Corrections to finite-size scaling in the lattice N-vector model for $N = \infty$

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We compute the corrections to finite-size scaling for the N-vector model on the square lattice in the large-N limit. We find that corrections behave as $\log L/L^2$. For tree-level improved Hamiltonians corrections behave as $1/L^2$. In general l-loop improvement is expected to reduce this behavior to $1/(L^2\log^l L)$. We show that the finite-size scaling limit and the perturbative limit do not commute in the calculation of the corrections to finite-size scaling. We present a detailed study of the corrections for the RP$^\infty$ model. [S0556-2821(98)03520-6]

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I. INTRODUCTION

In the study of statistical models it is extremely important to understand finite-size corrections. Indeed in experiments and in numerical work it is essential to take into account the finite size of the system in order to extract correct infinite-volume predictions from the data. Finite-size scaling (FSS) [1–5] concerns the critical behavior of systems in which one or more directions are finite, even though microscopically large, and it is therefore essential in the analysis of experimental data in many situations, for instance, for films of finite thickness. Numerically FSS can be used in a variety of ways to extract information on infinite-volume systems. A very interesting method to extract critical indices comparing data on lattices of different sizes was introduced by Nightingale [6], the so-called phenomenological renormalization group. Recently FSS has been used to obtain precise predictions at very large values of the correlation length from simulations on small lattices. This extrapolation technique was introduced by Lüscher, Weisz, and Wolff [7] and subsequently applied to many different models [8–13]: a careful theoretical analysis (see Sec. V A 2 of Ref. [13]) shows that the method is extremely convenient for asymptotically free theories and indeed one was able to simulate the $O(3)$ σ model [10] up to $\xi \approx 10^5$ and the SU(3) chiral model [12,13] up to $\xi \approx 4 \times 10^5$ using relatively small lattices ($L \leqslant 512$). In order to use these techniques reliably it is extremely important to have some theoretical prediction on the behavior of the corrections to FSS. One can use this information in two different ways. A possibility is to take advantage of the theoretical prediction to extrapolate the Monte Carlo data to the FSS limit—that still involves a limit $L \to \infty$—in the spirit of Ref. [7]. One also needs this information if one determines the FSS curve by comparing data from simulations on lattices of different sizes as proposed in Ref. [9]. For instance checking the absence (within error bars) of corrections to FSS for lattices of sizes $64 \leqslant L \leqslant 256$ is enough if the corrections vanish as $1/L^2$ while it can be totally misleading if

corrections behaving as $1/\log L$ are present (this is the case for instance of the four-state Potts model, see Ref. [14]).

A second topic that will be extensively discussed in this paper is the improvement of lattice Hamiltonians [15–19]. The idea behind all these attempts is to modify the lattice Hamiltonian with the addition of irrelevant operators in order to reduce lattice artifacts: in this way one hopes to have scaling and FSS at shorter correlation lengths. For general statistical models this is a nontrivial program (see, e.g., Ref. [20]). For asymptotically free theories the idea is much simpler to implement since in this case improvement can be discussed using perturbation theory.

In this paper we will study the problem of corrections to FSS and improvement in the context of the large-N N-vector model. This theory provides the simplest example for the realization of a non-Abelian global symmetry. Its two-dimensional version has been extensively studied because it shares with four-dimensional gauge theories the property of being asymptotically free in the weak-coupling perturbative expansion [21–23]. This picture predicts a nonperturbative generation of a mass gap that controls the exponential decay at large distance of the correlation functions.

Besides perturbation theory, the two-dimensional N-vector model can be studied using different techniques. It can be solved in the $N = \infty$ limit [24,25] and $1/N$ corrections can be systematically calculated [26–28]. An exact S matrix can be computed [29,30] and, using the thermodynamic Bethe ansatz, the exact mass gap of the theory in the limit $\beta \to \infty$ has been obtained [31,32]. The model has also been the object of extensive numerical work [10,33–36] mainly devoted to checking the correctness of the perturbative predictions [37–39]. In one dimension $O(N)$-invariant spin models with nearest-neighbor interactions are exactly soluble: a detailed investigation of the possible continuum limits and an exact computation of the FSS functions is reported in Ref. [40].

