrt{-2H}}m^I-S^{IJ}n_J, \quad J^{0I}=\frac{rV}{\sqrt{-2H}}n^I-S^{IJ}m_J. \quad (112)$$

The Casimir operator is expected to be independent of V since its value must be consistent with the gauge invariant treatment of the theory, as in Sec. II, where V does not appear. First note that the orbital parts drop out, and it takes the form

$$C_2=\frac{1}{2}S^{IJ}S_{IJ}+(S^{IJ}n_I m_J)^2-(S^{IJ}n_J)^2-(S^{IJ}m_J)^2=\frac{1}{2}\tilde{S}^{IJ}\tilde{S}_{IJ}. \quad (113)$$

It depends only on the components of spin that are perpendicular to both m_I, n_I, given by $\tilde{S}^{IJ}=(1/2i)[\tilde{\psi}^I,\tilde{\psi}^J]$, with $\tilde{\psi}^I=\psi^I-n^I n\cdot\psi-m^I m\cdot\psi$. In the classical theory fermions square to zero $\tilde{\psi}\cdot\tilde{\psi}=0$, then $C_2=0$, as expected from the classical version Eq. (28). Similarly, for the spinless theory the classical Casimir operator vanishes for any potential V.

When $\psi^{1'}$ is gauge fixed as in Eq. (109) the generators take the form

$$J^{0'0}=\frac{-rV}{\sqrt{-2H}}+\frac{1}{2rV}(\alpha\mathbf{r}\cdot\mathbf{p}-\sqrt{-2H})L_{ij}S^{ij}, \quad (114)$$

$$J^{0'1'}=\frac{-rV}{\sqrt{-2H}}\left(1+\frac{\mathbf{p}^2}{V}\right)+\frac{1}{2}(\beta-\alpha\mathbf{r}\cdot\mathbf{p})L_{ij}S^{ij}, \quad (115)$$

$$J^{0'i}=r\mathbf{p}^i-(1+\mathbf{r}\cdot\mathbf{p}\sqrt{-2H})\frac{S^{ij}\mathbf{r}_j}{r}-\frac{\mathbf{r}\cdot\mathbf{p}}{rV}(1+\beta\sqrt{-2H})S^{ij}\mathbf{p}_j, \quad (116)$$

$$J^{01'}=\mathbf{r}\cdot\mathbf{p}+\frac{1}{2}\alpha\sqrt{-2H}L_{ij}S^{ij}, \quad (117)$$

$$J^{0i}=\frac{V}{\sqrt{-2H}}\left(\mathbf{r}^i+\frac{\mathbf{r}\cdot\mathbf{p}}{V}\mathbf{p}^i\right)-\alpha(\mathbf{p}^2+V)S^{ij}\mathbf{r}_j \quad (118)$$

$$+\frac{1}{V}[\sqrt{-2H}-\beta(\mathbf{p}^2+V)]S^{ij}\mathbf{p}_j, \quad (119)$$

$$J^{1'i}=\frac{1}{V}\sqrt{-2H}\mathbf{r}\cdot\mathbf{p}\mathbf{p}^i-\alpha VS^{ij}\mathbf{r}_j-\beta S^{ij}\mathbf{p}_j \quad (120)$$

$$+\frac{V}{\sqrt{-2H}}\left(1+\frac{\mathbf{p}^2}{V}\right)\left(\mathbf{r}^i+\frac{\mathbf{r}\cdot\mathbf{p}}{V}\mathbf{p}^i\right), \quad (121)$$

$$J^{ij}=L^{ij}+S^{ij}, \quad (122)$$

where

$$L^{ij}=\mathbf{r}^i\mathbf{p}^j-\mathbf{r}^j\mathbf{p}^i, \quad S^{ij}=\frac{1}{2i}[\psi^i,\psi^j]. \quad (123)$$

Note that if we take $\alpha\mathbf{r}\cdot\mathbf{p}-\sqrt{-2H}=0$ and $rV=$constant, we find that the generator $J^{0'0}$ is simply related to the Hamiltonian. This was the case for the Coulomb potential and was used in the H-atom gauge for the choice of Hamiltonian. When V is not the Coulomb potential the relation between the generators J^{MN} and the Hamiltonian is not simple, and then the Hamiltonian is not easily diagonalized by algebraic means.

VII. ANTI–DE SITTER GAUGE

It is also possible to find gauges that correspond to particles in various curved spacetimes. We will ignore the fermions to keep it simple. As an example we consider the AdS spacetime. Consider the basis $X^M=(X^{0'},X^{1'},X^m)$ with $\eta^{1'1'}=-\eta^{0'0'}=1$, and $\eta^{mn}=$Minkowski. The Latin letter m denotes vector components in flat space, and we will reserve the Greek letter μ for vector components in curved space. Choose two gauges $X^{1'}=1$, $P^{1'}=0$, and solve the two constraints $X^2=X\cdot P=0$, and let $X^{0'}(x), X^m(x)$ be given in terms of x^μ in curved space

$$M=(0',1',m),$$

$$X^M=(\pm\sqrt{1+X_m^2(x)},1,X^m(x)), \quad (124)$$

$$P^M=\left(\frac{X^m(x)e_m^\mu(x)p_\mu}{\pm\sqrt{1+X_m^2(x)}},0,e_m^\mu(x)p_\mu\right). \quad (125)$$

