QED effective action in time dependent electric backgrounds

Gerald Dunne* and Theodore Hall†
Physics Department, University of Connecticut, Storrs, Connecticut 06269
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We apply the resolvent technique to the computation of the QED effective action in time dependent electric field backgrounds. The effective action has both real and imaginary parts, and the imaginary part is related to the pair production probability in such a background. The resolvent technique has been applied previously to spatially inhomogeneous magnetic backgrounds, for which the effective action is real. We explain how dispersion relations connect these two cases, the magnetic case which is essentially perturbative in nature, and the electric case where the imaginary part is nonperturbative. Finally, we use a uniform semiclassical approximation to find an expression for very general time dependence for the background field. This expression is remarkably similar in form to Schwinger's classic result for the constant electric background.
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I. INTRODUCTION

The effective action is an important tool in quantum electrodynamics (QED), and quantum field theory in general. For example, for fermions in a static magnetic background, the effective action yields (minus) the effective energy of the fermions in that background; while for fermions in an electric background, the effective action is complex and the imaginary part gives (half) the probability for fermion-antifermion pair production [1,2].

The computation of rates for pair production from the vacuum was initiated by Schwinger [1] who studied the constant field case and found that the rate is (exponentially) extremely small. Brezin and Itzykson [3] studied the more realistic case for alternating fields $\vec{E}(t)=(E\sin(\omega_0 t),0,0)$ but found negligible frequency dependence and still an unobservably low rate for realistic electric fields. Narozhnyĭ and Nikishov [4] obtained an expression for both the spinor QED and scalar QED effective action, as an integral over 3-momentum, for a time dependent field $\vec{E}(t)=(E\,\mathrm{sech}^2(t/\tau),0,0)$. Their approach was based on the well-known exact solvability of the Dirac and Klein-Gordon equations for such a background. This solvable case has also featured in the strong-field analysis of Cornwall and Tiktopoulos [5], the group-theoretic semiclassical approach of Balantekin *et al.* [6,7], the proper-time method of Chodos [8], and the *S*-matrix work of Gavrilov and Gitman [9]. Recent experimental work involving the SLAC accelerator and intense lasers has given renewed impetus to this subject, providing tantalizing hints that the critical fields required for direct vacuum pair production may be within reach [10,11].

In this paper we make several new contributions to this body of work. First, using the resolvent approach we present an expression for the exact effective action in the time-dependent background $\vec{E}(t)=(E\,\mathrm{sech}^2(t/\tau),0,0)$ that is a simple integral representation involving a single integral, rather than as an expression that must still be traced over all 3-momenta, as in [4,5,9]. Second, we use this explicit expression to make a direct comparison with independent results from the derivative expansion approximation [12,13]. Third, we show how the real and imaginary parts of the effective action are related by dispersion relations, connecting perturbative and nonperturbative expressions. Finally, we show how the uniform semiclassical approximation [3,6] fits into the resolvent approach, obtaining a simple semiclassical expression for the QED effective action in a general time dependent, but spatially uniform E field. This expression is remarkably similar to Schwinger's ''proper-time'' expression for the constant field case.

When the background field has constant field strength $F_{\mu\nu}$, it is possible to obtain an explicit expression for the exact effective action as an integral representation [1]. The physical interpretation of this expression depends upon the magnetic or electric character of the background, and this is reflected in how we expand the integral representation. In the case of a constant magnetic background, a simple perturbative expansion in powers of B yields

$$S_{\mathrm{eff}}=\frac{B^2TL^3}{2\pi^2}\sum_{n=1}^{\infty}\frac{\mathcal{B}_{2n+2}}{(2n+2)(2n+1)(2n)}\left(\frac{2B}{m^2}\right)^{2n}, \quad (1)$$

where the $\mathcal{B}_n$ are the Bernoulli numbers [14], and TL^3 is the space-time volume factor. In the case of a constant electric field background, the effective action is complex. The real part has a natural perturbative expansion which is just Eq. (1) with $B\to iE$, while the imaginary part is a sum over nonperturbative tunneling amplitudes

$$\mathrm{Re}(S_{eff})=-\frac{E^2TL^3}{2\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n\mathcal{B}_{2n+2}}{(2n+2)(2n+1)(2n)}\left(\frac{2E}{m^2}\right)^{2n} \quad (2)$$

$$\mathrm{Im}(S_{eff})=\frac{E^2TL^3}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{m^2\pi n}{E}}. \quad (3)$$

*Electronic address: dunne@hep.phys.uconn.edu
†Electronic address: tedhall@phys.uconn.edu

There are two clear motivations for studying the effective action in non-constant background fields. First, knowledge of the effective action for more general gauge fields is necessary for the ultimate quantization of the electromagnetic field. Second, realistic electromagnetic background fields do not have constant field strength, and so we would like to understand effective energies and pair production rates in more general backgrounds. However, it is, of course, not possible to evaluate the exact QED effective action for a completely arbitrary background. Thus we are led naturally to approximate expansion techniques. A common approach, known as the derivative expansion [15–17,12], involves a formal perturbative expansion about Schwinger's exactly solvable case of constant field strength. Unfortunately, this type of perturbative expansion is difficult to perform beyond first order, and is hard to interpret physically, even for magnetic-type backgrounds. This is even more problematic for electric-type backgrounds, for which we seek a *nonperturbative* expansion.

A complementary approach is to search for solvable examples that are more realistic than the constant field case, although still not completely general. A recent work [18] has found an exact, explicit integral representation for the (3+1)-dimensional QED effective action in a static but spatially inhomogeneous magnetic field of the form

$$\vec{B}(\vec{x})=\left(0,0,B\,\mathrm{sech}^2\left(\frac{x}{\lambda}\right)\right). \tag{4}$$

For fermions in this background field, there are three relevant scales: a magnetic field scale B, a width parameter λ characterizing the spatial inhomogeneity, and the fermion mass m. It is therefore possible to expand the exact effective action in terms of two independent dimensionless ratios of these scales, depending on the question of interest. For example, since $\lambda=\infty$ corresponds to the uniform background case, in order to compare with the derivative expansion we expand the exact S_{eff} as a series in $1/B\lambda^2$. It has been verified that the first two terms in this series agree precisely with independent derivative expansion results (there are no independent field theoretic calculations of higher order terms in the derivative expansion with which to compare). Furthermore, these and analogous results in 2+1 dimensions indicate that the derivative expansion is in fact an asymptotic series expansion [19,20].

Formally, one could change this magnetic-type result to an electric-type background by an appropriate analytic continuation $B\to iE$. However, it is not immediately clear how to obtain a *nonperturbative* expression [for example, something like Eq. (3)] for the imaginary part of the effective action. For constant background fields a simple dispersion relation provides this connection between the magnetic and electric cases, but for nonconstant fields the dispersion relations are more complicated. Understanding this connection, for non-constant backgrounds, is one of the main motivations for this paper.

This paper is organized as follows. In Sec. II we review briefly the constant field case, using Schwinger's proper time method. In Sec. III we review the resolvent method, which has been used to obtain exact integral representations for the effective action in the special nonuniform magnetic background (4). In Sec. IV we then use the resolvent method to evaluate the exact effective action for a time-dependent, but spatially uniform electric field

$$\vec{E}(\vec{x})=\left(E\,\mathrm{sech}^2\left(\frac{t}{\tau}\right),0,0\right). \tag{5}$$

In Sec. V we show how dispersion relations connect the magnetic and electric cases (4) and (5). In Sec. VI we review the derivative expansion for electric fields and in Sec. VII show its connection to the exact effective action of Sec. IV. In Sec. VIII we use a uniform semi-classical approximation to obtain a general (but semi-classical) expression for the pair production probability in a time-dependent electric background. The final section is devoted to some concluding comments.

II. SCHWINGER'S APPROACH

Integrating over the fermion fields gives the QED effective action for fermions in a background electromagnetic field

$$\begin{aligned} S_{eff}[A]&=-i\ln\det(i\not{D}-m)\\ &=-\frac{i}{2}\,\mathrm{tr}\ln(\not{D}^2+m^2). \end{aligned} \tag{6}$$

Here, the covariant derivative is $\not{D}=\gamma^\mu(\partial_\mu+iA_\mu)$ with the electric charge e absorbed into the gauge field A. In the calculations that follow we are implicitly subtracting off zero field contribution $S_{eff}[A=0]$.