FSS has been extensively studied in perturbation theory [7,41–45] and in the large-N limit where the scaling functions can be computed analytically as expansions in $1/N$ [41,46–48]. Here we will concentrate on the corrections to FSS for $N = \infty$ in two dimensions and we will compute the deviations from FSS for generic lattice interactions. We will

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show that in this limit FSS corrections behave generically as $\log L/L^2$. This is in agreement with a general renormalization-group argument that shows that corrections to FSS are controlled by the first subleading operator [49]. Tree-level improvement changes the behavior by a logarithm of L: these actions have corrections behaving as $1/L^2$. Subsequent improvement should reduce the corrections to $1/(L^2 \log L)$ and so on. Full $O(a^2)$ improvement to all orders of perturbation theory provides an action with corrections behaving as $\log L/L^4$.

In addition to the standard N-vector model we will also discuss a mixed N-vector–RP^{N-1} model [25,50]. There are two reasons why we decided to include this computation: first of all, for large values of N, the N-vector model shows many simplifying features: for instance, only the leading term of the β function does not vanish. For this reason one may expect that the behavior of the corrections for this class of models is far simpler than in generic models. Instead the mixed N-vector–RP^{N-1} model shows a more complex behavior and, for instance, the β function is nontrivial to all orders of perturbation theory. We find that in the mixed model the corrections behave as $(\log L/L^2) f(L)$ where $f(L)$ is a nontrivial function such that $f(\infty)$ is finite and that admits an asymptotic expansion in powers of $1/\log L$. The presence of powers $1/(L^2 \log^n L)$ is somewhat unexpected from the point of view of perturbation theory. We will show that this is related to the noncommutativity of the limits $L \to \infty$ and $\beta \to \infty$. In other words the perturbative limit $\beta \to \infty$ at L fixed followed by the limit $L \to \infty$ gives results that are different from the FSS limit. The commutativity of these two limits has been the object of intense debate. The standard wisdom is that the two limits are identical, but this point has been seriously questioned by Patrascioiu and Seiler [51] (for an answer to these criticisms see Ref. [52]) together with many other assumptions derived from perturbation theory [53]. Our calculation shows that, if the standard assumption is true, it is a result far from obvious: indeed the limits are *different* for the corrections to FSS.

A second motivation for studying this class of models is the problem of the critical limit of RP^{N-1} models. In Ref. [54] numerical results were presented indicating that RP^{N-1} models could have a critical limit different from that of the N-vector model. These results were subsequently questioned in Refs. [55,56]. In particular it was argued that numerical results could be misleading due to the presence of large corrections to scaling [56,57]. We wanted to understand if there is any sign of this phenomenon in the large-N limit. Our explicit calculation shows that RP^∞ has corrections that are larger than those of the N-vector model. Depending on the observable, for reasonable lattice sizes, we find an increase by a factor of 6–15. This is in qualitative agreement with the scenario of Ref. [56].

The paper is organized as follows. In Sec. II we define the models we consider, and compute various observables in the large-N limit. In Sec. III we discuss the corrections to FSS for the N-vector model and in Sec. IV we extend our results to the mixed N-vector–RP^{N-1} model. In Sec. V we present our conclusions.

In the Appendixes we report some general results on the FSS behavior of lattice sums. These results are of general interest and may be applied in many other contexts: in particular they may be used to study FSS properties of models that have a height (SOS) representation (see Refs. [58,59], and references therein). In Appendix A we define a set of basic functions that appear in all our results and we report some of their properties. We extend here the results of Ref. [60]. In Appendix B we give an algorithmic procedure that allows us to compute the expansion in powers of $1/L^2$ of any sum involving powers of the lattice propagator for a Gaussian model with arbitrary interaction in the FSS limit. As an example we report the explicit formulas that are needed in our main discussion. In Appendix C we report the asymptotic behavior of some lattice integrals. Preliminary results of this work were presented at the Lattice '96 Conference [61].

II. THE MODELS

In this paper we will study the FSS properties of the classical N-vector model on a square lattice with local translation- and parity-invariant ferromagnetic interactions. The Hamiltonian is given by

$$\mathcal{H} = -N \sum_{x,y} J(x-y) \, \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y, \qquad (2.1)$$

where the fields $\boldsymbol{\sigma}_x$ satisfy $\boldsymbol{\sigma}_x^2 = 1$. The partition function is simply

$$Z = \int \prod_x d\boldsymbol{\sigma}_x \, \delta(\sigma_x^2 - 1) e^{-\beta \mathcal{H}}. \qquad (2.2)$$