Note that $P^2=0$ has not been imposed yet, and there still is one more bosonic gauge freedom. Here $e_m^\mu(x)$ is the inverse of $e_\mu^m(x)$ defined by

$$e_\mu^m(x)=\partial_\mu X^m(x)-X^m(x)\frac{X(x)\cdot\partial_\mu X(x)}{1+X_m^2(x)}. \quad (126)$$

It is designed just such that p_μ has the meaning of canonical momentum when we insert this gauge in the action

$$S_0=\int_0^T d\tau\left(\partial_\tau X^M P^N \eta_{MN}-\frac{1}{2}A^{22}P\cdot P-0-0\right)$$

$$=\int_0^T d\tau\left(\dot{x}^\mu\cdot p_\mu-\frac{1}{2}A^{22}G^{\mu\nu}(x)p_\mu p_\nu\right). \tag{127}$$

The remaining part of the action imposes the constraint $P^2=0$,

$$G^{\mu\nu}(x)p_\mu p_\nu=0, \tag{128}$$

where the inverse metric $G^{\mu\nu}(x)$ follows from Eq. (125)

$$G^{\mu\nu}=e_m^\mu e_n^\nu\left(\eta^{mn}-\frac{X^n X^m}{1+X^2}\right). \tag{129}$$

Taking its inverse one finds $G_{\mu\nu}$

$$G_{\mu\nu}=e_\mu^m e_\nu^n(\eta_{nm}+X_n X_m)=\partial_\mu X\cdot\partial_\nu X-\frac{(X\cdot\partial_\mu X)(X\cdot\partial_\nu X)}{1+X^2}. \tag{130}$$

It turns out that this coincides with the metric obtained from the two conditions $X^2=0$ and $ds^2=dX\cdot dX=dx^\mu dx^\nu G_{\mu\nu}(x)$, using any parametrization for $X^{0'}(x),X^\mu(x)$, in the gauge $X^{1'}=1$:

$$G_{\mu\nu}(x)=\partial_\mu X^m(x)\partial_\nu X^n(x)\eta_{mn}-\partial_\mu X^{0'}(x)\partial_\nu X^{0'}(x), \tag{131}$$

$$-1=X^m(x)X^n(x)\eta_{mn}-X^{0'}(x)X^{0'}(x). \tag{132}$$

The last form (131) makes it evident that this is the AdS metric in $(d-1,1)$ dimensions. To construct it explicitly, one may choose any convenient function for $X^m(x)$, find the corresponding $X^{0'}(x)$ and insert it into Eq. (131). See below for some examples.

In the quantum theory the constraint is imposed on states $|\phi\rangle$. It is useful to consider the field $\phi(x)=\langle x|\phi\rangle$. The constraint equation becomes a differential equation on this field. The operators involved in the constraint must be ordered. A natural ordering corresponds to the Laplacian condition

$$\frac{1}{\sqrt{-G}}\partial_\mu[\sqrt{-G}G^{\mu\nu}(x)\partial_\nu\phi(x)]=0. \tag{133}$$

The effective field theory that gives this equation is

$$S_{\rm eff}=\frac{1}{2}\int d^d x\sqrt{-G}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{134}$$

We have seen that the OSp(1/2) gauge covariant action (11) is capable of describing curved spacetime as well. The underlying reason for this is the ability to choose time as a gauge in nonunique ways because we have more than one timelike coordinate in the $d+2$ dimensional spacetime. For each choice of time embedded in $d+2$ dimensions the corresponding canonical Hamiltonian looks different. In particular, the topology and geometry of the embedding in $d+2$ dimensions is different than the previous cases. Nevertheless these systems are OSp(1/2) gauge equivalent, or dual to each other, since they all correspond to the same action and same representation of SO$(d,2)$.

A. Case No. 1 for $d=2$

Consider the following AdS parametrization for $d=2$, which solves all the constraints $X^2=P^2=X\cdot P=X\cdot\psi=P\cdot\psi=0$ (including fermions) and gives an explicit metric. Here $\varepsilon=\mathrm{sgn}(p)=\pm 1$ is present to insure that the Hamiltonian is positive (see below). We can still choose two fermionic gauges, such as $\xi=\chi=0$ which make ψ^M trivial, however, we will not choose a fermionic gauge yet and see that at the end, gauge invariant quantities, such as the Hamiltonian and the conformal generators, do not depend on these fermions at the classical level (but they do at the quantum level as we will see):

$$M=[0',1',0,1],$$

$$X^M=[\varepsilon\cosh x\cos t,1,\varepsilon\cosh x\sin t,\sinh x], \tag{135}$$

$$P^M=\begin{bmatrix}\varepsilon_p\sinh x\cos t & & \varepsilon_p\sinh x\sin t & \\ & ,0, & & ,p\cosh x\\ -p\sin t & & +p\cos t & \end{bmatrix} \tag{136}$$

$$\psi^M=\begin{bmatrix}\varepsilon\xi\cos t & & \varepsilon\xi\sin t & \\ & \xi\cosh x-\chi\sinh x & & \\ +\xi\sinh x\sin t, & & ,-\xi\sinh x\cos t,\chi & \\ -\chi\cosh x\sin t & & +\chi\cosh x\cos t & \end{bmatrix}. \tag{137}$$

