On rotational excitations and axial deformations of BPS monopoles and Julia-Zee dyons

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It is shown that Julia-Zee dyons do not admit slowly rotating excitations. This is achieved by investigating the complete set of stationary excitations which can give rise to nonvanishing angular momentum. The relevant zero modes are parametrized in a gauge invariant way and analyzed by means of a harmonic decomposition. Since general arguments show that the solutions to the linearized Bogomol'nyi equations cannot contribute to the angular momentum, the relevant modes are governed by a set of electric and a set of non-self-dual magnetic perturbation equations. The absence of axial dipole deformations is also established. [S0556-2821(98)00122-2]

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I. INTRODUCTION

The main question addressed in this article is whether Julia-Zee dyons admit rotational excitations. The investigation of this problem was motivated by some surprising results which we recently obtained for a class of self-gravitating non-Abelian soliton and black hole configurations. In [1] we showed that the Bartnik-McKinnon solutions [2] admit slowly rotating excitations. A two-parameter family of axisymmetric excitations of the static black hole solutions to the Einstein-Yang-Mills system was established as well. In addition to the charged, rotating black holes found in [3], there also exists a branch of uncharged, rotating black holes, as well as a branch of stationary—but not static—black holes with vanishing Komar angular momentum [1].

On the other hand, the situation was shown to be completely different in the presence of scalar fields [4]. Slowly rotating generalizations of (self-gravitating) *solitons* were *excluded* for a relatively large class of theories with non-Abelian gauge fields coupled to Higgs fields. In particular, the results obtained in [4] apply to the 't Hooft–Polyakov monopole and its self-gravitating generalizations. For *black hole* solutions of gauge theories with Higgs fields the situation is again different: Rotating excitations of static black holes generically exist; they are, however, necessarily charged.

Since we are still lacking a deeper physical understanding of the facts mentioned above, we have been looking for other (not gravitating) examples which might help to find a clue. On the basis of our previous experience, we originally expected static *dyon* solutions to admit rotational excitations. The simplest examples are the Julia-Zee dyons, which are related to the Bogomol'nyi-Prasad-Sommerfield (BPS) monopole by a one-parameter family of hyperbolic rotations in internal space.

The problem of small fluctuations around BPS monopoles has been examined some time ago by Mottola [5], Adler [6], Weinberg [7], and completed in a comprehensive analysis by Akhoury *et al.* [8]. The main emphasis was placed on the study of normalizable zero modes in the *self-dual* sector, because these are relevant to the structure of multimonopole solutions. The moduli space of SU(2) monopole solutions carrying n units of magnetic charge was shown to be $4n$-dimensional [7,9]. (See also [10–12] for a generalization to arbitrary gauge groups and for further references.) As these studies are dealing with the self-dual sector, an investigation of the remaining zero modes seems to be necessary. This is also motivated by the following observations, which are obtained from general considerations.

The solutions to the linearized Bogomol'nyi equations—independently of whether or not they are physically acceptable—cannot give rise to a nonvanishing angular momentum. This is true for both BPS monopoles and Julia-Zee dyons.

The only excitations of BPS monopoles which can contribute to the angular momentum arise from perturbations of the time component, $\delta\Phi \equiv \delta A_t$, of the gauge potential; these will be called *electric* modes.

The only perturbations of Julia-Zee dyons which can contribute to the angular momentum are the electric ones and the non-self-dual magnetic ones. (A mode will be called *magnetic*, if $\delta\Phi$ vanishes, and *non-self-dual*, if it is a solution of the linearized field equations, but not of the linearized Bogomol'nyi equations.)

The full problem, including the non-self-dual fluctuations, was studied by Baake [13] in connection with the stability analysis of the t' Hooft–Polyakov monopole. In his work, Baake mainly focused on the *negative* fluctuation modes, the absence of which he was able to prove by applying the Jacoby criterion. Since we are not aware of any other work devoted to non-self-dual zero modes, we carry out a systematic, gauge invariant perturbation analysis in order to study the rotational excitations of BPS monopoles and Julia-Zee dyons. The emphasis in the present article is mainly placed on the methods. The main result is, unfortunately, negative: Neither BPS monopoles nor Julia-Zee dyons admit slowly rotating excitations.

A further motivation for studying non-self-dual rotational excitations is provided by a theorem due to Taubes [14], according to which not every finite energy solution to the field equations in the BPS limit has to satisfy the *first order* Bogomol'nyi equations. Hence, the existence of physically acceptable excitations orthogonal to the Bogomol'nyi sector is not *a priori* excluded. However, the results of the present work imply that all non-self-dual axisymmetric finite energy solutions, if they exist, are necessarily *disconnected* from the Julia-Zee dyons. This is, in fact, a weak version of the original conjecture [15] (the general form of which was disproved in Taubes' work [14]). It is, however, likely that configurations with unit winding number and discrete angular momenta exist. This is, for instance, the case for boson stars [16].

This paper is organized as follows: In Sec. II we briefly review the symmetry which connects the PBS monopole solution with the one-parameter family of Julia-Zee dyons. In Sec. III we show how to use this symmetry to reduce the perturbation analysis for Julia-Zee dyons to that for the PBS monopole. The main advantage of this consists in the fact that, after a hyperbolic rotation, the electric background field vanishes. This implies that—in the rotated system—the electric perturbations, $\delta\Phi$, do not couple to the magnetic ones. We then show that the non-self-dual magnetic perturbations are governed by a system of first order equations for a one-form, δB. The latter comprises the perturbations of the Higgs field, δH, and the perturbation of the three-dimensional gauge potential, δA, in a gauge invariant way.

In Sec. IV we present the decomposition of the gauge invariant perturbations $\delta\Phi$ and δB in terms of isospin harmonics. We also show that the expression for the angular momentum can be integrated, implying that only the boundary values of the perturbation amplitudes are relevant. The complete set of perturbation equations is derived in Sec. V. This consists of an even and an odd parity sector. Each sector comprises the electric equations for $\delta\Phi$, the magnetic equations for δB (governing the non-self-dual modes), and the inhomogeneous linearized Bogomol'nyi equations for δH and δA in terms of the source δB.

In Sec. VI we discuss the odd parity perturbations and present the solutions of the complete set of equations in closed form. As the odd parity modes cannot contribute to the angular momentum, we conclude from the solutions that there exist no physically acceptable axial dipole *deformations* of Julia-Zee dyons. The more interesting *even* parity modes are discussed in Sec. VII. We show how to use the explicitly known solutions to reduce the *magnetic* problem to a standard Schrödinger equation. We also prove that the *electric* perturbations are governed by exactly the same equation. Since the latter has a non-negative potential, we are able to present a rigorous discussion of all modes. It turns out that there exist solutions (both electric and magnetic) which give rise to finite angular momentum. However, none of these modes are regular.

II. BPS MONOPOLES AND JULIA-ZEE DYONS

We consider stationary solutions to the SU(2) Yang-Mills-Higgs (YMH) equations with gauge potential $A^{(4)}$ and Higgs triplet H in the BPS limit (i.e., without Higgs self-interaction). The dimensionally reduced YMH action becomes

$$S=\frac{1}{2}\int\{(F,F)+(\mathrm{D}H,\mathrm{D}H)-(\mathrm{D}\Phi,\mathrm{D}\Phi)-[\Phi,H]^2\}\mathrm{d}^3x, \tag{1}$$

where Φ and A parametrize the electric and the magnetic components of the gauge potential,

$$A^{(4)}=\Phi\,\mathrm{d}t+A. \tag{2}$$

The quantities F and D denote the field strength two-form and the gauge covariant derivative with respect to the three-dimensional magnetic potential A:

$$F=\mathrm{d}A+A\wedge A,\quad \mathrm{D}\Phi=\mathrm{d}\Phi+[A,\Phi],\quad \mathrm{D}H=\mathrm{d}H+[A,H]. \tag{3}$$

[For arbitrary Lie algebra valued p-forms α the inner product is defined by $(\alpha,\alpha)\mathrm{d}^3x=\mathrm{Tr}\{\alpha\wedge*\alpha\}$, where $*$ is the three-dimensional Hodge dual.]

The perturbation analysis for Julia-Zee (JZ) dyons will be simplified considerably by the fact that the dimensionally reduced action (1) is invariant under hyperbolic rotations in the (H,Φ) plane; that is, the transformation

$$\begin{pmatrix}H\\ \Phi\end{pmatrix}\to\begin{pmatrix}\cosh(\gamma) & \sinh(\gamma)\\ \sinh(\gamma) & \cosh(\gamma)\end{pmatrix}\begin{pmatrix}H\\ \Phi\end{pmatrix} \tag{4}$$

is a symmetry of the action (1).

In particular, a BPS monopole solution $H=H_{\mathrm{mon}}$, $\Phi=0$ with magnetic charge P_{mon} gives rise to a one-parameter family of JZ dyons, $H=\cosh(\gamma)H_{\mathrm{mon}}$, $\Phi=\sinh(\gamma)H_{\mathrm{mon}}$, with magnetic charge $P=\cosh(\gamma)P_{\mathrm{mon}}$ and electric charge $Q=\cosh(\gamma)\sinh(\gamma)P_{\mathrm{mon}}$. This is also seen from the field equations

$$*\mathrm{D}*F=[\Phi,\mathrm{D}\Phi]-[H,\mathrm{D}H], \tag{5}$$

$$*\mathrm{D}*\mathrm{D}H=[\Phi,[\Phi,H]], \tag{6}$$

$$*\mathrm{D}*\mathrm{D}\Phi=[H,[\Phi,H]], \tag{7}$$

which reduce to the monopole equations, $\mathrm{D}*F=-*[H_{\mathrm{mon}},\mathrm{D}H_{\mathrm{mon}}]$ and $\mathrm{D}*\mathrm{D}H_{\mathrm{mon}}=0$, for $H=\cosh(\gamma)H_{\mathrm{mon}}$ and $\Phi=\sinh(\gamma)H_{\mathrm{mon}}$.

It is worth recalling that the total energy is not invariant under the transformation (4). However, for fixed charges P and Q, defined by the flux integrals

$$P=\int\mathrm{Tr}\{H\,F\},\quad Q=\int\mathrm{Tr}\{H*\mathrm{D}\Phi\}, \tag{8}$$

over the two-sphere at infinity, the energy assumes its global minimum for the corresponding JZ dyon solution. This is seen as follows: Using the field equations to express P and Q as volume integrals of $\mathrm{Tr}\{\mathrm{D}H\wedge F\}$ and $\mathrm{Tr}\{\mathrm{D}H\wedge*\mathrm{D}\Phi\}$, respectively, the total energy may be expressed as follows [17,18]:

$$E=\frac{1}{2}\int\{(F)^2+(\mathrm{D}H)^2+(\mathrm{D}\Phi)^2+[H,\Phi]^2\}\mathrm{d}^3x$$

$$=\frac{1}{2}\int\Bigg\{(\mathrm{D}\Phi-\tanh(\gamma)\mathrm{D}H)^2+\left(*F-\frac{1}{\cosh^2(\gamma)}\,\mathrm{D}H\right)^2+[H,\Phi]^2\Bigg\}\mathrm{d}^3x+\frac{1}{\cosh(\gamma)}(Q\,\sinh(\gamma)+P), \tag{9}$$

where γ is arbitrary and $(F)^2$ is a shorthand for (F,F), etc. From this one finds the bound (assuming, without loss of generality, that Q and P are non-negative)

$$E\geqslant\sqrt{Q^2+P^2}=\cosh^2(\gamma)P_{\mathrm{mon}}, \tag{10}$$

where equality holds if and only if A, H, and Φ are subject to the first order equation $\mathrm{D}\Phi/\sinh(\gamma)=\mathrm{D}H/\cosh(\gamma)=*F$, which is exactly the Bogomol'nyi equation,

$$*F=\mathrm{D}H_{\mathrm{mon}}, \tag{11}$$

written in terms of the rotated fields $H=\cosh(\gamma)H_{\mathrm{mon}}$ and $\Phi=\sinh(\gamma)H_{\mathrm{mon}}$.