We will consider general *local* interactions. If $\hat{J}(p)$ is the Fourier transform of $J(x)$, locality and parity invariance imply that $\hat{J}(p)$ is a continuous function of p, even under $p \to -p$. We will require invariance under rotations of $\pi/2$, that is we will assume $\hat{J}(p)$ symmetric under interchange of p_1 and p_2. Redefining β we can normalize the couplings so that

$$\hat{J}(q) = \hat{J}(0) - \frac{q^2}{2} + O(q^4) \qquad (2.3)$$

for $q \to 0$. We also introduce the function

$$w(q) = -2[\hat{J}(q) - \hat{J}(0)], \qquad (2.4)$$

that behaves as q^2 for $q \to 0$. Finally we will require the theory to have the usual (formal) continuum limit: we will assume that the equation $w(q) = 0$ has only one solution for $-\pi \leq q_i \leq \pi$, namely, $q = 0$. We will need the small-q behavior of $w(q)$: we will assume in this limit the form

$$w(q) = \hat{q}^2 + \alpha_1 \sum_\mu \hat{q}_\mu^4 + \alpha_2 (\hat{q}^2)^2 + O(q^6), \qquad (2.5)$$

where α_1 and α_2 are arbitrary constants. Here $\hat{q}^2 = \hat{q}_1^2 + \hat{q}_2^2$, $\hat{q} = 2 \sin(q/2)$.

Let us give some examples we will use in the following. The standard N-vector model with Hamiltonian

$$H^{\text{std}} = -N \sum_{x\mu} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} \qquad (2.6)$$

corresponds to $w(q) = \hat{q}^2$ and thus we have $\alpha_1 = \alpha_2 = 0$. Other possibilities are the following.

(1) The Symanzik improved Hamiltonian [15]

$$H^{\text{Sym}} = -N \sum_{x\mu} \left(\frac{4}{3} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} - \frac{1}{12} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+2\mu} \right), \quad (2.7)$$

for which we have

$$w(q) = \hat{q}^2 + \frac{1}{12} \sum_{\mu} \hat{q}^4_{\mu} \qquad (2.8)$$

and $\alpha_1 = 1/12$ and $\alpha_2 = 0$.

(2) The ''diagonal'' Hamiltonian [62]

$$H^{\text{diag}} = -N \sum_{x} \left(\frac{2}{3} \sum_{\mu} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu} + \frac{1}{6} \sum_{\hat{d}} \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\hat{d}} \right), \qquad (2.9)$$

where $\hat{d}$ are the two diagonal vectors $(1, \pm 1)$, for which we have

$$w(q) = \hat{q}^2 - \frac{1}{6} \hat{q}^2_1 \hat{q}^2_2 \qquad (2.10)$$

and $\alpha_1 = 1/12$ and $\alpha_2 = -1/12$.

(3) The perfect Laplacian introduced in various works on the renormalization group [63] and recently revived in connection with the perfect actions [17]

$$\frac{1}{w(q)} = \frac{1}{3\kappa} + \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \frac{1}{(q_1+2\pi l_1)^2 + (q_2+2\pi l_2)^2}$$

$$\times \frac{\hat{q}^2_1 \hat{q}^2_2}{(q_1+2\pi l_1)^2 (q_2+2\pi l_2)^2} \qquad (2.11)$$

for which $\alpha_1 = 1/12$ and $\alpha_2 = (\kappa-4)/(12\kappa)$.

In general we will speak of tree-level improved Hamiltonians[1] whenever $\alpha_1 = \frac{1}{12}$, $\alpha_2 = 0$: in this case, for $q^2 \to 0$,

$$w(q) = q^2 + O(q^6). \qquad (2.13)$$

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[1] Properly speaking we should speak of $O(a^2)$ tree-level improved Hamiltonians. One can also consider $O(a^{2k})$ tree-level improved ones which are such that

$$w(q) = q^2 + O(q^{4+2k}) \qquad (2.12)$$

for $q^2 \to 0$. We do not consider them here since tree-level improvement beyond $O(a^2)$ does not have any effect on the corrections to FSS at order $1/L^2$. For a perturbative study of this class of Hamiltonians see Ref. [44]. Classically perfect Hamiltonians are tree-level improved to all orders in a [64].

The Hamiltonians (2.7) and (2.11) for $\kappa = 4$ are examples of tree-level improved Hamiltonians.