We need to evaluate the derivatives

$$dX^M = \begin{bmatrix} -\varepsilon\, dt\ \cosh x\ \sin t & & \varepsilon\, dt\ \cosh x\ \cos t & \\ & ,0, & & ,dx\ \cosh x \\ +\varepsilon p\, dx\ \sinh x\ \cos t & & +\varepsilon\, dx\ \sinh x\ \sin t & \end{bmatrix}, \tag{138}$$

$$d\psi^M|_{t=0} = \begin{bmatrix} \xi\, dt\ \sinh x & \xi\, dx\ \sinh x & \varepsilon \xi\, dt - \xi\, dx\ \cosh x & \\ , & , & & ,0 \\ -\chi\, dt\ \cosh x & -\chi\, dx\ \cosh x & +\chi\, dx\ \sinh x & \end{bmatrix}$$
$$+\begin{bmatrix} & d\xi\ \cosh x & -d\xi\ \sinh x & \\ \varepsilon\, d\xi, & , & & ,d_\chi \\ & -d\chi\ \sinh x & +d\chi\ \cosh x & \end{bmatrix}. \tag{139}$$

The metric in (t,x) space is obtained by computing

$$ds^2 = (dX)^2 = -dt^2\ \cosh^2 x + (dx)^2, \tag{140}$$

$$\psi \cdot d\psi = (\xi\chi - \chi\xi)(dt\ \varepsilon\ \cosh x - dx). \tag{141}$$

Although we have used $d\psi^M|_{t=0}$ in this computation for convenience, the result is valid for any t. This form also gives the Lagrangian in the second order form

$$L = \frac{1}{2A^{22}}(\partial_\tau X)^2 + \frac{i}{2}\psi \cdot \partial_\tau \psi$$
$$= \frac{1}{2A^{22}}[-(\partial_\tau t)^2\ \cosh^2 x + (\partial_\tau x)^2]$$
$$+ s(-\partial_\tau t \varepsilon\ \cosh x + \partial_\tau x), \tag{142}$$

where

$$s = \frac{1}{2i}(\xi\chi - \chi\xi). \tag{143}$$

Note that there is no kinetic term for the fermions ξ,χ hence they are not dynamical, and we will see that they are just gauge freedom. Alternatively, after making the gauge choice $t(\tau)=\tau$, the Lagrangian in the first order formalism is given directly by Eq. (11)

$$L = \partial_\tau X \cdot P + \frac{i}{2}\psi \cdot \partial_\tau \psi + 0 + 0$$
$$= \dot{x}(p+s) - H, \tag{144}$$

where

$$H = (p+s)\varepsilon\ \cosh x = |p+s| \cosh x. \tag{145}$$

The true canonical momentum is $P=p+s$, and to insure positivity of the Hamiltonian we choose ε to be

$$\varepsilon = \mathrm{sgn}(p+s). \tag{146}$$

We see that s is completely absorbed into the definition of the canonical momentum and it disappeared from the gauge invariant Hamiltonian. This means that we could have taken $s=0$ from the beginning as a gauge choice, thus preserving the definition of the canonical momentum as $P=p$.

The SO(2,2) generators are evaluated by inserting the gauge choice into the general expression at $\tau = t = 0$. If s is allowed from the beginning we find that it appears everywhere in the combination $p+s$. So, we set $s=0$ as a gauge choice. The result is

$$J^{0'0} = |p| \cosh x,\ \ J^{0'1'} = -|p| \sinh x,\ \ J^{0'1} = |p|, \tag{147}$$

$$J^{1'1} = p\ \cosh x,\ \ J^{01} = -p\ \sinh x,\ \ J^{1'0} = p. \tag{148}$$

These satisfy the SO(2,2) algebra at the classical level. By taking linear combinations we may construct the SO(2,2) $=\mathrm{SL}(2,R)_L \otimes SL(2,R)_R$ generators $J^{L,R}_{0,1,2}$ in the following form:

$$J_0^L \pm J_1^L = \frac{1}{2}(p - |p|)e^{\mp x},\ \ J_2^L = \frac{1}{2}(p - |p|), \tag{149}$$

$$J_0^R \pm J_1^R = \frac{1}{2}(p + |p|)e^{\mp x},\ \ J_2^R = \frac{1}{2}(p + |p|). \tag{150}$$

We see that either the left moving or the right moving generators must vanish in momentum space (but not in x space or other quantum space). The quadratic Casimir operators for both $\mathrm{SL}(2,R)_{L,R}$ vanish at the classical level:

$$C_2^{L,R} = (J_0^{L,R} + J_1^{L,R})(J_0^{L,R} - J_1^{L,R}) - (J_2^{L,R})^2 = 0. \tag{151}$$

B. Quantum ordering case No. 1

We now need to order the operators at the quantum level and make sure that the Casimir operator is consistent with the gauge invariance requirements at the quantum level. Recall that for the purely bosonic system Sp(2) gauge invariance we must have $C_2(\mathrm{SO}(d,2))=1-d^2/4$ and for the OSp(1/2) gauge invariance we must have $C_2(\mathrm{SO}(d,2))=-\frac{1}{8}(d+2)(d-1)$. For our case $d=2$ we must have $C_2(\mathrm{SO}(2,2))=0$ for the purely bosonic and $C_2(\mathrm{SO}(2,2))=-1/2$ for the fermionic cases. We see that the fermions must play a role.