III. LINEAR PERTURBATIONS OF DYONS

The perturbation analysis for the BPS monopole is simplified by the circumstance that the electric perturbation $\delta\Phi_{\mathrm{mon}}$ does not couple to the magnetic perturbations δH_{mon} and δA_{mon}. This is an immediate consequence of the fact that the BPS background configuration is nonelectric, $\Phi_{\mathrm{mon}}=0$.

Since the electric background field does not vanish for JZ dyons, the electric and the magnetic perturbations are coupled in this case. However, the linearity of the symmetry (4) implies that all linear perturbations of JZ dyons can be obtained from the linear perturbations of the BPS monopole after a hyperbolic rotation with parameter $\sinh(\gamma)=Q/P$. It is, therefore, sufficient to consider the perturbation analysis of the BPS monopole. Before doing so, we compute the various contributions to the angular momentum.

A. Angular momentum

The total angular momentum (along the symmetry axis) of a stationary YMH configuration is

$$J=\int T_{t\varphi}\mathrm{d}^3x, \tag{12}$$

where the relevant component of the stress-energy tensor in terms of the three-dimensional quantities is given by

$$T_{t\varphi}=\frac{1}{2}\,\mathrm{Tr}\{[\Phi,H]\mathrm{D}H-*(\mathrm{D}\Phi\wedge *F)\}_{\varphi}.$$

By virtue of the field equation (5) and the relations $\mathrm{Tr}\{\Phi[\Phi,\mathrm{D}\Phi]\}=0$ and $\mathrm{Tr}\{\Phi[H,\mathrm{D}H]\}=\mathrm{Tr}\{[\Phi,H]\mathrm{D}H]\}$, we also find (after integrating by parts)

$$T_{t\varphi}=-\frac{1}{2}(*\mathrm{d}\;\mathrm{Tr}\{\Phi *F\})_{\varphi}. \tag{13}$$

This shows that both the electric and the magnetic perturbations of JZ dyons contribute to the angular momentum, since

$$\delta T_{t\varphi}=-\frac{1}{2}(*\mathrm{d}\;\mathrm{Tr}\{\delta\Phi *F+\Phi *\delta F\})_{\varphi}.$$

(Note that the second term is absent if the electric background field vanishes, implying that only electric perturbations give rise to the angular momentum of a BPS monopole.) Since the dyon perturbations can be obtained from the monopole perturbations, we express the angular momentum in terms of the latter, using $\Phi=\sinh(\gamma)H_{\mathrm{mon}}$ and $\delta\Phi=\sinh(\gamma)\delta H_{\mathrm{mon}}+\cosh(\gamma)\delta\Phi_{\mathrm{mon}}$. With

$$\delta T_{t\varphi}=\cosh(\gamma)\,\delta T^{\mathrm{el}}_{t\varphi}+\sinh(\gamma)\,\delta T^{\mathrm{mg}}_{t\varphi}, \tag{14}$$

one finds

$$\delta T^{\mathrm{el}}_{t\varphi}=-\frac{1}{2}(*\mathrm{d}\;\mathrm{Tr}\{\delta\Phi_{\mathrm{mon}}*F\})_{\varphi}, \tag{15}$$

$$\delta T^{\mathrm{mg}}_{t\varphi}=-\frac{1}{2}(\delta *\mathrm{d}\;\mathrm{Tr}\{H_{\mathrm{mon}}*F\})_{\varphi}. \tag{16}$$

It is worthwhile noticing that both contributions to $\delta T_{t\varphi}$ are separately gauge invariant. This is obvious for the electric part, since this is proportional to the perturbation of a field which vanishes on the background, namely, Φ_{mon}. The same is true for the magnetic part, since the quantity $\mathrm{d}\,\mathrm{Tr}\{H_{\mathrm{mon}}*F\}$ vanishes as well for a PBS background configuration. (Use $*F=\mathrm{D}H_{\mathrm{mon}}$ to see this.) In fact, defining the one-form B according to

$$B=\mathrm{D}H_{\mathrm{mon}}-*F, \tag{17}$$

the magnetic contribution (16) to the angular momentum can be cast into the simple form

$$\delta T^{\mathrm{mg}}_{t\varphi}=-\frac{1}{2}(*\;\mathrm{Tr}\{\delta B\wedge *F\})_{\varphi}, \tag{18}$$

which is manifestly gauge invariant, since, by definition, B vanishes for the BPS background configuration.

The above expressions imply the following facts: First, the perturbation analysis for JZ dyons reduces to the perturbation analysis for BPS monopoles. Second, the electric *and* the magnetic perturbations of a BPS background contribute to the dyon angular momentum. Third, only the *non-self-dual* modes, that is, the magnetic perturbations with $\delta B\neq 0$, contribute to the dyon angular momentum.

The last statement reveals a fundamental difference between the perturbation theory of BPS monopoles and JZ dyons: Although the perturbation equations for JZ dyons can be reduced to the ones for the BPS monopole, the physical contents are quite different: While only electric perturbations can give rise to the angular momentum of a monopole configuration, magnetic perturbations need to be taken into account as well in the dyon case. Moreover, it is not sufficient to consider perturbations within the Bogomol'nyi sector, since the latter cannot contribute to the angular momentum of a dyon.

B. Linear perturbations of the BPS monopole

Since the perturbation analysis of the JZ dyons can be reduced to the one for the BPS monopole, we shall now focus on the latter. In the following we omit the subscript ''mon'' indicating the monopole fields, that is, we write δH for δH_{mon}, etc. Suppose that there is (at least) a one-parameter family of continuous deformations of the BPS monopole background, $*F=\mathrm{D}H$, $\Phi=0$. Then the tangent to this satisfies the linearized field equations. In order to linearize Eqs. (5) and (6), it is very convenient to introduce the one-form field B defined in Eq. (17). One may then write the first field equation in the form $\mathrm{D}B=\mathrm{D}^2H-\mathrm{D}*F=[H,*\mathrm{D}H-F]-[\Phi,*\mathrm{D}\Phi]$, whereas the second field equation becomes $\mathrm{D}*B=\mathrm{D}*\mathrm{D}H-\mathrm{D}F=*[\Phi,[\Phi,H]]$. Hence, Eqs. (5) and (6) assume the form

$$\mathrm{D}B-[H,*B]=-[\Phi,*\mathrm{D}\Phi], \tag{19}$$

and

$$\mathrm{D}*B=*[\Phi,[\Phi,H]], \tag{20}$$

respectively. The linearization of the field equations (7), (19), (20) is completely trivial, since both the electric field Φ and the magnetic one-form $B\equiv\mathrm{D}H-*F$ vanish for a BPS background. Hence, the linearized field equations involve only the gauge invariant perturbations $\delta\Phi$ and δB: One immediately finds the results

$$\text{electric perturbations:}\quad \mathrm{D}*\mathrm{D}\delta\Phi=*[H,[\delta\Phi,H]], \tag{21}$$

$$\text{magnetic perturbations:}\quad \mathrm{D}\delta B=[H,*\delta B],\quad \mathrm{D}*\delta B=0, \tag{22}$$

where δB is obtained from the definition (17), that is,

$$\delta B=\mathrm{D}\delta H-*\mathrm{D}\delta A-[H,\delta A]. \tag{23}$$

(Here and in the following all quantities without a ''δ'' refer to background fields.) Before we consider the harmonic analysis of Eqs. (21)–(23), we note the following:

The linearization of the Bogomol'nyi equation (11), $\delta B=0$, has been studied extensively in the literature. The solutions to $\delta B=0$ are, however, only a subset of the general magnetic perturbations. The full magnetic perturbations are governed by the second order equations for δA and δH, which are equivalent to the first order equations (22) for δB and the inhomogeneous equation (23). In particular, we have already argued above that only the nontrivial solutions $\delta B\neq 0$ to Eq. (22) can contribute to the angular momentum [see Eq. (18)].

In order to find the general magnetic perturbations, one proceeds in two steps: First, one has to solve the system (22) for δB. Once δB is known, it remains to solve the inhomogeneous linearized Bogomol'nyi equation (23) for δA and δH. This is achieved by using Green's method, also taking advantage of the explicitly known solutions to the homogeneous equations, $\delta B=0$, derived in [6] and [8].

Since the background BPS configuration has vanishing B, the magnetic perturbation δB is manifestly gauge invariant. This is also verified by using the general behavior of the perturbations δA and δH under gauge transformations generated by a Lie algebra valued scalar field χ:

$$\delta A\to\delta A+\mathrm{D}\chi,\quad \delta H\to\delta H+[H,\chi]. \tag{24}$$

Hence, $\delta\mathrm{D}H\to\delta\mathrm{D}H+[\mathrm{D}H,\chi]$, and $\delta F\to\delta F+[F,\chi]$, implying that $\delta B\to\delta B+[B,\chi]=\delta B$.

The second equation in Eq. (22) is a consistency condition for the first one: Indeed, applying D on the first equation and using $\mathrm{D}^2\delta B=[F,\delta B]$ on the left-hand side (LHS), and $[\mathrm{D}H,*\delta B]=[*\mathrm{D}H,\delta B]=[F,\delta B]$ on the right-hand side (RHS), yields the necessary condition $[H,\mathrm{D}*\delta B]=0$.

IV. HARMONIC ANALYSIS

Since the unperturbed BPS solution is spherically symmetric, we perform a multipole decomposition and rewrite the electric perturbation equations (21) and the magnetic ones (22), (23) as systems of ordinary differential equations with respect to the radial coordinate. Using these equations, we show that the angular momentum integral can be computed exactly. Hence, the total angular momentum arising from electric and magnetic perturbations is determined by the asymptotic behavior of the gauge invariant amplitudes $\delta\Phi$ and δB, respectively.

