Yang-Mills theory as a deformation of topological field theory, dimensional reduction, and quark confinement

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We propose a reformulation of Yang-Mills theory as a perturbative deformation of a novel topological (quantum) field theory. We prove that this reformulation of four-dimensional QCD leads to quark confinement in the sense of an area law of the Wilson loop. First, Yang-Mills theory with a non-Abelian gauge group G is reformulated as a deformation of a novel topological field theory. Next, a special class of topological field theories is defined by both Becchi-Rouet-Stora-Tyupin (BRST) and anti-BRST exact actions corresponding to the maximal Abelian gauge leaving the maximal torus group H of G invariant. Then we find topological field theory ($D>2$) has a hidden supersymmetry for a choice of maximal Abelian gauge. As a result, the D-dimensional topological field theory is equivalent to the ($D-2$)-dimensional coset G/H nonlinear sigma model in the sense of the Parisi-Sourlas dimensional reduction. After maximal Abelian gauge fixing, the topological property of the magnetic monopole and antimonopole of four-dimensional Yang-Mills theory is translated into that of an instanton and anti-instanton in a two-dimensional equivalent model. It is shown that the linear static potential in four dimensions follows from the instanton–anti-instanton gas in the equivalent two-dimensional nonlinear sigma model obtained from the four-dimensional topological field theory by dimensional reduction, while the remaining Coulomb potential comes from the perturbative part in four-dimensional Yang-Mills theory. The dimensional reduction opens a path for applying various exact methods developed in two-dimensional quantum field theory to study the nonperturbative problem in low-energy physics of four-dimensional quantum field theories. [S0556-2821(98)04920-0]

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I. INTRODUCTION AND MAIN RESULTS

In particle physics, perturbation theory is applicable if the coupling constant as an expansion parameter is small in the energy region considered. This is assured in the high-energy ultraviolet region of quantum chromodynamics (QCD) where the effective coupling constant is small due to asymptotic freedom [1]. On the other hand, in the infrared regime of QCD where the effective coupling is expected to be large, the perturbation theory loses its validity. The quark confinement is regarded as a typical example of indicating the difficulty of treating strongly coupled gauge theories. The conventional perturbation theory deals with the small deviation from the trivial gauge field configuration $\mathcal{A}_\mu=0$ which is a minimum of the action S.

In the last decade, various evidence about Abelian dominance and magnetic monopole dominance in the low-energy physics of QCD has been accumulated based on a Monte Carlo simulation of lattice QCD initiated by the work [2], see, e.g., Ref. [3] for a review. This urges us to reconsider if there may exist any perturbation theory appropriate for QCD with the expansion parameter being small even in the infrared region. There the expansion must be performed about a nontrivial gauge field configuration $\mathcal{A}_\mu\neq 0$ other than the trivial one $\mathcal{A}_\mu=0$. In gauge field theories, we know that there are soliton solutions called the vortex [4], magnetic monopole [5], and instanton [6,7]. They are candidates for such a nontrivial field configuration.

We know a few examples of such expansions around a nontrivial field configuration that have successfully led to the resolution of the strong coupling problem. An example is a proof of quark confinement by Polyakov [8] in three-dimensional compact U(1) gauge theory and three-dimensional compact quantum electrodynamics (QED) in the Georgi-Glashow model with gauge group SU(2). He considered the nontrivial minimum Ω_μ of the action given by the instanton (pseudoparticle). The field $\mathcal{A}_\mu$ is decomposed into $\Omega_\mu+Q_\mu$ and Q_μ is considered as a quantum fluctuation around Ω_μ. The integral over Q_μ is Gaussian and is exactly integrated out. The result is written as the sum over all possible configurations of instantons and anti-instantons. In the three-dimensional case, instanton (anti-instanton) is given by the magnetic monopole (antimonopole). Moreover, Seiberg and Witten [9] have shown that in the four-dimensional $N=2$ supersymmetric gauge theories, the nonperturbative contributions come only from the magnetic monopole or instanton in the prepotential which exactly determines the low-energy effective Abelian gauge theory. These examples show that the quark confinement is caused by the condensation of magnetic monopoles.

Recently, a reformulation of the Yang-Mills (YM) theory as a deformation of topological (quantum) field theory has been attempted [10–12], abbreviated T(Q)FT hereafter. The BF theory [12] as a topological field theory (TFT) can be regarded as a zero-coupling limit of YM theory [13–15]. A similar idea was proposed recently by Abe and Nakanishi [13] where two-dimensional BF theory is essentially equivalent to the zeroth-order approximation to YM theory in their framework of the newly proposed method of

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solving quantum field theory. In higher dimensions, however, the limit is singular due to the fact that the gauge symmetry in BF theory is larger than that in YM theory. In the last couple of years, considerable progress has been made to assure that YM theory can be obtained as a deformation (perturbation) of topological BF theory by Fucito, Martellini, and Zeni [15]. This reformulation is the first-order formulation of YM theory, called BF-YM theory [16,17]. They checked the area law behavior for the Wilson loop average and computed the string tension. In this formalism, an area law arises in a very simple geometrical fashion, as an higher linking number between the loop and surface.

In this paper, we reconsider YM theory from a topological point of view. First we reformulate YM theory as a deformation of a novel TFT. This is equivalent to saying that YM theory is described as a perturbation around the nontrivial field configuration Ω_μ given by TFT. This formulation of YM theory will be suitable for describing the low-energy region of YM theory, because the topological property does not depend on the details of the short-distance behavior of the theory and depends only on the global structure of the theory. In order for such a description to be successful, TFT must include the most essential or dominant degrees of freedom for describing the low-energy physics in question. The monopole dominance is a hint for the search of an appropriate TFT. The TFT we propose in this paper is different from the conventional TFT's of Witten type [10] or Schwarz type [11]. Witten type TFT starts from the gauge fixing condition of self-duality

$$\mathcal{F}_{\mu\nu}=\pm\tilde{\mathcal{F}}_{\mu\nu}, \quad \tilde{\mathcal{F}}_{\mu\nu}:=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma}, \tag{1.1}$$

corresponding to the instanton configuration in four-dimensional YM theory [6]. The total action can be written as Becchi-Rouet-Stora-Tyupin (BRST) transformation δ_B of some functional V composed of the fields and their ghosts:

$$S_{\rm tot}=[Q_B,V\}=\delta_B V. \tag{1.2}$$

On the other hand, Schwarz-type TFT has a nontrivial classical action $S_{\rm cl}$ which is metric independent (hence topological) with nontrivial gauge fixing. For example, BF theory and Chern-Simons theory belong to this type,

$$S_{\rm tot}=S_{\rm cl}+[Q_B,V'\}=S_{\rm cl}+\delta_B V'. \tag{1.3}$$

Our TFT tries to incorporate the magnetic monopole degrees of freedom as essential degrees of freedom for low-energy physics. For this, we use the the maximal Abelian gauge (MAG). In MAG, we find that the action is written in the form

$$S_{\rm tot}=\delta_B\bar{\delta}_B\mathcal{O}, \tag{1.4}$$

using the anti-BRST transformation $\bar{\delta}_B$ [18].

In a previous paper [17], we proved that the dual superconductor picture of quark confinement in QCD (proposed by Nambu, 't Hooft, and Mandelstam [19–21]) can be derived from QCD without any specific assumption. In order to realize the dual superconductor vacuum of QCD, we need to take the MAG. MAG is an example of Abelian projection proposed by 't Hooft [20]. The basic idea of Abelian projection is that the off-diagonal non-Abelian parts are made as small as possible. Imposing MAG, the gauge degrees of freedom corresponding to G/H is fixed and the residual gauge invariance for the maximal torus group H of the gauge group G remains unbroken. Under MAG, it is expected that the off-diagonal gluons (belonging to G/H) become massive and the low-energy physics of QCD is described by the diagonal Abelian part (belonging to H) alone. All the off-diagonal fields transform as charged fields under the residual Abelian gauge symmetry H and are expected to be massive. It is shown [17] that an Abelian-projected effective gauge theory (APEGT) of QCD is obtained by integrating out all the massive degrees of freedom in the sense of the Wilsonian renormalization group (RG) [22]. Therefore the resulting APEGT for $G=SU(2)$ is written in terms of the Abelian field variables only. In fact, the APEGT obtained in the previous paper is written in terms of the maximal Abelian U(1) gauge field a_μ, the dual Abelian gauge field b_μ, and the magnetic monopole current k_μ which couples to b_μ. This theory is an interpolating theory in the sense that it gives two dual descriptions of the same physics, say, quark confinement. APEGT tells us that the dual theory which is more suitable in the strong coupling region is given by the dual Ginzburg-Landau (GL) theory, i.e., the dual Abelian gauge Higgs model [4]. That is to say, monopole condensation provides the mass m_b for the dual gauge field and leads to the linear or confining static potential between quarks and the nonzero string tension σ is given by $\sigma\sim m_b^2$. APEGT is regarded as a low-energy effective theory of QCD in the distance scale $R>m_A^{-1}$ with m_A being the nonzero mass of the off-diagonal gluons. Consequently, the Abelian dominance [23,24] in the physics in the long distance $R>R_c:=m_A^{-1}$ will be realized in APEGT. A quite recent simulation by Amemiya and Suganuma [25] shows that the propagator of the off-diagonal charged gluon behaves as the massive gauge boson and provides the short-range interaction, while the diagonal gluon propagates long distance. For SU(2) YM theory, they obtain $m_A\cong 0.9$ GeV corresponding to $R_c=4.5$ fm. In fact, the massiveness of off-diagonal gluons is analytically derived as a by-product in this paper.

In our formulation of YM theory, the nonperturbative treatment of YM theory in the low-energy region can be reduced to that of TFT in the sense that any perturbation from TFT does not change essentially the result on low-energy physics obtained from TFT. Therefore, we can hope that the essential contribution for quark confinement is derived from TFT alone. In light of monopole dominance, the TFT should be constructed such that the monopole degrees of freedom are included as the most dominant topological configuration in TFT. If quark confinement is proved based on TFT, the monopole dominance will be naturally understood by this construction of TFT. Furthermore, this will shed light on a possible connection with the instanton configuration which is the only possible topological nontrivial configuration in four-dimensional Euclidean YM theory without partial gauge fixing.

The purpose of this paper is to prove quark confinement within the reformulation of four-dimensional QCD based on the criterion of area law for the Wilson loop [26] (see Sec. VI). Here the Wilson loop is taken to be planar and diagonal[1] in the maximal torus group H (as taken by Polyakov [8]). Although actual calculations are presented only for the SU(2) case, out strategy of proving quark confinement is also applicable to SU(N) case and more generally to arbitrary compact Lie group.

This paper is organized as follows. In Sec. II, the TFT is constructed from gauge-fixing and Faddeev-Popov terms. The action is written as a BRST exact form according to the standard procedure of BRST formalism. In other words, the TFT is written as a BRST transformation of a functional of the field variables including ghosts. Here we take the MAG as a gauge fixing condition. Then the MAG fixes the coset G/H of the gauge group G and leaves the maximal torus subgroup H unbroken. Consequently, YM theory is reformulated as a (perturbative) fluctuation around the nontrivial topological configuration given by TFT.

In Sec. III, it is shown that a version of MAG allows us to write the TFT in the form (1.4) which is both BRST and anti-BRST exact. This version of TFT is called MAG TFT hereafter. We find that MAG TFT has a hidden supersymmetry (SUSY) based on the superspace formulation [28–32] of BRST invariant theories [33,34]. The hidden SUSY plays quite a remarkable role in the next section.

In Sec. IV, it turns out that this choice of MAG leads to dimensional reduction in the sense of Parisi and Sourlas (PS) [28]. Consequently the D-dimensional MAG TFT is reduced to the equivalent $(D-2)$-dimensional coset G/H nonlinear sigma model (NLSM). This means the equivalence of the partition function in two theories. Furthermore, PS-dimensional reduction tells us that the calculation of correlation functions in D-dimensional TFT can be performed in the equivalent $(D-2)$-dimensional model if the arguments x_i lie on a certain $(D-2)$-dimensional subspace, because the correlation function coincides with the same correlation function calculated in the $(D-2)$-dimensional equivalent model defined on the subspace on which x_i lies,

$$\left\langle \prod_i \mathcal{F}_i(x_i) \right\rangle_{\text{MAG TFT}_D} = \left\langle \prod_i \mathcal{F}_i(x_i) \right\rangle_{G/H \ \text{NLSM}_{D-2}}. \tag{1.5}$$

In Sec. V, we study concretely the case of $G=SU(2)$ YM theory in four dimensions. In this case, $H=U(1)$ and the equivalent dimensionally reduced model is given by the two-dimensional O(3) nonlinear sigma model (NLSM). The two-dimensional NLSM on group manifolds or the principal chiral model is exactly solvable [35–46]. Therefore, the four-dimensional MAG TFT defined in this paper is exactly solvable. It is known that the two-dimensional $O(3)$ NLSM is renormalizable and asymptotic free [47,48]. Moreover, it has instanton solution as a topological soliton [49–53]. The instanton is a finite action solution of the field equation and obtained as a solution of the self-duality equation. The instanton (anti-instanton) solution is given by the holomorphic (antiholomorphic) function.

We show that the instanton (anti-instanton) configuration in two-dimensional $O(3)$ NLSM can be identified with the magnetic monopole (antimonopole) configuration in higher dimensions. Furthermore, the instanton (anti-instanton) configuration in two dimensions is considered as the projection of instanton (anti-instanton) solution of four-dimensional YM theory on the two-dimensional plane through dimensional reduction. From this observation, we can see intimate connection between magnetic monopole and instanton. In principle, the gluon propagator is calculable according to the exact treatment of the $O(3)$ NLSM. In the $O(3)$ NLSM, dynamical mass generation occurs and the correlation length becomes finite and all the excitations are massive [46]. This shows that the off-diagonal gluons are massive, $m_A \neq 0$. The mass is nonperturbatively generated and behaves as $m_A \sim \exp(-4\pi^2/g^2)$.

In Sec. VI, the planar diagonal Wilson loop in four-dimensional SU(2) MAG TFT is calculated in the two-dimensional equivalent model by making use of dimensional reduction. The actual calculation is done in the dilute-instanton-gas approximation [54–56] in two dimensions. This is very similar to the calculation of the Wilson loop in the Abelian Higgs model in two dimensions [57,58]. We can pursue this analogy further using the CP^1 formulation of the $O(3)$ NLSM. In CP^1 formulation, the residual U(1) symmetry is manifest and we can introduce the U(1) gauge field coupled to two complex scalar fields, whereas in the NLSM, the U(1) gauge invariance is hidden, since the field variable $\mathbf{n}(z)$ is gauge invariant. The CP^1 formulation indicates the correspondence of TFT to GL theory. As a result, the existence of a topological nontrivial configuration corresponding to the magnetic monopole and antimonopole in YM theory in MAG is sufficient to prove quark confinement in the sense of an area law of the diagonal Wilson loop.

At the end of the 1970s, two-dimensional NLSMs were extensively studied motivated by their similarity with four-dimensional YM theory. Some of the NLSMs exhibit renormalizability, asymptotic freedom, θ vacua, and an instanton solution. These analogies are not accidental in our view. Now this is understood as a consequence of dimensional reduction. The beta function in the two-dimensional $O(3)$ NLSM has been calculated by Polyakov [47]. This should coincide with the beta function of four-dimensional SU(2)/U(1) MAG TFT. Now we will be able to understand why the Migdal-Kadanoff approximate renormalization group (RG) scheme [59] yields reasonably good results.

It should be remarked that dimensional reduction is also possible for gauge fixings other than MAG. Such an example was proposed by Hata and Kugo [60] which is called the pure gauge model (PGM). However, the choice of MAG as a gauge-fixing condition is essential to prove quark confinement based on the nontrivial topological configuration, because MAG leads to the G/H NLSM by dimensional reduc-

[1]The full non-Abelian Wilson loop will be treated in a subsequent paper [27], see Sec. VII.

tion. The two-dimensional coset $SU(N)/U(1)^{N-1}$ NLSM can have a soliton solution as suggested by

$$\Pi_2[SU(N)/U(1)^{N-1}] = \mathbf{Z}^{N-1}. \tag{1.6}$$

However, the two-dimensional NLSM obtained from the PGM by dimensional reduction does not have any instanton solution, since

$$\Pi_2[SU(N)] = 0. \tag{1.7}$$

Therefore the PGM loses a chance of proving quark confinement based on the nontrivial topological configuration and more effort is needed to prove quark confinement based on the perturbative or nonperturbative treatment around the topologically trivial configuration [61,60,62–65]. Moreover, the MAG has a clear physical meaning which leads to the dual superconductor picture of QCD vacuum as shown in Ref. [17]. This is not the case in PGM. In fact, there is a claim [68] that the criterion of Kugo and Ojima for color confinement [66,67] is different from the Wilson criterion.

It is possible to extend our treatment to arbitrary compact Lie group G along the same lines as above, as long as the existence of the instanton solution is guaranteed by the nontrivial homotopy group, $\Pi_2(G/H) \neq 0$. Although the dilute-gas approximation is sufficient to deduce the linear potential, it is better to compare this result with those obtained by other methods. For this purpose, it is worth performing a $1/N$ expansion to know the result especially for $N>2$. The $O(N)$ and CP^{N-1} models have been extensively studied [69–77]. However, $SU(N)/U(1)^{N-1}$ is isomorphic to $O(N+1)$ or CP^{N-1} only when $N=2$, and the two-dimensional $O(N)$ NLSM has no instanton solution for $N>3$. To the author's knowledge, the $1/N$ analysis of the two-dimensional coset $SU(N)/U(1)^{N-1}$ NLSM has not been worked out, probably due to the fact that $SU(N)/U(1)^{N-1}$ is not a symmetric space in the sense of a Riemannian manifold [78].

It should be remarked that the resolution of quark confinement is not simply to show that the full gluon propagator behaves as $1/k^4$ in the infrared region as $k \to 0$. The correct picture of quark confinement must be able to explain the anisotropy (or directional dependence) caused by the existence of a widely separated quark-antiquark pair if we stand on the dual superconductivity scenario. This is necessary to deduce the QCD (hadron) string picture. Our proof of quark confinement is possible only when the two-dimensional plane on which a pair of quarks and anti-quarks exists is selected as a subspace of dimensional reduction. Hence this feature is desirable from the viewpoint of the string picture. In fact, the effective Abelian gluon propagator obtained from the dual description in APEGT shows such an anisotropy [17].

The dimensional reduction of TFT opens a path for analyzing nonperturbative problems in four-dimensional YM theory based on various technologies developed for two-dimensional field theories, such as the Bethe ansatz [46] and conformal field theory (CFT) [79,80]. They are intimately connected to the Wess-Zumino-Novikov-Witten model [81], non-Abelian bosonization [81,82], the quantum spin model [83], Chern-Simons theory [84], the induced potential in the path integral [85,86], and so on. The exact solubility is pulled up at the level of correlation function, not the field equation. This should be compared with the Hamiltonian reduction of the YM self-duality equation [87]. Furthermore, the APEGT obtained in MAG can have the same meaning as the low-energy effective theory of $N=2$ supersymmetric YM theory and QCD obtained by Seiberg and Witten [9]. This issue will be discussed in subsequent papers.

II. YANG-MILLS THEORY AS A DEFORMATION OF TOPOLOGICAL FIELD THEORY

First, we summarize the BRST formulation of YM theory in the manifestly covariant gauge and subsequently introduce the MAG. Next, we derive the TFT describing the magnetic monopole from the YM theory in MAG. The TFT is obtained from the gauge fixing part of the YM theory. Finally, the YM theory in MAG is reformulated as a (perturbative) deformation of the TFT.

A. Yang-Mills theory and gauge fixing

We consider the Yang-Mills (YM) theory with a gauge group $G=SU(N)$ on the D-dimensional space-time described by the action $(D>2)$

$$S_{\text{tot}} = \int d^D x (\mathcal{L}_{\text{QCD}}[\mathcal{A},\psi] + \mathcal{L}_{\text{GF}}), \tag{2.1}$$

$$\mathcal{L}_{\text{QCD}}[\mathcal{A},\psi] := -\frac{1}{2g^2}\text{tr}_G(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu}) + \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu[\mathcal{A}] - m)\psi, \tag{2.2}$$

where

$$\mathcal{F}_{\mu\nu}(x) := \sum_{A=1}^{N^2-1} \mathcal{F}^A_{\mu\nu}(x) T^A$$
$$:= \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) - i[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)], \tag{2.3}$$

$$\mathcal{D}_\mu[\mathcal{A}] := \partial_\mu - i\mathcal{A}_\mu. \tag{2.4}$$

The gauge fixing term $\mathcal{L}_{\text{GF}}$ is specified below. We adopt the following convention. The generators $T^A (A=1,\dots,N^2-1)$ of the Lie algebra $\mathcal{G}$ of the gauge group $G=SU(N)$ are taken to be Hermitian satisfying $[T^A,T^B] = i f^{ABC} T^C$ and normalized as $\text{tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$. The generators in the adjoint representation are given by $[T^A]_{BC} = -i f_{ABC}$. We define the quadratic Casimir operator as $C_2(G)\delta^{AB} = f^{ACD} f^{BCD}$. Let H be the maximal torus group of G and T^a be the generators in the Lie algebra $\mathcal{G} \setminus \mathcal{H}$ of the coset G/H where $\mathcal{H}$ is the Lie algebra of H.

