Phase structure of an SU(N) gauge theory with N_f flavors

Thomas Appelquist
Department of Physics, Yale University, New Haven, Connecticut 06511

Anuradha Ratnaweera
Institute of Fundamental Studies, Kandy, Sri Lanka

John Terning
Department of Physics, University of California, Berkeley, California 94720
and Theory Group, Lawrence Berkeley National Laboratory, Berkeley, California 94720

L. C. R. Wijewardhana
Department of Physics, University of Cincinnati, Cincinnati, Ohio 45221
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We investigate the chiral phase transition in SU(N) gauge theories as the number of quark flavors, N_f, is varied. We argue that the transition takes place at a large enough value of N_f so that it is governed by the infrared fixed point of the β function. We study the nature of the phase transition analytically and numerically, and discuss the spectrum of the theory as the critical value of N_f is approached in both the symmetric and broken phases. Since the transition is governed by a conformal fixed point, there are no light excitations on the symmetric side. We extend previous work to include higher order effects by developing a renormalization group estimate of the critical coupling. [S0556-2821(98)00722-X]

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I. INTRODUCTION

In an SU(N) gauge theory with N_f massless quarks, it is expected that both confinement and spontaneous chiral symmetry breaking take place provided that N_f is not too large. If, on the other hand, N_f is large enough, the theory is expected neither to confine nor break chiral symmetry. For example, if N_f is larger than $11N/2$ for quarks in the fundamental representation, asymptotic freedom (and hence confinement and chiral symmetry breaking) is lost. Even for a range of N_f below $11N/2$, the theory should remain chirally symmetric and deconfined. The reason is that an infrared fixed point is present [1,2], determined by the first two terms in the renormalization group (RG) beta function. By an appropriate choice of N and N_f, the coupling at the fixed point, α_*, can be made arbitrarily small [3], making a perturbative analysis reliable. Such a theory is massless and conformally invariant in the infrared. It is asymptotically free, but without confinement or chiral symmetry breaking.

As N_f is reduced, α_* increases. At some critical value of N_f (N_f^c) there will be a phase transition to the chirally asymmetric and confined phase. It is an important problem in the study of gauge field theories to determine N_f^c and to characterize the nature of the phase transition.

In a recent Letter [4], we suggested that the phase transition takes place at a large enough value of N_f^c so that the infrared fixed point α_* reliably exists and governs the phase transition. The transition was then analyzed using the ladder expansion of a gap equation, or equivalently the Cornwall-Jackiw-Tomboulis (CJT) effective potential [5]. It was argued that confinement effects can be neglected to estimate N_f^c and to determine the nature of the transition. It was then shown that the chiral order parameter vanishes continuously at $N_f \to N_f^c$ from below, but that the phase transition is not conventionally second order in that there is no effective, low energy Landau-Ginzburg Lagrangian, i.e. the correlation length does not diverge as the critical point is approached.

Once chiral symmetry breaking sets in, the quarks decouple at momentum scales below the dynamical mass leaving the pure gauge theory behind. The effective coupling then grows, leading to confinement at a scale on the order of the quark mass. Thus for N_f just below N_f^c, the fixed point is only an approximate feature of the theory governing momentum scales above the dynamically generated mass. This is adequate, however, since it is this momentum range that determines N_f^c and the character of the transition.

Our discussion of this phase transition paralleled an analysis of the chiral transition in $(2+1)$-dimensional gauge theories with N_f quarks [6]. Using a large N_f expansion it was found [7] that the effective infrared coupling runs to a fixed point proportional to $1/N_f$. As N_f is lowered this coupling strength exceeds the critical coupling necessary to produce spontaneous symmetry breaking. It was argued that this critical $1/N_f$ coupling lies in a range where the large N_f expansion is reliable [8]. These conclusions were also supported by lattice simulations [9]. It was then noted that as in the case of the $(3+1)$-dimensional SU(N) theory, this phase transition is not conventionally second order [6].

For QCD the study of the chiral phase transition as a function of N_f is of theoretical interest, but is unlikely to shed direct light on the physics of the real world. There remains the possibility, however, that if technicolor is the correct framework for electroweak symmetry breaking, the transition could be physically relevant. In a recent Letter [10], it was pointed out that in an SU(2) technicolor theory, a single family of techniquarks ($N_f=8$) leads to an infrared fixed point near the critical coupling for the chiral phase transition. This can provide a natural origin [11] for walking technicolor [12] and has other interesting phenomenological features.

In this paper, we explore further the features of the chiral phase transition as function of N_f. In Sec. II, we summarize the properties of an SU(N) gauge theory with N_f massless quarks, and describe the existence and properties of an infrared (IR) stable fixed point. In Sec. III, we review chiral phase transition lore in SU(N) gauge theories, both at zero temperature and finite temperature. We present our study of the chiral phase transition in Sec. IV. We examine the character of the phase transition by computing the quark-antiquark scattering amplitude for $N_f>N_f^c$ ($\alpha_*<\alpha_c$) in the RG improved ladder approximation. We observe that for $\alpha_*\to\alpha_c$ from below, there are no light scalar or pseudo-scalar degrees of freedom, showing that the phase transition is not conventionally second order. A light spectrum, in addition to the Goldstone bosons, does exist in the broken phase, and we describe what is currently known about it. In Sec. V, we include the effects of higher order contributions to both the RG β function and the estimate of the critical coupling, and then discuss the reliability of our results. In Sec. VI, we summarize our results, compare them to those from other recent studies of SU(N) theories, and make some comparisons of our work to the phase structure of supersymmetric gauge theories. In an Appendix, we discuss infrared and collinear divergences, and issues of gauge invariance arising in the study of the quark-antiquark scattering amplitude.

II. FEATURES OF AN SU(N) GAUGE THEORY WITH N_f FLAVORS

The Lagrangian of an SU(N) gauge theory is

$$\mathcal{L}=\bar{\psi}[i\partial\!\!\!/+g(\mu)A\!\!\!/^aT^a]\psi-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \tag{1}$$

where ψ is a set of N_f 4-component spinors, the T^a are the generators of SU(N), and $g(\mu)$ is the gauge coupling defined by integrating out momentum components above μ. With no quark mass, the quantum theory is invariant under the global symmetry group $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R\times\mathrm{U}(1)_{L+R}$.