A. Isospin harmonics

The basic fields H,Φ,A, the auxiliary field B, and their perturbations, are functions and one-forms with values in the Lie algebra su(2) of the gauge group SU(2). Let us start by considering such functions on the two-sphere S^2. A convenient basis, reducing the natural representation of SU(2), is obtained by taking the inner product of the vector spherical harmonics $\boldsymbol{Y}^L_{JM}$ with the basis $\boldsymbol{\tau}=\boldsymbol{\sigma}/(2i)$ of su(2) (where $\boldsymbol{\sigma}$ are the Pauli matrices)

$$C^L_{JM}(\vartheta,\varphi)=\boldsymbol{\tau}\cdot\boldsymbol{Y}^L_{JM}(\vartheta,\varphi). \tag{25}$$

The isospin harmonics C^L_{JM} have total angular momentum J and fixed parity $(-1)^L$. Instead of the $\boldsymbol{Y}^L_{JM}$ it is also usual to consider the basis $\boldsymbol{Y}^{(\lambda)}_{JM}$ (with$\lambda=0,\pm 1$). For $\lambda=0$ and $\lambda=1$ these vector harmonics are transverse, while they are longitudinal for $\lambda=-1$ (with respect to the radial unit direction $\hat{\boldsymbol{r}}$). The transverse harmonics $\boldsymbol{Y}^{(1)}_{JM}$ and $\boldsymbol{Y}^{(0)}_{JM}$ are also called electric and magnetic multipoles, respectively. They are obtained by applying certain differential operators on the ordinary spherical harmonics Y_{JM}, while the longitudinal harmonics are given by $\boldsymbol{Y}^{(-1)}_{JM}=\hat{\boldsymbol{r}}Y_{JM}$ (see, e.g., [19] or [20]). The formulas for the $\boldsymbol{Y}^{(\lambda)}_{JM}$ can readily be translated into the corresponding formulas for the isospin harmonics $C^{(\lambda)}_{JM}=\boldsymbol{\tau}\cdot\boldsymbol{Y}^{(\lambda)}_{JM}$ (with $\lambda=0,\pm1$). One finds

$$C^{(-1)}_{JM}=\tau_r Y_{JM},$$

$$C^{(0)}_{JM}=\frac{i}{\sqrt{J(J+1)}}\langle \mathrm{d}\tau_r,\hat{*}\mathrm{d}Y_{JM}\rangle,$$

$$C^{(+1)}_{JM}=\frac{1}{\sqrt{J(J+1)}}\langle \mathrm{d}\tau_r,\mathrm{d}Y_{JM}\rangle, \tag{26}$$

where $\tau_r=\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}$. Here $\langle\,,\,\rangle$ and $\hat{*}$ denote the inner product and the Hodge dual with respect to the standard metric on S^2. (Also note that the spherical components of the $\boldsymbol{\tau}$ obey the relations $\mathrm{d}\tau_r=\tau_\vartheta\mathrm{d}\vartheta+\tau_\varphi\sin\vartheta\mathrm{d}\varphi$ and $[\tau_\vartheta,\tau_\varphi]=\tau_r$; see Appendix A.) In terms of the isospin harmonics, the well-known relations between the vector harmonics $\boldsymbol{Y}^{L}_{JM}$ and $\boldsymbol{Y}^{(\lambda)}_{JM}$ become

$$C^{J+1}_{JM}=\frac{1}{\sqrt{2J+1}}[\sqrt{J}C^{(1)}_{JM}-\sqrt{J+1}C^{(-1)}_{JM}],$$

$$C^{J}_{JM}=C^{(0)}_{JM},$$

$$C^{J-1}_{JM}=\frac{1}{\sqrt{2J+1}}[\sqrt{J+1}C^{(1)}_{JM}+\sqrt{J}C^{(-1)}_{JM}]. \tag{27}$$

By construction, the isospin harmonics $C^{J}_{JM},C^{J\pm1}_{JM}$ are eigenfunctions of the spherical Laplacian, $\hat{\Delta}=\hat{*}\mathrm{d}\hat{*}\mathrm{d}$, and of the parity operator, $\hat{P}$:

$$\hat{\Delta}C^{L}_{JM}=-L(L+1)C^{L}_{JM}, \tag{28}$$

$$\hat{P}C^{L}_{JM}=(-1)^L C^{L}_{JM}, \tag{29}$$

where $L=J$, $J\pm1$. (The exterior derivatives of the isospin harmonics C^{J}_{JM} and $C^{J\pm1}_{JM}$ and their S^2 duals are particularly convenient for analyzing perturbations of Lie algebra valued one-forms [21,4]. For the general theory of monopole harmonics we refer to [22].)

B. Perturbation amplitudes

Since rotational modes are our primary concern in this article, we now focus on the sector $J=1$. For the $C^{(\lambda)}_{10}$ ($\lambda=0,\pm1$) we use (with some change of normalization) the letters X, Y, and Z. A convenient basis of $J=1$ isospin harmonics then is

$$X=\tau_r K,\quad \text{where}\ \ K\equiv\cos\vartheta,$$

$$\sqrt{2}Y=\langle \mathrm{d}\tau_r,\mathrm{d}K\rangle=-\tau_\vartheta\sin\vartheta,$$

$$\sqrt{2}Z=-\langle \mathrm{d}\tau_r,\hat{*}\mathrm{d}K\rangle=\tau_\varphi\sin\vartheta, \tag{30}$$

where X and Y span the even parity sector, while Z has odd parity. The su(2) valued electric perturbation function $\delta\Phi$ can, therefore, be expanded as $\delta\Phi=\delta\Phi^{\mathrm{even}}+\delta\Phi^{\mathrm{odd}}$, with

$$\delta\Phi^{\mathrm{even}}=\frac{1}{r}(\phi_- X+\phi_+ Y),$$

$$\delta\Phi^{\mathrm{odd}}=\frac{1}{r}(\tilde{\phi}Z). \tag{31}$$

[The factor $1/r$ is introduced for convenience; see, e.g., Eqs. (45), (46). Throughout this article, all amplitudes furnished with a tilde refer to the odd parity sector, which is relevant for deformations.] A similar expansion holds for δH; however, unlike $\delta\Phi$, δH is not gauge invariant; see Sec. V C and Appendix D.

Turning to Lie algebra valued one-forms, we note that the exterior derivatives of the basis functions X, Y, and Z can be expressed in terms of the derivatives of τ_r and $K=\cos\vartheta\propto Y_{1\,0}$. [This is a peculiarity of the $J=1$ harmonics, for which $\mathrm{d}C^{(0)}_{1M}=(\sqrt{2}\mathrm{d}Y+\mathrm{d}X)/\sqrt{3}=0$.] One finds

$$\mathrm{d}X=-\sqrt{2}\mathrm{d}Y=\tau_r\mathrm{d}K+K\mathrm{d}\tau_r,$$

$$\hat{*}\sqrt{2}\mathrm{d}Z=\tau_r\mathrm{d}K-K\mathrm{d}\tau_r. \tag{32}$$

As the parity operation commutes with the exterior differentiations and anticommutes with the Hodge dual, one can expand the su(2) valued magnetic perturbation one-form δB as $\delta B=\delta B^{\mathrm{even}}+\delta B^{\mathrm{odd}}$, with

$$\delta B^{\mathrm{even}}=\frac{1}{r^2}(b_- X+b_+ Y)\mathrm{d}r+\beta_1\tau_r\mathrm{d}K+\beta_2 K\mathrm{d}\tau_r,$$

$$\delta B^{\mathrm{odd}}=\frac{1}{r^2}(\tilde{b}Z)\mathrm{d}r+\tilde{\beta}_1\hat{*}\tau_r\mathrm{d}K+\tilde{\beta}_2\hat{*}K\mathrm{d}\tau_r, \tag{33}$$

where $\tilde{b}$, $b_\pm$, $\beta_{1,2}$, and $\tilde{\beta}_{1,2}$ depend on the radial coordinate r. (Again, a similar formula holds for δA. In contrast to δB, δA is not gauge invariant, implying that not all coefficients in the expansion of δA correspond to physical degrees of freedom; see Sec. V C and Appendix D).

At this point we also recall that the background gauge potential and Higgs field are parametrized in terms of two radial functions $w(r)$ and $h(r)$ (see Appendix A),

$$A=[1-w(r)]\hat{*}\mathrm{d}\tau_r,\quad H=h(r)\tau_r. \tag{34}$$

Since τ_r is an eigenfunction of the spherical Laplacian, $\mathrm{d}\hat{*}\mathrm{d}\tau_r=-2\tau_r\mathrm{d}\Omega$, the background field strength becomes $F=-\mathrm{d}w\wedge\hat{*}\mathrm{d}\tau_r+(w^2-1)\tau_r\mathrm{d}\Omega$. The BPS equations, $F=*\mathrm{D}H$, thus read

$$w' = wh, \quad r^2 h' = w^2 - 1, \tag{35}$$

with the globally regular solution

$$w(r) = \frac{r}{\sinh(r)}, \quad h(r) = \frac{1}{r} - \frac{\cosh(r)}{\sinh(r)}. \tag{36}$$

For later use we also note that the second order equation for h can be integrated, which yields the useful relation

$$h' = h^2 - \frac{2h}{r} - 1. \tag{37}$$

C. Integration of angular momentum

We now show that the total angular momentum $\delta J = \cosh(\gamma)\delta J^{\text{el}} + \sinh(\gamma)\delta J^{\text{mg}}$ can be expressed in terms of the values of the gauge invariant perturbations $\delta\Phi$ and δB at the origin and at infinity. According to Eqs. (15) and (18), the electric and the magnetic perturbations give rise to

$$\delta J^{\text{el}} = -\frac{1}{2}\int (*\mathrm{d}\,\mathrm{Tr}\{\delta\Phi^{\text{even}} * F\})_{\varphi}\mathrm{d}^3x \tag{38}$$

and

$$\delta J^{\text{mg}} = -\frac{1}{2}\int (*\mathrm{Tr}\{\delta B^{\text{even}} \wedge * F\})_{\varphi}\mathrm{d}^3x, \tag{39}$$

respectively. Here we have already used the fact that only the even parity sector contributes to the total angular momentum. In order to express the above integrands in terms of the radial amplitudes $\phi_{\pm}$, $b_{\pm}$, and $\beta_{1,2}$, we first note that the background field strength can be written in the simple form

$$*F = w'\mathrm{d}\tau_r + h'\tau_r \mathrm{d}r. \tag{40}$$

Taking advantage of the trace formulas $\mathrm{Tr}\{X\tau_r\} = -K/2$, $\mathrm{Tr}\{Y\tau_r\} = \mathrm{Tr}\{Z\tau_r\} = 0$, and $\mathrm{Tr}\{X\mathrm{d}\tau_r\} = 0$, $\mathrm{Tr}\{Y\mathrm{d}\tau_r\} = \hat{*}\ \mathrm{Tr}\{Z\mathrm{d}\tau_r\} = -dK/(2\sqrt{2})$, it is now not difficult to compute the above integrands from the expansions (31) and (33). One finds

$$*\mathrm{d}\,\mathrm{Tr}\{\delta\Phi^{\text{even}} * F\} = \frac{1}{2r^2}\left[rh'\phi_- - \frac{r^2}{\sqrt{2}}\left(\frac{w'\phi_+}{r}\right)'\right]\hat{*}\mathrm{d}K, \tag{41}$$

$$*\mathrm{Tr}\{\delta B^{\text{even}} \wedge * F\} = \frac{1}{2r^2}\left[r^2 h'\beta_1 - \frac{1}{\sqrt{2}}w' b_+\right]\hat{*}\mathrm{d}K. \tag{42}$$

With $K \equiv \cos\vartheta$ we have $\hat{*}\mathrm{d}K = -\sin^2\vartheta\mathrm{d}\varphi$, which shows that the above formulas yield the φ components appearing in the integrands of Eqs. (38) and (39). It is an interesting fact that the above brackets can be written as radial derivatives. This enables one to perform the angular momentum integrals and to express δJ^{el} and δJ^{mg} in terms of the values of $\delta\Phi^{\text{even}}$ and δB^{even} at the origin and at infinity. In order to see this, one has to use the perturbation equations in the harmonic decomposition, as given in the next section. Considering the magnetic part, one uses the first two equations in Eq. (49) to obtain $2(1-w^2)\beta_1 = b'_- + \sqrt{2}wb'_+$, which enables one to eliminate β_1 in Eq. (42). Also taking advantage of the background equation (35), one then has $[r^2 h'\beta_1 - w' b_+/\sqrt{2}] = -[b_- + \sqrt{2}wb_+]'/2$. A similar, but more complicated manipulation uses the second order equations (45) to write the electric contribution (41) in the desired form; see Appendix F. The two contributions (38) and (39) to the angular momentum finally become

$$\delta J^{\text{el}} = -\frac{\pi}{3}[(1-w^2-2rh)\phi_- + r^2 h\phi'_- + \sqrt{2}wrh\phi_+]_0^{\infty}, \tag{43}$$

$$\delta J^{\text{mg}} = -\frac{\pi}{3}[b_- + \sqrt{2}wb_+]_0^{\infty}. \tag{44}$$

V. PERTURBATION EQUATIONS

Using the expansions (31) and (33), as well as the tools developed in Appendixes B and C, it is now straightforward to write down the system of perturbation equations. This consists of Eq. (21) for the electric perturbations $\delta\Phi$, Eqs. (22) for the magnetic perturbations δB, and the inhomogeneous BPS Eqs. (23) for δH and δA.