In a classic paper [1], Schwinger computed the effective action for constant background fields. One expresses the logarithm through an integral representation, the "proper-time" representation:

$$\begin{aligned} S_{eff}&=-\frac{i}{2}\,\mathrm{tr}\ln(\not{D}^2+m^2)\\ &=\frac{i}{2}\int_0^\infty\frac{ds}{s}\,\mathrm{tr}\,e^{-s(\not{D}^2+m^2)}. \end{aligned} \tag{7}$$

Clearly, to proceed, we need information concerning the spectrum of the operator $\not{D}^2+m^2$.

For a constant magnetic background of strength B, we choose $A_\mu=(0,0,0,By)$ and the Dirac representation of the gamma matrices so that the operator becomes diagonal:

$$\not{D}^2+m^2=[\partial_0^2-\partial_x^2-\partial_y^2-(\partial_z+iBy)^2+m^2]\mathbf{1}+\begin{pmatrix} B & 0 & 0 & 0\\ 0 & -B & 0 & 0\\ 0 & 0 & B & 0\\ 0 & 0 & 0 & -B \end{pmatrix}. \tag{8}$$

The Dirac trace is trivial, and we are left with a harmonic oscillator system with eigenvalues

$$m^2 - k_0^2 + k_x^2 + 2B(n + \tfrac{1}{2} \pm \tfrac{1}{2}). \tag{9}$$

The remaining traces are straightforward, yielding the exact effective action for a constant magnetic field [1]

$$S_{eff} = \frac{BTL^3}{8\pi^2}\int_0^\infty \frac{ds}{s^2} e^{-m^2 s}\left(\coth Bs - \frac{1}{Bs} - \frac{Bs}{3}\right). \tag{10}$$

Here, the $1/Bs$ term is an explicit subtraction of $S_{eff}[0]$, while the $Bs/3$ term corresponds to a charge renormalization. A straightforward expansion of Eq. (10) yields expansion (1).

In a constant electric background the calculation is similar. Choosing $A_\mu = (0, Ex_0, 0, 0)$ and using the chiral representation for the gamma matrices, we find the operator $D\!\!\!/^2 + m^2$ diagonalizes:

$$D\!\!\!/^2 + m^2 = [\partial_0^2 - (\partial_x + iEt)^2 - \partial_y^2 - \partial_z^2 + m^2]\mathbf{1} + \begin{pmatrix} iE & 0 & 0 & 0 \\ 0 & iE & 0 & 0 \\ 0 & 0 & -iE & 0 \\ 0 & 0 & 0 & -iE \end{pmatrix}. \tag{11}$$

Once again, the Dirac trace is trivial, and we are left with a harmonic oscillator with imaginary frequency. Thus $D\!\!\!/^2 + m^2$ has complex eigenvalues

$$m^2 + 2iE(n + \tfrac{1}{2} \pm \tfrac{1}{2}) + k_y^2 + k_z^2. \tag{12}$$

The traces can be performed as before, yielding

$$S_{eff} = \frac{ETL^3}{8\pi^2}\int_0^\infty \frac{ds}{s^2} e^{-m^2 s}\left(\cot Es - \frac{1}{Es} + \frac{Es}{3}\right), \tag{13}$$

where we have subtracted the same vacuum contribution and charge renormalization terms.

Going from Eqs. (10) to (13), we note poles of the integrand have moved onto the contour of integration. This is the trademark of background electric fields and the ultimate source of the imaginary contribution. Regulating the poles with the standard principal parts prescription [1], we separate out the imaginary and real contributions to the effective action:

$$S_{eff} = i\frac{E^2TL^3}{8\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-m^2\pi n/E} + \frac{ETL^3}{8\pi^2}\mathcal{P}\int_0^\infty \frac{ds}{s^2}e^{-m^2 s}\left(\coth Es - \frac{1}{Es} - \frac{Es}{3}\right). \tag{14}$$

As before, it is straightforward to expand the integral and arrive at expansion (2), for the real part of the effective action.

III. RESOLVENT METHOD

Now consider a class of more general backgrounds—fields pointing in a given direction and depending on only one space-time coordinate. This is still far from the most general case; nevertheless, this class is sufficiently broad to study the effects of inhomogeneities, and yet simple enough to be analytically tractable.

In the magnetic case we choose

$$\vec{A} = (0,0,a_B(y)) \rightarrow \vec{B} = (a_B'(y),0,0) \tag{15}$$

while in the electric case we choose

$$\vec{A} = (a_E(t),0,0) \rightarrow \vec{E} = (a_E'(t),0,0). \tag{16}$$

In the magnetic case there is no time dependence and $A_0 = 0$, so we can perform the energy trace in Eq. (6). After an integration by parts in k_0, this reduces the evaluation of the effective action to a trace of a one-dimensional Green's function, or resolvent

$$S_{eff} = -iL\int \frac{dk_0}{2\pi}\sum_\pm \mathrm{tr}\frac{k_0^2}{\mathcal{D}_\pm(k_x,y,k_z) - k_0^2}, \tag{17}$$

where the one-dimensional operator $\mathcal{D}_\pm$ is

$$\mathcal{D}_\pm = m^2 + k_x^2 - \partial_y^2 + (k_z - a_B(y))^2 \pm a_B'(y). \tag{18}$$

In the electric case there is no y dependence and $A_y = 0$, so we can perform the k_y trace and obtain

$$S_{eff} = iL\int \frac{dk_y}{2\pi}\sum_\pm \mathrm{tr}\frac{k_y^2}{\mathcal{D}_\pm(t,k_x,k_z) + k_y^2} \tag{19}$$

which involves the resolvent of the operator

$$\mathcal{D}_\pm = m^2 + \partial_0^2(k_x - a_E(t))^2 + k_z^2 \pm i a_E'(t). \tag{20}$$

Thus, for both the magnetic and electric backgrounds in Eqs. (15),(16) the problem reduces to tracing the diagonal resolvents (17),(19) of a one-dimensional differential operator. This makes clear the advantage of the resolvent approach. For a typical background field we usually think of computing the effective action by some sort of summation over the spectrum of the appropriate Dirac operator. This is easy for constant fields because the spectrum is discrete [see Eqs. (9) and (12)]. But for non-constant fields the spectrum will typically have both discrete and continuous parts, which makes a direct summation extremely difficult. However, for one-dimensional operators, we do not have to use this eigenfunction expansion approach—we can alternatively express the resolvent as a product of two suitable independent solutions, divided by their Wronskian. This provides a simple and direct way to compute the effective action when the background field has the form as in Eqs. (15),(16).

This resolvent approach has been applied successfully to spatially inhomogeneous magnetic backgrounds

[19,20,18,12]. It has also been used previously by Chodos [8] in an analysis of the possibility of spontaneous chiral symmetry breaking for QED in time-varying background electric fields. In this paper we present a detailed analysis of the resolvent approach to the computation of the QED effective action in time-dependent electric backgrounds. We first check the resolvent approach by computing the effective action for a constant electric field. The constant electric field case follows the constant magnetic case very closely. Choosing $a_E(t)=Et$, the eigenfunctions of the operator (20) are parabolic cylinder functions. Taking independent solutions with the appropriate behavior at $t=\pm\infty$, we obtain the Green's function

$$\mathcal{G}(t,t')=-\frac{\Gamma[-\nu]}{\sqrt{4\pi iE}}D_\nu\left(\sqrt{\frac{2i}{E}}(Et-k_x)\right)\times D_\nu\left(-\sqrt{\frac{2i}{E}}(Et'-k_x)\right), \tag{21}$$

where we have defined $\nu=(m^2+k_y^2+k_z^2)/(2iE)\pm\frac{1}{2}+\frac{1}{2}$. The trace of the diagonal Green's function can be performed [14], yielding psi functions, where $\psi(u)=\Gamma'(u)/\Gamma(u)$ is the logarithmic derivative of the gamma function [14]. Thus the effective action is

$$\begin{aligned}S_{eff}&=-\frac{iL^3}{4\pi^3}\int_0^{ET}dk_x\int_{-\infty}^{\infty}k_y^2dk_ydk_z\sum_{\pm}\int_{-\infty}^{\infty}dx_0\mathcal{G}(x_0,x_0)\\&=-\frac{EL^3T}{4\pi^3}\int_{-\infty}^{\infty}k_y^2dk_ydk_z\sum_{\pm}\left(\psi\left(\frac{1}{2}-\frac{\nu}{2}\right)+\psi\left(-\frac{\nu}{2}\right)\right)\end{aligned} \tag{22}$$

$$=\frac{EL^3T}{8\pi^2}\int_0^\infty\frac{ds}{s^2}e^{-m^2s}\left(\cot Es-\frac{1}{Es}+\frac{Es}{3}\right). \tag{23}$$