For $G=SU(2)$, $T^A = (1/2)\sigma^A (A=1,2,3)$ with Pauli matrices σ^A and the structure constant is $f^{ABC} = \epsilon^{ABC}$. The indices $a,b,\dots,$ denote the off-diagonal parts of the matrix representation. The Cartan decomposition is given by

$$\mathcal{A}_\mu(x) = \sum_{A=1}^{3} \mathcal{A}_\mu^A(x) T^A := a_\mu(x) T^3 + \sum_{a=1}^{2} A_\mu^a(x) T^a. \tag{2.5}$$

Under gauge transformation, the gauge field $\mathcal{A}_\mu(x)$ transforms as

$$\mathcal{A}_\mu(x) \to \mathcal{A}_\mu^U(x) := U(x)\mathcal{A}_\mu(x)U^\dagger(x) + iU(x)\partial_\mu U^\dagger(x). \tag{2.6}$$

These gauge degrees of freedom are fixed by the procedure of gauge fixing. A covariant choice is given by the Lorentz gauge

$$F[\mathcal{A}] := \partial_\mu \mathcal{A}^\mu = 0. \tag{2.7}$$

The procedure of gauge fixing must be done in such a way that the gauge fixing condition is also preserved for the gauge rotated field $\mathcal{A}_\mu^U$, i.e., $F[\mathcal{A}^U]=0$. This is guaranteed by the Faddeev-Popov (FP) ghost term.

We formulate the theory based on the BRST formalism. In the BRST formalism, the gauge-fixing and FP part $\mathcal{L}_{\rm GF}$ is specified by a functional $G_{\rm gf}$ of the field variables through the relation

$$\mathcal{L}_{\rm GF} := -i\delta_B G_{\rm gf}[\mathcal{A}_\mu, \mathcal{C}, \bar{\mathcal{C}}, \phi], \tag{2.8}$$

where $\mathcal{C}, \bar{\mathcal{C}}$ are ghost, antighost fields and ϕ is the Lagrange multiplier field for incorporating the gauge fixing condition. Here δ_B denotes the nilpotent BRST transformation

$$\begin{aligned}
&\delta_B \mathcal{A}_\mu(x) = \mathcal{D}_\mu \mathcal{C}(x) := \partial_\mu \mathcal{C}(x) - i[\mathcal{A}_\mu(x), \mathcal{C}(x)],\\
&\delta_B \mathcal{C}(x) = i\frac{1}{2}[\mathcal{C}(x), \mathcal{C}(x)], \quad \delta_B \bar{\mathcal{C}}(x) = i\phi(x),\\
&\delta_B \phi(x) = 0,\\
&\delta_B \psi(x) = i\mathcal{C}(x)\psi(x).
\end{aligned} \tag{2.9}$$

The partition function of QCD is given by

$$Z_{\rm QCD}[J] := \int [d\mathcal{A}_\mu][d\mathcal{C}][d\bar{\mathcal{C}}][d\phi][d\psi][d\bar{\psi}] \times \exp\{i(S_{\rm tot}+S_J)\}, \tag{2.10}$$

where the source term is introduced as

$$S_J := \int d^D x \{ {\rm tr}[J^\mu \mathcal{A}_\mu + J_c \mathcal{C} + J_{\bar{c}} \bar{\mathcal{C}} + J_\phi \phi] + \bar{\eta}\psi + \eta\bar{\psi} \}. \tag{2.11}$$

In the BRST formalism, both the gauge-fixing and the FP terms are automatically produced according to Eq. (2.8). The most familiar choice of G is

$$G_{\rm gf} = {\rm tr}_{\mathcal{G}}\left[\bar{\mathcal{C}}\left(\partial_\mu \mathcal{A}^\mu + \frac{\alpha}{2}\phi\right)\right]. \tag{2.12}$$

This yields

$$\begin{aligned}
\mathcal{L}_{\rm GF} &:= -i\delta_B G_{\rm gf}[\mathcal{A}_\mu, \mathcal{C}, \bar{\mathcal{C}}, \phi]\\
&= {\rm tr}_{\mathcal{G}}\left[\phi \partial_\mu \mathcal{A}^\mu + i\bar{\mathcal{C}}\partial^\mu \mathcal{D}_\mu[\mathcal{A}]\mathcal{C} + \frac{\alpha}{2}\phi^2\right].
\end{aligned} \tag{2.13}$$

B. MAG and singular configuration

In a previous paper [17], we examined the maximal Abelian gauge as an example of Abelian projection [20]. For $G=SU(2)$, MAG is given by

$$F^\pm[A,a] := (\partial^\mu \pm i a^\mu) A_\mu^\pm = 0, \tag{2.14}$$

using the $(\pm,3)$ basis

$$\mathcal{O}^\pm := (\mathcal{O}^1 \pm i\mathcal{O}^2)/\sqrt{2}. \tag{2.15}$$

The simplest choice of $G_{\rm gf}$ for MAG in $(\pm,3)$ basis is given by

$$G_{\rm gf} = \sum_\pm \bar{C}^\mp \left(F^\pm[A,a] + \frac{\alpha}{2}\phi^\pm\right), \tag{2.16}$$

which is equivalently rewritten in the usual basis as

$$G_{\rm gf} = \sum_{a=1,2} \bar{C}^a \left(F^a[A,a] + \frac{\alpha}{2}\phi^a\right), \tag{2.17}$$

$$F^a[A,a] := (\partial^\mu \delta^{ab} - \epsilon^{ab3} a^\mu) A_\mu^b := D^{\mu ab}[a] A_\mu^b. \tag{2.18}$$

The basic idea of Abelian projection proposed by 't Hooft [20] is to remove as many non-Abelian degrees of freedom as possible, by partially fixing the gauge in such a way that the maximal torus group H of the gauge group G remains unbroken. Under the Abelian projection, $G=SU(N)$ gauge theory reduces to $H=U(1)^{N-1}$ Abelian gauge theory plus magnetic monopoles. Actually, the choice (2.14) for $G=SU(2)$ is nothing but the condition of minimizing the functional $\mathcal{R}[A]$ for the gauge rotated off-diagonal gluon fields A, i.e., $\min_U \mathcal{R}[A^U]$,

$$\begin{aligned}
\mathcal{R}[A] &:= \frac{1}{2}\int d^D x \{[A_\mu^1(x)]^2 + [A_\mu^2(x)]^2\}\\
&= \int d^D x A_\mu^+(x) A_\mu^-(x).
\end{aligned} \tag{2.19}$$

We can generalize the MAG to arbitrary group G as

$$\mathcal{R}[A] := \int d^D x \, {\rm tr}_{\mathcal{G}\setminus\mathcal{H}}\left[\frac{1}{2}\mathcal{A}_\mu(x)\mathcal{A}_\mu(x)\right], \tag{2.20}$$

where the trace is taken over the Lie algebra $\mathcal{G}\setminus\mathcal{H}$. Under the MAG, it is shown [17] that the integration of the off-diagonal gluon fields $A_\mu^a \in \mathcal{G}\setminus\mathcal{H}$ in SU(2) YM theory leads to the Abelian-projected effective gauge theory (APEGT) written in terms of the maximal Abelian U(1) gauge field a_μ, the dual U(1) gauge field b_μ, and the magnetic (monopole) current k_μ.

In the gauge transformation (2.6), the local gauge rotation $U(x)$ is performed in such a way that the gauge rotated field $\mathcal{A}_\mu^U(x)$ minimizes the functional $\mathcal{R}[\mathcal{A}^U]$ and hence satisfies the gauge-fixing condition (2.14). We define the magnetic current by

$$k_\mu(x) := \epsilon_{\mu\nu\rho\sigma}\partial^\nu f^{\rho\sigma}(x), \tag{2.21}$$

$$f_{\rho\sigma}(x) := \partial_\rho a_\sigma^U(x) - \partial_\sigma a_\rho^U(x), \tag{2.22}$$

using the Abelian part (diagonal part) extracted as

$$a_\mu^U(x) := \mathrm{tr}[T^3\mathcal{A}_\mu^U(x)]. \tag{2.23}$$

If the gauge field $\mathcal{A}_\mu(x)$ is not singular, the first piece $U(x)\mathcal{A}_\mu(x)U^\dagger(x)$ of $\mathcal{A}_\mu^U(x)$ is nonsingular and does not give rise to magnetic current. On the contrary, the second piece $\Omega_\mu(x)$,

$$\Omega_\mu(x) := iU(x)\partial_\mu U^\dagger(x) \tag{2.24}$$

does give the nonvanishing magnetic monopole current (see, e.g., Ref. [17]) for

$$a_\mu^\Omega(x) := \Omega_\mu^3(x) := \mathrm{tr}[T^3\Omega_\mu(x)]. \tag{2.25}$$

According to a Monte Carlo simulation on the lattice [3], the magnetic monopole part gives the most dominant contribution in various quantities characterizing the low-energy physics of QCD, e.g., string tension, chiral condensate, topological charge, etc.

Therefore, it is expected that the most important degrees of freedom for the low-energy physics comes from the second piece $\Omega_\mu(x)$ of $\mathcal{A}_\mu^U(x)$. Therefore, we decompose the YM theory into two pieces, i.e., the contribution from the part $\Omega_\mu(x)$ and the remaining part.

C. Magnetic monopole in non-Abelian gauge theory

First we recall the calculation of the Abelian (diagonal) field strength in four-dimensional YM theory. We introduce three local field variables corresponding to the Euler angles

$$(\theta(x),\varphi(x),\chi(x)),(\theta\in[0,\pi],\varphi\in[0,2\pi],\chi\in[0,2\pi]) \tag{2.26}$$

to write an element $U(x)\in SU(2)$ as

$$U(x) = e^{i\chi(x)\sigma_3/2}e^{i\theta(x)\sigma_2/2}e^{i\varphi(x)\sigma_3/2} = \begin{pmatrix} e^{(i/2)[\varphi(x)+\chi(x)]}\cos\frac{\theta(x)}{2} & e^{-(i/2)[\varphi(x)-\chi(x)]}\sin\frac{\theta(x)}{2} \\ -e^{(i/2)[\varphi(x)-\chi(x)]}\sin\frac{\theta(x)}{2} & e^{-(i/2)[\varphi(x)+\chi(x)]}\cos\frac{\theta(x)}{2} \end{pmatrix}. \tag{2.27}$$

In the usual convention of perturbation theory, we take

$$\Omega_\mu(x) := \frac{i}{g}U(x)\partial_\mu U^\dagger(x). \tag{2.28}$$

Note that the following identity [17] holds for Ω_μ:

$$\partial_\mu\Omega_\nu(x) - \partial_\nu\Omega_\mu(x) = ig[\Omega_\mu(x),\Omega_\nu(x)] + \frac{i}{g}U(x)[\partial_\mu,\partial_\nu]U^\dagger(x). \tag{2.29}$$

Then the diagonal part reads

$$f_{\mu\nu}^\Omega(x) := \partial_\mu\Omega_\nu^3(x) - \partial_\nu\Omega_\mu^3(x) = C_{\mu\nu}^{[\Omega]}(x) + \frac{i}{g}\{U(x)[\partial_\mu,\partial_\nu]U^\dagger(x)\}^{(3)}, \tag{2.30}$$

where $C_{\mu\nu}$ was introduced in a previous paper [17] as

$$C_{\mu\nu}^{[\Omega]} := (ig[\Omega_\mu,\Omega_\nu])^{(3)} = g\epsilon^{ab3}\Omega_\mu^a\Omega_\nu^b = ig(\Omega_\mu^+\Omega_\nu^- - \Omega_\mu^-\Omega_\nu^+). \tag{2.31}$$

Using the Euler angle expression for U, we obtain

$$i(\Omega_\mu^+\Omega_\nu^- - \Omega_\mu^-\Omega_\nu^+) = \frac{1}{g^2}\sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta), \tag{2.32}$$

which implies

$$C_{\mu\nu}[\Omega] = \frac{1}{g}\sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta). \tag{2.33}$$

Now we show that $C_{\mu\nu}^{[\Omega]}$ denotes the monopole contribution to the diagonal field strength $f_{\mu\nu}$. Note that $C_{\mu\nu}^{[\Omega]}$ is generated from the off-diagonal gluon fields $\Omega_\mu^1,\Omega_\mu^2$.

In four dimensions, the magnetic monopole charge is calculated from the magnetic current

$$k_\mu = \partial_\nu\tilde{f}_{\mu\nu}^\Omega, \quad \tilde{f}_{\mu\nu}^\Omega := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}f_{\rho\sigma}^\Omega, \tag{2.34}$$

as

$$g_m(V^{(3)}) = \int_{V^{(3)}} d^3\sigma_\mu k_\mu = \int_{V^{(3)}} d^3\sigma_\mu\partial_\nu\tilde{f}_{\mu\nu}^\Omega = \int_{S^{(2)}=\partial V^{(3)}} d^2\sigma_{\mu\nu}\tilde{f}_{\mu\nu}^\Omega. \tag{2.35}$$

We can identity the first and second parts of right-hand-side (RHS) of Eq. (2.30) with the the magnetic monopole and the Dirac string part respectively contained in the TFT_4 theory and hence the YM_4 theory [17]. This is clearly seen by the explicit calculation using Euler angles, since we can rewrite Eq. (2.30) as

$$f_{\mu\nu}^{\Omega} = -\frac{1}{g}\sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta) + \frac{1}{g}([\partial_\mu,\partial_\nu]\chi + \cos\theta[\partial_\mu,\partial_\nu]\varphi). \tag{2.36}$$

The magnetic monopole part is given by

$$g_m(V^{(3)}) = \frac{1}{2g}\int_{S^{(2)}} d^2\sigma_{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}\sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta), \tag{2.37}$$

while the Dirac string part is

$$g_{\mathrm{DS}}(V^{(3)}) = \frac{1}{2g}\int_{S^{(2)}} d^2\sigma_{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} \times ([\partial_\mu,\partial_\nu]\chi + \cos\theta[\partial_\mu,\partial_\nu]\varphi). \tag{2.38}$$

The first definition (2.37) of g_m gives the quantized magnetic charge [17]. The integrand is the Jacobian from S^2 to S^2 as will be shown in Sec. V and the Homotopy group reads

$$\Pi_2(\mathrm{SU}(2)/\mathrm{U}(1)) = \Pi_2(S^2) = \mathbf{Z}. \tag{2.39}$$

Then Eq. (2.37) gives the magnetic charge g_m satisfying the Dirac quantization condition,

$$g_m = \frac{2\pi n}{g}, \quad g g_m = 2\pi n (n \in Z). \tag{2.40}$$

In the second definition (2.38) of g_m, if we choose $\chi = -\varphi$ using residual U(1) gauge invariance, then the Dirac string appears on the negative Z axis, i.e., $\theta = \pi$. In this case, the surface integral reduces to the line integral around the string

$$g_{\mathrm{DS}}(V^{(3)}) = \frac{1}{2g}\int_{S^{(2)}} d\sigma_{\mu\nu}\epsilon_{\mu\nu\rho\sigma}[\partial_\rho,\partial_\sigma]\varphi(x) = -\frac{1}{2g}\int_{S^{(1)}} d\sigma_{\mu\nu\rho}\epsilon_{\mu\nu\rho\sigma}\partial_\rho\varphi(x). \tag{2.41}$$

This gives the same result (2.38) but with the minus sign, as suggested from the Homotopy group

$$\Pi_1(\mathrm{U}(1)) = \mathbf{Z}. \tag{2.42}$$

Actually, two description are equivalent, as can be seen from the relation

$$\Pi_2(\mathrm{SU}(2)/\mathrm{U}(1)) = \Pi_1(\mathrm{U}(1)). \tag{2.43}$$

If the contribution from $U(x)\mathcal{A}_\mu(x)U^\dagger(x)$ is completely neglected, i.e., $\mathcal{A}_\mu^U(x) \equiv \Omega_\mu(x) = iU(x)\partial_\mu U^\dagger(x)$, Eq. (2.29) implies

$$\mathcal{F}_{\mu\nu}^U(x) \equiv \frac{i}{g}U(x)[\partial_\mu,\partial_\nu]U^\dagger(x), \tag{2.44}$$

where the RHS is identified with the contribution from the Dirac string, see Ref. [17]. Note that the original YM theory does not have a magnetic monopole solution. However, if we partially fix the gauge $G/H = \mathrm{SU}(2)/\mathrm{U}(1)$ and retain the residual $H = \mathrm{U}(1)$ gauge, the theory can have a singular configuration. This is a reason why the magnetic monopole appears in YM theory which does not have a Higgs field. The existence of a Dirac string in the RHS of Eq. (2.44) reflects the fact that the field strength $\mathcal{F}_{\mu\nu}^U(x)$ contains the magnetic monopole contribution. We have obtained a gauge theory with magnetic monopole starting from YM theory. Therefore, MAG enables us to deduce the magnetic monopole without introducing the scalar field, in contrast to the 't Hooft-Polyakov monopole. See Ref. [17] for more details.

D. TFT and its deformation

Since the Dirac string does not contribute to the action, the topological nontrivial sector with a magnetic monopole in YM theory is described by the gauge-fixing and FP ghost terms alone (we forget the matter field for a while),

$$S_{\mathrm{TFT}}[\Omega_\mu,\mathcal{C},\bar{\mathcal{C}},\phi] = \int d^D x \mathcal{L}_{\mathrm{TFT}},$$

$$\mathcal{L}_{\mathrm{TFT}} := -i\delta_B G_{\mathrm{gf}}[\Omega_\mu,\mathcal{C},\bar{\mathcal{C}},\phi]. \tag{2.45}$$

This theory describes the topological field theory for the magnetic monopole, which is called MAG TFT hereafter. If we restrict the gauge rotation U(x) to the regular one, $\Omega_\mu(x)$ reduces to a pure gauge field $\mathcal{F}_{\mu\nu}^U(x) \equiv 0$ and hence the TFT is reduced to topological trivial theory. This model is called the pure gauge model (PGM) which has been studied by Hata, Kugo, Niigata, and Taniguchi [61–64]. However, PGM has only unphysical gauge modes and does not have physical modes. We consider that the topological objects must give the main contribution to the low-energy physics. From this viewpoint, the PGM is not interesting to us, since PGM cannot contain the topological nontrivial configuration as will be shown in the following.

In this paper, we take into account the topological nontrivial configuration involved in the theory (2.45) and extract the most important contribution in low-energy physics. We consider that $\Omega_\mu(x)$ gives the most important dominant contribution and the remaining contributions are treated as a perturbation around it. Whether this is efficient or not crucially depends on the choice of G_{gf}. For this purpose, MAG is most appropriate as will be shown later.

Our reformulation of YM theory proceeds as follows. First of all, we decompose the gauge field $\mathcal{A}_\mu(x)$ into the nonperturbative piece $\Omega_\mu(x)$ (including a topological nontrivial configuration) and the perturbative piece $U(x)\mathcal{A}_\mu(x)U^\dagger(x)$ (including only the topological trivial configuration). Next, we treat the original YM theory as a perturbative deformation of TFT written in terms of $\Omega_\mu(x)$ alone. Using the normalization of the field in perturbation theory, TFT is obtained from YM theory in the limit of vanishing coupling constant $g \to 0$. If we absorb the coupling constant g into the gauge field, TFT does not have an apparent coupling constant.

We expect that the TFT of describing the magnetic monopole gives the most dominant nonperturbative contributions in low-energy physics. In fact, the monopole dominance in low-energy physics of QCD has been confirmed by Monte Carlo simulations [3]. A similar attempt to reformulate YM theory as a deformation of topological BF theory was done by Martellini *et al.* [15]. The model is called BF-YM theory. A similar attempt was also made by Izawa [14] for the PGM using the BF formulation in three dimensions. Topological BF theory includes the topological nontrivial configuration. The APEGT for BF-YM theory can be constructed, see Ref. [17].

First, we regard the fields $\mathcal{A}_\mu$ and ψ as the gauge transformation of the fields $\mathcal{V}_\mu$ and Ψ (we use different characters to avoid confusions),

$$\mathcal{A}_\mu(x) := U(x)\mathcal{V}_\mu(x)U^\dagger(x) + \Omega_\mu(x),$$

$$\Omega_\mu(x) := \frac{i}{g}U(x)\partial_\mu U^\dagger(x) \tag{2.46}$$

$$\psi(x) := U(x)\Psi(x), \tag{2.47}$$

where $\mathcal{V}_\mu$ and Ψ are identified with the perturbative parts in the topological trivial sector.