The RG equation for the running gauge coupling is

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\alpha(\mu)&=\beta(\alpha)\\ &\equiv-b\alpha^2(\mu)-c\alpha^3(\mu)-d\alpha^4(\mu)-\dots,\end{aligned} \tag{2}$$

where $\alpha(\mu)=g^2(\mu)/4\pi$. With N_f flavors of quarks in the fundamental representation, the first two coefficients are given by

$$b=\frac{1}{6\pi}(11N-2N_f) \tag{3}$$

$$c=\frac{1}{24\pi^2}\left(34N^2-10NN_f-3\frac{N^2-1}{N}N_f\right). \tag{4}$$

These two coefficients are independent of the renormalization scheme. The theory is asymptotically free if $b>0$ ($N_f<\frac{11}{2}N$). At two loops, the theory has an infrared stable, nontrivial fixed point if $b>0$ and $c<0$. In this case the fixed point is at

$$\alpha_*=-\frac{b}{c}. \tag{5}$$

The fixed point coupling α_* can be made arbitrarily small by taking $(11N/2-N_f)/N$ to be small and positive [3]. This can be achieved either by going to large N and N_f with the ratio fixed, or by analytically continuing in N_f. With the coupling taken to run between zero in the ultraviolet and α_* in the infrared, the higher order terms in $\beta(\alpha)$ can then reliably be neglected. The theory is only weakly interacting in the infrared, so that there is no chiral symmetry breaking or confinement.

At two-loops the solution of the RG equation can be written as

$$b\ \log\left(\frac{q}{\mu}\right)=\frac{1}{\alpha}-\frac{1}{\alpha(\mu)}-\frac{1}{\alpha_*}\log\left(\frac{\alpha(\alpha(\mu)-\alpha_*)}{\alpha(\mu)(\alpha-\alpha_*)}\right), \tag{6}$$

where $\alpha=\alpha(q)$. For α, $\alpha(\mu)<\alpha_*$ we can introduce a scale defined by

$$\Lambda=\mu\ \exp\left[\frac{-1}{b\alpha_*}\log\left(\frac{\alpha_*-\alpha(\mu)}{\alpha(\mu)}\right)-\frac{1}{b\alpha(\mu)}\right], \tag{7}$$

so that

$$\frac{1}{\alpha}=b\ \log\left(\frac{q}{\Lambda}\right)+\frac{1}{\alpha_*}\log\left(\frac{\alpha}{\alpha_*-\alpha}\right). \tag{8}$$

Then for $q\gg\Lambda$ the running coupling displays the usual perturbative behavior:

$$\alpha\approx\frac{1}{b\ \log\left(\frac{q}{\Lambda}\right)}, \tag{9}$$

while for $q\ll\Lambda$ it approaches the fixed point α_*:

$$\alpha\approx\frac{\alpha_*}{1+\frac{1}{e}\left(\frac{q}{\Lambda}\right)^{b\alpha_*}}. \tag{10}$$

Thus for N_f in the range where an infrared fixed-point exists, Λ represents the intrinsic scale of the theory: above the scale Λ the coupling becomes asymptotically free, while below Λ the coupling rapidly approaches the infrared fixed-point.

It is interesting to note that the solution for $\alpha=\alpha(q)$ can be written generally as

$$\alpha=\alpha_*[W(q^{b\alpha_*}/e\Lambda^{b\alpha_*})+1]^{-1}, \tag{11}$$

where $W(x)=F^{-1}(x)$ with $F(x)=xe^{x}$ is the Lambert W function [13,14]. In the limit of small x, $W(x)\approx x$, giving Eq. (10) for $q\ll\Lambda$. In the limit of large x, $W(x)\approx\log x$, giving Eq. (9) for $q\gg\Lambda$.

III. CHIRAL SYMMETRY BREAKING

The physics of an SU(N) gauge theory, even at zero temperature, depends strongly on the number of massless flavors. As we have just noted, if $(11N/2-N_f)/N$ is small, the coupling remains small at all scales and the theory neither confines nor spontaneously breaks chiral symmetry. The quarks and gluons remain massless and the theory is governed by an infrared fixed point and is therefore conformally invariant in the infrared.

For N_f small compared to $11N/2$, the situation is quite different. With $N_f=0$, lattice simulations indicate that the theory confines producing a physical spectrum of massive glueballs. In the case of real-world QCD ($N=3$ with two light flavors), confinement and the spontaneous breakdown of the chiral symmetry from $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_{L+R}$ to $\mathrm{SU}(2)_{L+R}\times\mathrm{U}(1)_{L+R}$ are approximate experimental features, seen also in lattice simulations. Small N_f can also be explored by taking the large N limit with N_f fixed. There the chiral symmetry is $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$, the chromodynamic anomaly being irrelevant to leading order. It was was shown by Coleman and Witten [15] that under reasonable assumptions, confinement then necessarily implies the spontaneous breaking of $\mathrm{U}(N_f)_L\times\mathrm{U}(N_f)_R$ to $\mathrm{U}(N_f)_{L+R}$.

These two different phases of a zero-temperature SU(N) theory can be characterized by a simple chiral order parameter, the expectation value of the quark bilinear

$$M^i_j=\langle\overline{q^i_L}q^j_R\rangle, \tag{12}$$

a.k.a. the quark condensate. For some range of $(11N/2-N_f)/N$ small, the order parameter vanishes, while for N_f small compared to $11N/2$, it is non-vanishing. The location and character of the transition constitute an important and unresolved problem in the study of gauge field theories. This problem has been studied by the continuum gap equation method, by the consideration of instanton configurations, and by lattice simulations. After summarizing the results of the first approach here, we will comment on the other approaches and compare the results.

It is also interesting to compare this phase transition with the finite temperature transition of an SU(N) gauge theory. There, the transition is known to be second order [16] for $N_f=2$ and has been argued to be strongly first order [17] for $N_f\geqslant 3$. An important distinction between finite and zero temperature is that at finite temperature, the quarks are screened at distance scales large compared to the inverse temperature. This is because in Euclidean field theory at finite temperature, the integral over the energy is replaced by a sum over Matsubara frequencies given by $2n\pi T$ for bosons and $(2n+1)\pi T$ for fermions, where n is an integer. Only the $n=0$ bosons survive at large distances. Thus to characterize a finite temperature transition in which the order parameter vanishes continuously, it is not necessary to consider the quarks or fermionic bound states of quarks. This is not the case in the zero-temperature transition to be considered here. Furthermore, at zero temperature quarks experience long range interactions, which are screened at finite temperature. These differences have important consequences.

IV. THE GAP EQUATION WITH AN INFRARED FIXED POINT

We examine the chiral phase transition by making a set of simple assumptions whose validity we will examine later. First of all, we assume that the transition takes place at a value of N_f such that the infrared coupling is reliably governed by the two-loop fixed point described above. Even though this may not be a very small coupling, we assume that the transition may be studied by focusing on the underlying quark and gluon degrees of freedom, ignoring other bound states or resonances that might be formed. Next we assume that the transition is governed to first approximation by a gap equation in RG-improved ladder approximation. The most attractive channel then corresponds to the breaking pattern $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R\times\mathrm{U}(1)_{L+R}$ to $\mathrm{SU}(N_f)_{L+R}\times\mathrm{U}(1)_{L+R}$.