A. Electric perturbations

For the electric perturbations (31), governed by Eq. (21), one finds the differential equations

$$\begin{pmatrix}\phi''_- \\ \phi''_+\end{pmatrix} = \frac{1}{r^2}\begin{pmatrix} 2(w^2+1) & -2\sqrt{2}w \\ -2\sqrt{2}w & (w^2+1+r^2h^2)\end{pmatrix}\begin{pmatrix}\phi_- \\ \phi_+\end{pmatrix} \tag{45}$$

in the even parity sector, and

$$\tilde{\phi}'' = \frac{1}{r^2}(w^2+1+r^2h^2)\tilde{\phi} \tag{46}$$

in the odd parity sector. Here we have used Eq. (C1) to compute the LHS of Eq. (21), and $[\tau_r, X] = 0$, $[\tau_r, Y] = -Z$, $[\tau_r, Z] = Y$ to obtain the RHS: $[H,[\delta\Phi, H]] = h^2 r^{-1}(\phi_+ Y + \tilde{\phi}Z)$.

B. Magnetic perturbations: δB equations

In order to determine the magnetic perturbations δB, we first write the decomposition (33) in the form

$$\delta B = \frac{1}{r^2}b\mathrm{d}r + \hat{B}, \tag{47}$$

where the one-form $\hat{B}$ is tangential to S^2. In terms of b and $\hat{B}$, the magnetic perturbation equations (22) assume the form

$$[H,b]=\hat{*}\hat{D}\hat{B},$$

$$\hat{B}'=r^{-2}\hat{D}b-[H,\hat{*}\hat{B}],$$

$$b'=-\hat{*}\hat{D}\hat{*}\hat{B}. \tag{48}$$

Here we have used the fact that the unperturbed gauge potential has no radial component, implying the decomposition $D=dr\wedge\partial_r+\hat{D}$ for the covariant derivative (see Appendix B). Taking advantage of the formulas given in Appendix C, it is now not hard to obtain the sets of differential equations for the radial functions parametrizing $\delta B^{\rm even}$ and $\delta B^{\rm odd}$. One finds

$$b_-'=2(\beta_1+w\beta_2),\quad b_+'=-\sqrt{2}(w\beta_1+\beta_2),$$

$$r^2\beta_1'=b_--\frac{w}{\sqrt{2}}b_+,\quad r^2(\beta_2'+h\beta_2)=wb_--\frac{1}{\sqrt{2}}b_+,$$

$$hb_+=\sqrt{2}(\beta_2-w\beta_1) \tag{49}$$

for the even parity sector, and

$$\tilde{b}'=\sqrt{2}(w\tilde{\beta}_1-\tilde{\beta}_2),$$

$$r^2\tilde{\beta}_1'=\frac{w}{\sqrt{2}}\tilde{b},\quad r^2(\tilde{\beta}_2'+h\tilde{\beta}_2)=-\frac{1}{\sqrt{2}}\tilde{b},$$

$$0=\tilde{\beta}_1+w\tilde{\beta}_2,\quad h\tilde{b}=\sqrt{2}(w\tilde{\beta}_1+\tilde{\beta}_2) \tag{50}$$

for the odd parity sector. We note that both sectors contain constraint equations, reflecting the fact that the second equation in Eq. (22) is an integrability condition for the first one.

At this point we also note the following, somewhat surprising fact: The scalar magnetic amplitudes $b_\pm$ and $\tilde{b}$ satisfy the same set of second order equations (45), (46) as the electric amplitudes $\phi_\pm$ and $\tilde{\phi}$,

$$\begin{pmatrix} b_-'' \\ b_+'' \end{pmatrix}=\frac{1}{r^2}\begin{pmatrix} 2(w^2+1) & -2\sqrt{2}w \\ -2\sqrt{2}w & (w^2+1+r^2h^2)\end{pmatrix}\begin{pmatrix} b_- \\ b_+ \end{pmatrix}, \tag{51}$$

$$\tilde{b}''=\frac{1}{r^2}(w^2+1+r^2h^2)\tilde{b}. \tag{52}$$

This follows from the arguments given in Appendix B, and is also verified directly from the above equations. Using the odd parity equations (50) we have, for instance, $\tilde{b}''=\sqrt{2}(w'\tilde{\beta}_1+w\tilde{\beta}_1'-\tilde{\beta}_2')=[h^2+r^{-2}(w^2+1)]\tilde{b}$, where we have used the Bogomol'nyi equations (35) for the background fields h and w.

We also point out that not all solutions to the second order equations (51) and (52) satisfy the first order equations (49) and (50). In fact, it is not hard to see that the solution spaces defined by Eqs. (49) and (50) are three- and one-dimensional, respectively, rather than four- and two-dimensional.

C. Magnetic perturbations: Inhomogeneous BPS equations

In order to write out the inhomogeneous Bogomol'nyi equations (23), we need the harmonic decomposition of the fields δH and δA. The fact that the latter are not gauge invariant enables us to get rid of certain amplitudes. In Appendix D it is shown that—up to a pure gauge—the harmonic decompositions of δH and δA assume the form

$$\delta H^{\rm even}=\gamma_-X+\gamma_+Y,$$

$$\delta H^{\rm odd}=\tilde{\gamma}Z, \tag{53}$$

and

$$\delta A^{\rm even}=\alpha_1\hat{*}\tau_r dK+\alpha_2\hat{*}K d\tau_r,$$

$$\delta A^{\rm odd}=\tilde{\alpha}_1\tau_r dK+\tilde{\alpha}_2 K d\tau_r, \tag{54}$$

respectively. The radial functions $\gamma_\pm$, $\tilde{\gamma}$, $\alpha_{1,2}$, and $\tilde{\alpha}_{1,2}$ are gauge invariant, up to a one-dimensional set of residual gauge transformations in the even parity sector,

$$\gamma_-\to\gamma_-,\quad \gamma_+\to\gamma_++hc_3,$$

$$\alpha_1\to\alpha_1+\frac{w}{\sqrt{2}}c_3,$$

$$\alpha_2\to\alpha_2-\frac{1}{\sqrt{2}}c_3, \tag{55}$$

and a two-dimensional set of residual gauge transformations in the odd parity sector,

$$\tilde{\gamma}\to\tilde{\gamma}-hc_2,$$

$$\tilde{\alpha}_1\to\tilde{\alpha}_1+c_1-\frac{w}{\sqrt{2}}c_2,$$

$$\tilde{\alpha}_2\to\tilde{\alpha}_2+wc_1-\frac{1}{\sqrt{2}}c_2, \tag{56}$$

where c_1, c_2, and c_3 are arbitrary constants parametrizing the residual gauge freedom (see Appendix D). In terms of the gauge invariant source terms $b_\pm$, $\tilde{b}$, $\beta_{1,2}$, and $\tilde{\beta}_{1,2}$, and the (almost) gauge invariant amplitudes introduced above, the inhomogeneous linearized Bogomol'nyi equations (23) eventually become

$$r^2\gamma_-'+2(\alpha_1+w\alpha_2)=b_-,$$

$$r^2\gamma_+'/\sqrt{2}-(w\alpha_1+\alpha_2)=b_+/\sqrt{2},$$

$$\alpha_1'+\gamma_--w\gamma_+/\sqrt{2}=\beta_1,$$

$$\alpha_2'-h\alpha_2+w\gamma_--\gamma_+/\sqrt{2}=\beta_2, \tag{57}$$

in the even parity sector, and

$$r^2\tilde{\gamma}'/\sqrt{2}+(\tilde{\alpha}_2-w\tilde{\alpha}_1)=\tilde{b}/\sqrt{2},$$
$$-\tilde{\alpha}_1'+w\tilde{\gamma}/\sqrt{2}=\tilde{\beta}_1,$$
$$-\tilde{\alpha}_2'+h\tilde{\alpha}_2-\tilde{\gamma}/\sqrt{2}=\tilde{\beta}_2, \tag{58}$$

in the sector with odd parity.

For the vanishing RHS, the above equations are the linearized Bogomol'nyi equations, which have been studied in the literature. Using the background equations (35), it is easy to verify that the residual gauge mode $\gamma_-=0$, $\gamma_+=\sqrt{2}h$, $\alpha_1=w$, $\alpha_2=-1$ satisfies the homogeneous equations (57), while the residual gauge modes $\tilde{\gamma}=\sqrt{2}h$, $\tilde{\alpha}_1=w$, $\tilde{\alpha}_2=1$ and $\tilde{\gamma}=0$, $\tilde{\alpha}_1=1$, $\tilde{\alpha}_2=w$ are solutions to the homogeneous equations (58).

VI. ODD PARITY MODES

We shall now solve the perturbation equations. We start with the odd parity sector, for which all solutions can be obtained in closed form. We emphasize, however, that this sector is of minor importance, since the odd parity modes cannot contribute to the angular momentum. In Sec. VI A we compute the magnetic amplitudes δB^{odd}, which we use in Sec. VI B as source terms to obtain the perturbations δH^{odd} and δA^{odd}. In Sec. VI C we finally compute the electric perturbations $\delta\Phi^{\text{odd}}$.

A. Solutions to the δB equations

In order to compute the source term δB^{odd}, we have to solve Eqs. (50) for the amplitudes $\tilde{b}$ and $\tilde{\beta}_{1,2}$ defined in Eq. (33). Using the last two equations in Eq. (50) to express $\tilde{\beta}_1$ and $\tilde{\beta}_2$ in terms of $\tilde{b}$, the first equation becomes $\tilde{b}'/\tilde{b}=h(w+w^{-1})/(w-w^{-1})$, which is trivial to solve, since the numerator is the derivative of the denominator. Hence, the only solution to Eqs. (50) is

$$\tilde{b}=w-\frac{1}{w},\quad \sqrt{2}\tilde{\beta}_1=h,\quad \sqrt{2}\tilde{\beta}_2=-\frac{h}{w}. \tag{59}$$

Inserting this back into the expansion (33), and using the background equation (35) for h' and the formula (C1) for DZ, yield the simple result

$$\delta B^{\text{odd}}=\frac{1}{w}\,\mathrm{D}(hZ). \tag{60}$$

B. Solutions to the inhomogeneous BPS equations

Now that the source terms for the linearized Bogomol'nyi equations (58) are known, we can proceed and solve the inhomogeneous problem. Since the homogeneous equations admit three solutions, two of which are the residual gauge modes $\tilde{\gamma}=\sqrt{2}h$, $\tilde{\alpha}_1=w$, $\tilde{\alpha}_2=1$ and $\tilde{\gamma}=0$, $\tilde{\alpha}_1=1$, $\tilde{\alpha}_2=w$, we need to find the remaining solution of the homogeneous problem and a solution of the inhomogeneous equations. This is achieved by deriving a third order equation for $\tilde{\gamma}$. In fact, since $\tilde{\gamma}=0$ is a residual gauge mode of Eqs. (58), the differential equation for $\tilde{\gamma}$ will be of second, rather than third order. Moreover, using the second residual gauge mode, one eventually ends up with a first order equation. First, one easily finds, from Eqs. (58),

$$(r^2\tilde{\gamma}')'-h(r^2\tilde{\gamma}')-(w^2+1)\tilde{\gamma}=-h\tilde{b}, \tag{61}$$

where $\sqrt{2}(w\tilde{\beta}_1-\tilde{\beta}_2)=\tilde{b}'$ was used on the RHS. Now using the second residual gauge mode $\tilde{\gamma}^{\text{gauge}}=h$, the homogeneous part of the above equation can be cast into the following first order equation for $(\tilde{\gamma}/h)'$:

$$\left[\frac{h^2r^2}{w}\left(\frac{\tilde{\gamma}}{h}\right)'\right]'=0,$$

with the solution $\tilde{\gamma}\propto h\int w/(rh)^2$. The integration can be performed by using the relation $[w/(r^2h)]'=w/(rh)^2$, following from the background equation (37). Hence, the only non-gauge mode of the homogeneous equations (58) is

$$\tilde{\gamma}^{\text{hom}}=\frac{w}{r^2},\quad \sqrt{2}\tilde{\alpha}_1^{\text{hom}}=h-\frac{1}{r},\quad \sqrt{2}\tilde{\alpha}_2^{\text{hom}}=\frac{w}{r}. \tag{62}$$