Let us first deal with the purely bosonic case. For either the left or right movers we need Hermitian generators. There is ambiguity in the quantum ordering as illustrated by the following possible Hermitian quantum ordering of the classical $e^x p$:

$$e^{x/2}pe^{x/2},\;\; p^{1/2}e^xp^{1/2},\;\; p^{\lambda}e^{x/2}p^{1-2\lambda}e^{x/2}p^{\lambda},\ldots, \tag{152}$$

and similarly for $e^{-x}p$ ordering. If one reorders these to the first form we find

$$e^xp\rightarrow p^{\lambda}e^{x/2}p^{1-2\lambda}e^{x/2}p^{\lambda}=e^{x/2}\left(p^2+\frac{1}{4}\right)^{\lambda}p^{1-2\lambda}e^{x/2}, \tag{153}$$

$$e^{-x}p\rightarrow p^{\lambda'}e^{-x/2}p^{1-2\lambda'}e^{-x/2}p^{\lambda'} =e^{-x/2}\left(p^2+\frac{1}{4}\right)^{\lambda'}p^{1-2\lambda'}e^{-x/2}. \tag{154}$$

We may also take λ,λ' different from each other. In fact we find that as long as

$$\lambda+\lambda'=1 \tag{155}$$

the quantum ordered generators close correctly and they give the Casimir operator $C_2=0$. Thus, let us take $\lambda=\frac{1}{2}+\alpha$ and $\lambda'=\frac{1}{2}-\alpha$. Then we have

$$J_0\pm J_1=e^{\mp x/2}\left(p^2+\frac{1}{4}\right)^{(1/2)\mp\alpha}p^{\pm 2\alpha}e^{\mp x/2},\;\; J_2=p \tag{156}$$

with commutation rules

$$\begin{aligned}[J_0+J_1,J_0-J_1]&=e^{-x/2}\left(p^2+\frac{1}{4}\right)e^{x/2}-e^{x/2}\left(p^2+\frac{1}{4}\right)e^{-x/2}\\ &=\left(p-\frac{i}{2}\right)^2-\left(p+\frac{i}{2}\right)^2=-2ip\\ &=-2iJ_2\end{aligned} \tag{157}$$

and Casimir operator

$$\begin{aligned}C(\mathrm{SL}(2,R))&=\frac{1}{2}(J_0+J_1)(J_0-J_1)\\ &\quad+\frac{1}{2}(J_0-J_1)(J_0+J_1)-(J_2)^2\\ &=\frac{1}{2}e^{-x/2}\left(p^2+\frac{1}{4}\right)e^{x/2}\\ &\quad+\frac{1}{2}e^{x/2}\left(p^2+\frac{1}{4}\right)e^{-x/2}-p^2\\ &=\frac{1}{2}\left(p-\frac{i}{2}\right)^2+\frac{1}{8}+\frac{1}{2}\left(p+\frac{i}{2}\right)^2+\frac{1}{8}-p^2\\ &=0.\end{aligned} \tag{158}$$

In particular the values $\alpha=0,1/2$ yield interesting looking generators:

$$\alpha=0{:}\,J_0\pm J_1=e^{\mp x/2}\left(p^2+\frac{1}{4}\right)^{1/2}e^{\mp x/2},\;\; J_2=p, \tag{159}$$

$$\alpha=1/2{:}\begin{cases}J_0+J_1=e^{-x/2}pe^{-x/2},\;\; J_2=p,\\ J_0-J_1=e^{x/2}\left(p+\dfrac{1}{4p}\right)e^{x/2}.\end{cases} \tag{160}$$

There is no way to decide which of these versions one should use for our problem.

Next we return to the spinning case. The following modification of the purely bosonic generators give the desired result for the $\alpha=0,1/2$ cases:

$$\alpha=0{:}\,J_0\pm J_1=e^{\mp x/2}\left[\left(p^2+\frac{1}{4}\right)^{1/2}\pm\gamma\right]e^{\mp x/2},\quad J_2=p, \tag{161}$$

$$\alpha=1/2{:}\begin{cases}J_0+J_1=e^{-x/2}pe^{-x/2},\;\; J_2=p+\gamma,\\ J_0-J_1=e^{x/2}\left(p+\dfrac{1}{4p}+2\gamma\right)e^{x/2}.\end{cases} \tag{162}$$

Then the $\mathrm{SL}(2,R)$ algebra closes correctly and the Casimir operator is

$$C_2=-\gamma^2. \tag{163}$$

The choice $\gamma^2=1/2$ matches the spinning case.