The limits on the k_x trace can be motivated by the classical Lorentz interaction of the electron-positron pair after pair creation, and can be checked by the requirement that the zero field part cancels correctly. Note that the arguments of the psi functions appearing in the effective action (22) are complex. Thus we must be careful to use the correct integral representation of the ψ function in the analysis. A convenient representation for a complex argument is given in [21] as

$$\psi(z)=\log z-\frac{1}{2z}-\int_0^{\infty e^{i\beta}}dt\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)e^{-zt}$$
$$-\frac{\pi}{2}<\beta<\frac{\pi}{2};\quad -\left(\frac{\pi}{2}+\beta\right)<\arg z<\left(\frac{\pi}{2}+\beta\right). \tag{24}$$

Expression (23) is the same as Eq. (13), and the calculation proceeds exactly as before. But for the constant field case the resolvent method is unnecessarily complicated. The advantages of the resolvent method will become evident when applied to more complicated background fields, as is done in the remainder of this paper.

IV. EXACTLY SOLVABLE CASE

In this section we apply the resolvent method to a background gauge field $A^\mu=(0,E\tau\tanh(t/\tau),0,0)$. This gauge field corresponds to a single pulsed electric field in the x-direction $E_x(t)=E\,\mathrm{sech}^2(t/\tau)$. The electric field is spatially uniform but time-dependent; it vanishes at $t=\pm\infty$, peaks at $t=0$, and has a temporal width τ that is arbitrary. This field contains the constant field as a special case when we take $\tau\to\infty$. The resolvent expression (19) for the effective action gives

$$S_{eff}=i\frac{L^3}{4\pi^3}\int d^3k\,\mathrm{tr}\frac{k_y^2}{\partial_0^2+\left(k_x-E\tau\tanh\left(\frac{t}{\tau}\right)\right)^2+k_y^2+k_z^2+m^2\pm iE\,\mathrm{sech}^2\left(\frac{t}{\tau}\right)}. \tag{25}$$

The k_x momentum trace runs over $(-\infty,\infty)$ since we consider an infinite interaction time.

To determine the effective action we need the resolvent, which is constructed from solutions to the ordinary differential equation

$$\left[\partial_0^2+m^2+k_y^2+k_z^2+\left(k_x-E\tau\tanh\left(\frac{t}{\tau}\right)\right)^2\pm iE\,\mathrm{sech}^2\left(\frac{t}{\tau}\right)\right]\phi=0. \tag{26}$$

This can be converted, by the substitution $y=\frac{1}{2}[1+\tanh(t/\tau)]$, to a hypergeometric equation, with independent solutions

$$\phi_1=y^\alpha(1-y)^\beta{}_2F_1\left(\frac{i\tau}{2}(\alpha+\beta\pm 2E\tau),\frac{i\tau}{2}(\alpha+\beta+1\mp 2E\tau);1+i\tau\alpha;y\right)$$

$$\phi_2=y^\alpha(1-y)^\beta{}_2F_1\left(\frac{i\tau}{2}(\alpha+\beta\pm 2E\tau),\frac{i\tau}{2}(\alpha+\beta+1\mp 2E\tau);1+i\tau\beta;1-y\right), \tag{27}$$

where we have defined

$$y=\frac{1}{2}\left(1+\tanh\left(\frac{t}{\tau}\right)\right)$$

$$\alpha=(m^2+k_y^2+k_z^2+(E\tau+k_x)^2)^{1/2}$$

$$\beta=(m^2+k_y^2+k_z^2+(E\tau-k_x)^2)^{1/2}. \qquad (28)$$

The boundary conditions are a particle of energy α traveling forward in time and a particle of energy $-\beta$ traveling backward in time.

The diagonal resolvent is $\mathcal{G}(t,t)=\phi_1(t)\phi_2(t)/W[\phi_1,\phi_2]$, where $W[\phi_1,\phi_2]$ is the Wronskian. The trace over time once again yields psi functions (just as in the magnetic cases treated in [19,20]):

$$\begin{aligned}S_{eff}=&-\frac{L^3\tau}{4\pi^3}\sum_{\pm}\int_{-\infty}^{\infty}\frac{k_y^2d^3k}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\\&\times\left(\psi\left(1+\frac{i\tau}{2}(\alpha+\beta\mp2E\tau)\right)+\psi\left(\frac{i\tau}{2}(\alpha+\beta\pm2E\tau)\right)\right)\\=&-\frac{L^3}{4\pi^3}\sum_{\pm}\int_{-\infty}^{\infty}\frac{k_y^2d^3k}{4k_\perp}\frac{\partial\Omega_{(\pm)}}{\partial k_\perp}\\&\times\left(\psi\left(1+\frac{i}{2}\Omega_{\mp}\right)+\psi\left(\frac{i}{2}\Omega_{\pm}\right)\right)\\=&\frac{L^3}{4\pi^3}\frac{1}{2}\int d^3k\int_0^{\infty}\frac{ds}{s}(e^{-\Omega_+s}+e^{-\Omega_-s})\left(\cot s-\frac{1}{s}\right),\end{aligned} \qquad (29)$$

where we have defined $\Omega_+=(\tau/2)(\alpha+\beta+2E\tau)$ and $\Omega_-=(\tau/2)(\alpha+\beta-2E\tau)$.

Equation (29) is the exact effective action for this time-dependent background gauge field. Notice the close similarity to Schwinger's expression (23) for the constant background electric field. It is straightforward to check that taking $\tau\to\infty$ reduces Eq. (29) to the constant field result (23).

The effective action (29) has both real and imaginary parts. As described before for the constant field case, we regulate the integral using the principal part prescription to obtain the imaginary part

$$\begin{aligned}\mathrm{Im}(S_{eff})&=\frac{1}{2}\frac{L^3}{4\pi^3}\int d^3k\sum_{n=1}^{\infty}\frac{1}{n}(e^{-n\pi\Omega_+}+e^{-n\pi\Omega_-})\\&=-\frac{1}{2}\frac{L^3}{4\pi^3}\int d^3k\ln((1-e^{-\pi\Omega_+})(1-e^{-\pi\Omega_-}))\end{aligned} \qquad (30)$$

and real part of the exact effective action

$$\begin{aligned}\mathrm{Re}(S_{eff})&=\frac{1}{2}\frac{iL^3}{4\pi^3}\int d^3k\int_0^{\infty}\frac{ds}{s}(e^{-i\Omega_+s}+e^{-i\Omega_-s})\left(\coth s-\frac{1}{s}\right)\\&=\frac{1}{6}\frac{L^3}{4\pi^3}\int d^3k\frac{1}{\Omega_+}+\frac{L^3}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{(-1)^n\mathcal{B}_{2n+2}}{(2n+2)(2n+1)}\int d^3k\left(\frac{2}{\Omega_+}\right)^{2n+1},\end{aligned} \qquad (31)$$

where we have asymptotically expanded the integral over s in inverse powers of Ω_+. The first term can be regulated and absorbed by renormalization. In the second term the k integrals can be done to yield the integral representation

$$\mathrm{Re}(S_{eff}^{ren})=-\frac{2L^3\tau^3}{3\pi^2}\int_0^{\infty}\frac{dt}{e^{2\pi t}-1}\left(\frac{t-E\tau^2}{v_-}(m^2\tau^2-v_-^2)^{3/2}\sin^{-1}\left(\frac{v_-}{\tau m}\right)+(E\to-E)\right), \qquad (32)$$

where we have defined $v_-=(t^2-2tE\tau^2)^{1/2}$. This integral may be expanded as

$$\mathrm{Re}S_{eff}^{ren}=-\frac{L^3\tau m^4}{8\pi^{3/2}}\sum_{j=0}^{\infty}\frac{1}{\Gamma(j+1)}\left(\frac{1}{2E\tau^2}\right)^j\sum_{k=1}^{\infty}\frac{\Gamma(2k+j)\Gamma(2k+j-2)}{\Gamma(2k+1)\Gamma\left(2k+j+\frac{1}{2}\right)}(-1)^{k+j}\mathcal{B}_{2k+2j}\left(\frac{2E}{m^2}\right)^{2k+j}. \qquad (33)$$