In the broken phase, a common dynamical mass $\Sigma(p)$, with p the magnitude of a Euclidean momentum, will then be generated for all the N_f quarks. It can be taken to serve as the order parameter for the chiral phase transition, and is related simply to the quark condensate. Although this quantity, unlike the quark condensate, is gauge dependent, it is possible to extract gauge-independent information from it.

With only the quark and gluon degrees of freedom employed, an analysis of the gap equation leads to the conclusion that the chiral transition is one in which the order parameter vanishes continuously at the transition. Near the transition, $\Sigma(p)$ is small compared to the intrinsic scale Λ, and the equation can be linearized to study the momentum regime $\Sigma(p)<p<\Lambda$ that dominates the transition. At low momenta the running coupling $\alpha(k)$ appearing in the gap equation approaches its fixed point value α_*. It is well known that the gap equation has non-vanishing solution only when this coupling exceeds a gauge-invariant critical[1] value

$$\alpha_c\equiv\frac{\pi}{3C_2(R)}=\frac{2\pi N}{3(N^2-1)}. \tag{13}$$

It can be shown that when the coupling exceeds this critical value, the CJT effective potential [3] becomes unstable at the origin, indicating that a chirally-asymmetric solution is energetically favored and therefore represents the ground state of the theory.

Setting α_* equal to α_c gives an estimate [4] of the critical number of flavors

$$N_f^c=N\left(\frac{100N^2-66}{25N^2-15}\right), \tag{14}$$

[1] A more general definition [14] of the critical coupling is that the anomalous dimension of $\overline{\psi}\psi$ becomes 1.

above which there is no chiral symmetry breaking. Note that the ratio N_f^c/N is predicted to be very close to 4 for all N.

We next discuss the critical behavior at this transition. Since the infrared behavior is governed by the fixed point α_*, we can get a simplified look at the transition by taking the coupling to be constant and equal to $\alpha_* > \alpha_c$ in a momentum range up to some cutoff $\Lambda_* < \Lambda$. The well-known solution to this simplified model (often referred to in the literature as quenched QED) is a non-vanishing dynamical mass $\Sigma(p)$ falling monotonically as a function of p from some value $\Sigma(0)$ [19,20]. For $\alpha_* \to \alpha_c$ from above ($N_f \to N_f^c$ from below), $\Sigma(0)$ exhibits the behavior

$$\Sigma(0) \approx \Lambda_* \exp\left(\frac{-\pi}{\sqrt{\frac{\alpha_*}{\alpha_c} - 1}} \right). \tag{15}$$

Thus the order parameter $\Sigma(0)$ is predicted to vanish non-analytically as $\alpha_* \to \alpha_c$.

We expect a similar critical behavior in the full theory. After all, the intrinsic scale Λ introduced in Eq. (7), where $\alpha(\Lambda) \approx 0.78\alpha_*$, plays the role of an ultraviolet cutoff. Asymptotic freedom sets in beyond this scale and the dynamical mass function falls rapidly ($\sim 1/p^2$). Indeed we find that with a running coupling the critical behavior is exponential as above, but that the coefficient in the exponential depends on the details of physics at scales on the order of Λ. It is not universally $-\pi$.

This can be understood analytically in the following manner. Following Ref. [21], the gap equation can be converted to differential form with appropriate boundary conditions, and the solution to the linearized equation can be written as

$$\Sigma(p) = \frac{c\Sigma(0)^2}{p} \sin \int_{a\Sigma(0)}^{p} \frac{dk}{k} \sqrt{\alpha(k)/\alpha_c - 1} \tag{16}$$

for momenta p below the scale Λ_c at which $\alpha(\Lambda_c) = \alpha_c$, where c is chosen so that $\Sigma(\Sigma(0)) = \Sigma(0)$. We have dropped terms explicitly proportional to derivatives of $\alpha(k)$ since the coupling is near the fixed point in this range and we have taken the lower limit of the integral to be of order $\Sigma(0)$ [$a = \mathcal{O}(1)$]. For $k > \Lambda_c$, the solution takes a different form, expressible in terms of a hyperbolic sine function when the running is slow. The two solutions must match at $p = \Lambda_c$ and the upper solution must satisfy the ultraviolet boundary condition. Note that Λ_c/Λ vanishes like $(r-1)^{1/b\alpha_*}$ as $r \to 1$, where $r \equiv \alpha_*/\alpha_c$.

The matching condition at Λ_c says simply that

$$\int_{a\Sigma(0)}^{\Lambda_c} \frac{dk}{k} \sqrt{\alpha(k)/\alpha_c - 1} \tag{17}$$

takes on some value depending on the details of the upper solution. It can be seen to be finite in the limit $r \to 1$ and it must be less than π if the dynamical mass is to remain positive for all momenta. (Solutions with nodes also exist, but a computation of the vacuum energy [5,22] indicates that the nodeless solution represents the stable ground state.) Because $\alpha(k) \approx \alpha_*$ for small momenta, it can then be seen that $1/\log(\Lambda_c/\Sigma(0))$ vanishes like $\sqrt{r-1}$ as $r \to 1$. Since Λ_c/Λ behaves like $(r-1)^{1/b\alpha_*}$, it follows that $1/\log(\Lambda/\Sigma(0))$ also vanishes like $\sqrt{r-1}$ as $r \to 1$.

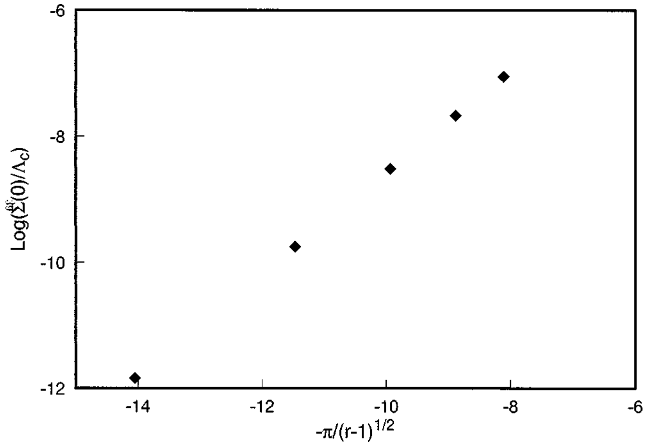

FIG. 1. Numerical solution of the Schwinger-Dyson equation with a running coupling possessing an infrared fixed point. Here Σ_0 is the dynamical mass and r is the ratio of the fixed point coupling to the critical coupling.