It may be of interest to note the following more general construction of $\mathrm{SL}(2,R)$. Instead of the form parametrized by α we can use a more general function $F(p)$

$$J_0\pm J_1=e^{\mp x/2}\left[\left(p^2+\frac{1}{4}\right)^{1/2}\right]F^{\pm 1}e^{\mp x/2},\;\; J_2=p \tag{164}$$

with Casimir operator $C_2=0$. Some choices of $F(p)$ are interesting. For example, taking $F=(p^2+\frac{1}{4})^{-1/2}$ yields

$$J_0+J_1=e^{-x},\ \ J_0-J_1=e^{x/2}\left(p^2+\frac{1}{4}\right)e^{x/2},\ \ J_2=p. \tag{165}$$

It is interesting to note that we can find a gauge choice for X^M,P^M that yields these SO(2,2) generators at $\tau=0$, namely, in Eqs. (135)–(137) replace everywhere $\cosh x$ by $c(x,p)=(1/2p)(p^2e^x+e^{-x})$, and $\sinh x$ by $s(x,p)=(1/2p)(p^2e^x+e^{-x})$ and then proceed the same way. Since $c^2(x,p)-s^2(x,p)=1$, the computation produces similar expressions, ending with the form (165). Finally, this may be modified with a parameter γ

$$J_0+J_1=e^{-x},\ \ J_0-J_1=e^{x/2}\left(p^2+\frac{1}{4}-\gamma^2\right)e^{x/2},\ \ J_2=p \tag{166}$$

to yield the Casimir operator $C_2=-\gamma^2$.

C. Case No. 2 for $d=2$

Consider the following AdS parametrization for $d=2$, which solves all the constraints. By using similar methods to case No. 1 we compute the metric, action, and SO(2,2) generators:

$$M=[0',1',0,1],$$

$$X^M=[-\csc x\ \cos t,1,-\csc x\ \sin t,-\cot x], \tag{167}$$

$$P^M=\begin{bmatrix} |p|\sin x\ \sin t & & -|p|\sin x\ \cos t & \\ & ,0, & & ,p \\ +p\ \cos x\ \cos t, & & +p\ \cos x \sin t & \end{bmatrix}, \tag{168}$$

$$\psi^M=\xi X^M+\chi\frac{P^M}{p}. \tag{169}$$

The metric is

$$ds^2=(dX)^2=\frac{1}{\sin^2 x}(-dt^2+dx^2), \tag{170}$$

$$\psi\cdot d\psi=s(-\varepsilon\ dt+dx), \tag{171}$$

where $s=(1/2i)(\xi\chi-\chi\xi)$. The quantity s is absorbed into the definition of true canonical momentum and it disappears. Thus we take it $s=0$ from the beginning, and compute the Lagrangian

$$L=\partial_\tau X\cdot P+\frac{i}{2}\psi\cdot\partial_\tau\psi+0+0$$

$$=\dot{x}p-H\ \ \text{with}\ \ H=|p|. \tag{172}$$

The SO(2,2) generators are

$$J^{0'0}=|p|,\ \ J^{1'0'}=p\ \cos x,\ \ J^{10'}=p\ \sin x, \tag{173}$$

$$J^{1'1}=p,\ \ J^{10}=|p|\cos x,\ \ \ J^{01'}=|p|\sin x. \tag{174}$$

These satisfy the SO(2,2) algebra at the classical level. The $SO(2,2)=SL(2,R)_L\otimes SL(2,R)_R$ generators $J_{0,1,2}^{L,R}$ are

$$J_0^R=\frac{1}{2}(|p|+p),\ \ J_1^R\pm iJ_2^R=\frac{1}{2}(|p|+p)e^{\pm ix}, \tag{175}$$

$$J_0^L=\frac{1}{2}(|p|-p),\ \ J_1^L\pm iJ_2^L=\frac{1}{2}(|p|-p)e^{\pm ix}. \tag{176}$$

We see that either the left moving or the right moving generators must vanish in momentum space (but not in x space or other quantum space). The quadratic Casimir for both $SL(2,R)_{L,R}$ vanishes at the classical level

$$C_2^{L,R}=(J_0^{L,R})^2-(J_i^{L,R}+iJ_2^{L,R})(J_1^{L,R}-iJ_2^{L,R})=0. \tag{177}$$

D. Quantum ordering case No. 2

We now need to order the operators at the quantum level and make sure that the Casimir operator is consistent with the gauge invariance requirements at the quantum level. For the bosonic case we must have $C_2(SO(d,2))=1-d^2/4=0$ and with fermions we must have $C_2(SO(d,2))=-\frac{1}{8}(d+2)(d-1)=-1/2$. Let us first deal with the purely bosonic case. For either the left or right movers we need Hermitian generators $J_{1,2,0}^{L,R}$, which implies $J_1^{L,R}-iJ_2^{L,R}=(J_1^{L,R}+iJ_2^{L,R})^\dagger$. The following ordering of operators is Hermitian for any real number α:

$$J_1\pm iJ_2=e^{\pm ix/2}\left(p^2-\frac{1}{4}\right)^{(1/2)\mp\alpha}p^{\pm2\alpha}e^{\pm ix/2},\ \ J_0=p. \tag{178}$$

The commutation rules close for any α

$$\begin{aligned}[J_1+iJ_2,J_1-iJ_2]&=e^{ix/2}\left(p^2-\frac{1}{4}\right)e^{-ix/2}\\ &\quad-e^{-ix/2}\left(p^2-\frac{1}{4}\right)e^{ix/2}\\ &=\left(p-\frac{1}{2}\right)^2-\left(p+\frac{1}{2}\right)^2=-2p\\ &=-2J_0\end{aligned} \tag{179}$$

and the Casimir operator is zero:

$$C(\mathrm{SL}(2,R))=(J_0)^2-\frac{1}{2}(J_1+iJ_2)(J_1-iJ_2)$$
$$-\frac{1}{2}(J_1-iJ_2)(J_1+iJ_2)$$
$$=p^2-\frac{1}{2}e^{ix/2}\left(p^2-\frac{1}{4}\right)e^{-ix/2}$$
$$-\frac{1}{2}e^{-ix/2}\left(p^2-\frac{1}{4}\right)e^{ix/2}$$
$$=p^2-\frac{1}{2}\left(p-\frac{1}{2}\right)^2+\frac{1}{8}$$
$$-\frac{1}{2}\left(p+\frac{1}{2}\right)^2+\frac{1}{8}$$
$$=0. \tag{180}$$