We now compare these results to previous analyses. The real part of the effective action is exactly the same as the effective action for the magnetic sech^2 background case [see Eqs. (10) and (18) in [18]), with the replacements $B\to iE$ and $\lambda T\to\tau L$. Thus, our naive expectation that this simple analytic continuation from a magnetic to an electric background is borne out. But in an electric background we are more interested in the imaginary part, which does not have this type of perturbative expansion. Rather, it has the nonperturbative form (30). This explains how it is possible to obtain both a perturbative and a nonperturbative expression, for the real and imaginary parts respectively, from the exact effective action (29). Balantekin *et al.* [7] have also computed this imaginary part of the effective action for a sech^2 electric field. Our result (30) agrees with their expression [see Eq. (3.29) of [7]), once theirs is symmetrized in $E\to -E$, as it must be to satisfy Furry's theorem. This difference is not important for the imaginary part, but it is crucial for the consistency of the dispersion relations which relate the real and imaginary parts, as we show in the next section.

V. DISPERSION RELATIONS

In the previous section we found an expression for the exact effective action for a particular background electric field. This effective action has both real and imaginary parts. Given the real or imaginary part of the effective action there exist dispersion relations which relate the two. Here, we exploit the cuts in the electron self-energy function to analytically continue it to the entire complex plane. We shall show that there exist simple dispersion relations between the real and imaginary parts of the effective action, both at the perturbative level and also at the level of the general expression (29).

A. Perturbative dispersion relations

Expand the imaginary part (30) of the effective action in powers of E^2

$$\mathrm{Im}(\mathrm{S}_{eff}) = \frac{1}{2}\frac{L^3}{4\pi^3}\sum_{n=0}^{\infty}\frac{1}{n}\int d^3k\left(e^{-n\pi\Omega_+}+e^{-n\pi\Omega_-}\right) \tag{34}$$

$$\begin{aligned}&=\frac{L^3}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n}\int d^3k\Bigg[e^{-2\pi n\tau\sqrt{\mu^2+k_x^2}}+\frac{E^2}{2}e^{-2\pi n\tau\sqrt{\mu^2+k_x^2}}\left(4n^2\pi^2\tau^4-\frac{2n\pi\mu^3\tau^3}{(\mu^2+k_x^2)^{3/2}}\right)\\
&\quad+E^4e^{-2\pi n\tau\sqrt{\mu^2+k_x^2}}\left(\frac{2n^4\pi^4\tau^8}{3}-\frac{2n^3\pi^3\mu^2\tau^7}{(\mu^2+k_x^2)^{3/2}}+\frac{n^2\pi^2\mu^4\tau^6}{2(\mu^2+k_x^2)^3}\right.\\
&\quad\left.+\frac{\pi n\nu^2\tau^5}{4(\mu^2+k_x^2)^{5/2}}-\frac{5\pi n\mu^2k_i^2\tau^5}{4(\mu^2+k_x^2)^{7/2}}\right)+\dots\Bigg].\end{aligned} \tag{35}$$

Consider first the E^2 term. Doing the angular integrals we obtain

$$\begin{aligned}[\mathrm{ImS}_{eff}]_{E^2}&=\frac{L^3}{4\pi^3}E^2\sum_{n=1}^{\infty}\frac{1}{2n}\int d^3k e^{-2\pi n\tau\sqrt{m^2+k^2}}2\pi n\tau\left(2n\pi\tau-\frac{\tau(m^2+k^2\sin^2\theta)}{(m^2+k^2)^{3/2}}\right)\\
&=\frac{L^3}{4\pi^3}\frac{4E^2\pi^2\tau^3}{3}\sum_{n=1}^{\infty}\int_0^{\infty}dk e^{-2\pi n\tau\sqrt{m^2+k^2}}\left(6\pi nk^2\tau-\frac{3k^2m^2+2k^4}{(m^2+k^2)^{3/2}}\right).\end{aligned} \tag{36}$$

With the substitution $q=2\sqrt{m^2+k^2}$ this becomes

$$\begin{aligned}[\mathrm{ImS}_{eff}]_{E^2}&=\frac{L^3}{4\pi^3}\frac{E^2\pi^2\tau^3}{3}\sum_{n=1}^{\infty}\int_{2m}^{\infty}dq e^{n\pi q/\lambda}(q^2-4m^2)^{1/2}\left(3n\pi\tau q-\frac{2}{q^2}(q^2+2m^2)\right)\\
&=\frac{L^3}{4\pi^3}\frac{E^2\pi^3\tau^4}{6}\int_{2m}^{\infty}dq q^2\mathrm{csch}^2\frac{\pi q\tau}{2}\left(1-\frac{4m^2}{q^2}\right)^{1/2}\left(1+\frac{2m^2}{q^2}\right)\\
&=\frac{L^3}{4\pi^3}4E^2\pi^4\tau^4\int_0^{\infty}dq q^2\mathrm{csch}^2\frac{\pi q\tau}{2}\mathrm{Im}\Pi(q^2).\end{aligned} \tag{37}$$

This expression agrees with the result of Itzykson and Zuber [22], where $\Pi(q^2)$ is the one-loop self-energy. They reduced the problem to a one-dimensional Lippman-Schwinger equation and expanded perturbatively to find the E^2 order term.

Along the real axis, there is a cut in the q^2 complex plane from $(-\infty,-2m]$ and from $[2m,\infty)$. To derive the dispersion relations we will need to consider an integral as $q^2\to\infty$. Since the electron self-energy does not go to zero as q^2, we need to add a linear convergence factor. The convergence factor gives a residue at the origin which will ultimately be absorbed by renormalization. This results in a once-subtracted dispersion relation as follows.

Apply Cauchy's integral theorem to a function $f(z)$ satisfying these properties. Let the contour be from $(-\infty,\infty)$ along the real axis and close with an arc of infinite radius in the upper half plane:

$$\frac{f(z)}{z}=\frac{1}{2\pi i}\oint_{C}\frac{f(\xi)d\xi}{\xi(\xi-z)}=\frac{f(0)}{2z}+\frac{1}{2\pi i}\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x')dx'}{x'(x'-z)}. \tag{38}$$

Now let the point z go to the real axis $z\to x+i\varepsilon$;

$$\frac{f(x)}{x}=\frac{f(0)}{x}+\frac{1}{\pi i}\mathrm{P}\int_{-\infty}^{\infty}\frac{f(x')dx'}{x'(x'-x)}. \tag{39}$$

Take the real and imaginary parts of Eq. (39) and assume that $f(z)$ satisfies the Schwarz reflection principle $f(z^*)=f^*(z)$:

$$\mathrm{Re}(f(x)-f(0))=\frac{x}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Im}f(x')dx'}{x'(x'-x)}=\frac{2x^2}{\pi}\mathrm{P}\int_{0}^{\infty}\frac{\mathrm{Im}f(x')dx'}{x'(x'^2-x^2)} \tag{40}$$

$$\mathrm{Im}(f(x)-f(0))=-\frac{x}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\mathrm{Re}f(x')dx'}{x'(x'-x)}=-\frac{2x}{\pi}\mathrm{P}\int_{0}^{\infty}\frac{\mathrm{Im}f(x')dx'}{x'(x'^2-x^2)}. \tag{41}$$

From the imaginary part of the electron self-energy in Eq. (37),

$$\mathrm{Im}\Pi(k)=\frac{1}{24\pi}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\left(1+\frac{2m^2}{k^2}\right)\Theta(k^2-4m^2), \tag{42}$$

we can obtain the real part.