This can also be seen in a direct, numerical solution of the integral gap equation. In Landau gauge and after Wick rotation to Euclidean space, this equation can be written in the form

$$\Sigma(p) = \frac{1}{4} \int \frac{dk^2}{M^2} \frac{k^2\Sigma(k)}{k^2 + \Sigma(k)^2} \frac{\alpha(M^2)}{\alpha_c} \tag{18}$$

where $M = \max(p,k)$ and the approximation $\alpha[(p-k)^2] \approx \alpha(M^2)$ has been made before doing the angular integration. We solve this equation with a numerical ultraviolet cutoff much larger than Λ and plot $\log(\Sigma(0)/\Lambda_c)$ versus $1/\sqrt{r-1}$ in Fig. 1. The result is insensitive to the numerical cutoff and exhibits straight line behavior as $r \to 1$. The slope of the line is 0.82π. If the theory is modified in some way at scales on the order of Λ, straight line behavior is still exhibited, but with a slope depending on the details of the modification. Thus the only feature of the critical behavior determined purely by the infrared, fixed point behavior is that $1/\log(\Lambda/\Sigma(0))$ vanishes like $\sqrt{r-1}$ as $r \to 1$.

Below the scale of the dynamical mass $\Sigma(p)$, the quarks decouple, leaving a pure gauge theory behind. One might worry that this would invalidate the above analysis since it relies on the fixed point which only exists when the quarks contribute to the β function. This is not a problem, however, since when $\Sigma(0) \ll \Lambda$, the dominant momentum range in the gap equation, leading to the above critical behavior (15), is $\Sigma(0) < p < \Lambda$. In this range, the quarks are effectively massless and the coupling does appear to be approaching an infrared fixed point. Below the scale $\Sigma(0)$ confinement sets in. The confinement scale can be estimated by noting that at the decoupling scale $\Sigma(0)$, the effective coupling constant is of order α_c. A simple estimate using the above expressions then shows that the confinement scale is roughly the same order as the chiral symmetry breaking scale, $\Sigma(0)$.

If N_f is reduced sufficiently below N_f^c so that α_* is not close to α_c, both $\Sigma(0)$ and the confinement scale become of order Λ. The linear approximation to the gap equation is then no longer valid and it is no longer the case that higher order contributions to the effective potential can be argued to be small. The methods of this paper are then no longer useful.

From the behavior of $\Sigma(0)$ near the transition, the corresponding behavior of the Goldstone boson decay constant, the quark condensate, and other physical scales can be estimated. We return to this question after considering further the nature of the chiral phase transition we have just described.

The smooth vanishing of the order parameter $\Sigma(0)$, Eq. (15), suggests that the chiral symmetry phase transition at $N_f = N_f^c$ ($\alpha_* = \alpha_c$) might be second order. In a second order transition, however, an infinite correlation length is associated with a set of scalar and pseudoscalar degrees of freedom, with vanishing masses, described by an effective Landau-Ginzburg Lagrangian. In the broken phase, the Goldstone bosons are massless and the other scalar masses vanish at the transition. There are no other light degrees of freedom. In the symmetric phase, the scalars and pseudoscalars form a degenerate multiplet. The situation here is quite different. We first demonstrate this by showing that in the symmetric phase, there are no light scalar and pseudoscalar degrees of freedom. We then comment more generally on the physics of the transition.

A. The symmetric phase

To search for light, scalar and pseudoscalar degrees of freedom in the symmetric phase, we examine the color-singlet quark-antiquark scattering amplitude in the same (RG-improved ladder) approximation leading to the above critical behavior. If the transition is second order, then poles should appear which move to zero momentum as we approach the transition. We take the incoming (Euclidean) momentum of the initial quark and antiquark to be $q/2$, but keep a non-zero momentum transfer by assigning outgoing momenta $q/2 \pm p$ for the final quark and antiquark. Any light scalar resonances should make their presence known by producing pole in the scattering amplitude (in the complex q^2 plane).

If the Dirac indices of the initial quark and antiquark are λ and ρ, and those of the final state quark and antiquark are σ and τ, then the scattering amplitude can be written for sufficiently small q as

$$T_{\lambda\rho\sigma\tau}(p,q) = \delta_{\lambda\rho}\delta_{\sigma\tau}\frac{1}{p^2}\,T(p,q) + \cdots, \tag{19}$$

where the dots indicate pseudoscalar, vector, axial-vector, and tensor components, and we have factored out $1/p^2$ to make $T(p,q)$ dimensionless. We contract Dirac indices so that we obtain the Schwinger-Dyson (SD) equation for the scalar s-channel scattering amplitude, $T(p,q)$, containing only t-channel gluon exchanges. If $p^2 \gg q^2$, then q^2 will simply act as an infrared cutoff in the loop integrations.

The SD equation in the scalar channel is:

$$T(p,q) = \frac{\alpha_*}{\alpha_c}\pi^2 + 4\pi^2\lambda\,\frac{p^2}{\Lambda_*^2} + \frac{\alpha_*}{4\alpha_c}\left(\int_{q^2}^{p^2}\frac{dk^2}{k^2}\,T(k,q) + \int_{p^2}^{\Lambda_*^2}\frac{dk^2}{k^2}\,T(k,q)\,\frac{p^2}{k^2}\right) + \lambda\int_{q^2}^{\Lambda_*^2}\frac{dk^2}{k^2}\,T(k,q)\,\frac{p^2}{\Lambda_*^2}. \tag{20}$$

For the purpose of this discussion we neglect the running of the gauge coupling α up to the scale Λ_*. This is a good approximation at the low momenta of interest here, where the coupling is near the infrared fixed point α_*. For convenience, we use Landau gauge ($\xi = 1$) where the quark wave function renormalization vanishes. The issue of gauge invariance is addressed in the Appendix. The first term in Eq. (20) is simply one gluon exchange, while the second term arises from a chirally symmetric, four-quark interaction, i.e. a Nambu–Jona-Lasinio (NJL) [23] interaction, which we have introduced here for purposes of this analysis. It allows us to make contact with the familiar study of light degrees of freedom in the NJL theory when it is near-critical.