[In order to verify that this solves the homogeneous part of Eqs. (58), one uses again the first order equation (37) for the background field h.] We may finally use the two solutions $\tilde{\gamma}^{\text{hom}}$ and $\tilde{\gamma}^{\text{gauge}}=h$ to solve the inhomogeneous equation (61) with source term $\mathcal{I}=-h\tilde{b}=-h(w-w^{-1})$:

$$\tilde{\gamma}^{\text{inh}}=\int \mathrm{d}r\,\frac{\mathcal{I}}{r^2}(\mu_{(1)}\tilde{\gamma}^{\text{gauge}}+\mu_{(2)}\tilde{\gamma}^{\text{hom}}), \tag{63}$$

with $\mu_{(1)}=\tilde{\gamma}^{\text{hom}}/W$ and $\mu_{(2)}=-\tilde{\gamma}^{\text{gauge}}/W$, where $W=\tilde{\gamma}^{\text{gauge}}(\tilde{\gamma}^{\text{hom}})'-\tilde{\gamma}^{\text{hom}}(\tilde{\gamma}^{\text{gauge}})'$ is the Wronskian of the two homogeneous solutions. A short computation yields $W=w/r^2$, and hence $\mu_{(1)}=1$, $\mu_{(2)}=-r^2h/w$. We thus end up with

$$\tilde{\gamma}^{\text{inh}}=\frac{w}{r^2}\left[\int\frac{r^2h}{w}\frac{hh'}{w}\,\mathrm{d}r-\frac{r^2h}{w}\int\frac{hh'}{w}\,\mathrm{d}r\right]. \tag{64}$$

This shows that the physical modes describing magnetic perturbations with $J=1$ and odd parity form a two-parameter family. In particular, the perturbations of the Higgs field become

$$\delta H^{\text{odd}}=(C_1\tilde{\gamma}^{\text{hom}}+C_2\tilde{\gamma}^{\text{inh}}). \tag{65}$$

[The arbitrary constant C_2 reflects the fact that the source terms δB are themselves solutions to a homogeneous set of equations, implying that the inhomogeneity in Eq. (61) is only fixed up to a multiplicative constant.] Since the self-dual solution $\tilde{\gamma}^{\text{hom}}$ diverges like $1/r^2$ near the origin, while the non-self-dual part $\tilde{\gamma}^{\text{inh}}$ diverges like $\int e^r/r$ at infinity, we conclude that there exist no small magnetic perturbations of BPS monopoles and JZ dyons with odd parity.

C. Solutions to the $\delta\Phi$ equations

The electric perturbations $\tilde{\phi}$ with odd parity are governed by Eq. (46). Since the magnetic amplitude $\tilde{b}$ fulfills the same second order equation, we immediately conclude from the solution (59) that

$$\tilde{\phi}^{(1)} = w - \frac{1}{w} \tag{66}$$

solves Eq. (46). [In fact, using $(w \pm w^{-1})' = h(w \mp w^{-1})$, one has $(w-w^{-1})'' = h'(w+w^{-1}) + h^2(w-w^{-1}) = [(w^2+1)/r^2 + h^2](w-w^{-1})$.] The second solution is given by $\tilde{\phi}^{(2)} = \tilde{\phi}^{(1)} \int [\tilde{\phi}^{(1)}]^{-2} \mathrm{d}r$. The integral can be carried out, and yields

$$\tilde{\phi}^{(2)} = \frac{1}{r}\left(w + \frac{1}{w}\right) - \frac{h}{w}. \tag{67}$$

[Using the background equations (35) it is not hard to verify that this is indeed the second solution to Eq. (46).] The electric perturbations with odd parity are, therefore,

$$\delta\Phi^{\mathrm{odd}} = \frac{1}{r}(C_1 \tilde{\phi}^{(1)} + C_2 \tilde{\phi}^{(2)}), \tag{68}$$

which remains finite for $r \to \infty$ only if $C_1 = C_2$. However, as $(\tilde{\phi}^{(1)} + \tilde{\phi}^{(2)})/r$ diverges like $1/r^2$ in the vicinity of the origin, we conclude that there exist no small electric perturbations of BPS monopoles and JZ dyons with odd parity.

VII. EVEN PARITY MODES

A. Solutions to the δB equations

In order to solve Eqs. (49), we first note that the equation for b_+' is a consequence of the remaining ones. Eliminating b_+ by using the last equation in Eq. (49), we obtain a system of three first order equations for b_-, β_1, and β_2. It is then straightforward to decouple these equations, which yields a third order equation for b_-. Since b_- enters this equation only via its derivatives, one concludes that $b_- = \mathrm{const}$ is a solution. In fact, one easily verifies that (any constant times)

$$b_-^{(0)} = 2, \quad b_+^{(0)} = \sqrt{2}\left(w + \frac{1}{w}\right), \quad \beta_1^{(0)} = -h, \quad \beta_2^{(0)} = \frac{h}{w} \tag{69}$$

solves Eqs. (49). In order to find the remaining two solutions, it is convenient to define the quantities

$$\Sigma = hb_- + (\beta_1 - w\beta_2),$$
$$\Delta = h^{-1}(\beta_1 + w\beta_2). \tag{70}$$

Since Σ and Δ vanish for the solution (69), it is possible to derive a system of two first order equations for these quantities. Using the three equations for b_-, β_1, and β_2, one finds after some manipulations $\Sigma' = 2h^2\Delta$ and $\Delta' = h^2 r^2 (w^2+1)^{-1}\Sigma$, which also yields the following second order equation for Σ:

$$\Sigma'' - 2\frac{h'}{h}\Sigma' - 2\frac{w^2+1}{r^2}\Sigma = 0. \tag{71}$$

Considering the quantity Σ/h, we obtain a Schrödinger equation with potential $P(r)$,

$$(\Sigma/h)'' = P(r)(\Sigma/h), \quad \text{where} \quad P(r) = 2\frac{w^2+1}{r^2} + \frac{(h^{-1})''}{h^{-1}}. \tag{72}$$

Having solved this equation, one obtains the magnetic amplitude b_- from the definition of Σ and the first equation in Eq. (49). In order to find b_+ one uses the last equation in Eq. (49) and solves Eqs. (70) for β_1 and β_2. This yields

$$b_- = \int \frac{\Sigma'}{h}\,\mathrm{d}r,$$
$$\sqrt{2}b_+ = \frac{1}{h}\left(w + \frac{1}{w}\right)(hb_- - \Sigma) - \frac{1}{2h^2}\left(w - \frac{1}{w}\right)\Sigma'. \tag{73}$$

The formulas for β_1 and β_2 in terms of Σ are not given here, since δB can be expressed in terms of b_- and b_+ alone. This is seen as follows: Using the relations (C1), the terms tangential to S^2 in the expansion (33) for δB^{even} can be written in the form

$$\beta_1 \tau_r \mathrm{d}K + \beta_2 K \mathrm{d}\tau_r$$
$$= \frac{1}{w^2-1}[(w\beta_2 - \beta_1)\hat{\mathrm{D}}X + \sqrt{2}(\beta_2 - w\beta_1)\hat{\mathrm{D}}Y]$$
$$= \frac{1}{w^2-1}[(hb_- - \Sigma)\hat{\mathrm{D}}X + (hb_+)\hat{\mathrm{D}}Y],$$

where we have used the last equation in Eq. (49) and the definition (70) to get rid of β_1 and β_2. Now using Eq. (73) for b_-, we have $\Sigma' = hb_-'$, and thus $(hb_- - \Sigma)' = h'b_-$, which enables us to write the term proportional to X in Eq. (33) as $r^{-2}b_-\mathrm{d}r = (w^2-1)^{-1}h'b_-\mathrm{d}r = (w^2-1)^{-1}\mathrm{d}(hb_- - \Sigma)$. Hence, the terms proportional to X and $\hat{\mathrm{D}}X$ combine to an exact covariant derivative, which finally yields the result

$$\delta B^{\mathrm{even}} = \frac{1}{w^2-1}\{\mathrm{D}[(hb_- - \Sigma)X] + b_+\mathrm{D}[hY]\}. \tag{74}$$

This shows that all three magnetic modes with even parity are obtained from Eqs. (72), (73), and (74). In particular, the trivial solution $\Sigma^{(0)} = 0$ of Eq. (72) gives rise to the solution (69), which yields

$$\delta B^{\mathrm{even},(0)} = \frac{2}{w^2-1}\left\{\mathrm{D}(hX) + \frac{1}{\sqrt{2}}\left(w + \frac{1}{w}\right)\mathrm{D}(hY)\right\}. \tag{75}$$

The two nontrivial solutions of Eq. (72) are not (yet) known in closed form. However, their qualitative behavior can be discussed rigorously: The potential $P(r)$ is positive definite for all finite values of r. [In fact, using the background equations to compute $(h^{-1})''$, one finds from Eq. (72)

$$P(r)=\frac{1}{r^2}+\left(\frac{h'}{h}\right)^2+\left(\frac{1}{r}+\frac{h'}{h}\right)^2, \tag{76}$$

which is manifestly non-negative, and vanishes only for $r\to\infty$.] As $h'/h=1/r-2r/15+\mathcal{O}(r^3)$ in the vicinity of the origin, we have $P(r)=6/r^2+\mathcal{O}(1)$, implying that the fundamental solutions behave like $\Sigma^{(1)}/h\propto 1/r^2$ and $\Sigma^{(2)}/h\propto r^3$. For $r\to\infty$ one has $P(r)=2/r^2+\mathcal{O}(1/r^3)$, which yields $\Sigma^{(1)}/h\propto 1/r$ and $\Sigma^{(2)}/h\propto r^2$. The monotonicity property of Σ/h, following from the positivity of the potential (76), enables one to conclude that the solution which diverges at the origin remains finite at infinity, and vice versa. [Also we have used global existence, following from the linearity of Eq. (72) and from the finiteness of the potential for $r\neq 0$.] Since h behaves like r near the origin and approaches the constant value -1 at infinity, the two nontrivial solutions of Eq. (72) behave as follows:

$$\Sigma^{(1)}\propto r^{-1},\quad \Sigma^{(2)}\propto r^4,\quad \text{for } r\to 0, \tag{77}$$

and

$$\Sigma^{(1)}\propto r^{-1},\quad \Sigma^{(2)}\propto r^2,\quad \text{for } r\to\infty. \tag{78}$$

In Sec. IV C we have shown that the angular momentum can be expressed as a boundary integral. The relevant quantity appearing in Eq. (44) is the difference of $(b_-+\sqrt{2}wb_+)$ between infinity and zero. For the solution $\Sigma^{(1)}$ this quantity remains finite at infinity, whereas it diverges logarithmically at the origin. (Note that the leading power, $1/r^2$, cancels in the above combination.) On the other hand, $(b_-+\sqrt{2}wb_+)$ remains bounded at the origin for the solution $\Sigma^{(2)}$, whereas it obviously diverges like r^2 at infinity. Hence, neither of the two nontrivial solutions to Eq. (72) gives rise to a finite angular momentum. The fact that we are able to decide this without solving the inhomogeneous equations (57) for δH and δA follows from the observation that the angular momentum depends on the magnetic perturbations only via the field δB. Moreover, only the boundary values of the gauge invariant quantity δB are needed to obtain the magnetic contribution to the total angular momentum.