$$\begin{aligned}\mathrm{Re}(\Pi(k)-\Pi(0))&=\frac{2k^2}{\pi}\frac{1}{24\pi}\mathrm{P}\int_{2m}^{\infty}\frac{dk'}{k'(k'^2-k^2)}\times\left(1-\frac{4m^2}{k'^2}\right)^{1/2}\left(1+\frac{2m^2}{k'^2}\right)\\&=\frac{1}{32\pi^{3/2}}\sum_{j=1}^{\infty}\frac{\Gamma(j+2)}{j\Gamma\left(j+\frac{5}{2}\right)}\left(\frac{k}{2m}\right)^{2j}.\end{aligned} \tag{43}$$

This is the kernel for the E^2 order real part of the effective action:

$$\begin{aligned}[\mathrm{ReS}_{eff}]_{E^2}&=\frac{L^3}{4\pi^3}4E^2\pi^4\tau^4\int_0^{\infty}dq\,q^2\mathrm{csch}^2\frac{\pi q\tau}{2}\times\mathrm{Re}(\Pi(q^2)-\Pi(0))\\&=\frac{L^2}{4\pi^3}\frac{4\pi^4E^2\tau^4}{32\pi^{3/2}}\sum_{j=1}^{\infty}\frac{\Gamma(j+2)}{j\Gamma(j+2)}\frac{1}{(2m)^{2j}}\times\int_{-\infty}^{\infty}dq\,q^{2j+2}\mathrm{csch}^2\frac{q\pi\tau}{2}\\&=\frac{E^2L^3\tau}{4\pi^{3/2}}\sum_{j=1}^{\infty}\frac{(-1)^j\Gamma(j+2)}{j\Gamma\left(j+\frac{5}{2}\right)}\mathcal{B}_{2j+2}\left(\frac{1}{m\tau}\right)^{2j}.\end{aligned} \tag{44}$$

This agrees with the $k=1$ term of Eq. (33), the real part of the full effective action to order E^2.

A similar analysis can be done for the E^4 contribution. Doing the angular integrals in the E^4 piece from Eq. (35) gives

$$\begin{aligned}[\mathrm{ImS}_{eff}]_{E^4}=\frac{L^3}{4\pi^3}\frac{4\pi^2\tau^5}{3}\sum_{n=1}^{\infty}\int_0^{\infty}k^2dk\,e^{-2\pi n\tau\sqrt{m^2+k^2}}&\times\left(2n^3\pi^3\tau^3-\frac{2n^2\pi^2\tau^2(2k^2+3m^2)}{(m^2+k^2)^{3/2}}\right.\\&+\frac{n\pi\tau(15m^4+20m^2k^2+8k^4)}{10(m^2+k^2)^3}\\&\left.+\frac{3m^4}{4(m^2+k^2)^{7/2}}\right).\end{aligned} \tag{45}$$

The substitution $q=2\sqrt{m^2+k^2}$ leads to

$$\begin{aligned}[\mathrm{ImS}_{eff}]_{E^4}=\frac{L^3}{4\pi^3}\frac{\pi^2\tau^5}{3}\sum_{n=1}^{\infty}\int_{2m}^{\infty}dq\,q(q^2-4m^2)^{1/2}&\times e^{-n\pi q\tau}\left(n^3\pi^3\tau^3-\frac{4n^2\pi^2\tau^2(q^2+2m^2)}{q^3}\right.\\&\left.+\frac{8n\pi\tau(q^4+2m^2q^2+6m^4)}{5q^6}+\frac{48m^4}{q^7}\right).\end{aligned} \tag{46}$$

Integrate by parts in the 1st, 2nd and 4th terms and collect terms:

$$\begin{aligned}[\mathrm{ImS}_{eff}]_{E^4}=-\frac{L^3}{4\pi^3}\frac{8\pi^3E^4m^4\tau^6}{3}&\times\int_0^{\infty}dq\,q^4\mathrm{csch}^2\frac{\pi q\tau}{2}\Theta(q^2-4m^2)\\&\times\frac{1}{q^8}\left(1-\frac{4m^2}{q^2}\right)^{-3/2}\left(3-\frac{10m^2}{q^2}\right).\end{aligned} \tag{47}$$

The dispersion relation for the E^4 term is derived in the same way except that no subtraction is needed.

$$\mathrm{Re}f(x)=\frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{x'dx'}{x'^2-x^2}\mathrm{Im}f(x'),$$

$$\mathrm{Im}f(x)=-\frac{2x}{\pi}\mathrm{P}\int_0^\infty \frac{dx'}{x'^2-x^2}\mathrm{Re}f(x'). \tag{48}$$

With the dispersion relations (48) we can immediately write down the complementary part of the effective action at order E^4:

$$\begin{aligned}[\mathrm{ReS}_{eff}]_{E^4}=&-\frac{L^3}{4\pi^3}\frac{8\pi^3E^4m^4\tau^6}{3}\int_0^\infty dq q^4 \mathrm{csch}^2\frac{\pi q\tau}{2}\frac{2}{\pi}\\ &\times \mathrm{P}\int_{2m}^\infty \frac{kdk}{k^2-q^2}\frac{1}{k^8}\left(1-\frac{4m^2}{k^2}\right)^{-3/2}\left(3-\frac{10m^2}{k^2}\right)\\ =&-\frac{2L^3E^4m^4\tau^9}{\pi^{3/2}}\sum_{j=0}^\infty \frac{(-1)^j}{\Gamma(j+1)}\frac{\Gamma(j+4)\Gamma(j+2)}{\Gamma(5)\Gamma(j+\frac{9}{2})}\\ &\times \mathcal{B}_{2j+4}\left(\frac{1}{m\tau}\right)^{2j+8}. \end{aligned} \tag{49}$$

Thus, the dispersion relations have enabled us to deduce the E^4 term of the real part (33) of the effective action, beginning with the E^4 term in the imaginary part.

Using dispersion relations we have shown how it is possible to go from a tunneling like expression to an asymptotic expansion at the first two orders in E^2. Recall that the real part for the exact effective action with a sech^2 background electric field (33) is an asymptotic expansion in *two* dimensionless scales $1/E\tau^2$ and $(E/m^2)^2$. Following steps similar to those taken above, we can find similar dispersion relations for the other expansion scale $1/E\tau^2$. These relations have been derived and are presented in [12].

B. All-orders dispersion relations

The above approach could be continued to higher orders in E^2, but the integrals become more difficult. Instead, we look for a dispersion relation connecting the full exact expressions for the real part (32) and the imaginary part (30) of the effective action. Begin with the imaginary part (30):

$$\mathrm{ImS}_{eff}=\frac{L^3}{4\pi^3}\frac{1}{2}\sum_{n=1}^\infty \frac{1}{n}\int d^3k\left[e^{-n\pi\tau\left(-2E\tau+\sqrt{\mu^2+(E\tau+k_x)^2}+\sqrt{\mu^2+(E\tau-k_x)^2}\right)}+(E\to -E)\right]. \tag{50}$$

Make the following substitution to unravel the exponents

$$\begin{aligned}2t=&2E\tau^2+\tau\sqrt{m^2+E^2\tau^2+k^2+2E\tau k\cos\theta}\\ &+\tau\sqrt{m^2+E^2\tau^2+k^2-2E\tau k\cos\theta}.\end{aligned} \tag{51}$$

Solve Eq. (51) for k,

$$k=\lambda\sqrt{\frac{(t-E\tau^2)^2(t^2-m^2\tau^2-2tE\tau^2)}{t(t-2E\tau^2)-E^2\tau^4\sin^2\theta}}, \tag{52}$$

substitute into Eq. (50), and do the angular integration

$$\begin{aligned}\mathrm{ImS}_{eff}=&\frac{L^3}{4\pi^3}\sum_{n=1}^\infty \frac{\pi}{n}\int_{E\tau^2+\sqrt{m^2\tau^2+E^2\tau^4}}^\infty dt e^{-2\pi nt}\frac{d}{dt}\\ &\times\int_0^{2\pi} d\theta\sin\theta\left(k^3(E)+k^3(-E)\right)\\ =&-\frac{L^3}{4\pi^3}\frac{4\pi^2m^4\tau}{3}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\\ &\times\left(\Theta(z_--1)z_-^3\frac{dz_-}{dt}\left(1-\frac{1}{z_-^2}\right)^{3/2}+(z_-\to z_+)\right),\end{aligned} \tag{53}$$

where we have defined $z_-=(1/m\tau)(t^2-2tE\tau^2)^{1/2}$.