For momenta $p^2 > q^2$, Eq. (20) can be converted to a differential equation:

$$p^4\frac{d^2}{(dp^2)^2}\,T = -\frac{\alpha_*}{4\alpha_c}\,T, \tag{21}$$

with appropriate boundary conditions determined from Eq. (20). The solutions of Eq. (21) have the form

$$T(p,q) = A\left(\frac{p^2}{\Lambda_*^2}\right)^{1/2+(1/2)\eta} + B\left(\frac{p^2}{\Lambda_*^2}\right)^{1/2-(1/2)\eta}, \tag{22}$$

where the coefficients A and B are functions of q^2/Λ_*^2, and for $\alpha_* < \alpha_c$,

$$\eta = \sqrt{1-\alpha_*/\alpha_c}. \tag{23}$$

The coefficients A and B can be determined by substituting the solution back into Eq. (20). This gives

$$A = \frac{-2\pi^2}{(1+\eta)^2}\,\frac{(1-\eta)\left(1-\dfrac{\lambda}{\lambda_*}\right)\left(\dfrac{q^2}{\Lambda_*^2}\right)^{-1/2+(1/2)\eta}}{1-\dfrac{\lambda}{\lambda_\alpha}+\left[\dfrac{\lambda}{\lambda_\alpha}-\left(\dfrac{1-\eta}{1+\eta}\right)^2\right]\left(\dfrac{q^2}{\Lambda_*^2}\right)^{\eta}}, \tag{24}$$

and

$$B=\frac{2\pi^2(1-\eta)\left(1-\frac{\lambda}{\lambda_\alpha}\right)\left(\frac{q^2}{\Lambda_*^2}\right)^{-1/2+(1/2)\eta}}{1-\frac{\lambda}{\lambda_\alpha}+\left[\frac{\lambda}{\lambda_\alpha}-\left(\frac{1-\eta}{1+\eta}\right)^2\right]\left(\frac{q^2}{\Lambda_*^2}\right)^\eta},\tag{25}$$

where

$$\lambda_\alpha\equiv\left[\frac{1}{2}+\frac{1}{2}\eta\right]^2,\tag{26}$$

and

$$\lambda_*\equiv\left[\frac{1}{2}-\frac{1}{2}\eta\right]^2.\tag{27}$$

If we denote the location of the poles of the functions A and B in the complex q^2 plane by q_0^2, we then have

$$|q_0^2|=\Lambda_*^2\left(\frac{|\lambda_\alpha-\lambda|}{|\lambda-\lambda_*|}\right)^{1/\eta}.\tag{28}$$

We see immediately that as $\lambda\to\lambda_\alpha$ [the critical Nambu–Jona-Lasinio (NJL) coupling] for $\alpha_*<\alpha_c$ the pole approaches the origin $q_0^2=0$, indicating the existence of light degrees of freedom. This is to be expected for a second order phase transition. As α_* is increased the corresponding particles become broad resonances [24]. Of course in this region our analysis is not complete, precisely because of the existence of the light scalar and pseudoscalar degrees of freedom. These light degrees of freedom must be incorporated into the analysis, for example they will have an effect on the two loop β function. Furthermore as discussed by Chivukula *et al.* [25] one generally expects that, with more than two flavors of quarks, as λ is tuned towards λ_α the theory undergoes a Coleman-Weinberg transition [26] to the chirally broken phase before λ reaches λ_α.

Now consider the limit $\eta\to0$ ($\alpha_*\to\alpha_c$), with $\lambda<1/4$, we have

$$|q_0^2|\to\Lambda_*^2\left(1+\frac{\eta}{1/4-\lambda}\right)^{1/\eta}$$
$$\to\Lambda_*^2\exp\left(\frac{4}{1-4\lambda}\right).\tag{29}$$

Thus we see that at $\alpha_*\to\alpha_c$, with $\lambda<1/4$, there are no poles in the complex q^2-plane with $q_0^2\ll\Lambda_*$. There are therefore no light scalar and pseudoscalar degrees of freedom to constitute an effective Landau-Ginzburg theory, so the chiral phase transition is not second order along the line $\alpha_*=\alpha_c$. This is in agreement with the analysis of Ref. [27].

Now imagine starting out with $\alpha_*<\alpha_c$ and $\lambda\approx\lambda_\alpha$, so that we have a light scalar resonance, and then dialing the parameters so that α_* increases and λ decreases in such a way that we approach the critical line $\alpha_*=\alpha_c$. We then see from Eqs. (28) and (27) that we must first cross the line $\lambda=\lambda_*$, and that as we approach this line, the mass of the scalar grows and actually diverges. Thus the scalar resonance disappears from the physical spectrum before we reach $\alpha_*=\alpha_c$. Even before we reach this point, the width of the scalars becomes as large as their mass, and they can no longer be considered resonances.

There is nothing special about the scalar and pseudoscalar channels in the above analysis. A similar analysis of the other channels, such as vector and axial-vector, would also reveal that there are no light excitations in the symmetric phase near the critical coupling α_c. That this should be the case is not surprising. With the transition governed by a long-range gauge force with an infrared fixed point, approximate conformal invariance should be exhibited at momentum scales small compared to Λ in the symmetric phase. (For further discussions on this point see Ref. [28].) Thus no light scales will be present, in contrast to phase transitions governed by short range forces as in the NJL or the finite temperature theories.

B. The broken phase

In the broken phase near the transition, one light scale, $\Sigma(0)$, appears. It is therefore natural (in the assumed absence of instanton effects) to expect that the entire physical spectrum of the theory will be set by $\Sigma(0)$ and scale to zero with it as $N_f\to N_f^c$ from below. This point has been stressed recently by Chivukula [29]. Thus there will clearly be no effective Landau-Ginzburg Lagrangian. No finite set of light degrees of freedom can be isolated in the broken phase in the limit $N_f\to N_f^c$, and no light degrees of freedom (other than quarks and gluons) exist in the symmetric phase.

Within this general picture, it is important to describe the spectrum of resonances in more detail. If, for example, a near-critical theory is the basis for a technicolor theory of electroweak symmetry breaking [10], then the light scale $\Sigma(0)$ will correspond to the electroweak scale and the spectrum of resonances at this scale will have a direct impact on precision electroweak measurements. In particular, the S parameter [30] will depend sensitively on this spectrum. An especially interesting question in this regard is whether parity doubling or even inversion of parity partners appears in this light spectrum as N_f^c is approached.

The Goldstone boson decay constant F_π is also proportional to $\Sigma(0)$. A simple dimensional estimate suggests that $F_\pi^2\approx N\Sigma^2(0)/16\pi^2$. Because of the dominance of the fixed point at scales below Λ, this is clearly a ''walking'' theory. If the coupling stays close to α_c then the dynamical mass $\Sigma(p)$ falls roughly like $1/p$ in this range. As a consequence, the condensate $\langle\overline{q_L^i}q_R^j\rangle$ is enhanced well above the value it would have in a QCD-like theory. A simple estimate gives $\langle\overline{q_L^i}q_R^j\rangle\approx N\Sigma(0)^2\Lambda/16\pi^2$.