In particular the value $\alpha=0$ yields the generators

$$\alpha=0:J_1\pm iJ_2=e^{\pm ix/2}\left(p^2-\frac{1}{4}\right)^{1/2}e^{\pm ix/2},\quad J_0=p. \tag{181}$$

There is no way to decide which of these α versions one should use for our problem.

Next we return to the spinning case. The following modification of the purely bosonic generators give the desired result for the $\alpha=0$ case

$$\alpha=0:J_1\pm iJ_2=e^{\pm ix/2}\left[\left(p^2-\frac{1}{4}\right)^{1/2}\pm\gamma\right]e^{\pm ix/2},\quad J_0=p. \tag{182}$$

Then the SL(2,R) algebra closes correctly and the Casimir operator is

$$C_2=-\gamma^2. \tag{183}$$

The choice $\gamma^2=1/2$ matches the spinning case.

In addition, there are also other orderings, such as

$$J_1+iJ_2=(p+\alpha)e^{ix}=(p+\alpha+1)e^{ix}, \tag{184}$$

$$J_1-iJ_2=e^{-ix}(p+\alpha^*)=e^{-ix}(p+\alpha^*+1), \tag{185}$$

$$J_0=p+\frac{1}{2}(1+\alpha+\alpha^*), \tag{186}$$

where α is a complex number to be determined by fixing the Casimir operator. The algebra closes and the Casimir operator is

$$C_2=\frac{1}{4}(-1+\alpha^2+\alpha^{*2}-2|\alpha|^2). \tag{187}$$

We may choose many possible values for α [e.g., $\alpha=(\cot\theta+i)/\sqrt{8}$] so that $C_2=-1/2$.

E. General d

As an example for general d consider the following choice of AdS gauge consistent with the general formula (124), parametrized in terms of $x^\mu=(t,\mathbf{r})$ and $p^\mu=(H,\mathbf{p})$

$$M=[0',1',0,i],$$
$$X^M=\left[\frac{r^2+1}{2r}\cos t,1,\frac{r^2+1}{2r}\sin t,\ \frac{r^2-1}{2r^2}\,\mathbf{r}\right], \tag{188}$$

$$P^M=\begin{bmatrix}\dfrac{r^2-1}{2r}\,\mathbf{r}\cdot\mathbf{p}\cos t & & \dfrac{r^2-1}{2r}\,\mathbf{r}\cdot\mathbf{p}\sin t & & \dfrac{2r^2}{r^2-1}\left(\mathbf{p}-\dfrac{\mathbf{r}}{r^2}\mathbf{r}\cdot\mathbf{p}\right)\\ & ,0, & & , & \\ -\dfrac{2rH}{r^2+1}\sin t & & +\dfrac{2rH}{r^2+1}\cos t & & +\dfrac{r^2+1}{2r^2}\,\mathbf{r}\cdot\mathbf{p}\mathbf{r}\end{bmatrix}. \tag{189}$$

The metric $G_{\mu\nu}$ is given by

$$ds^2=dX\cdot dX=-\left(\frac{r^2+1}{2r}\right)^2dt^2+\frac{1}{r^2}dr^2+\left(\frac{r^2-1}{2r}\right)^2(d\Omega)^2. \tag{190}$$

The classical Hamiltonian $H=p^0$ follows from $P^2=G_{\mu\nu}p^\mu p^\nu=0$:

$$H=\frac{r^2+1}{2}\ \sqrt{p_r^2+\left(\frac{2}{r^2-1}\right)^2\frac{1}{2}L^{ij}L_{ij}}. \tag{191}$$

The SO(d,2) generators $L^{MN}=X^MP^N-X^NP^M$ may now be constructed by inserting the gauge choice at $t=0$. The form is complicated and operator quantum ordering is difficult in this gauge. Therefore we will not go into details.

VIII. CONFORMAL GAUGE

The particle gauge in Eq. (29) may be modified by an overall multiplicative function $F(x)$

$$M=[+',-',\mu],$$

$$X^M=[1,x^2/2,x^\mu]F(x), \tag{192}$$

$$P^M=[0,x\cdot p,p^\mu]\,\frac{1}{F(x)}. \tag{193}$$

The $P^2=0$ constraint is yet to be imposed. The methods are similar to those used for the AdS gauge. The metric that corresponds to this gauge choice is

$$ds^2=dX^MdX_M=F^2dx^\mu dx_\mu. \tag{194}$$

Therefore F^2 plays the role of the conformal factor for an arbitrary conformal metric in d dimensions:

$$G_{\mu\nu}=F^2(x)\,\eta_{\mu\nu}. \tag{195}$$

The momentum constraint takes the form

$$P^2=G^{\mu\nu}p_\mu p_\nu=\frac{\eta^{\mu\nu}}{F^2(x)}p_\mu p_\nu=0. \tag{196}$$