A dispersion relation can be derived for the complex variable z_-. We regard the factor $(1-1/z_-^2)^{3/2}$ as the imaginary part of an analytic function defined along the whole real axis. Care must be taken since the function does not go to zero along the arc as $z_-\to\infty$; so we must insert a convergence factor. There is a dispersion relation giving the real part in terms of the imaginary part of a function with these characteristics:

$$\mathrm{Re}(f(z_-)-f(0))=\frac{2z_-^2}{\pi}\mathrm{P}\int_0^\infty \frac{\mathrm{Im}f(k)dk}{k(k^2-z_-^2)} \tag{54}$$

With Eq. (54) we can obtain the real part of the effective action:

$$\begin{aligned}\mathrm{ReS}_{eff}^{ren}=&-\frac{L^3\pi^2m^4\tau}{3\pi}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\\ &\times\left(z_-^3\frac{dz_-}{dt}\frac{2z_-^2}{\pi}\mathrm{P}\int_1^\infty \frac{\left(1-\frac{1}{k^2}\right)^{3/2}dk}{k(k^2-z_-^2)}+(z_-\to z_+)\right)\\ =&-\frac{m^4L^3\tau}{15\pi^2}\int_0^\infty \frac{dt}{e^{2\pi t}-1}\left(z_-^4\frac{dz_-^2}{dt}{}_2F_1(1,1;\tfrac{7}{2},z_-^2)\right.\\ &\left.+(z_-\to z_+)\right).\end{aligned} \tag{55}$$

In the last equation, we recognize the hypergeometric function ${}_2F_1$, which has another representation in terms of $\sin^{-1}$ [14]:

$$\mathrm{Re}S_{eff}^{ren}=-\frac{m^4L^3\tau}{\pi^2}\int_0^\infty\frac{dt}{e^{2\pi t}-1}\frac{2\lambda^2}{m^2}\times\left[(t-E\tau^2)\left(-\frac{2}{3}+\frac{8v_-^2}{9m^2\tau^2}+\frac{2m\tau}{3v_-}\left(1-\frac{v_-^2}{\tau^2m^2}\right)^{3/2}\times\sin^{-1}\frac{v_-}{\tau m}\right)+(E\to -E)\right]. \tag{56}$$

We drop terms independent of E (since these cancel against the vacuum subtraction) and get

$$\mathrm{Re}S_{eff}^{ren}=-\frac{2L^3}{3\pi^2\tau^3}\int_0^\infty\frac{dt}{e^{2\pi t}-1}\left(\frac{t-E\tau^2}{v_-}(m^2\tau^2-v_-^2)^{3/2}\times\sin^{-1}\frac{v_-}{\tau m}+(E\to -E)\right), \tag{57}$$

where $v_-=(t^2-2tE\tau^2)^{1/2}$ and $v_+=(t^2+2tE\tau^2)^{1/2}$. This expression for the real renormalized effective action is exactly the same as obtained by a direct computation (32). Thus the dispersion relations enable us to compute the real part, given the imaginary part. The reverse direction works similarly.

VI. DERIVATIVE EXPANSION IN (3+1)-DIMENSIONAL ELECTRIC FIELD

Schwinger solved the effective action exactly for constant background fields. To solve for more realistic fields one must use some perturbative expansion such as the derivative expansion. In the derivative expansion the fields are assumed to vary very slowly. We rewrite the trace in Eq. (7) as a supersymmetric quantum-mechanical path integral, expand the gauge field in a Taylor series about the constant case, and interpret the successive coefficients as successively increasing n-body interaction terms. This has been done for (2+1)-dimensional electric fields [12] and we may immediately generalize to 3+1 dimensions by making the substitution $m^2\to m^2+k_z^2$ and tracing over the additional momentum [18]. We obtain the zeroth and first orders of the derivative expansion for a spatially homogeneous electric field in 3+1 dimensions:

$$S=\frac{i}{2}\int d^4x\int_0^\infty\frac{ds}{s}\frac{e^{-m^2s}}{4i(\pi s)^2}\left[(Es\cot Es-1)+(\partial_0E)^2\left(\frac{s^2}{8E^4}\right)(Es\cot Es)'''\right]. \tag{58}$$

Regulating the s integral as before with the principal parts prescription, we easily separate the real and imaginary parts of the zeroth order derivative expansion term

$$\mathrm{Im}[S_{eff}]_0=\int d^4x\frac{E^2}{8\pi^3}\sum_{n=1}^\infty\frac{1}{n^2}e^{-m^2\pi n/E} \tag{59}$$

$$\mathrm{Re}[S_{eff}]_0=\int d^4x\mathcal{P}\int_0^\infty\frac{ds}{s}\frac{e^{-m^2s}}{8\pi^2s^2}(Es\cot Es-1). \tag{60}$$

We perform an asymptotic expansion of the integral over s and we obtain

$$\mathrm{Re}[S_{eff}]_0=-\int d^4x\frac{E^2}{2\pi^2}\sum_{n=1}^\infty\frac{(-1)^n\mathcal{B}_{2n+2}}{2n(2n+1)(2n+2)}\left(\frac{2E}{m^2}\right)^{2n}, \tag{61}$$

where $\mathcal{B}_\nu$ is the ν^{th} Bernoulli number. Equations (59) and (61) are the same as the corresponding equations for the constant field result (3) and (2), with the constant field E replaced by the time dependent field $E(t)$.

Now consider first derivative term in Eq. (58). Separating out the imaginary component is complicated by the fact that the triple derivative introduces fourth order poles along the real axis, while in the zeroth order term the poles are of first order. The exact effective action, containing both imaginary and real components, for the first order derivative term is

$$[S_{eff}]_1=\frac{i}{2}\int d^4x\int_0^\infty\frac{ds}{s}\frac{e^{-m^2s}}{4i(\pi s)^2}(\partial_0E)^2\frac{s^2}{8E^4}(sE\cot sE-1)'''$$
$$=-\frac{1}{64\pi^2}\int d^4x\frac{(\partial_0E)^2}{E^4}\sum_{n=1}^\infty\int_0^\infty\frac{ds}{s}\frac{e^{-m^2s}}{E^4\left(s-\frac{n\pi}{E}\right)^4}\frac{48n^2\pi^2E^4s(n^2\pi^2+s^2E^2)}{(Es+\pi n)^4}. \tag{62}$$

In this expression we clearly see the presence of the fourth order poles along the real axis. Regulating using the principal parts prescription, we get the imaginary part which is just a sum of 1/2 the residues:

$$\mathrm{Im}[\mathrm{S}_{eff}]_1 = -\frac{1}{64\pi^2}\int d^4x \frac{(\partial_0 E)^2}{E^4}\sum_{n=1}^{\infty}\frac{\pi}{3!}\left(\frac{e^{-m^2 s}}{s}\frac{48n^2\pi^2 s(n^2\pi^2+s^2E^2)}{(Es+\pi n)^4}\right)'''\Bigg|_{s\to\frac{\pi n}{E}}$$

$$=\frac{1}{64\pi}\int d^4x \frac{(\partial_0 E)^2}{E^4}\sum_{n=1}^{\infty}\frac{e^{-\frac{nm^2\pi}{E}}}{(\pi n)^3}(6E^3+6E^2m^2n\pi+3Em^4n^2\pi^2+m^6n^3\pi^3). \tag{63}$$

As before we asymptotically expand the integral in powers of E/m^2 and we find

$$\mathrm{Re}[\mathrm{S}_{eff}]_1 = \frac{m^6}{64\pi^2}\int d^4x \frac{(\partial_0 E)^2}{E^4}\sum_{n=1}^{\infty}\frac{(-1)^n \mathcal{B}_{2n+2}}{2n-1}\left(\frac{2E}{m^2}\right)^{2n+2}. \tag{64}$$

Note that in the spirit of the derivative expansion approximation, E means $E(t)$ in the expressions (58)–(64). In the next section we will specialize to the sech^2 electric field and compare with the exact result (29) for the effective action.