Finally, it is important to note that with the entire spectrum of physical states collapsing to zero with $\Sigma(0)$ at the transition, the analysis of the transition using only the quark and gluon degrees of freedom is open to question. It seems reasonable, however, to conjecture that these states will not be important at the momentum scales $\Sigma(0)<k<\Lambda$ dominating the transition. Some evidence for this is provided by estimates of higher order effects to which we now turn.

V. HIGHER ORDER ESTIMATES

We have so far analyzed the chiral symmetry breaking phase transition using the ladder gap equation, i.e. the Schwinger-Dyson (SD) equation with the lowest order kernel, and the running gauge coupling determined by the two-loop β function. In order to consider higher order effects we first develop a gauge-invariant technique to estimate the critical coupling without relying on the intricacies of the SD equation.

In Ref. [31], it was noted that to lowest order the SD criticality condition can be written in the form

$$\gamma(2-\gamma)=1, \tag{30}$$

where γ is the anomalous dimension of the quark mass operator. To all orders in perturbation theory, this condition is gauge invariant (since γ is gauge invariant) and is equivalent to the condition [18] $\gamma=1$ mentioned previously in the text. However if these conditions are truncated at a finite order in perturbation theory they lead to different results. We will take Eq. (30) to define the critical coupling order by order, since it allows us to reproduce the known leading order result.

Through three loops γ is given in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme by [32]

$$\gamma=\gamma_0\alpha+\gamma_1\alpha^2+\gamma_2\alpha^3+\cdots \tag{31}$$

where

$$\gamma_0=\frac{3C_2(R)}{2\pi} \tag{32}$$

$$\gamma_1=\frac{1}{16\pi^2}\left(3C_2(R)^2-\frac{10C_2(R)N_f}{3}+\frac{97C_2(R)N}{3}\right) \tag{33}$$

$$\gamma_2=\frac{1}{64\pi^3}\; 129C_2(R)^3-\frac{70C_2(R)N_f^2}{27}-\frac{129C_2(R)^2N}{2}+\frac{11413C_2(R)N^2}{54}+C_2(R)N_fN\left(-\frac{556}{27}-48\zeta(3)\right)+C_2(R)^2N_f(-46+48\zeta(3)). \tag{34}$$

Inserting this result in Eq. (30) and truncating to one-loop we find

$$2\gamma_0\alpha=1. \tag{35}$$

Solving for α we find a one-loop estimate of the critical coupling that agrees with standard result:

$$\alpha_c^{(1)}=\frac{\pi}{3C_2(R)}=\frac{2\pi N}{3(N^2-1)}. \tag{36}$$

At two-loops the critical condition is

$$2\gamma_0\alpha+2\gamma_1\alpha^2-\gamma_0^2\alpha^2=1. \tag{37}$$

Solving for α we find a two-loop estimate of the critical coupling:

$$\alpha_c^{(2)}=\frac{36\pi}{45C_2(R)-97N+10N_f}\pm\frac{\sqrt{24}\pi\sqrt{9C_2(R)+97N-10N_f}}{\sqrt{C_2(R)}[-45C_2(R)+97N-10N_f]}. \tag{38}$$

The + sign gives the positive root. We compare this with the one-loop estimate by taking N large and using the value $N_f\approx 4N$ corresponding to criticality:

$$\alpha_c^{(2)}\approx\frac{(\sqrt{11808}-72)\pi}{69N}\approx\frac{1.67}{N}. \tag{39}$$

Numerically it can be seen that the $\mathcal{O}(\alpha^2)$ terms in the criticality condition, Eq. (37), evaluated at $\alpha=\alpha_c^{(2)}$ are typically about 25% to 30% of the leading term for $N_f\approx 4N$. It can also be seen numerically that for $N_f\approx 4N$ the four-loop term [32] in γ is larger than the three-loop term, so it is not appropriate to go beyond two loops in this expansion for these values of N_f, and we should only use the three-loop term as an estimate of the error in our calculation.

Through three-loops, the β function is given by

$$\beta(\alpha)=-b\alpha^2-c\alpha^3-d\alpha^4$$

where b and c are given by Eqs. (3) and (4), and in the MS scheme,

$$d=\frac{1}{32\pi^2}\left(\frac{2857N^3-1415N^2N_f+79N(N_f)^2}{54}-\frac{205N}{18}C_2(R)N_f+\frac{11}{9}C_2(R)(N_f)^2+C_2(R)^2N_f\right). \tag{40}$$

Since the three-loop term is scheme dependent we cannot obtain a scheme independent answer without going to the same order in β and γ, so we will only use the three-loop term for error estimates.

In Table I we list some numerical results. We have computed the value of N_f^c for SU(N) gauge theories for values of N ranging form 2 to 10, showing the results at different orders in perturbation theory. In Sec. IV (using the leading order estimate of the critical coupling) it was shown that N_f^c goes like $4N$ for large N. We see that going to two loops in the criticality condition produces a small shift in this relation. We also list the estimated value of the critical coupling at one and two loops. We see that even though the percentage shift of the value of N_f^c is small, the higher order terms of the beta function make a significant contribution at the critical point. For N_c between 3 and 10 we estimate that the error in N_f^c at two-loops is about 12% from the truncation of the β function and about 10% from the truncation of γ, while for $N_c=2$ the errors are somewhat larger, around 14% from each. It is important to emphasize that these are simply numerical estimates of the next to leading contributions. Even

TABLE I. Estimates of N_f^c. The two numbers in parentheses give the order used in the critical condition on γ and the β function. The comparison of the (2,2) and (2,3) give an estimate of the error in truncating the β function at two-loops.

N_c	N_f^c (1,2)	N_f^c (2,2)	N_f^c (2,3)	$\alpha_c^{(1)}$	$\alpha_c^{(2)}$
2	7.86	8.27	7.12	1.4	1.11
3	11.9	12.4	10.9	0.785	0.595
4	15.9	16.6	14.6	0.559	0.412
5	20.0	20.8	18.3	0.436	0.317
6	24.0	24.9	22.	0.359	0.258
7	28.0	29.1	25.7	0.305	0.218
8	32.0	33.3	29.4	0.266	0.189
9	36.0	37.4	33.1	0.236	0.166
10	40.0	41.6	36.8	0.212	0.149

at large N, there is no obvious small parameter here leading to a controlled expansion. Thus the smallness of still higher order terms is not guaranteed.