Surprisingly enough, the third solution, given in closed form in Eq. (69), does give rise to a finite angular momentum, although the amplitude b_+ diverges at infinity. By virtue of Eq. (44) we have

$$\delta J^{\mathrm{mg}}=-\frac{\pi}{3}[b_-+\sqrt{2}wb_+]_0^\infty=\frac{2\pi}{3}. \tag{79}$$

In order to decide whether this is an acceptable perturbation, we have to compute the physical fields δH and δA.

B. Solutions to the inhomogeneous BPS equations

We now discuss the inhomogeneous linearized Bogomol'nyi equations (57). These can be written as a fourth order equation for either of the four variables $\gamma_\pm,\alpha_{1,2}$, parametrizing δH^{even} and δA^{even}. Since γ_- vanishes for the residual gauge mode of the system (57), the equation for γ_- is only of third, rather than fourth order. One finds

$$(r^2\gamma_-')''-2h(r^2\gamma_-')'-2(1+2w^2)\gamma_-'+4h(1-w^2)\gamma_- =\mathcal{I}[\delta B^{\mathrm{even}}], \tag{80}$$

where the inhomogeneity is an expression in terms of $b_\pm$ and $\beta_{1,2}$. Using Eqs. (49) for these amplitudes, it is possible to express $\mathcal{I}[\delta B^{\mathrm{even}}]$ in terms of b_+ and b_- alone:

$$\mathcal{I}[\delta B^{\mathrm{even}}]=\frac{2w}{1-w^2}[w(hb_-)'+\sqrt{2}(hb_+)']. \tag{81}$$

The solutions to the homogeneous problem, that is, the solutions to the linearized Bogomol'nyi equations are known in closed form [5,6,8]. In fact, they can be expressed in terms of the quantities w and h. Using the background relations (35), it is not difficult—and not particularly pleasant either—to verify that the three solutions of the homogeneous Eq. (80) are

$$\gamma_-^{(1)}=h'=\frac{1}{r^2}(w^2-1),$$

$$\gamma_-^{(2)}=\frac{1}{r^3}[w^2-(rh-1)],$$

$$\gamma_-^{(3)}=w^2+2(rh-1). \tag{82}$$

Concerning these solutions of the homogeneous linearized PBS equations, we note the following.

The amplitude $\gamma_-^{(2)}$ is not finite at the origin, while $\gamma_-^{(3)}$ becomes unbounded at infinity. Hence, neither $\gamma_-^{(2)}$ nor $\gamma_-^{(3)}$ gives rise to small perturbations of δH^{even}.

The third solution, $\gamma_-^{(1)}$, does give rise to an acceptable physical mode. The latter corresponds to a translation along the z axis. This is seen by differentiating the background field $H=\tau_r h$ with respect to $\partial_z=\cos\vartheta\partial_r-r^{-1}\sin\vartheta\partial_\vartheta$, which yields $\partial_z H=h'X+r^{-1}h\sqrt{2}Y$, and hence $\gamma_-=h'$. [Note that the coefficient in front of Y is not the amplitude γ_+ introduced in Eq. (53), since the latter was defined in a gauge where δA is tangential to S^2; see Eq. (54). The only quantity which is not affected by the corresponding gauge transformation is γ_-; see also Appendix D.]

The remaining two physical zero modes in the sector $J=1$ correspond to translations along the x and y axes. They do not occur in the above calculation, since, for reasons of symmetry, we have restricted the harmonic decompositions to the magnetic quantum number $M=0$.

We recall that none of the above solutions to the linearized BPS equations can contribute to the angular momentum, since only the field δB, describing the non-self-dual perturbations, enters the expression (18).

It remains to find the solutions to the inhomogeneous equation (80). Since all solutions to the homogeneous problem are known, we can apply standard methods to obtain the particular solution $\gamma_-^{\rm inh}$. For given inhomogeneity $\mathcal{I}$ one has

$$\gamma_-^{\rm inh} = \sum \gamma_-^{(k)} \int \mu_{(k)} \frac{\mathcal{I}}{r^2}\, dr, \tag{83}$$

where the three quantities $\mu_{(k)}$ are obtained from the homogeneous solutions $\gamma_-^{(k)}$ by

$$\mu_{(k)} = \frac{\varepsilon_{ijk}\gamma_-^{(i)}(\gamma_-^{(j)})'}{\varepsilon_{mn\ell}\gamma_-^{(m)}(\gamma_-^{(n)})'(\gamma_-^{(\ell)})''}. \tag{84}$$

A rather lengthy computation yields the value $-24w^2/r^4$ for the Wronskian in the denominator, and then

$$\mu_{(1)} = \frac{1}{4}(1 + r^2 + \sinh^2(r)),$$

$$\mu_{(2)} = \frac{1}{4}\left(\sinh(r)\cosh(r) - r - \frac{2}{3}r^3\right),$$

$$\mu_{(3)} = -\frac{1}{12}. \tag{85}$$

In the previous section we have argued that only the solution (69) of the equations (49) for $\delta B^{\rm even}$ gives rise to a finite angular momentum. Hence, it remains to compute $\gamma_-^{\rm inh}$ for the source term given by $b_-^{(0)} = 2$ and $b_+^{(0)} = \sqrt{2}(w + w^{-1})$. Using the expression (81), we immediately have

$$\mathcal{I}[\delta B^{\rm even(0)}] = -4\left(h^2 + \frac{2w^2+1}{r^2}\right). \tag{86}$$

The solution $\gamma_-^{\rm inh}$ is now obtained from Eqs. (83), (84), (85), and (86). An expansion in powers of r reveals that $\gamma_-^{\rm inh}$ diverges like $1/r$ in the vicinity of the origin. Since $\gamma_-^{(2)}$ diverges like $1/r^3$, while $\gamma_-^{(1)}$ and $\gamma_-^{(3)}$ are well behaved for $r \to 0$, there is no combination of $\gamma_-^{\rm inh}$ with a homogeneous solution (82) which remains bounded at the origin. Hence, we conclude that there exist no bounded magnetic perturbations $\delta H, \delta A$, which give rise to finite angular momentum.

C. Solutions to the $\delta\Phi$ equations

It remains to discuss the electric perturbations with even parity. The latter are governed by Eqs. (45) for ϕ_- and ϕ_+. We have already argued that three solutions of these equations coincide with the magnetic solutions $b_\pm^{(0)}$, $b_\pm^{(1)}$, and $b_\pm^{(2)}$, discussed in Sec. VII A. In order to find the remaining solution, it is convenient to write the system (45) in the form of a second order equation with two inhomogeneities. The manipulations by which this can be achieved are discussed in Appendix E. Introducing the quantity $\tilde{\Sigma}$ in the same way as in the magnetic case [see Eq. (73)],

$$\tilde{\Sigma}' = h\phi_-', \tag{87}$$

one eventually finds the equation

$$\tilde{\Sigma}'' - 2\frac{h'}{h}\tilde{\Sigma}' - 2\frac{w^2+1}{r^2}\tilde{\Sigma} = -k_3 h' - k_0 2\frac{w^2+1}{r^2}, \tag{88}$$

where k_0 and k_3 are integration constants. The homogeneous part of this equation coincides with the corresponding magnetic equation (71).

The particular solution for $k_3 = 0$ is $\tilde{\Sigma} = k_0\tilde{\Sigma}^{(0)}$, with $\tilde{\Sigma}^{(0)} = 1$. This yields $\phi_-^{(0)} = $ const, which coincides with the magnetic solution (69). The two solutions to the homogeneous problem, $\tilde{\Sigma}^{(1)}$ and $\tilde{\Sigma}^{(2)}$, say, coincide with the two remaining magnetic solutions obtained from the homogeneous equation (71). The additional solution, $\tilde{\Sigma} = k_3\tilde{\Sigma}^{(3)}$, which is not present in the magnetic case, is the particular solution for $k_0 = 0$. Again, this can be given in closed form by introducing the quantity $S = \tilde{\Sigma}/h^2$. Using again the background equation (37) for h, a short computation shows that the LHS of Eq. (88) assumes the form $(S'h^2)' + 2h'S$. Hence, the desired solution is $S = -k_3/2$, that is, $\tilde{\Sigma}^{(3)} = -h^2/2$. By virtue of Eq. (87) this yields $\phi_-^{(3)} \propto h$.

We thus conclude that the four electric perturbations with even parity are given by

$$\phi_-^{(0)} = 2, \quad \phi_+^{(0)} = \sqrt{2}\left(w + \frac{1}{w}\right), \tag{89}$$

$$\phi_-^{(3)} = 2h, \quad \phi_+^{(3)} = \frac{\sqrt{2}}{w}(rh)', \tag{90}$$

$$\phi_-^{(1,2)} = \int dr\, \frac{(\tilde{\Sigma}^{(1,2)})'}{h},$$

$$\phi_+^{(1,2)} = \frac{1}{\sqrt{2}}\left[\left(w + \frac{1}{w}\right)\phi_-^{(1,2)} - \frac{r^2(\phi_-^{(1,2)})''}{2w}\right], \tag{91}$$

where $\tilde{\Sigma}^{(1,2)}$ are the two nontrivial solutions to $\tilde{\Sigma}'' - 2(h'/h)\tilde{\Sigma}' - 2r^{-2}(w^2+1)\tilde{\Sigma} = 0$. The angular momentum is obtained from Eq. (43) and the above solutions. One finds

$$\delta J^{\rm el(0)} = \frac{2\pi}{3}[r^2h'(1-rh)]_0^\infty,$$

$$\delta J^{\rm el(3)} = \frac{2\pi}{3}[rh(h-rh')]_0^\infty,$$

$$\delta J^{\rm el(1,2)} = -\frac{\pi}{3}\left[r^2h\left(\frac{h'}{h}(rh-1)\phi_-^{(1,2)} + (\phi_-^{(1,2)})' - \frac{r}{2}(\phi_-^{(1,2)})''\right)\right]_0^\infty. \tag{92}$$

It is easy to see that the only combination of $\phi^{(0)}$ and $\phi^{(3)}$ which gives rise to finite angular momentum is their sum. However, since the amplitudes entering $\delta\Phi$ are ϕ_+/r and ϕ_-/r [see Eq. (31)], the perturbation $\delta\Phi$ obtained from $\phi^{(0)}+\phi^{(3)}$ diverges at the origin like $1/r$.

The behavior of $\tilde{\Sigma}^{(1)}$ given in Eqs. (77) and (78) implies that $\phi_-^{(1)}=\mathcal{O}(1/r^2)$ as $r\to 0$, and $\phi_-^{(1)}=\mathcal{O}(1/r)$ as $r\to\infty$. Using this in the above expression shows that the angular momentum is again finite, $|\delta J^{\mathrm{el}(1)}|<\infty$. As above, the perturbation $\delta\Phi$ is, however, not bounded at the origin.