Let $[dU]$ be the invariant Haar measure on the group G. Using the gauge invariance of the FP determinant $\Delta[\mathcal{A}]$ given by

$$\Delta[\mathcal{A}]^{-1} := \int [dU]\prod_x \delta[\partial^\mu \mathcal{A}_\mu^{U^{-1}}(x)],$$

$$\Delta[\mathcal{A}] = \Delta[\mathcal{A}^{U^{-1}}], \tag{2.48}$$

we can rewrite

$$\begin{aligned}1 &= \Delta[\mathcal{A}]\int [dU]\prod_x \delta[\partial^\mu \mathcal{A}_\mu^{U^{-1}}(x)]\\ &= \Delta[\mathcal{A}^{U^{-1}}]\int [dU]\prod_x \delta[\partial^\mu \mathcal{A}_\mu^{U^{-1}}(x)]\\ &= \Delta[\mathcal{V}]\int [dU]\prod_x \delta[\partial^\mu \mathcal{V}_\mu(x)]\\ &\cong \int [d\gamma][d\bar{\gamma}][d\beta]\\ &\quad\times \exp\left\{ i\int d^Dx(\mathrm{tr}_{\mathcal{G}}\{\beta\partial^\mu\mathcal{V}_\mu + i\bar{\gamma}\partial^\mu\mathcal{D}_\mu[\mathcal{V}]\gamma\})\right\}\\ &= \int [d\gamma][d\bar{\gamma}][d\beta]\\ &\quad\times \exp\left\{ i\int d^Dx[-i\tilde{\delta}_B\tilde{G}_{\mathrm{gf}}(\mathcal{V}_\mu,\gamma,\bar{\gamma},\beta)]\right\},\end{aligned} \tag{2.49}$$

where

$$\tilde{G}_{\mathrm{gf}}(\mathcal{V}_\mu,\gamma,\bar{\gamma},\beta) := \mathrm{tr}_{\mathcal{G}}(\bar{\gamma}\partial^\mu\mathcal{V}_\mu). \tag{2.50}$$

Here we have introduced new ghost field γ, antighost field $\bar{\gamma}$, and the multiplier field β which are subject to a new BRST transformation $\tilde{\delta}_B$,

$$\begin{aligned}\tilde{\delta}_B\mathcal{V}_\mu(x) &= \mathcal{D}_\mu[\mathcal{V}]\gamma(x) := \partial_\mu\gamma(x) - i[\mathcal{V}_\mu(x),\gamma(x)],\\ \tilde{\delta}_B\gamma(x) &= i\frac{1}{2}[\gamma(x),\gamma(x)],\\ \tilde{\delta}_B\bar{\gamma}(x) &= i\beta(x),\\ \tilde{\delta}_B\beta(x) &= 0,\\ \tilde{\delta}_B\Psi(x) &= \gamma(x)\Psi(x).\end{aligned} \tag{2.51}$$

Then the partition function can be rewritten as

$$\begin{aligned}Z_{\mathrm{QCD}}[J] &= \int [dU][d\mathcal{C}][d\bar{\mathcal{C}}][d\phi]\\ &\quad\times\int [d\mathcal{V}_\mu][d\gamma][d\bar{\gamma}][d\beta][d\Psi][d\bar{\Psi}]\\ &\quad\times\exp\left\{ i\int d^Dx\{-i\delta_B G_{\mathrm{gf}}[\Omega_\mu + U\mathcal{V}_\mu U^\dagger,\mathcal{C},\bar{\mathcal{C}},\phi]\right.\\ &\quad\left. +\mathcal{L}_{\mathrm{QCD}}[\mathcal{V},\Psi] - i\tilde{\delta}_B\tilde{G}_{\mathrm{gf}}(\mathcal{V}_\mu,\gamma,\bar{\gamma},\beta)\} + iS_J\right\},\end{aligned} \tag{2.52}$$

where

$$\begin{aligned}S_J &= \int d^Dx\{\mathrm{tr}_{\mathcal{G}}[J^\mu(\Omega_\mu + U\mathcal{V}_\mu U^\dagger)\\ &\quad + J_c\mathcal{C} + J_{\bar{c}}\bar{\mathcal{C}} + J_\phi\phi] + \bar{\eta}U\Psi + \eta\bar{\Psi}U^\dagger\}.\end{aligned} \tag{2.53}$$

The correlation functions of the original fundamental field $\mathcal{A}_\mu, \psi, \bar{\psi}$ is obtained by differentiating $Z[J]$ with respect to the source $J_\mu, \bar{\eta}, \eta$. The integration over the fields $(U,\mathcal{C},\bar{\mathcal{C}},\phi)$ should be treated nonperturbatively. The perturbative expansion around TFT means performing an integration over the new fields $(\mathcal{V}_\mu,\gamma,\bar{\gamma},\beta)$ after power-series expansions in the coupling constant g.

Assume that a choice of G_{gf} allows the separation of the variable in such a way that

$$\begin{aligned}&-i\delta_B G_{\mathrm{gf}}[\Omega_\mu + U\mathcal{V}_\mu U^\dagger,\mathcal{C},\bar{\mathcal{C}},\phi]\\ &\quad = -i\delta_B G_{\mathrm{gf}}[\Omega_\mu,\mathcal{C},\bar{\mathcal{C}},\phi] + i\mathcal{V}_\mu^A\mathcal{M}_\mu^A[U]\\ &\qquad + \frac{i}{2}\mathcal{V}_\mu^A\mathcal{V}_\mu^B\mathcal{K}^{AB}[U].\end{aligned} \tag{2.54}$$

In the next section, we show that the MAG satisfies the condition (2.54) and obtain the explicit form for $\mathcal{M}_\mu, \mathcal{K}$. Then, under the condition (2.54), the partition function is rewritten as

$$Z_{\rm QCD}[J] := \int [dU][d\mathcal{C}][d\bar{\mathcal{C}}][d\phi] \times \exp\Big\{ iS_{TFT}[\Omega_\mu, \mathcal{C}, \bar{\mathcal{C}}, \phi] + iW[U; J^\mu, \bar{\eta}, \eta] + i\int d^Dx\, {\rm tr}_{\mathcal{G}}[J^\mu \Omega_\mu + J_c \mathcal{C} + J_{\bar{c}} \bar{\mathcal{C}} + J_\phi \phi]\Big\}, \tag{2.55}$$

where $W[U; J^\mu, \bar{\eta}, \eta]$ is the generating functional of the connected correlation function of $\mathcal{V}_\mu$ in the perturbative sector given by

$$e^{iW[U;J^\mu,\bar{\eta},\eta]} := \int [d\mathcal{V}_\mu][d\gamma][d\bar{\gamma}][d\beta][d\Psi][d\bar{\Psi}] \times \exp\Big\{ iS_{\rm PQCD}[\mathcal{V}_\mu, \Psi, \gamma, \bar{\gamma}, \beta] + i\int d^Dx \Big[\mathcal{V}_\mu^A \mathcal{J}_\mu^A + \frac{i}{2}\mathcal{V}_\mu^A \mathcal{V}_\mu^B \mathcal{K}^{AB}[U] + {\rm tr}_{\mathcal{G}}(\bar{\eta} U \Psi + \eta \bar{\Psi} U^\dagger)\Big]\Big\}, \tag{2.56}$$

where

$$\mathcal{J}_\mu^A := (U^\dagger J^\mu U)^A + i\mathcal{M}_\mu^A[U]. \tag{2.57}$$

Here PQCD denotes the perturbative QCD (topological trivial sector) defined by the action $S_{\rm PQCD}$,

$$S_{\rm PQCD}[\mathcal{V}_\mu, \Psi, \gamma, \bar{\gamma}, \beta] := \int d^Dx \{ \mathcal{L}_{\rm QCD}[\mathcal{V}_\mu, \Psi] - i\tilde{\delta}_B \tilde{G}_{\rm gf}[\mathcal{V}_\mu, \gamma, \bar{\gamma}, \beta]\}. \tag{2.58}$$

The deformation $W[U; J^\mu, \bar{\eta}, \eta]$ should be calculated according to the ordinary perturbation theory in the coupling constant g. When there is no external source for quarks, we have

$$iW[U; J^\mu, 0, 0] := \ln\Big\langle \exp\Big\{ i\int d^Dx \Big[\mathcal{J}_\mu^A(x)\mathcal{V}_\mu^A(x) + \frac{i}{2}\mathcal{V}_\mu^A(x)\mathcal{V}_\mu^B(x)\mathcal{K}^{AB}(x)\Big]\Big\}\Big\rangle_{\rm PQCD}$$
$$= \frac{1}{2}g^2 \int d^Dx \int d^Dy \langle \mathcal{V}_\mu^A(x)\mathcal{V}_\nu^B(y)\rangle^c_{\rm PQCD} \times \{\mathcal{J}_\mu^A(x)\mathcal{J}_\nu^B(y) - \delta^D(x-y)\delta_{\mu\nu}\mathcal{K}^{AB}[U](x)\} + O(g^4\mathcal{J}^4). \tag{2.59}$$

Therefore, $W[U; J^\mu, 0, 0]$ is expressed as a power series in the coupling constant g and goes to zero as $g \to 0$. It turns out that the QCD is reduced to TFT in the vanishing limit of coupling constant. Thus QCD has been reformulated as a deformation of TFT. In a similar way, we can reformulate QED as a deformation of TFT, see Ref. [88].

III. MAXIMAL ABELIAN GAUGE AND HIDDEN SUPERSYMMETRY

The purpose of this section is to give some prerequisites which are necessary in order to understand the dimensional reduction discussed in the next section. First of all, we give a special version of the MAG which leads to the dimensional reduction of the TFT part obtained from YM theory in MAG. Using the BRST δ_B and anti-BRST $\bar{\delta}_B$ transformations, the action of the MAG TFT is written in the form

$$S_{\rm TFT} = \int d^Dx\, \delta_B \bar{\delta}_B \mathcal{O}(x). \tag{3.1}$$

Second, we introduce the superfield formalism. The $(D+2)$-dimensional superspace $X=(x^\mu, \theta, \bar{\theta})$ is defined by introducing two Grassmannian coordinates $\theta, \bar{\theta}$ in addition to the ordinary (bosonic) D-dimensional coordinates $x^\mu (\mu = 1, \ldots, D)$. We define the supersymmetry transformations and study the property of the superfield which is invariant under the supersymmetry transformation.

Third, we give a geometrical meaning of the BRST and anti-BRST transformations in the superspace. The gauge field $\mathcal{A}_\mu(x)$ is extended into a superfield $\mathcal{A}(X)$ as the connection one-form in the superspace. The merit of this formalism lies in the fact that we can also give a geometrical meaning to the FP ghost and antighost fields; actually the FP ghost and antighost fields can be identified as connection fields in the superspace. Furthermore, BRST transformation is rewritten as a geometrical condition, the horizontal condition. Consequently, the BRST transformation δ_B (anti-BRST transformation $\bar{\delta}_B$) of the field variable coincides with the derivative $\partial/\partial\theta$ $(\partial/\partial\bar{\theta})$ in the direction $\theta(\bar{\theta})$. Taking into account that the differentiation is equivalent to the integration for the Grassmannian variable, we can write the MAG TFT in a manifestly supersymmetric covariant form

$$S_{\rm TFT} = \int d^Dx \int d\theta \int d\bar{\theta}\, \mathcal{O}(x, \theta, \bar{\theta}), \tag{3.2}$$

where $\mathcal{O}(x)$ is extended to the superfield $\mathcal{O}(X) = \mathcal{O}(x, \theta, \bar{\theta})$ and $\mathcal{O}(x, \theta, \bar{\theta})$ has OSp$(D/2)$ invariant form. This implies the existence of the hidden supersymmetry in MAG TFT which is an origin of the dimensional reduction shown in the next section.

A. Choice of MAG

In the previous section, we considered the simplest MAG condition (2.16) which leads to

$$\mathcal{L}_{\rm GF}=\phi^a F^a[A,a]+\frac{\alpha}{2}(\phi^a)^2+i\bar{C}^a D^{\mu ab}[a]D^{bc}_{\mu}[a]C^c - i\bar{C}^a[A^a_\mu A^{\mu b}-A^c_\mu A^{\mu c}\delta^{ab}]C^b+i\bar{C}^a\epsilon^{ab3}F^b[A,a]C^3. \tag{3.3}$$

Note that we can take a more general form for $G_{\rm gf}$ [62,17],

$$G_{\rm gf}=\sum_{\pm}\bar{C}^{\mp}\left(F^{\pm}[A,a]+\frac{\alpha}{2}\phi^{\pm}\right)+\zeta C^3\bar{C}^+\bar{C}^- + \eta\sum_{\pm}(\pm)\bar{C}^3\bar{C}^{\pm}C^{\mp}. \tag{3.4}$$

In what follows, we choose a specific form

$$G'_{\rm gf}=\sum_{\pm}\bar{C}^{\mp}(F^{\pm}[A,a]-\phi^{\pm})-2C^3\bar{C}^+\bar{C}^-, \tag{3.5}$$

which corresponds in Eq. (3.4) to

$$\alpha=-2,\quad \zeta=-2,\quad \eta=0. \tag{3.6}$$

Then the gauge fixing part $\mathcal{L}_{\rm GF}=-i\delta_B G_{\rm gf}$ has an additional contribution

$$\mathcal{L}'_{\rm GF}=\mathcal{L}_{\rm GF}-\zeta\sum_{\pm}(\pm)C^3\bar{C}^{\mp}\phi^{\pm}-\zeta\bar{C}^+\bar{C}^-C^+C^- =\mathcal{L}_{\rm GF}-\zeta\sum_{a,b}i\epsilon^{ab3}C^3\bar{C}^a\phi^b-\zeta\bar{C}^+\bar{C}^-C^+C^-. \tag{3.7}$$

The four-ghost interaction term is generated. This is a general feature of nonlinear gauge fixing.[2] Separating the ϕ^a-dependent terms and integrating out the field ϕ^a, we obtain

$$S_{\rm GF}=\int d^Dx\Big[-\frac{1}{2\alpha}(F^a[A,a]+J^a_\phi+\zeta i\epsilon^{ab3}C^3\bar{C}^b)^2 + i\bar{C}^aD^{\mu ab}[a]D^{bc}_\mu[a]C^c-i\bar{C}^a(A^a_\mu A^{\mu b}-A^c_\mu A^{\mu c}\delta^{ab}) \times C^b+i\bar{C}^a\epsilon^{ab3}F^b[A,a]C^3-\zeta\bar{C}^+\bar{C}^-C^+C^- + A^a_\mu J^{\mu a}\Big], \tag{3.9}$$

where we have included the source term $J^a_\phi\phi^a+J^a_\mu A^a_\mu$. Thus the action is summarized as

$$S_{\rm GF}=\int d^Dx\Big[\frac{1}{2g^2}A^a_\mu Q^{ab}_{\mu\nu}A^b_\nu+i\bar{C}^aD^{\mu ac}[a]D^{cb}_\mu[a]C^b + A^a_\mu\left(G^a_\mu+\frac{1}{\alpha}D^{\mu ab}[a]J^b_\phi+J^{\mu a}\right)+\frac{1}{2\alpha}(\zeta\epsilon^{ab3}C^3\bar{C}^b)^2 - \zeta\bar{C}^+\bar{C}^-C^+C^- -\frac{i\zeta}{\alpha}J^b_\phi\epsilon^{ab3}C^3\bar{C}^a-\frac{1}{2\alpha}(J^a_\phi)^2\Big], \tag{3.10}$$

$$Q^{ab}_{\mu\nu}:=-2ig^2(\bar{C}^aC^b-\bar{C}^cC^c\delta^{ab})\delta_{\mu\nu}+\frac{1}{\alpha}D_\mu[a]^{ac}D_\nu[a]^{cb}, \tag{3.11}$$

$$G^c_\mu:=i\left(\frac{\zeta}{\alpha}-1\right)D_\mu[a]^{cb}(\epsilon^{ab3}C^3\bar{C}^a), \tag{3.12}$$

where $G^c_\mu(x)\equiv 0$ for the choice of Eq. (3.6).

An advantage of the choice (3.5) is that $G'_{\rm gf}$ is written as the anti-BRST exact form

$$G'_{\rm gf}=\bar{\delta}_B\left(\frac{1}{2}(A^a_\mu)^2+iC^a\bar{C}^a\right)=\bar{\delta}_B\left(A^+_\mu A^-_\mu+i\sum_{\pm}C^{\pm}\bar{C}^{\mp}\right), \tag{3.13}$$

where $\bar{\delta}_B$ is the nilpotent anti-BRST transformation [18],

$$\begin{aligned}
&\bar{\delta}_B\mathcal{A}_\mu(x)=\mathcal{D}_\mu\bar{\mathcal{C}}(x):=\partial_\mu\bar{\mathcal{C}}(x)-i[\mathcal{A}_\mu(x),\bar{\mathcal{C}}(x)],\\
&\bar{\delta}_B\mathcal{C}(x)=i\bar{\phi}(x),\\
&\bar{\delta}_B\bar{\mathcal{C}}(x)=i\frac{1}{2}[\bar{\mathcal{C}}(x),\bar{\mathcal{C}}(x)],\\
&\bar{\delta}_B\bar{\phi}(x)=0,\\
&\bar{\delta}_B\psi(x)=i\bar{\mathcal{C}}(x)\psi(x),\\
&\phi(x)+\bar{\phi}(x)=[\mathcal{C}(x),\bar{\mathcal{C}}(x)],
\end{aligned} \tag{3.14}$$

where $\bar{\phi}$ is defined in the last equation. The BRST and anti-BRST transformations have the following properties:[3]

$$(\delta_B)^2=0,\quad (\bar{\delta}_B)^2=0,\quad \{\delta_B,\bar{\delta}_B\}:=\delta_B\bar{\delta}_B+\bar{\delta}_B\delta_B=0. \tag{3.16}$$

Hence, we obtain

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[2]Such a term is necessary to renormalize the YM theory in MAG, since the MAG is nonlinear gauge-fixing. This is reflected in the fact that the U(1) invariant four-ghost interaction term $\bar{C}^+\bar{C}^-C^+C^-$ is produced through the expansion of $\ln\det Q$ (see Ref. [17]),

$$(\bar{C}^aC^b-\bar{C}^cC^c\delta^{ab})(\bar{C}^bC^a-\bar{C}^dC^d\delta^{ba}) =-2\bar{C}^1C^1\bar{C}^2C^2=-2\bar{C}^+\bar{C}^-C^+C^-. \tag{3.8}$$

[3]The operation δ_B or $\bar{\delta}_B$ on the product of two quantities is given by

$$\delta(XY)=(\delta X)Y\mp X\delta Y,\quad \delta=\delta_B,\bar{\delta}_B, \tag{3.15}$$

where the $+(-)$ sign is taken for a bosonic (fermionic) quantity X.

$$\mathcal{L}_{\rm GF}=i\,\delta_B\bar{\delta}_B\left(\frac{1}{2}(A_\mu^a)^2+iC^a\bar{C}^a\right)$$
$$=i\,\delta_B\bar{\delta}_B\left(A_\mu^+A_\mu^-+i\sum_{\pm}C^{\pm}\bar{C}^{\mp}\right), \tag{3.17}$$

which is invariant under the BRST and anti-BRST transformations,

$$\delta_B\mathcal{L}_{\rm GF}=0=\bar{\delta}_B\mathcal{L}_{\rm GF}. \tag{3.18}$$

Thus the MAG TFT action can be written as

$$S_{\rm TFT}=\int d^Dx\,i\,\delta_B\bar{\delta}_B\left(\frac{1}{2}[\Omega_\mu^a(x)]^2+iC^a(x)\bar{C}^a(x)\right) \tag{3.19}$$
$$=\int d^Dx\,i\,\delta_B\bar{\delta}_B\left(\Omega_\mu^+(x)\Omega_\mu^-(x)+i\sum_{\pm}C^{\pm}(x)\bar{C}^{\mp}(x)\right) \tag{3.20}$$
$$=\int d^Dx\;\; i\,\delta_B\bar{\delta}_B{\rm tr}_{\mathcal{G}\setminus\mathcal{H}}\left(\frac{1}{2}[\Omega_\mu(x)]^2+i\mathcal{C}(x)\bar{\mathcal{C}}(x)\right). \tag{3.21}$$

For our choice of MAG, we find for Eq. (2.54)

$$\mathcal{M}_\mu^A[U]:=\delta_B\bar{\delta}_B[(UT^AU^\dagger)^a\Omega_\mu^a],$$
$$\mathcal{K}^{AB}[U]:=\delta_B\bar{\delta}_B[(UT^AU^\dagger)^a(UT^BU^\dagger)^a], \tag{3.22}$$

where we have used

$$\delta_B\mathcal{V}_\mu(x)=0=\bar{\delta}_B\mathcal{V}_\mu(x). \tag{3.23}$$

The BRST and anti-BRST transformations for U are

$$\delta_BU(x)=i\mathcal{C}(x)U(x),\quad \bar{\delta}_BU(x)=i\bar{\mathcal{C}}(x)U(x). \tag{3.24}$$

This reproduces the usual BRST and anti-BRST transformations of the gauge field $\Omega_\mu:=iU\partial_\mu U^\dagger$.