This is seen also by inserting the gauge into the action (11), which becomes (in the absence of fermions)

$$S=\int d\tau\left(\partial_\tau x^\mu p_\mu-\frac{1}{2}A^{22}\frac{\eta^{\mu\nu}}{F^2(x)}p_\mu p_\nu\right). \tag{197}$$

The quantum ordered version of the constraint is applied on states $|\phi\rangle$ or $\phi(x)=\langle x|\phi\rangle$. A good guess is that the quantum ordering should correspond to the Laplacian for the metric $G_{\mu\nu}$

$$\frac{1}{\sqrt{-G}}\partial_\mu[\sqrt{-G}G^{\mu\nu}\partial_\nu\phi(x)]=\frac{1}{F^d}\partial_\mu[F^{d-2}\eta^{\mu\nu}\partial_\nu\phi(x)]=0. \tag{198}$$

The effective field theory that gives this equation is [for the spinless Sp(2) gauge theory]

$$S_{\rm eff}=\int d^dx\,F^{d-2}(x)\,\partial_\mu\bar{\phi}\partial_\nu\phi\,\eta^{\mu\nu}. \tag{199}$$

This is modified to a Dirac equation for the theory with spin [OSp(1/2) gauge theory]

$$S_{\rm eff}=\int d^dx\,F^{d-1}(x)\bar{\Psi}\gamma^\mu\partial^\nu\Psi\,\eta_{\mu\nu}. \tag{200}$$

The SO(d,2) generators $J^{MN}=X^MP^N-X^NP^M+S^{MN}$ are unaltered at the classical level since the $F(x)$ factor cancels. However, at the quantum level, they need to be quantum ordered so that they are Hermitian according to the dot product in curved backgrounds

$$\langle\phi|\phi\rangle=\frac{1}{2}\int d^{d-1}x\,F^{d-2}(\bar{\phi}i\partial_0\phi-i\partial_0\bar{\phi}\phi), \tag{201}$$

$$\langle\Psi|\Psi\rangle=\int d^{d-1}x\,F^{d-1}\bar{\Psi}\gamma_0\Psi. \tag{202}$$

After the quantum ordering one should check the Casimir operator $C_2(\mathrm{SO}(d,2))$ and verify that it is consistent with Eq. (28) for the fermionic theory, and with $C_2=1-d^2/4$ for the bosonic theory, as follows.

To find the correct order of operators consider the condition $\langle J^{+'-'}\phi|\phi\rangle=\langle\phi|J^{+'-'}\phi\rangle$ or $\langle J^{+'-'}\Psi|\Psi\rangle=\langle\Psi|J^{+'-'}\Psi\rangle$. We find that we must have

$$J^{+'-'}=\frac{1}{2}(x\cdot p+p\cdot x)+is_0+\frac{i}{2}(d-2)x\cdot\partial\ln F(x). \tag{203}$$

The first term $\frac{1}{2}(x\cdot p+p\cdot x)$ is the Hermitian ordering for a dot product with naive integration measure. The quantum correction is_0, was already present in flat space due to hermiticity with a more involved dot product [$s_0=1$ for ϕ, and $s_0=1/2$ for Ψ; see [1] and Eq. (33)]. The last term is required in the conformal curved background F with the dot products given above. The proof of Hermiticity uses the conservation of the current $J^\mu=(i/2)F^{d-2}(\bar{\phi}\partial^\mu\phi-\partial^\mu\bar{\phi}\phi)$ or $F^{d-1}\bar{\Psi}\gamma^\mu\Psi$, i.e., $\partial_\mu J^\mu=0$, that follows from the equation of motion (i.e., constraint). This expression for $J^{+'-'}$ may be rewritten in the form

$$J^{+'-'}=\frac{1}{2}(x\cdot\tilde{p}+\tilde{p}\cdot x)+is_0, \tag{204}$$

where $\tilde{p}^\mu$ is the following order of operators:

$$\tilde{p}^\mu=F^{(1/2)(d-2)}(x)p^\mu F^{-(1/2)(d-2)}(x)$$
$$=p^\mu+i(d-2)x\cdot\partial\ln F(x). \tag{205}$$

We find that the rest of the generators J^{MN} are also Hermitian provided we use the result for flat space (33) and replace everywhere $p^\mu\to\tilde{p}^\mu$. The generators in curved conformal space are then given in terms of those in flat space by the prescription

$$J^{MN}_{\rm conf}(x,\tilde{p})=F^{(1/2)(d-2)}(x)J^{MN}_{\rm flat}(x,p)F^{-(1/2)(d-2)}(x). \tag{206}$$

Then the Casimir operator becomes

$$C_2(\mathrm{SO}(d,2))_{\mathrm{conf}} = F^{(1/2)(d-2)}(x) C_2(\mathrm{SO}(d,2))_{\mathrm{flat}} F^{-(1/2)(d-2)}(x) = C_2(\mathrm{SO}(d,2))_{\mathrm{flat}}, \tag{207}$$

where the last step holds since $C_2(\mathrm{SO}(d,2))_{\mathrm{flat}}$ is independent of x or p [see Eq. (35)]. The same is true for all higher Casimir operators because the orbital part x,p drops out [19]. This proves again that the quantum theory in the conformal gauge has the same quantum Hilbert space as all other gauges.