The solution $\phi_-^{(2)}$ diverges like r^2 as $r\to\infty$. However, the leading order terms in the expression for $\delta J^{\mathrm{el}(2)}$ cancel, and so do the next-to-leading order terms. Hence, like $\delta J^{\mathrm{el}(0)}$ and $\delta J^{\mathrm{el}(3)}$, $\delta J^{\mathrm{el}(2)}$ diverges only with the first power of r, implying that there exist linear combinations of $\phi_-^{(2)}$ with $\phi_-^{(0)}$ (or $\phi_-^{(3)}$) which give rise to finite angular momentum. Since $\phi_-^{(0)}/r$ is not bounded at the origin, while $\phi_-^{(2)}/r$ is bounded, only linear combinations of $\phi_-^{(2)}$ with $\phi_-^{(3)}$ need to be considered. However, the latter give rise to perturbations $\delta\Phi$ which are not bounded at infinity. (Note that $\phi_+^{(2)}/r$ behaves like e^r/r, whereas $\phi_+^{(3)}/r$ grows like e^r.)

We therefore conclude that all electric perturbations of BPS monopoles and JZ dyons which give rise to finite angular momentum are either unbounded at the origin or at infinity.

VIII. CONCLUSIONS

In this article we have presented a gauge invariant approach to the stationary perturbations of Julia-Zee dyons and BPS monopoles. Restricting our attention to axisymmetric perturbations, we have found three sets of modes in each parity sector.

Electric perturbations: These are manifestly gauge invariant, since the electric background field vanishes after a hyperbolic rotation. There exist even parity perturbations with finite angular momentum; however, these are either not well behaved at the origin or at infinity. The same is true of the odd parity perturbations, which give rise to axial deformations.

Non-self-dual magnetic perturbations: These are perturbations which satisfy the linearized field equations, but are not at the same time subject to the linearized Bogomol'nyi equations. As the corresponding background field vanishes, the non-self-dual magnetic perturbations are also gauge invariant. As in the electric case, there exist even parity perturbations with finite angular momentum. However, neither these nor the odd parity modes are well behaved.

Self-dual magnetic perturbations: These have been investigated before and are known to be physically not acceptable, apart from the translational modes. Moreover, general considerations show that self-dual modes cannot contribute to the angular momentum. The fact that all solutions to the linearized BPS equations are known in closed form is, however, very useful to reconstruct the physical fields δH and δA for the non-self-dual modes.

In conclusion, we would like to emphasize that the distinguished properties of the BPS background are very critical to the methods developed in this article. Whether the main result—the fact that there exist no rotational excitations of Julia-Zee dyons and BPS monopoles—generalizes to more general background configurations is an open problem. In particular, the effect of a Higgs potential and the coupling to gravity need to be investigated for the excitations of Julia-Zee dyons.

APPENDIX A: BPS BACKGROUND

The appropriate way to treat axial perturbations in gauge theories is by using the isospin harmonics introduced in Sec. IV A. It is, therefore, suited to write the background fields in terms of the spherical su(2) basis $\tau_r,\tau_\vartheta,\tau_\varphi$, defined by

$$\tau_r=\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}},\quad \mathrm{d}\tau_r=\tau_\vartheta\mathrm{d}\vartheta+\tau_\varphi\sin\vartheta\mathrm{d}\varphi, \tag{A1}$$

where $\hat{\boldsymbol{r}}\equiv\boldsymbol{r}/r$ is the radial unit direction, and $\boldsymbol{\tau}=\boldsymbol{\sigma}/(2i)$. The commutation relations of the Pauli matrices imply $[\tau_r,\tau_\vartheta]=\tau_\varphi$ (and cyclic), from which one obtains the formulas

$$[\tau_r,\mathrm{d}\tau_r]=-\hat{*}\mathrm{d}\tau_r,\quad [\mathrm{d}\tau_r,\hat{*}\mathrm{d}\tau_r]=0, \tag{A2}$$

$$[\mathrm{d}\tau_r,\mathrm{d}\tau_r]=[\hat{*}\mathrm{d}\tau_r,\hat{*}\mathrm{d}\tau_r]=2\tau_r\mathrm{d}\Omega, \tag{A3}$$

where $\hat{*}$ denotes the Hodge dual with respect to the standard metric of the two-sphere S^2. It is helpful to recall that the radial unit direction $\hat{\boldsymbol{r}}$ is a vector valued eigenfunction of the spherical Laplacian with eigenvalue 2, implying

$$\mathrm{d}\hat{*}\mathrm{d}\tau_r=-2\tau_r\mathrm{d}\Omega. \tag{A4}$$

In terms of τ_r, the "Witten ansatz" for the spherically symmetric connection one-form assumes the simple form

$$A=[1-w(r)]\hat{*}\mathrm{d}\tau_r, \tag{A5}$$

since $\hat{*}\mathrm{d}\tau_r=r^{-2}(\boldsymbol{r}\times\boldsymbol{\tau})\cdot\mathrm{d}\boldsymbol{r}$. Using the commutation relations (A3), the gauge covariant derivatives of τ_r, $\mathrm{d}\tau_r$, and $\hat{*}\mathrm{d}\tau_r$ become

$$\mathrm{D}\tau_r=w\,\mathrm{d}\tau_r,\quad \mathrm{D}\mathrm{d}\tau_r=0, \tag{A6}$$

$$\mathrm{D}\hat{*}\mathrm{d}\tau_r=-2w\tau_r\mathrm{d}\Omega. \tag{A7}$$

The Bogomol'nyi equations, $*F=\mathrm{D}H$, can easily be written out by using Eq. (A7), the ansatz $H=h(r)\tau_r$, and the formulas $*(\mathrm{d}r\wedge\hat{*}\mathrm{d}\tau_r)=-\mathrm{d}\tau_r$, $*\mathrm{d}\Omega=\mathrm{d}r/r^2$ for the three-dimensional Hodge dual. One finds

$$*F=w'\mathrm{d}\tau_r+\frac{w^2-1}{r^2}\tau_r\mathrm{d}r, \tag{A8}$$

$$\mathrm{D}H=hw\,\mathrm{d}\tau_r+h'\tau_r\mathrm{d}r, \tag{A9}$$

which yields the well-known first order equations (35) for $w(r)$ and $h(r)$.

APPENDIX B: THE 2+1 DECOMPOSITION

The axial perturbation equations for a static, spherically symmetric su(2) valued function involve the three-dimensional gauge covariant Laplacian with respect to the gauge potential (A5). As the latter is tangential to S^2, the three-dimensional gauge covariant derivative operator is

$$\mathrm{D} = \mathrm{d}r \wedge \partial_r + \hat{\mathrm{D}}, \tag{B1}$$

where

$$\hat{\mathrm{D}} = \hat{\mathrm{d}} \cdot + [A, \cdot], \quad \text{with} \quad \hat{\mathrm{d}} = \mathrm{d}\vartheta \wedge \partial_\vartheta + \mathrm{d}\varphi \wedge \partial_\varphi . \tag{B2}$$

For an arbitrary Lie algebra valued function f we thus have

$$*\mathrm{D}*\mathrm{D}\left(\frac{f}{r}\right) = \frac{1}{r}\left(\partial_r^2 + \frac{1}{r^2}\,\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}\right) f, \tag{B3}$$

where the factor $1/r$ is introduced for convenience. (Here we have used $*\mathrm{d}r = r^2 \mathrm{d}\Omega$ and $*\hat{\mathrm{D}}f = -\mathrm{d}r \wedge \hat{*}\hat{\mathrm{D}}f$.) The above formula enables us to immediately write down the $2+1$ decomposition of the electric perturbation equation (21). With $f = r\,\delta\Phi$ this becomes

$$\left(\partial_r^2 + \frac{1}{r^2}\,\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}\right)(r\,\delta\Phi) = -[H,[H, r\,\delta\Phi]]. \tag{B4}$$

The $2+1$ decomposition of the (first order) magnetic equations (22) with respect to the ansatz

$$\delta B = \frac{1}{r^2} b\,\mathrm{d}r + \hat{B} \tag{B5}$$

was given in Sec. V B; see Eqs. (48). [Here $\hat{B}$ denotes an su(2) valued one-form tangential to S^2, and b is an su(2) valued scalar field.] We owe the proof of the assertion that b is subject to the same second order equation as the scalar electric perturbation $\delta\Phi$. In order to see this, one applies $\hat{*}\hat{\mathrm{D}}\hat{*}$ on the second, and ∂_r on the third equation in (48). A short calculation yields

$$\left(\partial_r^2 + \frac{1}{r^2}\,\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}\right) b = -\hat{*}[(\hat{\mathrm{D}}H - \hat{*}A'), \hat{B}] - [H, \hat{*}\hat{\mathrm{D}}\hat{B}].$$

The $2+1$ decomposition of the Bogomol'nyi equation gives $\hat{\mathrm{D}}H = \hat{*}A'$, implying that the first commutator on the RHS vanishes. By virtue of the first equation in Eq. (48), the second commutator becomes $[H,[H,b]]$, which yields the result

$$\left(\partial_r^2 + \frac{1}{r^2}\,\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}\right) b = -[H,[H,b]]. \tag{B6}$$

Hence, the equation (B4) for the scalar electric perturbation, $r\,\delta\Phi$, coincides with the second order equation (B6) for the scalar part of the magnetic perturbation, $b \equiv r^2(dr, B)$.

APPENDIX C: HARMONIC ANALYSIS

By virtue of the above decompositions, the task of writing out the perturbation equations reduces to the problem of computing the gauge covariant derivative $\hat{\mathrm{D}}$ of su(2) valued functions and one-forms over S^2. We have already argued in Sec. IV A that the $J=1$ sector is spanned by the three scalar harmonics X, Y, and Z, defined in terms of τ_r and $K \equiv \cos\vartheta$ [see Eq. (30)], and the four one-forms $\mathrm{d}X = -\sqrt{2}\mathrm{d}Y$, $\hat{*}\mathrm{d}X$, $\mathrm{d}Z$, and $\hat{*}\mathrm{d}Z$ [see Eq. (32)]. Instead of the latter, it is very convenient to use the linear combinations $\tau_r \mathrm{d}K, K\mathrm{d}\tau_r$ and their duals. The entire harmonic decomposition is then obtained from the formulas

$$\hat{\mathrm{D}}X = \tau_r \mathrm{d}K + wK\mathrm{d}\tau_r,$$

$$\sqrt{2}\hat{\mathrm{D}}Y = -w\tau_r \mathrm{d}K - K\mathrm{d}\tau_r,$$

$$\sqrt{2}\hat{\mathrm{D}}Z = w\tau_r \hat{*}\mathrm{d}K - K\hat{*}\mathrm{d}\tau_r \tag{C1}$$

for the covariant derivatives of the scalar basis, and the relations

$$\hat{*}\hat{\mathrm{D}}(\tau_r \mathrm{d}K) = w\sqrt{2}Z,$$

$$\hat{*}\hat{\mathrm{D}}(K\mathrm{d}\tau_r) = -\sqrt{2}Z,$$

$$\hat{*}\hat{\mathrm{D}}(\tau_r \hat{*}\mathrm{d}K) = w\sqrt{2}Y - 2X,$$

$$\hat{*}\hat{\mathrm{D}}(K\hat{*}\mathrm{d}\tau_r) = \sqrt{2}Y - 2wX \tag{C2}$$

for the covariant derivatives of the basis one-forms. [The equation for $\hat{\mathrm{D}}X$ and Eqs. (C2) are immediate consequences of Eq. (A7), while the derivations of the expressions for $\hat{\mathrm{D}}Y$ and $\hat{\mathrm{D}}Z$ require slightly more work.]