In order to study the finite-size-scaling properties we must specify the geometry. We will consider here a square lattice of size $L \times T$ or a strip of width L with periodic boundary conditions in the finite direction(s). The large-N limit of this model is well known [24]. The theory is parametrized by a mass parameter $m^2_{L,T}$ related to β by the gap equation

$$\beta = \frac{1}{LT} \sum_{n_1,n_2} \frac{1}{w(p) + m^2_{L,T}} \equiv \mathcal{I}_{L,T}(m^2_{L,T}), \qquad (2.14)$$

where $p_1 = 2\pi n_1/L$, $p_2 = 2\pi n_2/T$ and the sum extends over $0 \leq n_1 \leq L-1$ and $0 \leq n_2 \leq T-1$. The two-point *isovector* Green's function is then given by

$$G_V(x-y;L,T) \equiv \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle = \frac{1}{\beta} \frac{1}{LT} \sum_{n_1 n_2} \frac{e^{ip\cdot(x-y)}}{w(p) + m^2_{L,T}}. \qquad (2.15)$$

All other Green's functions are obtained from $G_V(x;L,T)$ using the factorization theorem

$$\langle (\boldsymbol{\sigma}_{x_1} \cdot \boldsymbol{\sigma}_{y_1})(\boldsymbol{\sigma}_{x_2} \cdot \boldsymbol{\sigma}_{y_2}) \cdots (\boldsymbol{\sigma}_{x_n} \cdot \boldsymbol{\sigma}_{y_n}) \rangle$$

$$= \langle (\boldsymbol{\sigma}_{x_1} \cdot \boldsymbol{\sigma}_{y_1}) \rangle \langle (\boldsymbol{\sigma}_{x_2} \cdot \boldsymbol{\sigma}_{y_2}) \rangle \cdots \langle (\boldsymbol{\sigma}_{x_n} \cdot \boldsymbol{\sigma}_{y_n}) \rangle. \qquad (2.16)$$

In particular we will consider the *isotensor* (spin-two) two-point function

$$G_T(x-y;L,T) \equiv \langle (\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y)^2 \rangle - \frac{1}{N} = \langle \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y \rangle^2 + O(1/N). \qquad (2.17)$$

Beside the standard N-vector model we will also discuss a mixed N-vector–RP^{N-1}-model [25,50,54,65]. We will restrict our attention to nearest-neighbor interactions since only in this case the model is easily solvable in the large-N limit. The Hamiltonian is given by

$$H^{\text{mix}} = -N \sum_{\langle xy \rangle} \left[(1-r) \boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y + \frac{r}{2} (\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_y)^2 \right], \quad (2.18)$$

where the sum is extended over all links $\langle xy \rangle$ and r is a free parameter varying between 0 and 1. For $r = 0$ we have the nearest-neighbor N-vector model, while $r = 1$ corresponds to the RP^{N-1} model. Notice that for $r = 1$ the theory is invariant under local transformations $\boldsymbol{\sigma}_x \to \epsilon_x \boldsymbol{\sigma}_x$, $\epsilon_x = \pm 1$. Therefore for RP^{N-1} only isotensor observables are relevant. The limit we consider here corresponds to $N \to \infty$ with r fixed. We mention that this is not the only case in which the model is solvable: a different large-N limit is considered in Ref. [65].

Also in this case the theory is parametrized by a mass parameter $m_{L,T}^2$ related[2] to β by [50]

$$\beta = \frac{4I_{L,T}(m_{L,T}^2)^2}{4I_{L,T}(m_{L,T}^2) + r[m_{L,T}^2 I_{L,T}(m_{L,T}^2) - 1]}, \quad (2.19)$$

where

$$I_{L,T}(m_{L,T}^2) \equiv \frac{1}{LT} \sum_{n_1,n_2} \frac{1}{\hat{p}^2 + m_{L,T}^2}, \quad (2.20)$$

with $p_1 = 2n_1\pi/L$, $p_2 = 2n_2\pi/T$.

The isovector Green's function is given by

$$G_V(x-y;L,T) = \frac{1}{I_{L,T}(m_{L,T}^2)} \frac{1}{LT} \sum_{n_1,n_2} \frac{e^{ip\cdot(x-y)}}{\hat{p}^2 + m_{L,T}^2}. \quad (2.21)$$

All other correlations are obtained using Eq. (2.16). In the N-vector case we can use the gap equation to substitute $I_{L,T}(m_{L,T}^2)$ with β.