B. Superspace formulation

Now we explain the superspace formulation based on Refs. [28–33]. We introduce a $(D+2)$-dimensional superspace $\mathcal{M}$ with coordinates

$$X^M:=(x^\mu,\theta,\bar{\theta})\in\mathcal{M},\quad x\in R^D, \tag{3.25}$$

where x^μ denotes the coordinate of the D-dimensional Euclidean space and θ and $\bar{\theta}$ are anti-Hermitian Grassmann numbers satisfying

$$\theta^2=0,\quad \bar{\theta}^2=0,\quad \{\theta,\bar{\theta}\}:=\theta\bar{\theta}+\bar{\theta}\theta=0,$$
$$\theta^\dagger=-\theta,\quad \bar{\theta}^\dagger=-\bar{\theta}. \tag{3.26}$$

We define the inner product of two vectors by introducing the superspace (covariant) metric tensor η_{MN} with components

$$\eta_{\mu\nu}=\delta_{\mu\nu},\quad \eta_{\theta\bar{\theta}}=-\eta_{\bar{\theta}\theta}=-2/\gamma,\quad \text{others}=0. \tag{3.27}$$

The contravariant metric tensor is defined by $\eta^{MN}\eta_{NL}=\delta_L^M$. Note that η_{MN} is not symmetric. We introduce the covariant supervector,

$$X_M=\eta_{MN}X^N \tag{3.28}$$

and the quadratic form

$$X^MX_M=X^M\eta_{MN}X^N=x^2+(4/\gamma)\bar{\theta}\theta. \tag{3.29}$$

Note that X^MX_M and X_MX^M are different, because the metric tensor is not symmetric,

$$X^MX_M\neq X_MX^M=\eta_{MN}X^NX^M=x^2-(4/\gamma)\bar{\theta}\theta. \tag{3.30}$$

Integrations over $\bar{\theta}$ and θ are defined by

$$\int d\theta=\int d\bar{\theta}=0,\quad \int d\theta\theta=\int d\bar{\theta}\bar{\theta}=i \tag{3.31}$$

or

$$\int d\theta d\bar{\theta}\begin{pmatrix}1\\ \theta\\ \bar{\theta}\\ \theta\bar{\theta}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{3.32}$$

Supersymmetry transformations are simply rotations in the superspace leaving invariant the quadratic form

$$\eta_{MN}X_1^MX_2^N=x_1^\mu x_2^\mu+(2/\gamma)(\bar{\theta}_1\theta_2-\theta_1\bar{\theta}_2). \tag{3.33}$$

This corresponds to the orthosymplectic supergroup OSp(D/2). It contains the rotation in R^D, i.e., the D-dimensional orthogonal group O(D) which leaves x^2 invariant and the symplectic group OSp(2) of transformations leaving $\theta\bar{\theta}$ invariant. In addition, OSp(D/2) includes transformations that mix the commuting and anticommuting variables,

$$x^\mu\to x^{\mu\prime}:=x^\mu+2\bar{a}^\mu\xi\theta+2a^\mu\xi\bar{\theta},$$
$$\theta\to\theta':=\theta+\gamma a^\mu x_\mu\xi,$$
$$\bar{\theta}\to\bar{\theta}':=\bar{\theta}-\gamma\bar{a}^\mu x_\mu\xi, \tag{3.34}$$

where a, $\bar{a}$ are arbitrary D vectors and ξ is an anticommuting c-number ($\xi^2=\{\xi,\theta\}=\{\xi,\bar{\theta}\}=0$). We call this transformation $\tau(a,\bar{a})$.

Any object $A^M=(A^\mu,A^\theta,A^{\bar{\theta}})$ which transforms similar to the supercoordinate under OSp(D/2) is defined to be a (contravariant) supervector. If A_1^M and A_2^M are two such supervectors, then the inner product

$$A_1^M A_{2M} = A_2^M A_{1M} = A_1^\mu A_{2\mu} + (2/\gamma)(A_1^{\bar\theta} A_2^\theta + A_2^{\bar\theta} A_1^\theta) \tag{3.35}$$

is invariant under superrotations. We define the the partial derivatives to be covariant supervectors in superspace,

$$\partial_M := \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \bar\theta} \right) := (\partial_\mu, \partial_\theta, \partial_{\bar\theta}). \tag{3.36}$$

Then the super-Laplacian defined by

$$\partial^M \partial_M := \Delta_{SS} = \partial^\mu \partial_\mu + \gamma \partial_{\bar\theta} \partial_\theta, \tag{3.37}$$

is an invariant.

Introducing a grading $p(M)$ for each coordinate X^M as

$$p(\mu) = 0, \quad p(\theta) = p(\bar\theta) = 1, \tag{3.38}$$

the coordinates obey the graded commutation relations

$$X^M X^N - (-1)^{p(M)p(N)} X^N X^M = 0. \tag{3.39}$$

Similarly, objects F_{MN} which transform as $A_1^M A_2^N$ are defined to be (contravariant) supertensors and $F_M^M := F^{MN} \eta_{MN}$ is an invariant. The metric tensor defined above is a supertensor. The metric has another invariant called the supertrace in addition to the trace $\eta^{MN} \eta_{MN}$,

$$\mathrm{str}(\eta) = (-1)^{p(M)} \eta_M^M. \tag{3.40}$$

We introduce the superfield $\Phi(x,\theta,\bar\theta)$ as

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= \Phi_0(x) + \theta \bar\Phi_1(x) + \bar\theta \Phi_2(x) + \bar\theta\theta \Phi_3(x) \\ &= \Phi_0(x) + \theta \partial_\theta \Phi_0(x) + \bar\theta \partial_{\bar\theta} \Phi_0(x) + \bar\theta\theta \partial_\theta \partial_{\bar\theta} \Phi_0(x),\end{aligned} \tag{3.41}$$

where Φ_i are complex-valued functions, Φ_i: $\mathbf{R}^D \to \mathbf{C}$ $(i=0,1,2,3)$. It should be noted that all component fields Φ_i transform according to the same representation of $O(D)$. Hence, in this formulation of superspace, supersymmetry transformations mix fields obeying different statistics, but with identical spin.

For any superfield Φ, the supertransformation τ acts as

$$\begin{aligned}[\tau(a,\bar a)\Phi](x,\theta,\bar\theta) &= \Phi(x,\theta,\bar\theta) + [\gamma a^\mu x_\mu \Phi_1(x) - \gamma \bar a^\mu x_\mu \Phi_2(x)]\xi \\ &+ [-2\partial_\mu \Phi_0(x) \bar a^\mu + \gamma \bar a^\mu x_\mu \Phi_3(x)]\theta\xi \\ &+ [-2\partial_\mu \Phi_0(x) a^\mu + \gamma a^\mu x_\mu \Phi_3(x)]\bar\theta\xi \\ &+ 2[\partial_\mu \Phi_1(x) a^\mu - \partial_\mu \Phi_2(x) \bar a^\mu]\bar\theta\theta\xi.\end{aligned} \tag{3.42}$$

If the superfield Φ is invariant by τ for all $a, \bar a \in R^D$, the term with ξ of the RHS of this equation must be zero for all $a, \bar a \in R^D$. Hence,

$$\Phi_1(x) \equiv 0 \equiv \Phi_2(x), \quad \frac{2}{\gamma} \partial_\mu \Phi_0(x) = x_\mu \Phi_3(x). \tag{3.43}$$

This implies that $\Phi_0(x)$ is a function only of $x^2 := x^\mu x^\mu$. Then we can write $\Phi_0(x) = f(x^2)$ for a function f: $[0,\infty) \to \mathbf{C}$ and $\Phi_3(x) = (4/\gamma) f'(x^2)$.

Therefore, if the superfield $\mathcal{O}(X)$ is supersymmetric, then there exists a function f: $[0,\infty) \to \mathbf{C}$ such that

$$\mathcal{O}(x,\theta,\bar\theta) = f(x^2) + (4/\gamma)\bar\theta\theta f'(x^2) = f(x^2 + (4/\gamma)\bar\theta\theta). \tag{3.44}$$

C. Geometric meaning of BRST transformation in superspace

We define the connection one form (superspace vector potential) $\mathcal{A}(X)$ and its curvature (superspace field strength) $\mathcal{F}(X)$ in the superspace, $X^M := (x^\mu, \theta, \bar\theta) \in \mathcal{M}$,

$$\begin{aligned}\mathcal{A}(X) &:= \mathcal{A}_M(X) dX^M \\ &= \mathcal{A}_\mu(x,\theta,\bar\theta) dx^\mu + \mathcal{C}(x,\theta,\bar\theta) d\theta + \bar{\mathcal{C}}(x,\theta,\bar\theta) d\bar\theta, \\ \mathcal{F}(X) &:= \tilde d \mathcal{A}(X) + \frac{1}{2}[\mathcal{A}(X), \mathcal{A}(X)] \\ &= -\frac{1}{2} \mathcal{F}_{NM}(X) dX^M dX^N, \\ \mathcal{A}_M(X) &:= \mathcal{A}_M^A(X) T^A, \\ dX^M &:= (dx^\mu, d\theta, d\bar\theta),\end{aligned} \tag{3.45}$$

where $\tilde d$ is the exterior differential in the superspace,

$$\tilde d := d + \delta + \bar\delta := \frac{\partial}{\partial x^\mu} dx^\mu + \frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \bar\theta} d\bar\theta. \tag{3.46}$$

These definitions are compatible when

$$\begin{aligned} dx^M dx^N &= -(-1)^{p(M)p(N)} dx^N dx^M, \\ (x^M, \partial_M) dx^N &= (-1)^{p(M)p(N)} dx^N (x^M, \partial_M). \end{aligned} \tag{3.47}$$

The supergauge transformation is given by

$$\begin{aligned}\mathcal{A}(X) \to \mathcal{A}'(X) &:= U(X)\mathcal{A}(X)U^\dagger(X) + iU^\dagger(X) dx^M \partial_M U(X), \\ U(X) &:= \exp[i\omega^A(X) T^A].\end{aligned} \tag{3.48}$$

In what follows, we show that the superfields $\mathcal{A}_\mu(X)$, $\mathcal{C}(X)$, $\bar{\mathcal{C}}(X)$ are respectively identified with a generalization of $\mathcal{A}_\mu(x)$, $\mathcal{C}(x)$, $\bar{\mathcal{C}}(x)$ into the superspace. First, we require that

$$\mathcal{A}_\mu(x,0,0) = \mathcal{A}_\mu(x), \quad \mathcal{C}(x,0,0) = \mathcal{C}(x), \quad \bar{\mathcal{C}}(x,0,0) = \bar{\mathcal{C}}(x), \tag{3.49}$$

and impose the *horizontal condition* [33] for any M,

$$\mathcal{F}_{M\theta}(X) = \mathcal{F}_{M\bar\theta}(X) = 0, \tag{3.50}$$

which is equivalent to set

$$\mathcal{F}(X)=\frac{1}{2}\mathcal{F}_{\mu\nu}(X)dx^\mu dx^\nu. \tag{3.51}$$

By solving the horizontal condition, the dependence of the superfield $A_M(x,\theta,\bar{\theta})$ on $\theta,\bar{\theta}$ is determined as follows. The horizontal condition (3.51) is rewritten as

$$(d+\delta+\bar{\delta})(\mathcal{A}^1+\mathcal{C}^1+\bar{\mathcal{C}}^1)+\frac{1}{2}[\mathcal{A}^1+\mathcal{C}^1+\bar{\mathcal{C}}^1,\mathcal{A}^1+\mathcal{C}^1+\bar{\mathcal{C}}^1]=d\mathcal{A}^1+\frac{1}{2}[\mathcal{A}^1,\mathcal{A}^1]. \tag{3.52}$$

where we have defined the one form

$$\mathcal{A}^1:=\mathcal{A}_\mu(x,\theta,\bar{\theta})dx^\mu,\quad \mathcal{C}^1:=\mathcal{C}(x,\theta,\bar{\theta})d\theta,\quad \bar{\mathcal{C}}^1:=\bar{\mathcal{C}}(x,\theta,\bar{\theta})d\bar{\theta}. \tag{3.53}$$

By comparing both sides of Eq. (3.52), we obtain

$$\partial_\theta\mathcal{A}_\mu(X)=\partial_\mu\mathcal{C}(X)-i[\mathcal{A}_\mu(X),\mathcal{C}(X)],$$

$$\partial_\theta\mathcal{C}(X)=i\frac{1}{2}[\mathcal{C}(X),\mathcal{C}(X)],$$

$$\partial_{\bar{\theta}}\mathcal{A}_\mu(X)=\partial_\mu\bar{\mathcal{C}}(X)-i[\mathcal{A}_\mu(X),\bar{\mathcal{C}}(X)],$$

$$\partial_{\bar{\theta}}\bar{\mathcal{C}}(X)=i\frac{1}{2}[\bar{\mathcal{C}}(X),\bar{\mathcal{C}}(X)],$$

$$\partial_\theta\bar{\mathcal{C}}(X)+\partial_{\bar{\theta}}\mathcal{C}(X)=-\{\mathcal{C}(X),\bar{\mathcal{C}}(X)\}, \tag{3.54}$$

where we have used that $d\theta d\theta\neq 0$ and $d\theta,\theta$ anticommute with $\mathcal{C}$. For the components which cannot be determined by the horizontal condition alone, we use the following identification:

$$\partial_\theta\bar{\mathcal{C}}(x,0,0):=i\phi(x),\quad \partial_{\bar{\theta}}\mathcal{C}(x,0,0):=i\bar{\phi}(x). \tag{3.55}$$

This corresponds to $\mathcal{F}_{\theta\bar{\theta}}=0$ and gives

$$i\phi(x)+i\bar{\phi}(x)+\{\mathcal{C}(x),\bar{\mathcal{C}}(x)\}=0. \tag{3.56}$$

From these results, it turns out that the derivatives in the direction of $\theta,\bar{\theta}$ give respectively the BRST and the anti-BRST transformations

$$\frac{\partial}{\partial\theta}=\delta_B,\quad \frac{\partial}{\partial\bar{\theta}}=\bar{\delta}_B, \tag{3.57}$$

where we define the derivative as the left derivative. This implies that the BRST and anti-BRST charges $Q_B,\bar{Q}_B$ are the generators of the translations in the variables $\theta,\bar{\theta}$.

Thus the superfields are determined as

$$\begin{aligned}\mathcal{A}_\mu(x,\theta,\bar{\theta})&=\mathcal{A}_\mu(x)+\theta\mathcal{D}_\mu\mathcal{C}(x)+\bar{\theta}\mathcal{D}_\mu\bar{\mathcal{C}}(x)\\&\quad+\bar{\theta}\theta(i\mathcal{D}_\mu\phi(x)+\{\mathcal{D}_\mu\mathcal{C}(x),\bar{\mathcal{C}}(x)\}),\\ \mathcal{C}(x,\theta,\bar{\theta})&=\mathcal{C}(x)+\theta\left(-\frac{1}{2}[\mathcal{C},\mathcal{C}](x)\right)+\bar{\theta}i\bar{\phi}(x)\\&\quad+\bar{\theta}\theta[i\bar{\phi}(x),\mathcal{C}(x)],\\ \bar{\mathcal{C}}(x,\theta,\bar{\theta})&=\bar{\mathcal{C}}(x)+\theta i\phi(x)+\bar{\theta}\left(-\frac{1}{2}[\bar{\mathcal{C}},\bar{\mathcal{C}}](x)\right)\\&\quad+\bar{\theta}\theta[-i\phi(x),\bar{\mathcal{C}}(x)].\end{aligned} \tag{3.58}$$

The nonvanishing components of $\mathcal{F}_{\mu\nu}$ have

$$\begin{aligned}\mathcal{F}_{\mu\nu}(x,\theta,\bar{\theta})&=\mathcal{F}_{\mu\nu}(x)+\theta[\mathcal{F}_{\mu\nu}(x),\bar{\mathcal{C}}(x)]+\bar{\theta}[\mathcal{F}_{\mu\nu}(x),\mathcal{C}(x)]\\&\quad+\bar{\theta}\theta(i[\mathcal{F}_{\mu\nu}(x),\phi(x)]\\&\quad+\{[\mathcal{F}_{\mu\nu}(x),\mathcal{C}(x)],\bar{\mathcal{C}}(x)\}).\end{aligned} \tag{3.59}$$

For the matter field $\varphi(x)$, we define the superfield $\varphi(X)$ and its covariant derivative as

$$\varphi(X):=\varphi(x)+\theta\varphi_1(x)+\bar{\theta}\varphi_2(x)+\bar{\theta}\theta\varphi_3(x), \tag{3.60}$$

$$\tilde{\mathcal{D}}[\mathcal{A}]\varphi(X):=[\tilde{d}+\mathcal{A}(X)]\varphi(X). \tag{3.61}$$

The horizontal condition for the matter field is

$$\mathcal{D}_M\varphi(X)dX^M=\mathcal{D}_\mu\varphi(X)dx^\mu, \tag{3.62}$$

which implies

$$\mathcal{D}_\theta\varphi(X)=0=\mathcal{D}_{\bar{\theta}}\varphi(X). \tag{3.63}$$

From this, we have, for example,

$$\begin{aligned}\varphi_1(x,0,0)&=\partial_\theta\varphi(x,0,0)\\&=-\mathcal{A}_\theta(X)\varphi(X)|_{\theta=\bar{\theta}=0}=-\mathcal{C}(x)\varphi(x)=\delta_B\varphi(x).\end{aligned} \tag{3.64}$$

Accordingly, all the field variables obey the relation

$$\Phi(x,\theta,\bar{\theta})=\Phi(x)+\theta[\delta_B\Phi(x)]+\bar{\theta}[\bar{\delta}_B\Phi(x)]+\bar{\theta}\theta[\bar{\delta}_B\delta_B\Phi(x)]. \tag{3.65}$$

Let $\Phi_1(X)$ and $\Phi_2(X)$ be two superfields corresponding to $\phi_1(x)$ and $\phi_2(x)$, respectively. It is easy to show that the following formula holds:

$$\begin{aligned}\Phi_1(x,\theta,\bar{\theta})\Phi_2(x,\theta,\bar{\theta})&=\phi_1(x)\phi_2(x)+\theta\delta_B[\phi_1(x)\phi_2(x)]+\bar{\theta}\bar{\delta}_B[\phi_1(x)\phi_2(x)]\\&\quad+\bar{\theta}\theta\bar{\delta}_B\delta_B[\phi_1(x)\phi_2(x)].\end{aligned} \tag{3.66}$$

Thus, for any (elementary or composite) field $\mathcal{O}(x)$, we can define the corresponding superfield $\mathcal{O}(x,\theta,\bar\theta)$ using BRST and anti-BRST transformations as

$$\mathcal{O}(x,\theta,\bar\theta)=\mathcal{O}(x)+\theta\delta_B\mathcal{O}(x)+\overline{\theta\delta_B}\mathcal{O}(x)+\bar\theta\theta\bar\delta_B\delta_B\mathcal{O}(x). \tag{3.67}$$

In the superspace $\mathcal{M}$, the BRST and anti-BRST transformations correspond to the translation of θ and $\bar\theta$ coordinates, respectively.

For the Grassmann number, the integration $\int d\theta(\int d\bar\theta)$ is equivalent to the differentiation $d/d\theta(d/d\bar\theta)$. Hence the BRST δ_B and anti-BRST δ_B transformation has the following correspondence:

$$\delta_B\leftrightarrow\frac{d}{d\theta}\leftrightarrow\int d\theta,\quad \bar\delta_B\leftrightarrow\frac{d}{d\bar\theta}\leftrightarrow\int d\bar\theta. \tag{3.68}$$

This implies

$$\int d\theta d\bar\theta\mathcal{O}(x,\theta,\bar\theta)=-\mathcal{O}_3(x)=-\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta}\mathcal{O}(x,\theta,\bar\theta)$$
$$=-\bar\delta_B\delta_B\mathcal{O}(x)=\delta_B\bar\delta_B\mathcal{O}(x). \tag{3.69}$$

Therefore, if the Lagrangian (density) of the form $\delta_B\bar\delta_B\mathcal{O}(x)$ is given for an operator $\mathcal{O}$, the operator $\mathcal{O}$ can be extended into the superfield $\mathcal{O}(x,\theta,\bar\theta)$ in the superspace,

$$\int d^Dx\,\delta_B\bar\delta_B\mathcal{O}(x)=\int d^Dx\int d\theta d\bar\theta\mathcal{O}(x,\theta,\bar\theta). \tag{3.70}$$

D. MAG TFT as a supersymmetric theory

The operator

$$\mathcal{O}(x):=-i\,\mathrm{tr}_{\mathcal{G}\setminus\mathcal{H}}\left(\frac{1}{2}[\mathcal{A}_\mu(x)]^2+i\mathcal{C}(x)\bar{\mathcal{C}}(x)\right), \tag{3.71}$$

has a corresponding superfield given by

$$\mathcal{O}(X):=\frac{-i}{2}\mathrm{tr}_{\mathcal{G}\setminus\mathcal{H}}\{[\mathcal{A}_\mu(X)]^2+2i\mathcal{C}(X)\bar{\mathcal{C}}(X)\}, \tag{3.72}$$

where we have chosen

$$\frac{1}{\gamma}:=\frac{i}{2}. \tag{3.73}$$

The superfield $\mathcal{O}(X)$ is written in OSp$(D/2)$ invariant form,

$$\mathcal{O}(X)=\frac{-i}{2}\mathrm{tr}_{\mathcal{G}\setminus\mathcal{H}}[\eta_{NM}\mathcal{A}^M(X)\mathcal{A}^N(X)]. \tag{3.74}$$

Thus the action of MAG TFT can be written in the manifestly superspace covariant form

$$S_{\mathrm{TFT}}=\int d^Dx\int d\theta d\bar\theta\frac{-i}{2}\mathrm{tr}_{\mathcal{G}\setminus\mathcal{H}}[\eta_{NM}\Omega^M(x,\theta,\bar\theta)\Omega^N(x,\theta,\bar\theta)]. \tag{3.75}$$

IV. DIMENSIONAL REDUCTION OF TOPOLOGICAL FIELD THEORY

After giving a basic knowledge for the dimensional reduction of Parisi and Sourlas in the supersymmetric model, we apply this mechanism to MAG TFT. We show that D-dimensional MAG TFT is reduced to the $(D-2)$-dimensional coset G/H nonlinear σ model (NLSM). This implies that a class of correlation functions in D-dimensional MAG TFT can be calculated in the equivalent $(D-2)$-dimensional coset NLSM.