As an illustration we compute $\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}(r\,\delta\Phi)$, where we use the expansion (31) to write $r\,\delta\Phi = \phi_- X + \phi_+ Y + \tilde{\phi} Z$. For the first term we find, for instance,

$$\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}(\phi_- X) = \phi_- \hat{*}\hat{\mathrm{D}}[\tau_r \hat{*}\mathrm{d}K + wK\hat{*}\mathrm{d}\tau_r]$$

$$= [2\sqrt{2}wY - 2(w^2+1)]\phi_- .$$

A similar computation for the second and third term gives the result

$$\hat{*}\hat{\mathrm{D}}\hat{*}\hat{\mathrm{D}}(r\,\delta\Phi) = [-2(w^2+1)\phi_- + 2\sqrt{2}w\phi_+]X$$

$$+[2\sqrt{2}w\phi_- - (w^2+1)\phi_+]Y$$

$$-[(w^2+1)\tilde{\phi}]Z, \tag{C3}$$

which, together with the $2+1$ decomposition formula (B4) and $[H,[H,(r\,\delta\Phi)] = -(\phi_+ Y + \tilde{\phi} Z)$, yields the desired perturbation equations (45) and (46).

APPENDIX D: GAUGE TRANSFORMATIONS

In this Appendix we show that there exists a gauge for which the perturbations δH and δA assume the expansions (53) and (54), respectively. We also establish that the coefficients are gauge invariant, up to the residual gauge transformations given in Eqs. (55) and (56). For simplicity, we focus on the even parity sector; the manipulations for the odd parity sector are completely analogous. The general expansions for δH^{even} and δA^{even} are

$$\delta H^{\text{even}} = \overline{\gamma}_- X + \overline{\gamma}_+ Y, \tag{D1}$$

$$\delta A^{\text{even}} = \overline{\alpha}_0 Z \mathrm{d}r + \overline{\alpha}_1 \tau_r \hat{*} \mathrm{d}K + \overline{\alpha}_2 K \hat{*} \mathrm{d}\tau_r, \tag{D2}$$

where the bars have been introduced to tell the amplitudes apart from the ones introduced in Eqs. (53) and (54). Under a gauge transformation with an su(2) valued function χ one has

$$\delta H \to \delta H + [H, \chi],$$

$$\delta A \to \delta A + \mathrm{D}\chi, \tag{D3}$$

where, as usual, H is the background Higgs field and D the covariant derivative with respect to the background potential A. The strategy is to write δH and δA as sums of a pure gauge and an (almost) gauge invariant part. For δA this is achieved by a partial integration of the radial part, and by using the expressions (C1) for the covariant derivatives of the isospin basis. The radial part of δA^{even} can be written as

$$\overline{\alpha}_0 Z \mathrm{d}r = \mathrm{D}\left[Z \int \overline{\alpha}_0 \mathrm{d}r \right] - \hat{\mathrm{D}} Z \int \overline{\alpha}_0 \mathrm{d}r,$$

where we have used the fact that $\mathrm{D}Z = \hat{\mathrm{D}}Z$. (Recall that $\mathrm{D} = \mathrm{d}r \wedge \partial_r + \hat{\mathrm{D}}$, and that the isospin harmonics are defined over S^2.) Now using the expression (C1) for $\hat{\mathrm{D}}Z$ brings δA^{even} into the desired form:

$$\delta A^{\text{even}} = -\mathrm{D}\overline{\chi} + \left[\overline{\alpha}_1 - \frac{w}{\sqrt{2}} \int \overline{\alpha}_0 \mathrm{d}r \right] \tau_r \hat{*} \mathrm{d}K + \left[\overline{\alpha}_2 + \frac{1}{\sqrt{2}} \int \overline{\alpha}_0 \mathrm{d}r \right] K \hat{*} \mathrm{d}\tau_r, \tag{D4}$$

where

$$\overline{\chi} \equiv -Z \int \overline{\alpha}_0 \mathrm{d}r. \tag{D5}$$

In order to separate a pure gauge term from δH^{even}, we use $[\tau_r, Z] = Y$ and $H = h\tau_r$ to write

$$\delta H^{\text{even}} = -[H, \overline{\chi}] + \overline{\gamma}_- X + \left[\overline{\gamma}_+ - h \int \overline{\alpha}_0 \mathrm{d}r \right] Y, \tag{D6}$$

with $\overline{\chi}$ according to Eq. (D5). Hence, after a gauge transformation with $\overline{\chi}$, the general perturbations (D1) and (D2) assume the forms (53) and (54), respectively, where the coefficients are related as follows:

$$\gamma_- = \overline{\gamma}_-, \quad \gamma_+ = \overline{\gamma}_+ - h \int \overline{\alpha}_0 \mathrm{d}r,$$

$$\alpha_1 = \overline{\alpha}_1 - \frac{w}{\sqrt{2}} \int \overline{\alpha}_0 \mathrm{d}r, \quad \alpha_2 = \overline{\alpha}_2 + \frac{1}{\sqrt{2}} \int \overline{\alpha}_0 \mathrm{d}r. \tag{D7}$$

It is clear from the above reasoning, and not hard to verify, that the amplitudes without bars are gauge invariant, up to residual gauge transformations with

$$\chi_0 = c_1 X + c_2 Y + c_3 Z, \tag{D8}$$

where c_1, c_2 and c_3 are arbitrary constants. Since only the last term is relevant to the even parity sector, we have $D\chi_0^{\text{even}} = c_3(w\tau_r \hat{*} \mathrm{d}K - K \hat{*} \mathrm{d}\tau_r)/\sqrt{2}$ and $[H, \chi_0^{\text{even}}] = c_3 h Y$. Using this in the transformation laws (D3) for the even parity perturbations (53) and (54), we conclude that γ_- is gauge invariant, while γ_+, α_1 and α_2 transform according to Eqs. (55) under the residual gauge transformations. A completely analogous reasoning establishes the transformation laws (56) for the odd parity sector.

APPENDIX E: EVEN PARITY ELECTRIC PERTURBATIONS

In this Appendix we briefly show how the two coupled second order equations (45) for ϕ_- and ϕ_+ can be translated into the inhomogeneous second order equation (88) for $\tilde{\Sigma}$, defined by $\tilde{\Sigma}' = h\phi_-'$. The procedure involves two integrations. The first integration is achieved by the observation that Eqs. (45) can be cast into the form

$$\sqrt{2}\phi_-'' = -2\sqrt{2}\,\frac{w}{r^2}(\sqrt{2}\phi_+ - \mu\phi_-), \tag{E1}$$

$$\phi_+''\mu - \phi_+\mu'' = -2\sqrt{2}\,\frac{w}{r^2}(\sqrt{2}\phi_+ - \mu\phi_-), \tag{E2}$$

where we have introduced the shorthand $\mu \equiv w + 1/w$. Since the RHS of the above equations are equal, and since both LHS are exact derivatives $[\phi_+''\mu - \phi_+\mu'' = [\mu^2(\phi_+/\mu)']']$, an integration yields the following first order relation between ϕ_- and ϕ_+:

$$\frac{1}{\sqrt{2}}\left(\frac{\phi_+}{\mu}\right)' = -\frac{\phi_-' - k_3}{\mu^2}, \tag{E3}$$

where k_3 is an integration constant. We now solve Eq. (E2) for ϕ_+/μ, perform a derivative, and use the result on the LHS of Eq. (E3). This yields the following second order equation for ϕ_-':

$$\left(\frac{r^2\phi''_-}{w^2+1}\right)'-2\phi'_-=4\left(\frac{w}{w^2+1}\right)^2(\phi'_--k_3), \tag{E4}$$

which shows that $\phi_-=\text{const}$ is a solution of the system (E1), (E2).

Our aim is to integrate Eq. (E4) once more. In order to see that this is possible, we introduce the variable $\tilde{\Sigma}$ according to definition (87), and note that the term in front of (ϕ'_--k_3) can be written in the form $[(w^2-1)/(w^2+1)]'/h$. Hence, with

$$\tilde{\Sigma}'\equiv h\phi'_-,\quad a\equiv\frac{w^2-1}{w^2+1}, \tag{E5}$$

Eq. (E4) assumes the form

$$\left[\frac{a}{h'}\left(\frac{\tilde{\Sigma}'}{h}\right)'\right]'-2\frac{\tilde{\Sigma}'}{h}=\frac{a'}{h}\left(\frac{\tilde{\Sigma}'}{h}-k_3\right). \tag{E6}$$

It is not hard to perform the differentiations and to rewrite this third order equation for $\tilde{\Sigma}$ in the form

$$\left[\left(\frac{a}{h'}\right)\tilde{\Sigma}'''+\left(\frac{a}{h'}\right)'\tilde{\Sigma}''\right]-2\left[\left(\frac{a}{h}\right)\tilde{\Sigma}''+\left(\frac{a}{h}\right)'\tilde{\Sigma}'\right]-[2\tilde{\Sigma}'-k_3a']=0,$$

where each of the three pairs is manifestly an exact derivative. Integrating the above expression and multiplying the result with h'/a eventually yields

$$\tilde{\Sigma}''-2\frac{h'}{h}\tilde{\Sigma}'-2\frac{h'}{a}\tilde{\Sigma}=-2k_0\frac{h'}{a}-k_3h', \tag{E7}$$

where k_0 is a further integration constant. Since $a=r^2h'/(w^2+1)$, this is the desired inhomogeneous second order equation (88). We recall that the four parameter family of solutions to Eq. (E7) is

$$\tilde{\Sigma}=\sum k_i\tilde{\Sigma}^{(i)}, \tag{E8}$$

where the sum runs from 0 to 3, and where $\tilde{\Sigma}^{(0)}=1$, $\tilde{\Sigma}^{(3)}=-h^2/2$, and $\tilde{\Sigma}^{(1,2)}$ are the two (nontrivial) solutions to the homogeneous part of Eq. (E7). The four independent solutions (89)–(91) to the original system (45) are finally obtained from Eq. (87).

APPENDIX F: ELECTRIC CONTRIBUTION TO THE ANGULAR MOMENTUM

In Sec. IV C we have argued that the total angular momentum can be expressed in terms of the perturbation amplitudes at the origin and at infinity. While we have established this result for the magnetic contribution (39), we still owe the proof of the formula (43) for the electric part (38). In order to show that the bracket in the integrand in Eq. (41) is an exact radial derivative, we first perform a partial integration in both terms, which yields

$$rh'\phi_--\frac{r^2}{\sqrt{2}}\left(\frac{w'\phi_+}{r}\right)'=\left[rh\phi_--\frac{r}{\sqrt{2}}w'\phi_+\right]'-h(r\phi_-)'+\sqrt{2}wh\phi_+. \tag{F1}$$

In order to show that the last two terms on the RHS combine to an exact derivative, we use the first perturbation equation in Eq. (45) to express ϕ_+ in terms of ϕ_- and ϕ''_-. Also using the background equations for w and h, we then have

$$\begin{aligned}-h(r\phi_-)'+\sqrt{2}wh\phi_+&=-\frac{r^2h}{2}\phi''_--rh\phi'_-+w^2h\phi_-\\&=-\left(\frac{r^2h}{2}\phi_-\right)''+[w^2-1+rh]\phi'_-+[2w^2h+(rh)']\phi\\&=\left[-\left(\frac{r^2h}{2}\phi_-\right)'+(w^2-1+rh)\phi_-\right]'.\end{aligned}$$

Using this on the RHS of Eq. (F1) gives the desired formula,

$$rh'\phi_--\frac{r^2}{\sqrt{2}}\left(\frac{w'\phi_+}{r}\right)'=-\frac{1}{2}[(1-w^2-2rh)\phi_-+r^2h\phi'_-+\sqrt{2}wrh\phi_+]', \tag{F2}$$

which was used in Sec. IV C to establish the result (43).

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