Similarly, one may go through all the previous gauges that are closely associated with the particle gauge. These include the light cone gauge (36) and the timelike gauge (93). It would be interesting to study the modifications in the presence of the conformal factor F for these cases since this generates new representations of SO$(d,2)$ for each choice of F.

IX. OSp$(n/2)$ GAUGE THEORY AND HIGHER SPINS

We can generalize the OSp(1/2) theory by adding more copies of the fermions ψ_a^M with $a=1,2,\ldots,n$. This provides the possibility of describing particles and other systems with higher spins. Thus, consider the fundamental representation $\Phi_I^M=(\psi_a^M,X_i^M)$ of OSp$(n/2)$, with $X_1^M=X^M$ and $X_2^M=P^M$ as before. Introduce the gauge fields

$$A^{IJ}=\begin{pmatrix} B^{[ab]} & F^{ai} \\ \varepsilon_{ij}F^{jb} & A^{ij}\end{pmatrix}, \quad A,B=\text{Bose}, \quad F=\text{Fermi}, \tag{208}$$

where $B^{[ab]}$ is the antisymmetric SO(n) gauge field and A^{ij} is the symmetric Sp(2) gauge field, as before. There are also $2n$ fermionic gauge fields F^{ai}. The local OSp(n/2) gauge invariant Lagrangian is

$$S_0=\frac{1}{2}\int_0^T d\tau (D_\tau \Phi_I^M) g^{IJ}\Phi_J^N \eta_{MN}, \quad g^{IJ}\text{:OSp metric}$$
$$=\int_0^T d\tau \left[\begin{array}{c} X_2\cdot\partial_\tau X_1+\frac{i}{2}\psi_a\cdot\partial_\tau\psi_a-\frac{1}{2}A^{ij}X_i\cdot X_j \\ +iF^{ia}X_i\cdot\psi_a-\frac{1}{2}B^{ab}\psi_a\cdot\psi_b \end{array}\right]. \tag{209}$$

As before, the constraints have nontrivial solutions provided there are two times, and the global symmetry is SO$(d,2)$.

Covariant quantization can be carried out as before. In order to have OSp$(n/2)$ singlets all of its Casimir operators must vanish. Then we find that the quadratic Casimir operator of SO$(d,2)$ must have the special value

$$C_2(\mathrm{SO}(d,2))=\frac{1}{8}(n-2)(d+2)(d+n-2). \tag{210}$$

This is consistent with the $n=0$ case of Refs. [1,2] and the $n=1$ case treated in this paper.

The SO$(d-1,1)$ Lorentz covariant particle gauge is easy to analyze:

$$M=(+',-',\mu),$$
$$X^M=(1,x^2/2,x^\mu), \tag{211}$$
$$P^M=(0,x\cdot p,p^\mu), \quad p^2=0, \tag{212}$$
$$\psi_a^M=(0,x\cdot\psi_a,\psi_a^\mu), \quad p\cdot\psi_a=0=\psi_{[a}\cdot\psi_{b]}. \tag{213}$$

The remaining constraints and gauge symmetries are those of worldline supergravity with n supersymmetries. These were studied in Ref. [20]. From the analysis in Refs. [20] and [6] we know that this system describes massless spinning particles. The effective fields that represent them are the analogues of gauge fields, i.e., forms that couple to p-branes (with $p=n/2-1$), and their fermionic generalizations

$$A_{\mu_1\mu_2\cdots\mu_{n/2}}(x), \quad n=\text{even}, \tag{214}$$

$$\Psi_{\alpha\mu_1\mu_2\cdots\mu_{(n-1)/2}}(x), \quad n=\text{odd}. \tag{215}$$

When written in this form, the constraints generate the appropriate field equations that remove the ghosts and give the correct counting of degrees of freedom in d dimensions.

The SO$(d,2)$ generators in the particle gauge have the same form as Eq. (33), but with

$$s^{\mu\nu}=\frac{1}{2i}(\psi_a^\mu\psi_a^\nu-\psi_a^\nu\psi_a^\mu), \quad \frac{1}{2}s^{\mu\nu}s_{\mu\nu}=\frac{n}{8}d(d+n-2), \tag{216}$$

in the gauge invariant sector of SO(n) singlets $(\psi_{[a}\cdot\psi_{b]}=0)$. The quadratic Casimir operator is given in Eq. (35):

$$C_2=-\frac{d^2}{4}+s_0^2+\frac{1}{2}s^{\mu\nu}s_{\mu\nu}. \tag{217}$$

So, now we need

$$s_0=\left(1-\frac{n}{2}\right) \tag{218}$$

in order to agree with the requirements that followed from OSp$(n/2)$ gauge invariance given in Eq. (210). This value of s_0 gives the following dimensions for the fields $A_{\mu_1\mu_2\cdots\mu_{n/2}}(x)$, $\Psi_{\alpha\mu_1\mu_2\cdots\mu_{(n-1)/2}}(x)$:

$$iJ^{+'-'}(A \text{ or } \Psi)=d/2-s_0=\frac{1}{2}(d+n-2). \tag{219}$$

This agrees with the $n=0,1$ cases which we have already studied explicitly in several forms.

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