A. Parisi and Sourlas dimensional reduction

Now we split the D-dimensional Euclidean space into two subsets

$$x=(z,\hat x)\in R^D,\quad z\in\mathbf{R}^{D-2},\quad \hat x\in\mathbf{R}^2. \tag{4.1}$$

The relation (3.44) holds for any D. Hence, for supersymmetric operator $\mathcal{O}(X)$, we obtain

$$\mathcal{O}(x,\theta,\bar\theta)=f[z,\hat x^2+(4/\gamma)\bar\theta\theta]\equiv f(z,\hat x^2)+\frac{4}{\gamma}\bar\theta\theta\frac{d}{d\hat x^2}f(z,\hat x^2). \tag{4.2}$$

Therefore, for supersymmetric model, we find[4]

$$\begin{aligned}S_{\mathrm{GF}}&=\int d^Dx\int d\theta\int d\bar\theta\mathcal{O}(x,\theta,\bar\theta)\\&=\int d^{D-2}z\int d^2\hat x\int d\theta\int d\bar\theta\,\frac{4}{\gamma}\bar\theta\theta\frac{d}{d\hat x^2}f(z,\hat x^2)\\&=-\frac{4}{\gamma}\int d^{D-2}z\int d^2\hat x\frac{d}{d\hat x^2}f(z,\hat x^2)\\&=-\frac{4}{\gamma}\int d^{D-2}z\int_0^\infty\pi dr^2\frac{d}{dr^2}f(z,r^2)\\&=\frac{4\pi}{\gamma}\int d^{D-2}zf(z,0)\\&=\frac{4\pi}{\gamma}\int d^{D-2}z\mathcal{O}_0[(z,0),0,0],\end{aligned} \tag{4.3}$$

[4]An alternative derivation is as follows. By integration by parts, we find for $D>2$

$$\begin{aligned}\int d^Dxf'(x^2)&=S_D\int_0^\infty r^{D-1}drf'(r^2)\\&=-S_D\frac{D-2}{2}\int_0^\infty dr^2(r^2)^{D/2-2}f(r^2)\\&=-\pi\int d^{D-2}xf(x^2),\end{aligned}$$

where $S_D=2\pi^{D/2}/\Gamma(D/2)$ is the area of the unit sphere in D-dimensional space.

where we have assumed $f(z,\infty)\equiv\mathcal{O}_0[(z,\infty),0,0]=0$ and used the notation of Eq. (3.41).

This shows the dimensional reduction by two units. The supersymmetric D-dimensional model is equivalent to a purely bosonic model in $D-2$ dimensions. This fact[5] was first discovered by Parisi and Sourlas (PS) [28].

The correlation function in supersymmetric theory are generated by the partition function in the presence of external sources,

$$Z_{\rm SUSY}[\mathcal{J}]:=\int[d\Phi]\exp\left\{-\int d^Dx\,d\theta\,d\bar{\theta}[\mathcal{L}_{\rm SUSY}[\Phi]-\Phi(x,\theta,\bar{\theta})\mathcal{J}(x,\theta,\bar{\theta})]\right\}, \tag{4.4}$$

where we write all the fields by Φ collectively for the supersymmetric Lagrangian $\mathcal{L}_{\rm SUSY}[\Phi]$. Restricting the source to a $(D-2)$-dimensional subspace,

$$\mathcal{J}(x,\theta,\bar{\theta})=J(z)\,\delta^2(\hat{x})\,\delta(\theta)\,\delta(\bar{\theta}), \tag{4.5}$$

and taking the derivatives of $Z_{SS}[J]$ with respect to $J(z)$, we obtain the correlation functions of the superspace theory which are restricted to the $(D-2)$-dimensional subspace. These are identical to the correlation functions of the corresponding $(D-2)$-dimensional quantum theory,

$$Z_{\rm SUSY}[\mathcal{J}]=Z_{D-2}[J], \tag{4.6}$$

where $Z_{D-2}[J]$ is the generating functional for $(D-2)$-dimensional theory,

$$Z_{D-2}[J]:=\int[d\Phi_0]\times\exp\left\{-\int d^{D-2}z\left[\frac{4\pi}{\gamma}\mathcal{L}_0[\Phi_0]-\Phi_0(z)J(z)\right]\right\}. \tag{4.7}$$

When PS-dimensional reduction occurs, the three-way equivalence is known among (1) a field theory in a superspace of D commuting and two anticommuting dimensions, (2) the corresponding $(D-2)$-dimensional quantum field theory, and (3) the D-dimensional classical stochastic theory, namely, the stochastic average of the D-dimensional classical theory in the presence of random external sources. The final point has not yet been made clear in this paper.

B. Dimensional reduction of TFT to NLSM

The action (3.75) of TFT is manifestly invariant by all supertransformations. Therefore, D-dimensional MAG TFT is dimensionally reduced to the $(D-2)$-dimensional model in the sense of Parisi and Sourlas. From Eqs. (3.75) and (4.3), the equivalent $(D-2)$-dimensional theory is given by

$$S_{\rm NLSM}=2\pi\int d^{D-2}z\,{\rm tr}_{\mathcal{G}\setminus\mathcal{H}}\left[\frac{1}{2}\delta_{\mu\nu}\Omega^\mu(z)\Omega^\nu(z)\right] \tag{4.8}$$

$$=-\frac{\pi}{g^2}\int d^{D-2}z\,{\rm tr}_{\mathcal{G}\setminus\mathcal{H}}[U(z)\partial^\mu U^\dagger(z)U(z)\partial^\mu U^\dagger(z)]$$

$$=\frac{\pi}{g^2}\int d^{D-2}z\,{\rm tr}_{\mathcal{G}\setminus\mathcal{H}}[\partial^\mu U(z)\partial^\mu U^\dagger(z)]. \tag{4.9}$$

Thus the D-dimensional MAG TFT is reduced to the $(D-2)$-dimensional G/H nonlinear σ model (NLSM) whose partition function is given by

$$Z_{\rm NLSM}:=\int[dU]\exp\{-S_{\rm NLSM}[U]\}, \tag{4.10}$$

where we have dropped the ghost contribution $iC(z)\bar{C}(z)$. The correlation functions of the D-dimensional TFT coincide with the same correlation function calculated in the equivalent $(D-2)$-dimensional NLSM if the arguments x_i are located on the $(D-2)$-dimensional subspace,

$$\left\langle\prod_i\mathcal{F}_i(x_i)\right\rangle_{GMAG\ {\rm TFT}_D}=\left\langle\prod_i\mathcal{F}_i(x_i)\right\rangle_{G/H\ {\rm NLSM}_{D-2}}\quad\text{if}\quad x_i\in\mathbf{R}^{D-2}. \tag{4.11}$$

C. Gluon propagator and mass gap

The propagator of ${\rm NLSM}_{D-2}$ in momentum representation is obtained by taking $\hat{p}=p_\theta=p_{\bar{\theta}}=0$ in the supersymmetric quantity,

$$\frac{1}{2\pi i}\int d^Dx\,d\theta\,d\bar{\theta}e^{ip_\mu x_\mu-p_{\bar{\theta}}\theta+p_\theta\bar{\theta}}\times\eta_{NM}\langle\Omega_M^a(x,\theta,\bar{\theta})\Omega_N^b(0,0,0)\rangle_{{\rm TFT}_D}\big|_{\hat{p}=p_\theta=p_{\bar{\theta}}=0}$$

$$=\int d^{D-2}z\ \ e^{ip_k\cdot z_k}\delta_{ij}\langle\Omega_i^a(z)\Omega_j^b(0)\rangle_{{\rm NLSM}_{D-2}}$$

$$=\frac{g^2}{\pi}\delta_{ab}\delta_{ij}\left\{[1+u(p_k^2)]\frac{p_ip_j}{p_k^2}-u(p_k^2)\delta_{ij}\right\},\quad(p_i,p_j,p_k\in\mathbf{R}^{D-2}). \tag{4.12}$$

From OSp(D/2) invariance, we have

$$\frac{1}{2\pi i}\int d^Dx\,d\theta\,d\bar{\theta}e^{ip_\mu x_\mu-p_{\bar{\theta}}\theta+p_\theta\bar{\theta}}\langle\Omega_M^a(x,\theta,\bar{\theta})\Omega_N^b(0,0,0)\rangle_{{\rm TFT}_D}$$

$$=\frac{g^2}{\pi}\delta_{ab}\left\{[1+u(p_L^2)]\frac{p_Mp_N}{p_L^2}-u(p_L^2)\delta_{MN}\right\}, \tag{4.13}$$

where

[5]The dimensional reduction was first shown order by order in perturbation theory (i.e., diagram by diagram) for scalar field [28] and gauge field [29,30] theories. Later, the nonperturbative proof of dimensional reduction was given at least for scalar field theories [31,32]. We followed the presentation of [32] in this paper.

$$p_L^2 := p_\mu^2 + 2ip_{\bar\theta}p_\theta = p_k^2 + \hat{p}^2 + 2ip_{\bar\theta}p_\theta. \tag{4.14}$$

By setting $M=\mu$, $N=\nu$ and differentiating both sides of Eq. (4.13) by $\partial^2/\partial p_\theta \partial p_{\bar\theta}$, we obtain the propagator in D-dimensional TFT,

$$\frac{1}{2\pi i}\int d^D x e^{ip_\mu x_\mu}\langle \Omega_\mu^a(x)\Omega_\nu^b(0)\rangle_{\mathrm{TFT}_D} = \frac{g^2}{\pi}\delta_{ab}\{v(p)\delta_{\mu\nu} + (\delta_{\mu\nu} - p_\mu p_\nu)v'(p^2)\}, \tag{4.15}$$

where

$$v(p^2) := \frac{1+u(p^2)}{p^2}, \quad p^2 := p_\mu^2. \tag{4.16}$$

We compare Eq. (4.15) with Eq. (4.12) following Ref. [60]. If the particle spectrum has a mass gap in $D-2$ dimensions (4.12), then the function $v(p^2)$ is analytic around $p^2=0$ and hence there is no massless particle at all in the channel $A_\mu^a = \Omega_\mu^a$ in D dimensions (4.15).

Dimensional reduction shows the equivalence of the correlation functions at special coordinates, $\hat{x} = \theta = \bar\theta = 0$ or $\hat{p} = p_\theta = p_{\bar\theta} = 0$. It should be remarked that the spectra of particles in the channel U(x) differ between D- and $(D-2)$-dimensional models. It is worthwhile to remark that PS-dimensional reduction implies neither the equivalence of the state vector spaces nor the equivalence of the S matrices between the original model and the dimensionally reduced model.

The existence of mass gap in two-dimensional O(3) NLSM has been shown in Refs. [46,42]. Therefore, the off-diagonal gluons $A_\mu^a = \Omega_\mu^a (a=1,2)$ in four-dimensional SU(2) MAG TFT have a nonzero mass, $m_A \neq 0$. Although this was assumed in the previous study of APEGT of YM theory [17], it was supported by Monte Carlo simulation [25]. If we restrict the YM theory to the TFT part, the existence of a nonzero gluon mass has just been proven. This will also hold in the full YM theory, since the perturbation is not sufficient to diminish this mass to yield massless gluons.

V. NONLINEAR σ MODEL, INSTANTON, AND MONOPOLE

In the previous section we showed that, thanks to dimensional reduction, the calculation of correlation functions in TFT_D is reduced to that in the NLSM_{D-2}. In what follows, we restrict our considerations to SU(2) YM theory. In this section we study the correspondence between O(3) NLSM_2 and SU(2) MAG TFT_4, especially focusing on the topological nontrivial configurations. It is well known that the two-dimensional O(3) NLSM has instanton solutions. We find that the instanton in two dimensions corresponds to the magnetic monopole in four dimensions. This correspondence is utilized to prove quark confinement in the next section.

A. NLSM from TFT

For concreteness, we consider the case of $G=\mathrm{SU}(2)$. The case of $G=\mathrm{SU}(N)$, $N>2$ will be separately discussed in the next section.

First of all, we define

$$R_\mu(x) := iU^\dagger(x)\partial_\mu U(x) = R_\mu^A(x)T^A = \frac{-1}{2}\begin{pmatrix} \partial_\mu\varphi(x) + \cos\theta(x)\partial_\mu\chi(x) & -e^{-i\chi(x)}[i\partial_\mu\theta(x) - \sin\theta(x)\partial_\mu\chi(x)] \\ e^{+i\chi(x)}[i\partial_\mu\theta(x) + \sin\theta(x)\partial_\mu\chi(x)] & -[\partial_\mu\varphi(x) + \cos\theta(x)\partial_\mu\chi(x)] \end{pmatrix}, \tag{5.1}$$

and

$$L_\mu(x) := iU(x)\partial_\mu U(x)^\dagger = L_\mu^A(x)T^A = \frac{1}{2}\begin{pmatrix} \partial_\mu\chi(x) + \cos\theta(x)\partial_\mu\varphi(x) & -e^{+i\chi(x)}[i\partial_\mu\theta(x) + \sin\theta(x)\partial_\mu\varphi(x)] \\ e^{-i\chi(x)}[i\partial_\mu\theta(x) - \sin\theta(x)\partial_\mu\varphi(x)] & -[\partial_\mu\chi(x) + \cos\theta(x)\partial_\mu\varphi(x)] \end{pmatrix}, \tag{5.2}$$

where we have used the Euler angles θ, φ, χ and the fundamental representation

$$T^A = \frac{1}{2}\sigma^A, \quad \sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.3}$$

Note that R_μ and L_μ are Hermitian, $R_\mu^\dagger = R_\mu$, $L_\mu^\dagger = L_\mu$.

For later purposes, it is convenient to write various quantities in terms of Euler angle variables,

$$L_\mu^\pm(x) := \frac{1}{\sqrt{2}}(L_\mu^1 \pm iL_\mu^2) = \frac{\pm i}{\sqrt{2}}e^{\mp i\chi(x)}[\partial_\mu\theta(x) \mp i\sin\theta(x)\partial_\mu\varphi(x)], \tag{5.4}$$

$$L_\mu^3(x) = \partial_\mu \chi(x) + \cos\theta(x)\partial_\mu\varphi(x), \tag{5.5}$$

and

$$\begin{aligned} &L_\mu^1(x)L_\mu^1(x) + L_\mu^2(x)L_\mu^2(x) \\ &= 2L_\mu^+(x)L_\mu^-(x) \\ &= \partial_\mu\theta(x)\partial_\mu\theta(x) + \sin^2\theta(x)\partial_\mu\varphi(x)\partial_\mu\varphi(x), \end{aligned} \tag{5.6}$$

$$L_\mu^3(x)L_\mu^3(x) = [\partial_\mu\chi(x) + \cos\theta(x)\partial_\mu\varphi(x)]^2. \tag{5.7}$$

The O(3) NLSM is defined by introducing a three-dimensional unit vector $\mathbf{n}(x)$ on each point of space-time, $\mathbf{n}$: $\mathbf{R}^d \to S^2 (d := D-2)$,

$$\mathbf{n}(x) := \begin{pmatrix} n^1(x) \\ n^2(x) \\ n^3(x) \end{pmatrix} := \begin{pmatrix} \sin\theta(x)\cos\varphi(x) \\ \sin\theta(x)\sin\varphi(x) \\ \cos\theta(x) \end{pmatrix}. \tag{5.8}$$

The direction of the unit vector in internal space is specified by two angles $\theta(x), \varphi(x)$ at each point $x \in R^d$. Note that

$$\mathbf{n}(x)\cdot\mathbf{n}(x) := \sum_{A=1}^{3} n^A(x)n^A(x) = 1, \tag{5.9}$$

$$\mathbf{n}(x)\cdot\partial_\mu\mathbf{n}(x) = 0. \tag{5.10}$$

Using

$$\partial_\mu\mathbf{n}(x) := \begin{pmatrix} \cos\theta(x)\cos\varphi(x)\partial_\mu\theta(x) - \sin\theta(x)\sin\varphi(x)\partial_\mu\varphi(x) \\ \cos\theta(x)\sin\varphi(x)\partial_\mu\theta(x) + \sin\theta(x)\cos\varphi(x)\partial_\mu\varphi(x) \\ -\sin\theta(x)\partial_\mu\theta(x) \end{pmatrix}, \tag{5.11}$$

we find[6]

$$\frac{1}{2}[(\Omega_\mu^1(x))^2 + (\Omega_\mu^2(x))^2] = \frac{1}{2g^2}\{[L_\mu^1(x)]^2 + [L_\mu^2(x)]^2\} = \frac{1}{2g^2}\partial_\mu\mathbf{n}(x)\cdot\partial_\mu\mathbf{n}(x) \tag{5.14}$$

$$= \frac{1}{2g^2}\{[\partial_\mu\theta(x)]^2 + \sin^2\theta(x)[\partial_\mu\varphi(x)]^2\}. \tag{5.15}$$

Following the argument in the previous section, we conclude that the SU(2)/U(1) MAG TFT in D dimensions (TFT_D) is ''equivalent'' to O(3) NLSM in $D-2$ dimensions (NLSM_{D-2}) with the action

$$S_{\mathrm{NLSM}} = \int d^{D-2}x \frac{\pi}{2g^2}\partial_\mu\mathbf{n}(x)\cdot\partial_\mu\mathbf{n}(x). \tag{5.16}$$

Both the action (5.16) and the constraint (5.9) are invariant under global $O(3)$ rotation in internal space. The vector $\mathbf{n}$ is related to U through the adjoint orbit parametrization (see, e.g., Ref. [86] for a more rigorous mathematical presentation) as

$$n^A(x)T^A = U^\dagger(x)T^3U(x), \quad n^A(x) = \mathrm{tr}[U(x)T^AU^\dagger(x)T^3] \quad (A=1,2,3). \tag{5.17}$$

The residual U(1) invariance corresponds to a rotation about the vector $\mathbf{n}$. In other words, $\mathbf{n}$ is a U(1) gauge-invariant quantity and the NLSM is a theory written in terms of a gauge invariant quantity alone. In fact, under the transformation $U \to e^{i\theta T^3}U$, n^A is invariant. Then the U(1) part in the Haar measure is factored out. This can be seen as follows.

In general, the action of NLSM is determined as follows. The infinitesimal distance in the group manifold $\mathrm{SU}(2)/\mathrm{U}(1) \cong S^2$ is given by

$$ds^2 = g_{ab}(\Phi)d\Phi^a d\Phi^b = R^2[(d\theta)^2 + \sin^2\theta(d\varphi)^2]. \tag{5.18}$$

[6]It is easy to see that we can write an alternative form for the action

$$(\mathbf{n}\times\partial_\mu\mathbf{n})\cdot(\mathbf{n}\times\partial_\mu\mathbf{n}) = \partial_\mu\mathbf{n}\cdot\partial_\mu\mathbf{n}, \tag{5.12}$$

where the explicit form is written as

$$\mathbf{n}(x)\times\partial_\mu\mathbf{n}(x) := \begin{pmatrix} -\sin\varphi(x)\partial_\mu\theta(x) - \sin\theta(x)\cos\theta(x)\cos\varphi(x)\partial_\mu\varphi(x) \\ \cos\varphi(x)\partial_\mu\theta(x) - \sin\theta(x)\cos\theta(x)\sin\varphi(x)\partial_\mu\varphi(x) \\ \sin^2\theta(x)\partial_\mu\varphi(x) \end{pmatrix}. \tag{5.13}$$

This implies that the metric g_{ab} and its determinant g are given by

$$g_{\theta\theta}=R^2, \quad g_{\varphi\varphi}=R^2\sin^2\theta, \quad g=\det(g_{ab})=R^4\sin^2\theta. \tag{5.19}$$

Hence the corresponding action of NLSM is given by

$$S=\int d^dx g_{ab}[\Phi(x)]\partial_\mu\Phi^a(x)\partial_\mu\Phi^b(x), \tag{5.20}$$

where coordinates $x^\mu,\mu=1,\cdots,d$ span a d-dimensional flat space-time and the fields $\Phi^a(a=1,2)$ are coordinates in two-dimensional Riemann manifold $\mathcal{M}$ called the target space. The symmetric matrix $g_{ab}(\Phi)$ is the corresponding metric tensor. Indeed, this action (Lagrangian) agrees with Eq. (5.15) for $\Phi^a=(\theta,\varphi)$. Consequently, the integration measure is given by

$$\begin{aligned} d\mu(\Phi)&:=\prod_{x\in R^d}\sqrt{g(\Phi(x))}d\Phi^1 d\Phi^2 \\ &=\prod_{x\in R^d}R^2\sin\theta(x)d\theta(x)d\varphi(x). \end{aligned}\tag{5.21}$$

This is the area element of two-dimensional sphere of radius R. Thus the partition function is defined by

$$Z_{\rm NLSM}:=\int[d\mu(\mathbf{n})]\prod_{x\in R^d}\delta[\mathbf{n}(x)\cdot\mathbf{n}(x)-1]\exp(-S_{\rm NLSM}), \tag{5.22}$$

$$d\mu(\mathbf{n})=\prod_{x\in R^d}\sin\theta(x)d\theta(x)d\varphi(x). \tag{5.23}$$

The constraint (5.9) is removed by introducing the Lagrange multiplier field $\lambda(x)$ as

$$\begin{aligned} S_{\rm NLSM}=&\int d^{D-2}x \\ &\times\left[\frac{\pi}{2g^2}\partial_\mu\mathbf{n}(x)\cdot\partial_\mu\mathbf{n}(x)+\lambda(x)[\mathbf{n}(x)\cdot\mathbf{n}(x)-1]\right]. \end{aligned}\tag{5.24}$$

For this action, the field equation is

$$\partial_\mu\partial_\mu\mathbf{n}(x)+\lambda(x)\mathbf{n}(x)=0. \tag{5.25}$$

Using the constraint and this field equation, we see

$$\lambda(x)=\lambda(x)\mathbf{n}(x)\cdot\mathbf{n}(x)=-\mathbf{n}(x)\cdot\partial_\mu\partial_\mu\mathbf{n}(x). \tag{5.26}$$

Therefore λ is eliminated from the field equation

$$\partial_\mu\partial_\mu\mathbf{n}(x)-[\mathbf{n}(x)\cdot\partial_\mu\partial_\mu\mathbf{n}(x)]\mathbf{n}(x)=0. \tag{5.27}$$

B. Instanton solution

Instantons are solutions of field equations with a nonzero but finite action. For this, the field $\mathbf{n}(x)$ must satisfy

$$\partial_\mu\mathbf{n}(x)\to 0 \quad (r\to\infty), \tag{5.28}$$

namely, $\mathbf{n}(x)$ approach the same value $\mathbf{n}^{(0)}$ at infinity where $\mathbf{n}^{(0)}$ is any unit vector in internal space, $\mathbf{n}^{(0)}\cdot\mathbf{n}^{(0)}=1$.