VI. SUMMARY AND CONCLUSIONS

In this paper, we have explored features of the chiral phase transition in SU(N) gauge theories. We have argued that the transition takes place at a relatively large value of N_f ($N_f^c \approx 4N$) where the infrared coupling is determined by a fixed point accessible in the loop expansion of the β function, and that the transition can be studied using a ladder gap equation. Our higher order estimates suggest that the estimate of N_f^c is good to about 20%. To phrase things in physical terms, the effect of the light quarks is to screen the long range force, eventually disordering the system and taking it to the symmetric phase. That the transition takes place at a relatively large value of N_f means that the quarks are relatively ineffective at long range screening.

With an infrared fixed point governing the transition, the order parameter vanishes in a characteristic exponential fashion and all physical scales vanish in the same way. There is no finite set of light degrees of freedom that can be identified to form an effective, Landau-Ginzburg theory. In the symmetric phase ($N_f > N_f^c$), no light degrees of freedom are formed as $N_f \to N_f^c$. Thus the transition is continuous but not conventionally second order. The validity of the approach is considered by estimating higher order terms in both the β function and the anomalous dimension of the mass operator.

In Ref. [33], it was noted that single instanton effects in a theory with an infrared fixed point seem capable of triggering a chiral phase transition at similarly large values of N_f/N. A detailed computation was carried out only for an SU(2) gauge theory but the analysis indicated that this could be the case at larger values of N as well.

It is interesting to compare our results with the phase structure of supersymmetric SU(N) theories where exact results are available [34]. In such theories there is also a large range of N_f where the theory is asymptotically free and an infrared fixed point occurs. A transition to a strongly coupled phase occurs at $N_{\mathrm{f,SUSY}}^c = 3N/2$. Thus it seems plausible that infrared fixed points are fairly generic in asymptotically free gauge theories with a large number of flavors. One prominent difference between the supersymmetric and nonsupersymmetric cases is that the strongly coupled phase $N+1 < N_f \leq N_{\mathrm{f,SUSY}}^c$ does not have chiral symmetry breaking or confinement for $N > 3$. However a class of supersymmetric chiral gauge theories (with antisymmetric tensor fields) have been found [35] where the theory does go from an infrared fixed point to confinement upon the removal of one flavor.

The results of this paper can be contrasted with preliminary lattice work [36] and the instanton liquid model [37] which suggest that the chiral transition takes place at much smaller values of N_f contrary to earlier lattice results [38]. The transition would then be an intrinsically strong coupling phenomenon inaccessible to the methods used here. The quarks would have to be much more effective at long range screening than indicated by the gap equation, disordering the system even in the presence of a strong, attractive long range force. Further work on all these approaches will be required to help to resolve this difference.

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APPENDIX: GAUGE INVARIANCE AND COLLINEAR DIVERGENCES

We first discuss the gauge dependence of the quark-antiquark scattering amplitude used in Sec. IV to demonstrate the absence of light excitations in the symmetric phase. We will then discuss the presence of collinear divergences in this amplitude. To demonstrate gauge invariance to leading order, we follow the analysis of [39]. As was done before we will take the incoming (Euclidean) momentum of the initial quark and antiquark to be $q/2$, and have a non-zero momentum transfer by assigning outgoing momenta $q/2 \pm p$ for the final quark and antiquark. The SD equation in the scalar channel (and in a covariant gauge with gauge parameter ξ) is:

$$T(p,q) = \frac{g^2 Z_1^2(p,q)}{4\alpha_\xi Z_3(p)}\pi + \frac{4\pi^2 \lambda Z_4(p,q) p^2}{\Lambda_*^2} + \frac{\pi p^2}{\alpha_\xi} \int \frac{d^4k}{(2\pi)^4} \frac{g^2 Z_1^2(p,k)}{Z_3(p-k)(p-k)^2} \frac{T(k,q)}{k^2 Z_2^2(k)} + \frac{4\pi^2 p^2}{\Lambda_*^2} \int \frac{d^4k}{(2\pi)^4} \lambda Z_4(p,k) \frac{T(k,q)}{k^2 Z_2^2(k)}. \quad \text{(A1)}$$

The renormalization factors Z_1, Z_2, Z_3, and Z_4 correspond to the gauge vertex, the quark wavefunction, the gauge boson wavefunction, and the four-quark vertex respectively; and

$$\alpha_\xi = \frac{\pi}{(3+\xi)C_2(R)}. \tag{A2}$$

Using the definition of the renormalized couplings

$$g_R(p,k) = \frac{g Z_1(p,k)}{\sqrt{Z_3(p-k)Z_2(k)Z_2(p)}} \tag{A3}$$

$$\lambda_R(p,k) = \frac{\lambda Z_4(p,k)}{Z_2(k)Z_2(p)} \tag{A4}$$

and the approximations

$$\frac{g_R^2(p,k)}{4\pi} \approx \frac{g^2}{4\pi}\frac{Z_1(\max(p,k))}{Z_3(\max(p,k))Z_2(k)Z_2(p)} \equiv \alpha(\max(p,k)) \tag{A5}$$

and

$$\lambda_R(p,k) \approx \lambda \frac{Z_4(\max(p,k))}{Z_2(k)Z_2(p)} \equiv \lambda(\max(p,k)) \tag{A6}$$

we can perform the angular integrations to obtain

$$T(p,q) = \frac{\alpha(p)Z_2(p)Z_2(q)}{\alpha_\xi}\pi^2 + 4\pi^2\lambda(p)Z_2(p)Z_2(q)\,\frac{p^2}{\Lambda_*^2} + \frac{1}{4\alpha_\xi}\left(\int_{q^2}^{p^2}\frac{dk^2}{k^2}\alpha(p)\,\frac{Z_2^2(p)}{Z_2^2(k)}\,T(k,q) + \int_{p^2}^{\Lambda_*^2}\frac{dk^2}{k^2}\alpha(k)T(k,q)\,\frac{p^2}{k^2}\right) + \int_{q^2}^{p^2}\frac{dk^2}{k^2}\lambda(p)\,\frac{Z_2^2(p)}{Z_2^2(k)}\,T(k,q)\,\frac{p^2}{\Lambda_*^2} + \int_{p^2}^{\Lambda_*^2}\frac{dk^2}{k^2}\lambda(k)T(k,q)\,\frac{p^2}{\Lambda_*^2}. \tag{A7}$$