VII. DERIVATIVE EXPANSION IN AN EXACTLY SOLVABLE CASE

For the electric field

$$E_1(t) = E\,\mathrm{sech}^2\left(\frac{t}{\tau}\right) \tag{65}$$

the exact effective action is Eq. (29), with explicit real and imaginary parts in Eqs. (33) and (30), respectively. In order to compare with the derivative expansion results in Eqs. (59), (61), (63) and (64), we still need to perform the t integrals in these expressions, with $E(t)=E\,\mathrm{sech}^2(t/\tau)$.

A. Comparison of the real part

Insert the electric field (65) into the real part of the zero order derivative expansion effective action (60) and do the t integral using the formula (3.512.2) from Gradshteyn and Rhyzhik [14],

$$\int_0^{\infty}\frac{\sinh^{\mu}x}{\cosh^{\nu}x}dx = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}\right)}{2\Gamma\left(\frac{\nu+1}{2}\right)}, \tag{66}$$

and we obtain

$$\mathrm{Re}[\mathrm{S}_{eff}]_0 = -\frac{\tau L^3 m^4}{8\pi^{3/2}}\sum_{n=1}^{\infty}\frac{\Gamma(2n-2)\Gamma(2n)}{\Gamma(2n+1)\Gamma\left(2n+\frac{1}{2}\right)}\times(-1)^n\mathcal{B}_{2n}\left(\frac{2E}{m^2}\right)^{2n}. \tag{67}$$

This is precisely the leading term, as an expansion in $1/E\tau^2$, of the exact effective action (33). Similarly, for the real part of the first order derivative term (64) in the expansion of the of the effective action, doing the t integral yields

$$\mathrm{Re}[\mathrm{S}_{eff}]_1 = \frac{L^3 m^2}{8\pi^{3/2}\tau}\sum_{n=1}^{\infty}\frac{\Gamma(2n+1)\Gamma(2n-1)}{\Gamma(2n+1)\Gamma\left(2n+\frac{3}{2}\right)}\times(-1)^n\mathcal{B}_{2n+2}\left(\frac{2E}{m^2}\right)^{2n}. \tag{68}$$

This is precisely the next-to-leading term in expansion (33) of the exact result.

This agreement is as expected for the field (65), each order in the derivative expansion introduces an extra factor of $1/\tau^2$. These results provide strong evidence that expansion (33) of the exact result is an all-orders derivative expansion, as in the magnetic case [19,20]. However, as in the magnetic case, we note that this is an asymptotic expansion.

B. Comparison of the imaginary part

For the imaginary piece we follow a different approach to make the comparison. Inserting the $E(t)=\mathrm{sech}^2(t/\tau)$ into the zero-order and first-order expressions (59) and (63) leads to the probability integral, which cannot be computed explicitly. Instead, we expand the imaginary part of the exact effective action (30) in inverse powers of τ, and transform the momentum integrals into a form which can be compared directly with the derivative expansion answers (59) and (63).

Recall the imaginary part (30) of the exact effective action

$$\mathrm{Im}(\mathrm{S}_{eff}) = \frac{L^3}{4\pi^3}\frac{1}{2}\int d^3k\sum_{n=1}^{\infty}\frac{1}{n}(e^{-n\pi\Omega_+}+e^{-n\pi\Omega_-}), \tag{69}$$

where Ω_+ and Ω_- are defined as

$$\Omega_+ = \tau(\alpha+\beta+2E\tau) \qquad \Omega_- = \tau(\alpha+\beta-2E\tau) \tag{70}$$

and α and β are defined in Eq. (28). We can ignore the Ω_+ term in the derivative expansion, $\tau\to\infty$, since it is suppressed by an exponential factor $e^{-4E\tau}$ relative to the Ω_- piece. Make the transformation

$$2t = \tau[-2E\tau+\sqrt{\mu^2+(E\tau+k_x^2)^2}+\sqrt{\mu^2+(E\tau-k_x)^2}] \tag{71}$$

and solve for k_x

$$k_x=\frac{(t+\tau^2E)\sqrt{t^2-\mu^2\tau^2+2t\tau^2E}}{t^{1/2}\tau\sqrt{t+2\tau^2E}}. \tag{72}$$

The integral is now

$$\mathrm{Im}(S_{eff})=\frac{L^3}{2\pi^2}\sum_{n=1}^{\infty}\int dk_y dk_z\int_{t_0}^{\infty}e^{-2\pi nt}k_x(t), \tag{73}$$

where the lower limit on the integration is $t_0=-\tau^2E+\tau\sqrt{\mu^2+E^2\tau^2}$. Make another transformation to the coordinate z

$$z=\frac{1}{\mu\tau}\sqrt{t^2+2t\tau^2E}\qquad \frac{dt}{dz}=\frac{\tau z\mu^2}{\sqrt{\mu^2z^2+\tau^2E^2}} \tag{74}$$

and the integral becomes

$$\mathrm{Im}(S_{eff})=\frac{L^3\tau}{2\pi^2}\sum_{n=1}^{\infty}\int dk_y dk_z\mu^2 \times\int_1^{\infty}dz\sqrt{z^2-1}e^{-2\pi n(-\tau^2E+\tau\sqrt{\mu^2z^2+\tau^2E^2})} \tag{75}$$

which can be expanded in inverse powers of τ:

$$\mathrm{Im}(S_{eff})=\frac{L^3\tau}{2\pi^2}\sum_{n=1}^{\infty}\int dk_y dk_z\mu^2 \times\int_1^{\infty}dz\sqrt{z^2-1}e^{-\frac{n\pi z^2\mu^2}{E}}\left(1+\frac{n\pi z^4\mu^4}{4E^3\tau^2}+\dots\right). \tag{76}$$

Complete the integral over k_y and k_z in the leading term

$$\mathrm{Im}[S_{eff}]_0=\frac{L^3\tau E^2}{4\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_1^{\infty}\frac{dz}{z^4\sqrt{z^2-1}}e^{-\frac{n\pi m^2}{E}z^2} \tag{77}$$

$$=\frac{L^3\tau E^2}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{n\pi m^2}{E}}\Psi\left(\frac{1}{2},-1,\frac{n\pi m^2}{E}\right), \tag{78}$$

where Ψ is the confluent hypergeometric function defined in 6.5(2) of [21]. Gavrilov and Gitman [9] have found, by other methods, the zeroth order term for this field configuration and obtain precisely Eq. (78). In order to compare with the zeroth order derivative expansion result (59) we substitute $z=\cosh(t/\tau)$ in Eq. (77) to obtain

$$\mathrm{Im}(S_{eff})_\tau=\int d^4x\frac{E^2(t)}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{n\pi m^2}{E(t)}}, \tag{79}$$

where $E(t)=E\,\mathrm{sech}^2(t/\tau)$. This is precisely the imaginary part of the zeroth order term of the derivative expansion (59).

Similarly, perform the integrals over k_y and k_z in the next-to-leading order term in Eq. (76):

$$\mathrm{Im}[S_{eff}]_1=\frac{L^3}{8\pi\tau E^2}\sum_{n=1}^{\infty}\frac{1}{\pi^3n^3}\int_1^{\infty}dz\sqrt{\frac{z^2-1}{z^4}}e^{-\frac{n\pi m^2}{E}z^2} \times(6E^3+6E^2m^2n\pi z^2+3Em^4n^2\pi^2z^4+m^6n^3\pi^3z^6). \tag{80}$$

To compare with the first order derivative expansion result (63), we make the same substitution $z=\cosh(t/\tau)$ to obtain

$$\mathrm{Im}[S_{eff}]_1=\frac{1}{64\pi}\int d^4x\frac{(\partial_0E)^2}{E^4}\sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}e^{-\frac{n\pi m^2}{E}} \times(6E^3+6E^2m^2n\pi+3E^3m^4n^2\pi^2+m^6n^3\pi^2), \tag{81}$$

where here E means $E(t)=E\,\mathrm{sech}^2(t/\tau)$. This is the same result we obtained for the first derivative term of the imaginary part of the effective action (63). As with the real part of the effective action, successive terms in inverse powers of τ from Eq. (76) correspond to increasing orders of the derivative expansion.