It is important to remark that the coset SU(2)/U(1) is isomorphic to the two-dimensional surface $S^2[S^n\cong SO(n+1)/SO(n)]$,

$$SU(2)/U(1)\cong S^2:=S^2_{\rm int}. \tag{5.29}$$

Moreover, by one-point compactification (i.e., adding a point of infinity) the two-dimensional plane can be converted into the two-dimensional sphere

$$R^2\cup\{\infty\}\cong S^2=S^2_{\rm phy}. \tag{5.30}$$

This implies that any finite action configuration $\mathbf{n}(x)$ is just a mapping from $S^2_{\rm phy}$ to $S^2_{\rm int}$. The mapping can be classified by homotopy theory. The O(3) NLSM_2 has instanton and anti-instanton solutions, because the homotopy group is nontrivial,

$$\Pi_2[SU(2)/U(1)]=\Pi_2(S^2)=Z. \tag{5.31}$$

The instanton (topological soliton) is characterized by the integer-valued topological charge Q. This is seen as follows.

The mathematical identity

$$\begin{aligned} \partial_\mu\mathbf{n}\cdot\partial_\mu\mathbf{n}=&\frac{1}{2}(\partial_\mu\mathbf{n}\pm\epsilon_{\mu\rho}\mathbf{n}\times\partial_\rho\mathbf{n})\cdot(\partial_\mu\mathbf{n}\pm\epsilon_{\mu\sigma}\mathbf{n}\times\partial_\sigma\mathbf{n}) \\ &\pm\epsilon_{\mu\nu}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}) \end{aligned}\tag{5.32}$$

implies

$$\partial_\mu\mathbf{n}\cdot\partial_\mu\mathbf{n}\geq\pm\epsilon_{\mu\nu}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}). \tag{5.33}$$

Hence the action has a lower bound

$$S_{\rm NLSM}:=\int d^{D-2}x\frac{\pi}{2g^2}\partial_\mu\mathbf{n}(x)\cdot\partial_\mu\mathbf{n}(x)\geq S_Q:=\frac{4\pi^2}{g^2}|Q|, \tag{5.34}$$

where Q is the Pontryagin index (winding number) defined by

$$Q:=\frac{1}{8\pi}\int d^2x\,\epsilon_{\mu\nu}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}). \tag{5.35}$$

The Euclidean action $S_{\rm NLSM}$ of NLSM is minimized when the inequality (5.33) is saturated. This happens if and only if

$$\partial_\mu\mathbf{n}=\pm\epsilon_{\mu\nu}\mathbf{n}\times\partial_\nu\mathbf{n}. \tag{5.36}$$

Any field configuration that satisfies Eq. (5.36) as well as the constraint (5.9) will minimize the action and therefore automatically satisfies the extremum condition given by the field equation (5.27). The converse is not necessarily true. Note that Eq. (5.36) is a first-order differential equation and easier to solve than the field equation (5.27) which is a second-order differential equation.

Now we proceed to construct the topological charge:

$$\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n} = \sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta)\mathbf{n} = \sin\theta\frac{\partial(\theta,\varphi)}{\partial(x^\mu,x^\nu)}\mathbf{n}, \tag{5.37}$$

where $\partial(\theta,\varphi)/\partial(x^\mu,x^\nu)$ is the Jacobian of the transformation from coordinates (x^μ,x^ν) on $S^2_{\rm phy}$ to $S^2_{\rm int}$ parametrized by (θ,φ) where μ,ν are any pair from $1,\ldots,D$. Using

$$\mathbf{n}\cdot(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) = \sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta) = \sin\theta\frac{\partial(\theta,\varphi)}{\partial(x^\mu,x^\nu)}, \tag{5.38}$$

it is easy to see that Q is an integer, since

$$\begin{aligned} Q &:= \frac{1}{8\pi}\oint_{S^2_{phy}} d^2x\,\epsilon_{\mu\nu}\sin\theta\frac{\partial(\theta,\varphi)}{\partial(x^\mu,x^\nu)} \\ &= \frac{1}{4\pi}\oint_{S^2_{phy}} d\sigma_{\mu\nu}\sin\theta\frac{\partial(\theta,\varphi)}{\partial(x^\mu,x^\nu)} \\ &= \frac{1}{4\pi}\int_{S^2_{int}}\sin\theta d\theta d\varphi, \end{aligned} \tag{5.39}$$

where $S^2_{\rm int}$ is a surface of a unit sphere with area 4π. Hence Q gives a number of times the internal sphere $S^2_{\rm int}$ is wrapped by a mapping from the physical space $S^2_{\rm phys}$ to the space of fields $S^2_{\rm int}$.

The instanton equation (5.36) can be rewritten as

$$\partial_1 n = \mp i(n\partial_2 n_3 - n_3\partial_2 n), \quad \partial_2 n = \pm i(n\partial_1 n_3 - n_3\partial_1 n),$$
$$n := n_1 + i n_2. \tag{5.40}$$

By changing the variables (stereographic projection from north pole),

$$w_1(x) := \frac{n_1(x)}{1-n_3(x)}, \quad w_2(x) := \frac{n_2(x)}{1-n_3(x)}, \tag{5.41}$$

the instanton equation reads

$$\partial_1 w = \mp i\partial_2 w, \quad w := w_1 + i w_2. \tag{5.42}$$

This is equivalent to the Cauchy-Riemann equation

$$\frac{\partial w_1(z)}{\partial x_1} = \pm\frac{\partial w_2}{\partial x_2}, \quad \frac{\partial w_1(z)}{\partial x_2} = \mp\frac{\partial w_2(z)}{\partial x_1}, \quad z := x_1 + i x_2. \tag{5.43}$$

For the upper (lower) signs, w is an analytic function of $z^*(z)$. Any analytic function $w(z), w(z^*)$ is a solution of instanton equation and also of the field equation. Note that w is not an entire function and allows isolated poles in $w(z)$, while cuts are prohibited by the single-valuedness of $n_a(x)$. The divergence $w\to\infty$ corresponds to $n_3=1$, i.e., the north pole in $S^2_{\rm int}$. The Euler angles are related to the new variables as

$$w_1 := \tan\frac{\theta}{2}\cos\varphi, \quad w_2 := \tan\frac{\theta}{2}\sin\varphi,$$
$$w = \frac{n_1 + i n_2}{1-n_3} = e^{i\varphi}\cot\frac{\theta}{2}, \tag{5.44}$$

corresponding to the stereographic projection from the north pole.[7]

By using the new variables, we obtain the expressions for the topological charge

$$Q = \frac{i}{2\pi}\int_{S^2}\frac{dw dw^*}{(1+ww^*)^2} = \frac{i}{2\pi}\int_{S^2}\frac{dx_1 dx_2}{(1+|w|^2)^2} \times\left(\frac{\partial w}{\partial x_1}\frac{\partial w^*}{\partial x_2} - \frac{\partial w}{\partial x_2}\frac{\partial w^*}{\partial x_1}\right), \tag{5.46}$$

and an action

$$\int d^2x\frac{1}{2}\partial_\mu \mathbf{n}\cdot\partial_\mu \mathbf{n} = \int_{S^2}\frac{dx_1 dx_2}{(1+|w|^2)^2}\left(\frac{\partial w}{\partial x_1}\frac{\partial w^*}{\partial x_1} + \frac{\partial w}{\partial x_2}\frac{\partial w^*}{\partial x_2}\right). \tag{5.47}$$

A typical instanton solution with topological charge $Q = n$ is given by

$$w(z) = [(z-z_0)/\rho]^n, \tag{5.48}$$

where the constants ρ and z_0 is regarded as the size and location of the instanton. The theory has the translational and scale invariance ($x\to x-a$ and $x\to\rho x$, respectively), since the solution exists for arbitrary ρ and z_0, but neither the action nor the topological charge depend on these constants. The parameters ρ, z_0 are called collective coordinates.

C. One instanton solution

The one instanton solution at the origin $z_0 = 0$,

$$w(z) = z/\rho, \tag{5.49}$$

implies a solution for the $O(3)$ vector,

$$n_1 = \frac{2\rho x_1}{|z|^2+\rho^2}, \quad n_2 = \frac{2\rho x_2}{|z|^2+\rho^2}, \quad n_3 = \frac{|z|^2-\rho^2}{|z|^2+\rho^2},$$
$$|z|^2 := x_1^2 + x_2^2. \tag{5.50}$$

This solution is regarded as representing a monopole or a projection of the four-dimensional instanton onto the two-dimensional plane in the following sense. First, we observe

[7]The stereographic projection from the south pole is

$$w_1 := \cot\frac{\theta}{2}\cos\varphi, \quad w_2 := \cot\frac{\theta}{2}\sin\varphi,$$
$$w = \frac{n_1 + i n_2}{1+n_3} = e^{i\varphi}\tan\frac{\theta}{2}. \tag{5.45}$$

that the field of an instanton at infinity points in the positive 3 direction $\mathbf{n}^{(0)}$ while the field at the origin points in the opposite direction,

$$|z|=0 \rightarrow \mathbf{n}=(0,0,-1) \equiv -\mathbf{n}^{(0)},$$
$$|z|=\rho \rightarrow \mathbf{n}=(x_1/\rho, x_2/\rho, 0), \tag{5.51}$$
$$|z|=\infty \rightarrow \mathbf{n}=(0,0,1) \equiv \mathbf{n}^{(0)}.$$

If we identify the plane with the sphere S^2 by stereographic projection from north pole, the north (south) pole of S^2 corresponds to the infinity point (the origin) and equator to the circle $|z|=\rho$. Therefore, one instanton solution (5.49) looks similar to a magnetic monopole (or a sea urchin). The winding number Q of this configuration is determined by the area of the sphere divided by 4π, i.e., $Q=1$. Thus the one instanton has winding number $+1$ (the one anti-instanton has $Q=-1$). Equivalently, this denotes the magnetic charge $g_m=1$.

An alternative interpretation is possible as follows. The configuration (5.50) leads to

$$\mathbf{n}(z)\cdot[\partial_i \mathbf{n}(z)\times \partial_j \mathbf{n}(z)] = -\epsilon_{ij}\frac{4\rho^2}{(|z|^2+\rho^2)^2}. \tag{5.52}$$

This should be compared with the four-dimensional instanton solution in the nonsingular gauge,

$$\mathcal{A}_\mu^A(x)=\eta_{A\mu\nu}\frac{2x_\nu}{x^2+\rho^2}, \quad \mathcal{F}_{\mu\nu}^A(x)=-\eta_{A\mu\nu}\frac{4\rho^2}{(x^2+\rho^2)^2},$$
$$x^2=x_1^2+\cdots+x_4^2, \tag{5.53}$$

which implies

$$\mathcal{A}_i^3(z)=\epsilon_{ij}\frac{2x_j}{|z|^2+\rho^2}, \quad \mathcal{F}_{ij}^3(z)=-\epsilon_{ij}\frac{4\rho^2}{(|z|^2+\rho^2)^2}, \tag{5.54}$$

where we have used $\eta_{3ij}=\epsilon_{3ij}=\epsilon_{ij}$. Therefore, the instanton solution (5.52) in two dimensions is equal to the projection of the field strength $\mathcal{F}_{12}^3$ of the four-dimensional instanton solution (in the nonsingular gauge) onto a two-dimensional plane. Therefore it is expected that there is an interplay between the instanton and the monopole in four dimensions. However, this does not imply that the four-dimensional instanton configuration play the dominant role in the confinement. The degrees of freedom responsible for the confinement is the magnetic monopole which has complete correspondence with the two-dimensional instantons as shown in the next subsection furthermore.

D. Instanton and magnetic monopole

By dimensional reduction, we can convert the calculation of correlation functions in MAG TFT_D into that in NLSM_{D-2}, if all the arguments sit on the $(D-2)$-dimensional subspace. Euler angle expression yields

$$\mathbf{n}\cdot(\partial_\mu \mathbf{n}\times\partial_\nu \mathbf{n}) = C_{\mu\nu}[\Omega] = \sin\theta(\partial_\mu\theta\partial_\nu\varphi - \partial_\mu\varphi\partial_\nu\theta). \tag{5.55}$$

Hence we obtain an alternative expression for the winding number,

$$Q := \frac{1}{8\pi}\int d^2z\, \epsilon_{\mu\nu}C_{\mu\nu}[\Omega] = \frac{1}{4\pi}\int d^2\sigma_{\mu\nu}C_{\mu\nu}[\Omega]. \tag{5.56}$$

We can define the topological charge density by

$$\epsilon_{\mu\nu}C_{\mu\nu}[\Omega] = \epsilon_{\mu\nu}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}). \tag{5.57}$$

From Eqs. (2.30), (2.37), and (5.55), if we restrict μ,ν to two dimensions, the monopole contribution in four dimensions corresponds to the instanton contribution in two dimensions. However, the monopole current defined by the divergence of the dual field strength $*f_{\mu\nu}$ cannot be calculated in the dimensionally reduced model, since all the derivatives are not necessarily contained in two-dimensional space. However, if the four-dimensional diagonal field strength $f_{\mu\nu}^\Omega$ is self-dual,

$$*f_{\mu\nu}^\Omega = f_{\mu\nu}^\Omega, \tag{5.58}$$

the monopole charge in four dimensions completely agrees with the winding number (instanton charge) in two dimensions

$$g_m = Q. \tag{5.59}$$

The intimate relationship between the magnetic monopole and instantons may be a reflection of this observation. Intuitively speaking, the magnetic monople and antimonople currents piercing the surface of the (planar) Wilson loop corresponds to the instanton and anti-instanton in the dimensionally reduced two-dimensional world. In order to derive the area law of the Wilson loop, the currents must pierce the surface uniformly. In this sense, the monopole current condensation must occur in four dimensions. The dimensional reduction of TFT implies self-duality at the level of the correlation function,

$$\langle \mathcal{F}_{\mu\nu}\rangle_{\mathrm{TFT}_4} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\langle\mathcal{F}_{\rho\sigma}\rangle_{\mathrm{TFT}_4}, \tag{5.60}$$

since both sides coincide with the same correlation function in the dimensionally reduced two-dimensional model.

If we define

$$h_\mu := -\frac{1}{g}\frac{1}{1\mp n^3}(\mathbf{n}\times\partial_\mu\mathbf{n})^3 = -\frac{1}{g}\frac{1}{1\mp n^3}\epsilon^{ab3}n^a\partial_\mu n^b, \tag{5.61}$$

$$a_\mu := \mathcal{A}_\mu^A n^A + h_\mu, \tag{5.62}$$

then we obtain the field strength,

$$f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu = \partial_\mu(\mathcal{A}_\nu^A n^A) - \partial_\nu(\mathcal{A}_\mu^A n^A) - \frac{1}{g}\epsilon^{ABC} n^A \partial_\mu n^B \partial_\nu n^C. \tag{5.63}$$

This field strength is regular everywhere and does not contain the Dirac string. This is nothing but the field strength of 't Hooft-Polyakov monopole, since n^A is obtained from T^3 by gauge rotation (5.17), $n^A(x)T^A = U^\dagger(x)T^3U(x)$. The Euler angle expression

$$h_\mu := -\frac{1}{g}\frac{\sin^2\theta\partial_\mu\varphi}{1\mp\cos\theta} \tag{5.64}$$

is constructed from the instanton (vortex) solution in two dimensions by the stereographic projection.

The expression for the instanton charge in two-dimension is equivalent to the magnetic charge in four dimensions, because

$$K_\mu := \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon^{ABC}\partial_\nu n^A \partial_\rho n^B \partial_\sigma n^C = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon^{ABC}\partial_\nu(n^A \partial_\rho n^B \partial_\sigma n^C) = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\partial_\nu[\mathbf{n}\cdot(\partial_\rho\mathbf{n}\times\partial_\sigma\mathbf{n})], \tag{5.65}$$

we have

$$g_m := \int d^3x K_0 := \frac{1}{8\pi}\int d^3x \epsilon_{ijk}\partial_i[\mathbf{n}\cdot(\partial_j\mathbf{n}\times\partial_k\mathbf{n})] = \frac{1}{8\pi}\int d^2\sigma_i \epsilon_{ijk}[\mathbf{n}\cdot(\partial_j\mathbf{n}\times\partial_k\mathbf{n})] = \frac{1}{8\pi}\int_{S^2} d^2x \epsilon_{jk}[\mathbf{n}\cdot(\partial_j\mathbf{n}\times\partial_k\mathbf{n})] = Q. \tag{5.66}$$

The magnetic current is topologically conserved $\partial_\mu K_\mu = 0$ without an equation of motion.

We can also define the three-dimensional topological current

$$J_\mu = \frac{1}{8\pi}\epsilon_{\mu\rho\sigma}\epsilon^{ABC}(n^A\partial_\rho n^B \partial_\sigma n^C) = \frac{1}{8\pi}\epsilon_{\mu\rho\sigma}[\mathbf{n}\cdot(\partial_\rho\mathbf{n}\times\partial_\sigma\mathbf{n})]. \tag{5.67}$$

Then Q is obtained from

$$Q := \int d^2x J_0. \tag{5.68}$$

This is related to Hopf invariant and Chern-Simons theory [89]. The details will be presented in a forthcoming paper.

VI. WILSON LOOP AND LINEAR POTENTIAL

First of all, in order to see explicitly that the dimensionally reduced two-dimensional NLSM has U(1) gauge invariance (corresponding to the residual H symmetry), we study the CP^1 formulation of O(3) NLSM. The CP^1 formulation shows gauge structure more clearly than the O(3) NLSM and helps us to see the analogy of NLSM with the $(1+1)$-dimensional Abelian Higgs model, i.e., the GL model. The CP^{N-1} model can have an instanton solution for any N, whereas the O(N) NLSM cannot have one for $N>3$. For SU(N) YM theory in MAG, the dimensionally reduced $SU(N)/U(1)^{N-1}$ NLSM has instanton solutions for any N. The instanton solution of O(3) $NLSM_2$ is identified as a vortex solution.

Next, we give the relationship among three theories; the CP^1 model, O(3) NLSM and TFT. It turns out that the calculation of the Wilson loop in four-dimensional TFT is reduced to that in the two-dimensional CP^1 model owing to dimensional reduction.

In Sec. VI C, we will show that summing up the contribution of instanton and anti-instanton configurations to the Wilson loop in the $NLSM_2$ or CP^1 model leads to quark confinement in four-dimensional TFT and YM theory in the sense of an area law of the Wilson loop. We emphasize that the coset G/H is quite important for the existence of the instanton and that the coset structure is a consequence of the MAG together with dimensional reduction. We find that the magnetic monopole in four dimensions corresponds to an instanton (or vortex) in two dimensions. Finally, we discuss some extensions of the proof of quark confinement for the general gauge group and in higher-dimensional cases.