In order to get a gauge invariant result, it is helpful to divide the scattering amplitude by the gauge dependent normalization factors of the four quark legs, so we introduce

$$\tilde{T}(p,q) = \frac{T(p,q)}{Z_2(p)Z_2(q)}. \tag{A8}$$

We then have

$$\tilde{T}(p,q) = \frac{\alpha_*}{\alpha_\xi}\pi^2 + 4\pi^2\lambda\,\frac{p^2}{\Lambda_*^2} + \frac{\alpha_*}{4\alpha_\xi}\left(\int_{q^2}^{p^2}\frac{dk^2}{k^2}\frac{Z_2(p)}{Z_2(k)}\,\tilde{T}(k,q) + \int_{p^2}^{\Lambda_*^2}\frac{dk^2}{k^2}\frac{Z_2(k)}{Z_2(p)}\,\tilde{T}(k,q)\,\frac{p^2}{k^2}\right) + \lambda\left(\int_{q^2}^{p^2}\frac{dk^2}{k^2}\frac{Z_2(p)}{Z_2(k)}\,\tilde{T}(k,q)\,\frac{p^2}{\Lambda_*^2} + \int_{p^2}^{\Lambda_*^2}\frac{dk^2}{k^2}\frac{Z_2(k)}{Z_2(p)}\,\tilde{T}(k,q)\,\frac{p^2}{\Lambda_*^2}\right), \tag{A9}$$

where we have used the fact that $\alpha(p)$ approaches a fixed point for $p \ll \Lambda$. Here we will be satisfied with a result to leading order in α_*, neglecting terms suppressed by α_*^2, λ^2, and $\alpha_*\lambda$. With this approximation we can also neglect the running of λ. This is actually not a bad approximation, since in the infrared $\lambda(p)$ approaches a fixed-point given by Eq. (27). Now the RG solution for the quark wavefunction renormalization is:

$$Z_2(p) = \left(\frac{\Lambda_*^2}{p^2}\right)^{\gamma}, \tag{A10}$$

where

$$\gamma = \frac{\alpha_* C_2(R)\xi}{4\pi} + \mathcal{O}(\alpha_*^2). \tag{A11}$$

Next we substitute the form

$$\tilde{T}(p,q) = A\left(\frac{p^2}{\Lambda_*^2}\right)^{1/2+(1/2)\eta} + B\left(\frac{p^2}{\Lambda_*^2}\right)^{1/2-(1/2)\eta}, \tag{A12}$$

into Eq. (A9). Integrating this equation we see that to leading order in α_* the ξ dependent terms take the form

$$\frac{\alpha_*}{4\alpha_\xi\left(\frac{1}{2}-\frac{1}{2}\eta+\gamma\right)\left(\frac{1}{2}+\frac{1}{2}\eta+\gamma\right)} \approx 1 + \mathcal{O}(\alpha_*^2). \tag{A13}$$

So our solution for the scattering amplitude [Eqs. (24) and (25)] and the conclusion that there are no light scalar degrees of freedom as one approaches the critical point from the symmetric side of the critical curve are gauge invariant results to leading order.

We next discuss the collinear divergences present in $T(p,q)$. Consider the differential cross-section for the scattering of the quark and antiquark at $\mathcal{O}(\alpha^3)$. If the invariant amplitude at $\mathcal{O}(\alpha^2)$ is given by $\mathcal{M}$, then from Eqs. (22)–(25) we have, to next-to-leading order,

$$|\mathcal{M}|^2 \approx \frac{9\pi^2\alpha^2 C_2(R)^2}{p^4} + \frac{27\pi\alpha^3 C_2(R)^3}{2p^4}\left[1+\ln\left(\frac{p^2}{q^2}\right)\right], \tag{A14}$$

The differential cross section is:

$$d\sigma_0 = (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)|\mathcal{M}|^2 \times \frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}, \tag{A15}$$

which gives

$$\frac{d\sigma_0}{dq_1 d\Omega_1 dq_2 d\Omega_2} = \frac{1}{(2\pi)^2}\delta^{(4)}(p_1+p_2-q_1-q_2) \times |\mathcal{M}|^2 \frac{E_1E_2}{4}. \tag{A16}$$

This is not, however, a physically observable cross-section. To obtain a physically observable cross-section we must combine this with the differential cross-section where a collinear gluon (with momentum k and implicit summation on the gauge index a) is emitted:

$$d\sigma_{1g} = (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)|\mathcal{M}^a|^2 \times \frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}\frac{d^3k}{(2\pi)^3 2k}, \tag{A17}$$

A physical experiment cannot separately resolve the collinear gluon and quark, so it is appropriate to frame the discussion in terms of the momentum of the observed jet (we consider first the case where k is approximately collinear with q_2, so $q_j = q_2 + k$). Changing variables we have

$$\frac{d\sigma_{1g}}{dq_1 d\Omega_1 dq_j d\Omega_j} = \frac{1}{(2\pi)^2}\delta^{(4)}(p_1+p_2-q_1-q_j)\,\frac{E_1E_j}{4} \times \int \frac{d^3k}{(2\pi)^3 2k}\frac{(E_j-k)}{E_j}\overline{|\mathcal{M}^a|^2}. \tag{A18}$$

Thus, to see the cancellation of the collinear divergence we must add $|\mathcal{M}|^2$ to the final integral in Eq. (A18).

In order to project out the scalar channel of the gluon emission amplitude, we must contract the amplitude with $\delta_{\rho\lambda}/4$ and $[\gamma^\alpha,\gamma^\beta]_{\sigma\tau}/16$, where ρ and λ (σ and τ) are the Dirac indices of the initial (final) quark and antiquark. We then have

$$\mathcal{M}^a = -\frac{ig^3 C_2(R)}{p^2 q_j^2}\frac{3}{4}(\epsilon^\alpha q_j^\beta - \epsilon^\beta q_j^\alpha)T^a, \tag{A19}$$

where ϵ^α is the gluon polarization vector. Squaring and summing over gluons and gluon polarizations we have:

$$\overline{|\mathcal{M}^a|^2} = -\frac{g^6 C_2(R)^3}{p^4 q_j^2}\frac{27}{8}. \tag{A20}$$

Putting the gluon on shell ($k^2=0$), and performing the integration (with the requirement that the gluon momentum k be within a small cone of opening angle δ around the quark momentum q_2) we have

$$\int \frac{d^3k}{(2\pi)^3 2k}\frac{(E_j-k)}{E_j}\overline{|\mathcal{M}^a|^2} \approx -\frac{27g^6C_2(R)^3}{8p^2}\int_0^{E_j}\frac{dk\,k^2(E_j-k)}{(2\pi)^2 2k}\int_0^\delta \frac{\theta d\theta}{q_2^2+(E_j-k)k\theta^2} \approx -\frac{27\pi\alpha^3 C_2(R)^3}{4p^4}\ln\left(\frac{E_j^2\delta^2}{q_2^2}\right), \tag{A21}$$

where we have only kept terms which diverge as $q_2^2 \to 0$. When combined with the integration over the region of phase space corresponding to k being approximately collinear with q_1, and setting $q_1=q_2=q$, we see that these terms cancel with the $\ln(q^2)$ dependence in Eq. (A14), as expected [40].

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