VIII. EXACT SEMI-CLASSICAL ACTION FOR MORE GENERAL FIELDS

As discussed in Sec. III, the resolvent method is a useful technique for evaluating the exact effective action when the Dirac operator can be reduced to an effectively one-dimensional operator. In this section we show how a generalized WKB expansion can then be used to obtain an exact semi-classical effective action for background electric fields with more general time dependence than the $E(t)=E\,\mathrm{sech}^2(t/\tau)$ example considered in the previous two sections.

Assume the background gauge field has only one component in the x-direction $A_\mu=(0,a(t),0,0)$. According to Eq. (19), we seek the Green's functions

$$-(\hbar^2\partial_0^2+\mu^2+\phi^2(t)\pm i\hbar\phi'(t))\mathcal{G}^{\pm}_{k_\perp}(t,t')=\delta(t-t'), \tag{82}$$

where $\mu^2=m^2+k_y^2+k_z^2$, and $\phi=a(t)-k_x$.

In the uniform semiclassical approximation [6], one begins by looking for solutions $\psi(t)=K(t)U(S(t))$. The familiar WKB approximation of quantum mechanics consists of the choice $\psi(t)=Ke^{iS(t)}$. Instead, a uniform semiclassical approximation is obtained by choosing U to be a parabolic cylinder function. Define U to satisfy

$$-\hbar^2\frac{\partial^2U}{\partial S^2}-(S^2+i\eta\hbar)U(S)=\Omega U(S) \tag{83}$$

to which independent solutions are

$$D_\nu\left(\pm\frac{1+i}{\sqrt{\hbar}}S(t)\right), \tag{84}$$

where η goes as the sign of ϕ', and $\nu=\frac{1}{2}(\eta-1-i\Omega/\hbar)$. Now take $K=(S')^{-1/2}$. Then the general differential equation (82) becomes a differential equation relating K and S:

$$\hbar^2\frac{1}{K}\frac{\partial^2 K}{\partial t^2}-\left(\frac{\partial^2 S}{\partial t^2}\right)(\Omega+i\eta\hbar+S^2)+(\mu^2+\phi^2)\pm i\hbar\phi'=0. \tag{85}$$

Expand $S(t)\approx S_0(t)+\hbar S_1(t)$ and collect the zeroth order terms in $\hbar$:

$$\mu^2+\phi^2(t)=(\Omega+S_0^2)\left(\frac{\partial S_0}{\partial t}\right)^2. \tag{86}$$

The WKB expansion is a good approximation when the zeroth order term outsizes the first order term $1\gg|S_1/S_0|$. At points t' where $S_0(t')\to 0$ the approximation does not work unless we require $S_1(t')\to 0$ as well. Then apply L'Hôpital's rule

$$1\gg\left|\frac{S_1}{S_0}\right|=\left|\frac{S_1'}{S_0'}\right|=|S_1'|\left|\left(\frac{\Omega+S_0^2}{\mu^2+\phi^2}\right)^{1/2}\right| \tag{87}$$

and we see the generalized WKB will be an appropriate expansion if the turning points of the numerator $S_0(t_0)=i\sqrt{\Omega}$ and $S_0(t_0^*)=-i\sqrt{\Omega}$ are the same as those of the denominator $\phi(t_0)=i\mu$ and $\phi(t_0^*)=-i\mu$. Using the turning points, we can integrate Eq. (86) and find the quantity Ω [6]:

$$\int_{t_0}^{t_0^*}dt\sqrt{\mu^2+\phi^2(t)}=\int_{t_0}^{t_0^*}dt\frac{dS_0}{dt}\sqrt{\Omega+S_0^2}=\int_{i\sqrt{\Omega}}^{-i\sqrt{\Omega}}dS_0\sqrt{\Omega+S_0^2}=-\frac{i\Omega\pi}{2}. \tag{88}$$

Given the wavefunctions, we can express the Green's function as

$$\mathcal{G}_{k_\perp}^{\pm(\eta)}(t,t')=-\frac{\Gamma(-\nu)}{2\sqrt{\pi}}e^{-i\pi/4}\frac{1}{S'}D_\nu\left(\frac{1+i}{\sqrt{\hbar}}S(t)\right)\times D_\nu\left(-\frac{1+i}{\sqrt{\hbar}}S(t')\right). \tag{89}$$

The resolvent approach then gives the effective action as

$$S_{eff}=i\frac{L^3}{4\pi^3}\int k_y^2d^3k\frac{1}{2}\sum_{\pm E}2\sum_{\pm(\eta)}\int_{-\infty}^{\infty}dx_0\mathcal{G}_{k_\perp}^{\pm(\eta)}(t,t), \tag{90}$$

where we explicitly summed signs of the electric field to satisfy Furry's theorem.

Now make the semiclassical approximation by replacing $S(t)$ by $S_0(t)$. The semiclassical Green's function is

$$\mathcal{G}_{k_\perp}^{\pm(\eta),sc}(t,t')=-\frac{\Gamma(-\nu)}{4\sqrt{\pi}}e^{-\frac{i\pi}{4}}\frac{1}{k_\perp}\frac{\partial\Omega}{\partial k_\perp}S_0'D_\nu\left(\frac{1+i}{\sqrt{\hbar}}S_0(t)\right)D_\nu\left(-\frac{1+i}{\sqrt{\hbar}}S_0(t')\right), \tag{91}$$

where we have used the identity [see Eq. (86)] that $1/S_0'=(1/2k_\perp)\,(\partial\Omega/\partial k_\perp)\,S_0'$. The t integral in the trace of the diagonal resolvent can be converted to an integral over S_0, giving

$$S_{eff}^{sc}=-\frac{L^3}{(2\pi)^3}\int_{-\infty}^{\infty}k_y^2d^3k\sum_{\pm E}\frac{1}{2k_\perp}\frac{\partial\Omega}{\partial k_\perp}\left(\Psi\left(\frac{i}{2}\Omega\right)+\Psi\left(1+\frac{i}{2}\Omega\right)\right)=\frac{L^3}{(2\pi)^3}\frac{1}{2}\int d^3k\int_0^\infty\frac{ds}{s}(e^{-\Omega(E)s}+e^{-\Omega(-E)s})\left(\cot s-\frac{1}{s}\right). \tag{92}$$

This expression is the exact (but semiclassical) effective action for an electric background field that is spatially uniform, but has general time dependence $\phi'(t)$. The function Ω is given by Eq. (88). It is interesting to note how similar this general expression is to Schwinger's exact expression (13) for the constant background field case.

In the exactly solvable case studied in the previous two sections $\phi(t)=E\tau\tanh(t/\tau)$. In this case the integral (88) for Ω can be done exactly and we arrive at the exact expression (29) derived before with the resolvent method. The fact that the uniform semiclassical approximation is actually exact in this case is due to the supersymmetry underlying the uniform semiclassical approximation in this system. In general cases that are not exactly solvable, the expression (92) still gives the semiclassical answer. For example, a periodic background gauge field $A_\mu=(0,(E/\omega_0)\cos(\omega_0 t),0,0)$ is not an exactly solvable case. However, the expression (92) immediately gives the semiclassical result of Brezin and Itzykson [3] [see Eq. (44) of their paper] for the imaginary part of the effective action in an alternating electric field.

IX. CONCLUSIONS

In conclusion, we have used the resolvent approach to compute the exact QED effective action for the time dependent electric field background $\vec{E}(t)=(E\,\mathrm{sech}^2(t/\tau),0,0)$. The result is a simple integral representation involving a single integral, just as in Schwinger's proper-time result for the constant electric field case. We then used this exact result to investigate the dispersion relations relating the real and imaginary parts of the effective action. This explains the connection between the nonperturbative form of the imaginary part, and the perturbative form of the real part. It is this perturbative real part that should be compared with results for magnetic backgrounds. In addition, we made an asymptotic expansion of the exact answer in powers of $1/E\tau^2$, and showed that the first two terms agree with (independent) results from the derivative expansion. Finally, we showed how the uniform semiclassical approach of Balantekin *et al.* is incorporated into the resolvent approach, yielding a simple semiclassical expression that encodes both the real and imaginary parts of the effective action. The challenge now is to use these results for the effective action to obtain realistic estimates of pair production rates in electric fields with practically attainable strength and time dependence.

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