A. CP^{N-1} model and instanton solution

The CP^{N-1} model is described by the N complex scalar field $\phi_a(x)(a=1,\ldots,N)$ and the action of the $d=(D-2)$-dimensional CP^{N-1} model is given by

$$S_{CP}[\phi] = \frac{\beta}{2}\int d^dx\{\partial_\mu\phi^*(x)\cdot\partial_\mu\phi(x) + [\phi^*(x)\cdot\partial_\mu\phi(x)][\phi^*(x)\cdot\partial_\mu\phi(x)]\}, \tag{6.1}$$

where there is the constraint

$$\phi^*(x)\cdot\phi(x) := \sum_{a=1}^N \phi_a^*(x)\phi_a(x) = 1. \tag{6.2}$$

By introducing an auxiliary vector field V_μ, the CP^{N-1} model can be equivalently rewritten as

$$S_{CP} = \frac{\beta}{2}\int d^dx\{\partial_\mu\phi^*(x)\cdot\partial_\mu\phi(x) + V_\mu^2(x) - 2V_\mu(x)[i\phi^*(x)\cdot\partial_\mu\phi(x)]\}. \tag{6.3}$$

In fact, integrating out the V_μ field in Eq. (6.3) recovers Eq. (6.1). Here V_μ corresponds to the composite operator

$$V_\mu(x) = i\phi^*(x)\cdot\partial_\mu\phi(x). \tag{6.4}$$

This is real and $\phi^*(x)\cdot\partial_\mu\phi(x)$ is pure imaginary, since from the constraint,

$$\phi^*(x)\cdot\partial\phi(x) + \partial\phi^*(x)\cdot\phi(x) = 2\mathrm{Re}[\phi^*\cdot\partial\phi(x)] = 0. \tag{6.5}$$

Then, using the constraint (6.2), the CP^{N-1} model can be further rewritten as

$$S_{\mathrm{CP}}[\phi,V] = \frac{\beta}{2}\int d^dx\{D_\mu[V]\phi^*(x)\}\cdot\{D_\mu[V]\phi(x)\}, \tag{6.6}$$

$$D_\mu[V]\phi(x) := (\partial_\mu + iV_\mu)\phi(x). \tag{6.7}$$

The partition function is defined by

$$Z_{\mathrm{CP}} := \int [dV_\mu][d\phi][d\phi^*]\prod_{x\in R^d}\delta[\phi(x)\cdot\phi(x)-1] \times \exp(-S_{\mathrm{CP}}[\phi,V]). \tag{6.8}$$

Here $D_\mu[V]$ is actually interpreted as the covariant derivative, because the Lagrangian is invariant under the U(1) gauge transformation

$$\phi_a(x) \to \phi_a(x)' := \phi_a(x)e^{i\Lambda(x)},$$
$$V_\mu(x) \to V_\mu(x)' := V_\mu(x) - \partial_\mu\Lambda(x), \tag{6.9}$$

where Λ is independent of the index a and

$$D_\mu\phi_a(x) \to [D_\mu\phi_a(x)]e^{i\Lambda(x)}. \tag{6.10}$$

By this property, this model is called the CP^{N-1} model (the target space is the complex projective space). Note that

$$\mathrm{CP}^{N-1} \cong \mathrm{U}(N)/\mathrm{U}(1)/\mathrm{U}(N-1) \cong \mathrm{SU}(N)/\mathrm{U}(N-1). \tag{6.11}$$

The CP^{N-1} model has global $\mathrm{SU}(N)$ symmetry and the U(1) subgroup of this $\mathrm{SU}(N)$ is a local gauge symmetry. Hence the CP^{N-1} model is U(1) gauge theory for any N. However, V_μ is an auxiliary vector field and does not represent independent degrees of freedom, since the kinetic term is absent. Apart from this fact, the CP^{N-1} model is similar to the Abelian Higgs model or scalar quantum electrodynamics. It is known that the kinetic term of V_μ is generated through radiative correction, see Ref. [57].

The constraint is included in the action by introducing the Lagrange multiplier field λ as

$$S_{\mathrm{CP}} = \frac{\beta}{2}\int d^dx\{[D_\mu[V]\phi^*(x)]\cdot[D_\mu[V]\phi(x)] + \lambda(x)[\phi^*(x)\cdot\phi(x)-1]\}. \tag{6.12}$$

The field equation is

$$D_\mu[V]D_\mu[V]\phi(x) + \lambda(x)\phi(x) = 0. \tag{6.13}$$

The multiplier field is eliminated using

$$\lambda(x) = \lambda(x)\phi^*(x)\cdot\phi(x) = -\phi^*(x)\cdot D_\mu[V]D_\mu[V]\phi(x), \tag{6.14}$$

to yield

$$D_\mu[V]D_\mu[V]\phi(x) - \{\phi^*(x)\cdot D_\mu[V]D_\mu[V]\phi(x)\}\phi(x) = 0. \tag{6.15}$$

Instantons are finite action solutions of field equations. The finiteness of the action requires the boundary condition

$$D_\mu\phi_a := \partial_\mu\phi_a + iV_\mu\phi_a \to 0 \quad \text{as} \quad r := |\mathbf{x}| \to \infty. \tag{6.16}$$

Separating ϕ_a into the modulus and the angular part,

$$\phi_a(x) := |\phi_a(x)|e^{i\Theta_a(x)}, \tag{6.17}$$

the boundary condition yields

$$V_\mu = i\frac{\partial_\mu\phi_a}{\phi_a} = i\frac{\partial_\mu|\phi_a|}{|\phi_a|} - \partial_\mu\Theta_a. \tag{6.18}$$

Here V_μ must be real and independent of a. Hence, $\partial_\mu|\phi_a| = 0$ and $\partial_\mu\Theta_a$ is independent of a. This means $|\phi_a| = \phi_0$ for a fixed complex vector with $(\phi_0)^*\cdot\phi_0 = 1$ and $\Theta_a = \Theta(\varphi)$ for a common phase angle $\Theta(\varphi)$ which can depend on φ parametrizing a circle, S^1_{phy}. Consequently, the boundary condition is given by

$$\phi_a(x) \to \phi_0 e^{i\Theta(\varphi)}, \quad V_\mu \to -\partial_\mu\Theta(\varphi), \tag{6.19}$$

where the allowed values of the phase Θ form a circle S^1_{int}. The mapping Θ from S^1 to S^1 is characterized by an winding number

$$Q := \frac{1}{2\pi}\int_{S^1_{\mathrm{phy}}} d\varphi \frac{d\Theta}{d\varphi}, \tag{6.20}$$

which has an integral value corresponding to the fact that

$$\Pi_1(S^1) = \mathbf{Z}. \tag{6.21}$$

Although the global $\mathrm{SU}(N)$ rotations can continuously change the value of ϕ_0, this freedom does not introduce any further homotopy classification.

The winding number can be rewritten in terms of V_μ as follows. From Eq. (6.4),

$$V_\varphi = \frac{i}{r}\phi^*\cdot\frac{\partial\phi}{\partial\varphi} \to -\frac{1}{r}\frac{d\Theta}{\partial\varphi}. \tag{6.22}$$

This leads to

$$Q = -\frac{1}{2\pi}\int_{S^1_{phy}} d\varphi r V_\varphi = -\frac{1}{2\pi}\int_{S^1_{phy}} d\ell\cdot V = -\frac{1}{2\pi}\int d^2x\,\epsilon_{\mu\nu}\partial_\mu V_\nu, \tag{6.23}$$

where the integrand is a pure divergence. Using the constraint, we can show that this is rewritten as

$$Q=\int d^2x\,\epsilon_{\mu\nu}(D_\mu\phi)^*(D_\nu\phi). \tag{6.24}$$

From the identity

$$(D_\mu\phi)\cdot(D_\mu\phi)=\frac{1}{2}(D_\mu\phi\pm i\epsilon_{\mu\nu}D_\nu\phi)^*\cdot(D_\mu\phi\pm i\epsilon_{\mu\nu}D_\nu\phi)\mp i\epsilon_{\mu\nu}(D_\mu\phi)^*(D_\nu\phi) \tag{6.25}$$

we obtain

$$(D_\mu\phi)\cdot(D_\mu\phi)\geqslant\mp i\epsilon_{\mu\nu}(D_\mu\phi)^*(D_\nu\phi). \tag{6.26}$$

Hence a lower bound of the action is obtained,

$$S_{\rm CP}\geqslant\frac{\pi^2}{g^2}|Q|:=S_Q. \tag{6.27}$$

The action has the minimum value when the inequality is saturated,

$$D_\mu\phi_a=\pm i\epsilon_{\mu\nu}D_\nu\phi_a. \tag{6.28}$$

This is a self-duality equation which is analogous to the self-duality equation of YM theory. This equation is first order (partial differential) equation and easier to solve than the field equation. Solution of this equation automatically satisfies the field equation, but the converse is not necessarily true.

To solve Eq. (6.28), we introduce the gauge invariant field

$$\omega_a(x):=\phi_a(x)/\phi_1(x)\quad(a=1,\dots,N). \tag{6.29}$$

The covariant derivative is eliminated by substituting $\phi_a(x)=\omega_a(x)\phi_1(x)$ into Eq. (6.28),

$$\partial_\mu\omega_a(x)=\pm i\epsilon_{\mu\nu}\partial_\nu\omega_a. \tag{6.30}$$

This is nothing but the Cauchy-Riemann equation. For the minus (plus) sign, each ω_a is an analytic function of $z:=x_1+ix_2\,(z^*=x_1-ix_2)$.

The expression for V_μ in terms of ω is

$$V_\mu=\frac{i}{2|\omega|^2}(\omega^*\cdot\partial_\mu\omega-\omega\cdot\partial_\mu\omega^*)=\frac{i}{2}(\hat\omega^*\cdot\partial_\mu\hat\omega-\hat\omega\cdot\partial_\mu\hat\omega^*),\quad\hat\omega:=\omega/|\omega|,$$
$$|\omega|:=(\omega^*\cdot\omega)^{1/2}=|\phi_1|^{-1}. \tag{6.31}$$

Taking into account the Cauchy-Riemann relations, we obtain

$$V_\mu=\pm\epsilon_{\mu\nu}\frac{\omega^*\cdot\partial_\nu\omega+\omega\cdot\partial_\nu\omega^*}{2|\omega|^2}=\pm\epsilon_{\mu\nu}\partial_\nu\ln|\omega|. \tag{6.32}$$

The topological charge is expressed as

$$Q=-\frac{1}{2\pi}\int d^2x\,\epsilon_{\mu\nu}\partial_\mu V_\nu=\pm\frac{1}{4\pi}\int d^2x\,\partial_\mu\partial_\mu\ln|\omega|^2. \tag{6.33}$$

An example of the one instanton solution is given by

$$\omega(z)=u+[(z-z_0)/\rho]v, \tag{6.34}$$

where u,v are any pair of orthonormal complex vectors satisfying

$$u_1=v_1=1,\quad u^*\cdot u=v^*\cdot v=1,\quad u^*\cdot v=0. \tag{6.35}$$

Here the constant ρ,z_0 represent the size and location (in the z plane) of instanton. Reflecting the scale and translational invariance of the action, we can choose arbitrary values for ρ,z_0. The solution (6.34) is inverted to become

$$\phi_a(z)=\frac{\rho u_a+(z-z_0)v_a}{(\rho^2+|z-z_0|^2)^{1/2}}. \tag{6.36}$$

As $z\to\infty$, this solution satisfies the boundary condition with a phase angle $\Theta(\varphi)=\varphi$,

$$\phi_a(z)\to(z/|z|)u=e^{i\varphi}u. \tag{6.37}$$

Hence this solution leads to $Q=1$, the single instanton. The anti-instanton is obtained by replacing z by z^*.

Using the solution (6.34), the vector potential (6.32) and its field strength reads

$$V_\mu=\pm\epsilon_{\mu\nu}\frac{x_\nu}{|x|^2+\rho^2},\quad|x|^2=x_1^2+x_2^2, \tag{6.38}$$

$$V_{\mu\nu}:=\partial_\mu V_\nu-\partial_\nu V_\mu=\mp\epsilon_{\mu\nu}\frac{2\rho^2}{(|x|^2+\rho^2)^2}. \tag{6.39}$$

Note that V_μ tends to a pure U(1) gauge field configuration at infinity,

$$V_\mu\to\pm\epsilon_{\mu\nu}\frac{x_\nu}{|x|^2}=\partial_\mu\Theta,\quad\Theta:=\arctan\frac{x_2}{x_1}. \tag{6.40}$$

Hence V_μ denotes the vortex with a center at $x=0$. This is consistent with Eq. (6.23). This implies that the magnetic field of the magnetic current induces the (quantized) current around it on a plane perpendicular to the magnetic field. This is regarded as the dual of the usual Ampere law where the electric current induces the magnetic field around it,

$$I=\frac{1}{2\pi}\oint_C V=\frac{1}{2\pi}\oint_C d\Theta=n,\quad V:=V_\mu dx^\mu. \tag{6.41}$$

In two dimensions the dual of the vector is again the vector. The two descriptions are dual to each other.

The solution (6.39) should be compared with the four-dimensional instanton solution in the nonsingular gauge (5.54). The instanton solution (6.39) in two dimensions is regarded as the projection of the four-dimensional counterpart (5.54) on the two-dimensional plane. However, this does not imply that the four-dimensional instanton configuration play the dominant role in the confinement. The degrees of freedom responsible for the confinement is the magnetic monopole which has complete correspondence with the two-dimensional instantons. This has been shown in Secs. V B and V C.

B. CP^1 model, $O(3)$ NLSM, and TFT

The CP^1 model is locally isomorphic to the O(3) NLSM with the identification

$$n^A(x) := \frac{1}{2}\phi_a^*(x)(\sigma^A)_{ab}\phi_b(x) \quad (a,b=1,2) \tag{6.42}$$

or

$$n^1 = \mathrm{Re}(\phi_1^*\phi_2), \quad n^2 = \mathrm{Im}(\phi_1^*\phi_2),$$

$$n^3 = \frac{1}{2}(|\phi_1|^2 - |\phi_2|^2). \tag{6.43}$$

Actually, the constraint is satisfied, $n^A n^A = (|\phi_1|^2 + |\phi_2|^2)^2 = 1$. Hence the CP^1 model has three independent parameters, whereas O(3) vector **n** has two. One of three parameters in the CP^1 model is unobservable, since a global change of the phase does not lead to any observable effect. In fact, **n** is invariant under the U(1) gauge transformation. It is possible to show that the Lagrangian (6.1) for $N=2$ reduces to O(3) NLSM. The CP^{N-1} has instantons for arbitrary $N \geqslant 2$, while O(N) NLSM does not have them for $N>3$. The map from the CP^1 model to O(3) NLSM is identified with a Hopf map $H: S^3 \to S^2$ where S^3 denotes the unit three sphere embedded in R^4 by $|\phi_1|^2 + |\phi_2|^2 = 1$. In the language of mathematics, S^3 is a U(1) bundle over S^2, see e.g., Ref. [89].

The field variables of CP^1 model is written in terms of Euler angles,

$$\phi_1 = \sqrt{2S}\exp\left[\frac{i}{2}(\varphi + \chi)\right]\cos\frac{\theta}{2},$$

$$\phi_2 = \sqrt{2S}\exp\left[-\frac{i}{2}(\varphi - \chi)\right]\sin\frac{\theta}{2}, \tag{6.44}$$

which satisfies the constraint $\phi_a^*\phi_a = 2S$. Indeed, substitution of Eq. (6.44) into Eq. (6.43) leads to

$$n^1 = 2\mathrm{Re}\ (\phi_1\phi_2^*) = 2S\sin\theta\cos\varphi,$$

$$n^2 = 2\mathrm{Im}\ (\phi_1\phi_2^*) = 2S\sin\theta\sin\varphi,$$

$$n^3 = |\phi_1|^2 - |\phi_2|^2 = 2S\cos\theta. \tag{6.45}$$

This is nothing but the Schwinger-Wigner representation of the spin S operator in terms of two Bose creation and annihilation operators $\phi_a^\dagger, \phi_a$. In the path integral formalism, they are not operators, but c numbers.

Substituting Eq. (6.44) into Eq. (6.4) yields

$$V_\mu(x) = i\phi^*(x)\cdot\partial_\mu\phi(x) = -S[\partial_\mu\chi + \cos\theta\partial_\mu\varphi] = -SL_\mu^3. \tag{6.46}$$

Hence, the vector field V_μ is equivalent to Ω_μ^3 when μ is restricted to $\mu = 1,\ldots,d$. Furthermore,

$$\partial_\mu\phi^*(x)\cdot\partial_\mu\phi(x) = \frac{S}{2}[(L_\mu^1)^2 + (L_\mu^2)^2 + (L_\mu^3)^2]. \tag{6.47}$$

Owing to the dimensional reduction, the D-dimensional SU(2) MAG TFT is equivalent to the $d=(D-2)$-dimensional CP^1 model,

$$S_{CP^1} = \frac{\beta}{2}\int d^dx\{[L_\mu^1(x)]^2 + [L_\mu^2(x)]^2\}, \quad \beta := \frac{\pi}{g^2}. \tag{6.48}$$

Consequently, when the Wilson loop has the support on the $(D-2)$-dimensional subspace $R^d \subset R^D$, then the diagonal Wilson loop in D-dimensional SU(2) MAG TFT

$$W_C[a^\Omega] := \exp\left(iq\oint_C a_\mu^\Omega(z)dz^\mu\right), \quad z\in R^d \tag{6.49}$$

$$a_\mu^\Omega(x) := \mathrm{tr}[T^3\Omega_\mu(x)] = L_\mu^3(x), \tag{6.50}$$

corresponds to the Wilson loop in $d=(D-2)$-dimensional CP^1 model,

$$W_C[V] := \exp\left(iq\oint_C V_\mu(z)dz^\mu\right) = \exp\left(\frac{i}{2}q\int_S V_{\mu\nu}(z)d\sigma^{\mu\nu}\right). \tag{6.51}$$

C. Area law for the diagonal Wilson loop

Now we evaluate the Wilson loop expectation value to obtain the static potential for two widely separated charges $\pm q$ (in a θ vacuum). We define the diagonal Wilson loop operator [8] for a closed loop C by

$$W_C[a^U] := \exp\left(iq\oint_C a_\mu^U(x)dx^\mu\right), \quad a_\mu^U(x) := \mathrm{tr}[T^3\mathcal{A}_\mu^U(x)]. \tag{6.52}$$

According to the Stokes theorem, this is equal to

$$W_C[a^U] = \exp\left(\frac{i}{2}q\int_S f_{\mu\nu}^U(x)d\sigma^{\mu\nu}\right) \tag{6.53}$$

for any surface S with a boundary C. We restrict the loop C to be planar, otherwise, we could receive any benefit of dimensional reduction to calculate the Wilson loop expectation. In what follows we calculate the contribution from Ω_μ, namely, the topological contribution alone. Then the dimensional reduction implies

$$\langle W_C[a^\Omega]\rangle_{\mathrm{MAG\ TFT}_4} = \langle W_C[a^\Omega]\rangle_{\mathrm{O(3)\ NLSM}_2},$$

$$a_\mu^\Omega(x) := \mathrm{tr}[T^3\Omega_\mu(x)]. \tag{6.54}$$

Following the procedure in Sec. II, we regard other contributions as perturbative deformation $W[U;J^\mu,0,0]$, see Eq. (2.59).

According to Sec. V C (or VI B) for $G=\mathrm{SU}(2)$, the Wilson loop in two-dimensional O(3) NLSM is rewritten as

$$W_C[a^\Omega]=\exp\left(i\frac{2\pi q}{g}\int_S d^2x\frac{1}{8\pi}\epsilon_{\mu\nu}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n})\right). \tag{6.55}$$

Note that the integrand is the density of instanton number as shown in the previous section. This implies that the Wilson loop $W_C[a^\Omega]$ (6.55) counts the number of instanton–anti-instanton (or vortex–antivortex in CP^1 formulation) existing in the area S bounded by the loop C in the O(3) NLSM. The Wilson loop expectation value is written as

$$\langle W_C[a^\Omega]\rangle_{\mathrm{O(3)\ NLSM}_2}=\frac{\int d\mu(\mathbf{n})\delta(\mathbf{n}\cdot\mathbf{n}-1)e^{-S_{\mathrm{NLSM}}+i\theta Q}W_C[a^\Omega]}{\int d\mu(\mathbf{n})\delta(\mathbf{n}\cdot\mathbf{n}-1)e^{-S_{\mathrm{NLSM}}+i\theta Q}}=:\frac{I_2^\theta}{I_1^\theta}, \tag{6.56}$$

where we have included the topological term $i\theta Q$.[8] Inclusion of topological term $i\theta Q$ in the action is equivalent to consider the θ vacuum defined by

$$|\theta\rangle:=\sum_{n=-\infty}^{+\infty}e^{in\theta}|n\rangle. \tag{6.57}$$

The action with a topological angle θ is written as

$$S_{\mathrm{NLSM}}^\theta=S_{\mathrm{NLSM}}-i\theta Q=(n_++n_-)S_1-i\theta(n_+-n_-),$$
$$S_1(g)=\frac{4\pi^2}{g^2}. \tag{6.58}$$

We regard Eq. (6.56) as the average of the instanton number Q inside S over all the instanton–anti-instanton ensembles generated from the action of NLSM.

In the following, we use the dilute instanton-gas approximation as a technique to calculate Eq. (6.56). This method is well known, see, e.g., Chap. 11 of Rajaraman [58] or Chap. 7 of Coleman [57]. (We will give the Wilson loop calculation based on other methods elsewhere.) We first classify the configurations of the field **n** that contribute to the tunneling amplitude of instantons $\langle n|e^{-HT}|0\rangle$ according to the number of well-separated instantons n_+ and anti-instantons n_- such that $Q=n=n_+-n_-$. Then we sum over all configurations with n_+ instantons and n_- anti-instantons, all widely separated. In the dilute-gas approximation, the calculation of tunneling amplitude is reduced to that of a single instanton (anti-instanton) contribution $n\to n+1$ $(n\to n-1)$. The term with $n_+=1,\ n_-=0$ (or $n_+=0,\ n_-=1$) is given by

$$\langle n=\pm1|e^{-HT}|0\rangle=\int d\mu(\rho)\int d^2x\times\exp[-S_1(g)]\exp(\pm i\theta)$$
$$=BL_1L_2\exp[-S_1(g)]\exp(\pm i\theta). \tag{6.59}$$

Here $T=L_1$ or L_2 and the prefactor BL_1L_2 comes from integration of the collective coordinates, i.e., the size and position of the instanton,

$$\int d\mu(\rho)\int d^2x=BL_1L_2,\quad B\sim O(m_A^2), \tag{6.60}$$

where L_1L_2 is the (finite but large) volume of two-dimensional space and B is a normalization constant of order m_A^2, because instanton size is proportional to the inverse mass m_A^{-1} of off-diagonal gluons. In order to know the precise form of B, we must determine the measure $\mu(\rho)$ for the collective coordinate ρ, see Refs. [90–94].

In the dilute-gas approximation, the denominator I_1^θ is calculated as

$$I_1^\theta:=\langle\theta|e^{-HT}|\theta\rangle=\sum_{n_+,n_-=0}^{\infty}\frac{(BL_1L_2)^{n_++n_-}}{n_+!n_-!}\exp[-(n_++n_-)S_1(g)+i\theta(n_+-n_-)]$$
$$=\sum_{n_+=0}^{\infty}\frac{1}{n_+!}(BL_1L_2e^{-S_1(g)+i\theta})^{n_+}\sum_{n_-=0}^{\infty}\frac{1}{n_-!}(BL_1L_2e^{-S_1(g)-i\theta})^{n_-}$$
$$=\exp[BL_1L_2e^{-S_1(g)+i\theta}+BL_1L_2e^{-S_1(g)-i\theta}]$$
$$=\exp[2(BL_1L_2)\cos\theta e^{-S_1(g)}], \tag{6.61}$$

[8]Note that the nonzero θ is not essential to show the area law of the Wilson loop in the following. We can put $\theta=0$ in the final results (6.65) and (6.66).

where there is no constraint on the integers n_+ or n_-, since we are summing over all $Q=n_+-n_-$. The sum is precisely the grand partition function for a classical perfect gas (i.e., noninteracting particles)[9] containing two species of particles with equal chemical potential $e^{-S_1(g)}$ and volume measured in units of B. The energy (action) for a configuration with n_+ and n_- members of each species is $(n_++n_-)S_1(g)$ while the entropy of the configuration is $\ln[(BL_1L_2)^{n_++n_-}/n_+!n_-!]$.

The configuration of instanton and anti-instanton is not an exact solution of the equation of motion. However, the dominant term is given by the configuration for which the free energy (energy minus entropy) is smallest. For large coupling the action of a given field configuration decreases as g^{-2} while the entropy which is obtained as the log of the volume of function space occupied by the configuration is less sensitive to g. Thus for moderate or strong coupling the entropy of a field configuration can be more important than its action. The exact multi-instanton solutions are essentially of no relevance in constructing the vacuum state because they have so little entropy. In fact, the sum over all terms with either n_+ or n_- equal to zero is exponentially small compared to the complete sum for large T [56]. When g is small, the instanton gas is extremely dilute. For larger g instantons and anti-instantons come closer together.

When $\theta=0$, the most dominant term in this sum is given for large T at

$$n_+=n_-=BL_1L_2e^{-S_1(g)}, \tag{6.62}$$

and as $T\to\infty$ the entire sum comes essentially from this term alone. The important lessons learned from Ref. [56] are (i) the dominant term contains both instantons and anti-instantons and cannot be computed by a strict saddle-point method that relies on exact solutions to the (Euclidean) equation of motion and (ii) the dominant term is not the one for which the classical action $\exp[-S]$ is minimum.

The calculation of the numerator I_2^θ reduces to the construction of a system in a θ vacuum outside the loop and that in a $\theta+2\pi q$ vacuum inside the loop. Let $A(C)$ be the area enclosed by the loop C. In the dilute-gas approximation, the numerator is

$$\begin{aligned} I_2^\theta=&\sum_{n_+^{\rm in},n_-^{\rm in}=0}^{\infty}\frac{[BA(C)]^{n_+^{\rm in}+n_-^{\rm in}}}{n_+^{\rm in}!n_-^{\rm in}!}\exp\left[-(n_+^{\rm in}+n_-^{\rm in})S_1(g)+i\left(\theta+\frac{2\pi q}{g}\right)(n_+^{\rm in}-n_-^{\rm in})\right]\\ &\times\sum_{n_+^{\rm out},n_-^{\rm out}=0}^{\infty}\frac{\{B[L_1L_2-A(C)]\}^{n_+^{\rm out}+n_-^{\rm out}}}{n_+^{\rm out}!n_-^{\rm out}!}\exp[-(n_+^{\rm out}+n_-^{\rm out})S_1(g)+i\theta(n_+^{\rm out}-n_-^{\rm out})]\\ =&\exp\left\{2B\left[A(C)\cos\left(\theta+\frac{2\pi q}{g}\right)+[L_1L_2-A(C)]\cos\theta\right]e^{-S_1(g)}\right\}. \end{aligned} \tag{6.63}$$

Here we decomposed the sum inside the Wilson loop and outside it. The decomposition $n_\pm=n_\pm^{\rm in}+n_\pm^{\rm out}$ is meaningful only when the loop C is sufficiently large and the instanton size is negligible compared with the size of the loop C so that the overlapping of the instanton and anti-instanton with the loop is neglected (this is equivalent to neglecting the perimeter decay part of the Wilson loop). Then we can write

$$W_C[a^\Omega]=\exp\left[\frac{2\pi q}{g}i(n_+^{\rm in}-n_-^{\rm in})\right]. \tag{6.64}$$

Finally we notice that the volume dependence disappears in the ratio I_2^θ/I_1^θ. The above derivation is very similar to the two-dimensional Abelian Higgs model, see Ref. [55].

In the vacuum with the topological angle θ, therefore, the Wilson loop expectation value has

$$\langle W_C[a^\Omega]\rangle=\exp\left\{-2Be^{-S_1}\left[\cos\theta-\cos\left(\theta+\frac{2\pi q}{g}\right)\right]A(C)\right\}. \tag{6.65}$$

The Wilson loop integral exhibits an area law. If we take the rectangular Wilson loop, the static quark potential is derived. If q/g is an integer, the potential vanishes because the vacuum is periodic in θ with period 2π. The integral charge is screened by the formation of neutral bound states. When q is not an integral multiples of an elementary charge g, the static quark potential $V(R)$ is given by the linear potential with string tension σ,

$$V(R)=\sigma R,\quad \sigma=2Be^{-S_1}\left[\cos\theta-\cos\left(\theta+\frac{2\pi q}{g}\right)\right], \tag{6.66}$$

[9]By the fermionization method, the noninteracting instanton and anti-instanton system can be rewritten as the free massive fermion models with two flavors. From this viewpoint, including the interactions between instantons and anti-instantons is equivalent to introducing the four-fermion interaction of Thirring type [94]. By bosonization, the interacting fermionic model is converted into the sine-Gordon-like bosonic model [94]. The Wilson loop calculation from this point of view will be given in a forthcomming paper.

where $B \sim m_A^2$ and $S_1 = \exp(-4\pi^2/g^2)$ is the action for one instanton. It should be remarked that the confining potential is very much a nonperturbative quantum effect caused by instantons, because the linear potential has a factor $e^{-S_1/\hbar}$ (if we had retained $\hbar$ dependence) which is exponentially small in $\hbar$ and vanishes as $\hbar \to 0$. This is a crucial difference between the linear potential (6.66) and the linear Coulomb potential in two dimensions.

On the other hand, the four-dimensional Coulomb potential is calculated by perturbation theory [96] (see Ref. [27]),

$$V(R) = -\frac{C_2}{4\pi}\frac{g^2}{R} + \text{const.} \tag{6.67}$$

Therefore, we arrive at the conclusion that the total static quark potential in four-dimensional YM theory is given by [27]

$$V(R) = \sigma R - \frac{C_2}{4\pi}\frac{g^2}{R} + \text{const.} \tag{6.68}$$

The two-dimensional O(N+1) NLSM is asymptotic free and the β function [47] is given by

$$\beta(g) := \mu\frac{dg(\mu)}{d\mu} = -\frac{N-1}{8\pi^2}g^3 + O(g^5), \tag{6.69}$$

where g is the renormalized coupling constant and μ the renormalization scale (mass) parameter. By dimensional transmutation as in QCD, the mass and the ''string tension'' of NLSM should be given by [42]

$$m \sim \Lambda \exp\left(-\int^g \frac{dg}{\beta(g)}\right), \quad \sigma \sim \Lambda^2 \exp\left(-2\int^g \frac{dg}{\beta(g)}\right). \tag{6.70}$$

For the β function (6.69), this implies for $N=2$

$$\sigma \sim \Lambda^2 \exp\left(-\frac{4\pi^2}{g^2}\right), \tag{6.71}$$

in agreement essentially with the above result (6.66). In this case, the scale Λ of the theory is given by the off-diagonal gluon mass m_A. This result does not agree with four-dimensional SU(N) YM theory in which

$$\beta(g) = -\frac{b_0}{16\pi^2}g^3 + O(g^5), \quad b_0 = \frac{11N}{3} > 0, \tag{6.72}$$

because we have taken into account only the MAG TFT part of YM theory and neglected an additional contribution coming from the perturbative part [note that the correspondence of SU(N) YM theory to O(N+1) NLSM is meaningful only for $N=2$]. By integrating out the off-diagonal gluons $A_\mu^\pm$ in MAG TFT (3.10), we can obtain the APEGT of MAG TFT, as performed for YM theory in the previous paper [17]. The APEGT of MAG TFT is given by the $H=U(1)$ gauge theory with the running coupling $g(\mu)$ governed by β function (6.69),

$$S_{\text{APEGT}} = \int d^4x \left[-\frac{1}{4g^2(\mu)}f_{\mu\nu}f^{\mu\nu}\right], \tag{6.73}$$

where ghost interactions and higher derivative terms are neglected.

The naive instanton calculus given above can be improved by including the correction around the instanton solutions following the works [92–95]. Although we have identified the two-dimensional space with the sphere in the above, instanton solutions exist also for the torus [97,98] and the cylinder [99]. However, the torus only admits multi-instantons with topological charge two or more (no single-instanton solution).

D. Importance of coset G/H

In our approach, it is important to choose the coset G/H so that $\Pi_2(G/H) \neq 0$, because for any compact connected Lie group G,

$$\Pi_2(G) = 0, \tag{6.74}$$

where the two-dimensional NLSM fails to contain the instanton. The MAG naturally leads to such a coset G/H NLSM. This is a reason why the PGM based on G cannot contain nontrivial topological structure and dynamical degrees of freedom except for unphysical gauge modes, although the authors of Refs. [61,60] tried to include the physical modes as perturbation of PGM. It would be interesting to clarify the relationship between the Wilson criterion of quark confinement and color confinement criterion by Kugo and Ojima [66] and Nishijima [67]. This issue is reserved for future investigations.

E. Generalization to SU(N)

The above consideration can be generalized to the more general case $G = \text{SU}(N)$. Using

$$\Pi_1(\text{SU}(N)) = 0, \tag{6.75}$$

we obtain

$$\Pi_2(\text{SU}(N)/\text{U}(1)^{N-1}) = \Pi_1(\text{U}(1)^{N-1}) = \mathbf{Z}^{N-1}. \tag{6.76}$$

This formula guarantees the existence of the instanton and anti-instanton solution in the $\text{SU}(N)/\text{U}(1)^{N-1}\text{NLSM}_2$ model obtained from $\text{SU}(N)\text{MAG TFT}_4$ by dimensional reduction. Therefore, the whole strategy adopted in this paper to prove the quark confinement will be valid for SU(N) gauge theory in four dimensions. The origin of instantons in the dimensionally reduced model is the monopole in the original model, as suggested by the mathematical formula (6.76).

In order to study the case $N=3$ in more detail, it would be efficient to perform the $1/N$ expansion to the $\text{SU}(N)/\text{U}(1)^{N-1}\text{NLSM}_2$ model.

F. Higher-dimensional cases

Our strategy of proving quark confinement in D dimensions is based on the existence of instanton solutions in the dimensionally reduced $(D-2)$-dimensional NLSM. This can be generalized to arbitrary dimension, $D>4$. Remember the mathematical formula for the Homotopy group

$$\Pi_n(\mathrm{SU}(2)/\mathrm{U}(1)) = \Pi_n(S^2)(n := D-2>2), \quad (6.77)$$

and

$$\Pi_3(S^2) = \mathbf{Z} \quad (D=5),$$
$$\Pi_4(S^2) = \mathbf{Z}_2 \quad (D=6),$$
$$\Pi_5(S^2) = \mathbf{Z}_2 \quad (D=7), \ldots . \quad (6.78)$$

This provides the possibility of proving quark confinement based on instantons and anti-instantons even for $D>4$ dimensions.

G. Exact results in two dimensions

The classical O(3) NLSM in $1+1$ dimensions is characterized by an infinite number of conserved quantities and by Bäcklund transformations for generating solutions. The quantized O(3) NLSM is asymptotically free and the conserved quantities exist free of anomalies [47]. An exact factorized S matrix has been constructed using the existence of the infinite conserved quantities [37].

It is known [81] that the σ model

$$S = \frac{1}{4\lambda^2}\int d^2x \,\mathrm{tr}(\partial_\mu U^{-1}\partial^\mu U) + k\Gamma(U), \quad (6.79)$$

with a Wess-Zumino (WZ) term

$$\Gamma(U) := \frac{1}{24\pi}\int d^3x \,\epsilon^{\alpha\beta\gamma}\mathrm{tr}[L_\alpha L_\beta L_\gamma], \quad L_\mu := U^{-1}\partial_\mu U, \quad (6.80)$$

becomes massless and possesses an infrared stable fixed point when

$$\lambda^2 = \frac{4\pi}{k} \quad (k=1,2,\ldots). \quad (6.81)$$

At these special values of k, the model (6.79) is called the level k Wess-Zumino-Novikov-Witten (WZNW) model. The familiar σ model corresponds to $k=0$ case where the theory is asymptotically free and massive. The WZNW model is invariant under the conformal transformation and with respect to infinite-dimensional current (Kac-Moody) algebra.

The σ model with arbitrary coupling λ can be solved exactly by means of the Bethe ansatz technique [46]. However, the computation of correlation function remain beyond the powers of the Bethe ansatz method. Although the conformal field theory approach [79] is restricted to the fixed-point case, it provides much more detailed information about the theory including the correlation functions [80]. We can calculate exactly all correlation functions in rational conformal field theories which include the WZNW and minimal models as subsets. The off-critical theory can be considered as perturbation of conformal theories by a suitable relevant field. The perturbed field theory is called a deformation and corresponds to the renormalization group trajectory starting from the corresponding fixed point. The integrable deformation [40] among all possible deformations gives integrable perturbed field theory and factorized scattering theory.

The NLSM with a topological angle θ is integrable at two particular points $\theta=0$ and $\theta=\pi$ [8,39,42]. At $\theta=0$ the correlation length is finite and all the excitations are massive. The spectrum consists of a single O(3) triplet of massive particles with a nonperturbatively generated mass $m \sim r_0^{-1}e^{-2\pi/g^2}$. On the other hand, at $\theta=\pi$ the scale invariant behavior is observed in the IR limit, infinite correlation length. The large-distance asymptotics is described by $\mathrm{SU}(2)\times\mathrm{SU}(2)$ WZNW theory at level $k=1$. So the NLSM at $\theta=\pi$ can be considered as an interpolating trajectory ending up at the IR fixed point characterized by level 1 CFT.

The three-dimensional Chern-Simons gauge theory is a topological field theory in the sense that the integrand of the action is a total derivative and it is generally covariant without any metric tensor. If we quantize CS theory and take a time slice, one dimension is lost, and the theory becomes a two-dimensional conformal field theory. The correlation function in CS theory are purely topological invariants and the correlation functions over Wilson lines gives invariant knot polynomial [84]. The knot theory can describe all known rational conformal field theories. All the exact results in two dimensions mentioned above will be utilized to understand more quantitatively the quark confinement in four-dimensional QCD by dimensional reduction.

VII. DISCUSSION

In this paper we have considered one of the most important problems in modern particle physics: quark confinement in four-dimensional QCD. In order to prove quark confinement in QCD, we have suggested to use a TQFT which is extracted from the YM theory in the MAG. This TQFT describes the dynamics of magnetic monopole and antimonopole in YM theory in MAG. We have proposed a reformulation of QCD in which QCD can be considered as a perturbative deformation of the TQFT. In other words, in this reformulation the nonperturbative dynamics of QCD is saturated by the TQFT we proposed, as far as the issue of quark confinement is concerned. Needless to say, additional nonperturbative dynamics responsible for quark confinement could possibly come from the self-interaction among the gluon fields reflecting the non-Abelian nature of the gauge group. However, additional nonperturbative contributions to quark confinement are expected to be rather few, if any. This claim is strongly supported by the recent numerical simulations [24,3] of lattice gauge theory with the maximal Abelian gauge fixing, since the magnetic monopole dominance as well as the Abelian dominance in low-energy physics of QCD has been observed in this gauge for various quantities including the string tension. See Sec. IV E of [27] for more discussion.

The idea of reformulating the gauge theory as a deformation of a TQFT also works for Abelian gauge theory [88]. In the Abelian case, on the other hand, there is no self-interaction for the gauge field. Hence, using the similar reformulation of Abelian gauge theory, we can prove the existence of the quark (fractional charge) confinement phase in the strong coupling region of four-dimensional QED [88] without worrying about any additional nonperturbative effect. This result implies the existence of non-Gaussian fixed point in QED.

In this reformulation, the dimensional reduction occurs as a result of the supersymmetry hidden in the TQFT. Hence the calculation of the Wilson loop in four-dimensional QCD is reduced to that in two-dimensional NLSM. It should be remarked that this equivalence between TQFT_4 and NLSM_2 is exact.

In this paper we have used the instanton calculus to calculate the Wilson loop in two dimensions. We have shown that the area law of the Wilson loop is derived from naive instanton calculus, i.e., the dilute instanton-gas approximation. The improvement of the instanton calculus can be performed along the lines shown in Refs. [92–94]. (In the Abelian case, the improvement can be easily performed and the result is reinterpreted in terms of the vortex; see [88].) The two-dimensional instanton (anti-instanton) is considered as the intersection of the magnetic monopole (antimonopole) current with the two-dimensional space (plane). This implies that the quark confinement in QCD is caused by condensation of magnetic monopole and antimonopole (currents), together with the previous result [17]. Therefore, these results support the scenario of quark confinement proposed by Nambu, 't Hooft, and Mandelstam, i.e., the dual superconductor picture of QCD vacuum.

Note that we have used the instanton calculus merely to see the correspondence between the two-dimensional instanton and four-dimensional magnetic monopole (current), we need not to use the instanton calculus for exactly calculating the Wilson loop in two-dimensional NLSM. We can use other methods too, e.g., fermionization [88]. There is some hope to perform the calculation exactly, since the two-dimensional O(3) NLSM is exactly soluble [42,43].

Our formulation is also able to estimate the perturbative correction around the nonperturbative (topologically nontrivial) background without *ad hoc* assumptions. As an example, a calculation of static potential is given in Ref. [27] where the perturbative Coulomb potential is reproduced in addition to the linear potential part coming from the TQFT. The relationship between the full non-Abelian Wilson loop and the diagonal Abelian Wilson loop can be given based on the non-Abelian Stokes theorem [27]. Consequently, Ref. [27] completes (together with the results of this paper) the proof of area decay of the full non-Abelian Wilson loop *within* the reformulation of four-dimensional QCD as a perturbative deformation of TQFT.

The advantage of this reformulation is that one can in principle check whether this reformulation is reliable or not, since the calculations of the Wilson loop (and therefore string tension) are reduced to calculations in a two-dimensional NLSM. In fact, one can check by direct numerical simulation whether the string tension obtained from the diagonal Wilson loop in two-dimensional NLSM saturates that of the full non-Abelian Wilson loop in four-dimensional QCD, as proposed in Ref. [100]. This is nothing but the test of Abelian dominance and magnetic monopole dominance through the dimensionally reduced two-dimensional model. Such simulations will prove or disprove the validity of the reformulation of QCD proposed in this paper.

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