In this paper we will study the finite-size-scaling properties of various quantities. We define the vector and tensor susceptibilities

$$\chi_V(L,T) = \sum_x G_V(x;L,T), \quad (2.22)$$

$$\chi_T(L,T) = \sum_x G_T(x;L,T). \quad (2.23)$$

Using the explicit expressions for the two-point functions we get

$$\chi_V(L,T) = \frac{1}{\mathcal{I}_{L,T}(m_{L,T}^2) m_{L,T}^2}, \quad (2.24)$$

$$\chi_T(L,T) = \frac{1}{[\mathcal{I}_{L,T}(m_{L,T}^2)]^2} \frac{1}{LT} \sum_{n_1 n_2} \frac{1}{[w(p) + m_{L,T}^2]^2}. \quad (2.25)$$

We want also to define a quantity behaving as a correlation length. In an infinite lattice there are essentially two possibilities. One can define the exponential correlation length from the large-x behavior of a given two-point function[3] $G(x;\infty)$: one considers a wall-wall correlation function

$$G^{(w)}(y;\infty) = \sum_x G[(x,y);\infty], \quad (2.26)$$

and then defines

$$\xi^{(\exp)}(\infty) = -\lim_{y\to+\infty} \frac{y}{\log G^{(w)}(y;\infty)}. \quad (2.27)$$

The mass gap $\mu(\infty)$ is the inverse of $\xi^{(\exp)}(\infty)$. A second possibility is the second-moment correlation length $\xi^{(m)}(\infty)$ that is defined by

$$[\xi^{(m)}(\infty)]^2 = \frac{1}{4} \frac{\sum_x |x|^2 G(x;\infty)}{\sum_x G(x;\infty)}. \quad (2.28)$$

The factor $\frac{1}{4}$ has been introduced in order to have $\xi^{(m)}(\infty) = \xi^{(\exp)}(\infty)$ for Gaussian models.

We must now give the definitions in finite volume. Of course the exponential correlation can only be defined in a strip. In this case we can still use the definition (2.27). For the second-moment correlation length we can use any definition that becomes equivalent to Eq. (2.28) in the limit $L,T\to\infty$. We will consider here three different definitions: given a two-point function $G(x;L,T)$ and its Fourier transform $\hat{G}(p;L,T)$ we define

$$[\xi^{(m,1)}(L,T)]^2 = \frac{1}{2\hat{p}_{0x}^2} \left[\frac{\hat{G}(0;L,T)}{\hat{G}(p_{0x};L,T)} - 1 \right]$$
$$+ \frac{1}{2\hat{p}_{0y}^2} \left[\frac{\hat{G}(0;L,T)}{\hat{G}(p_{0y};L,T)} - 1 \right], \quad (2.29)$$

$$[\xi^{(m,2)}(L,T)]^2 = \frac{L^2}{8\pi^2} \left(1 - \frac{\hat{G}(p_{0x};L,T)}{\hat{G}(0;L,T)} \right)$$
$$+ \frac{T^2}{8\pi^2} \left(1 - \frac{\hat{G}(p_{0y};L,T)}{\hat{G}(0;L,T)} \right), \quad (2.30)$$

$$[\xi^{(m,3)}(L,T)]^2 = \frac{1}{4\hat{G}(0;L,T)} \sum_{i=1-\lfloor(L+1)/2\rfloor}^{\lfloor L/2 \rfloor} \sum_{j=1-\lfloor(T+1)/2\rfloor}^{\lfloor T/2 \rfloor}$$
$$\times (i^2+j^2) G[(i,j);L,T], \quad (2.31)$$

where $p_{0x} = (2\pi/L,0)$, $p_{0y} = (0,2\pi/T)$, and $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. The third definition evidently coincides with Eq. (2.28) for $L,T\to\infty$. To verify the correctness of the other two definitions notice that Eq. (2.28) can be rewritten as

$$[\xi^{(m)}(\infty)]^2 = -\frac{1}{4\hat{G}(0;\infty)} \sum_\mu \left. \frac{\partial^2}{\partial p_\mu^2} \hat{G}(p;\infty) \right|_{p=0}. \quad (2.32)$$

Expanding in $1/L^2$ it is easy to verify that both $\xi^{(m,1)}(L,T)$ and $\xi^{(m,2)}(L,T)$ converge to Eq. (2.28) for $L,T\to\infty$. Essentially Eqs. (2.29) and (2.30) are definitions in which one

approximates the second derivative of $\hat{G}$ with the difference at two nearby points. Thus these definitions converge to $\xi^{(m)}(\infty)$ as $1/L^2$ [notice that $G(x;L,T) \to G(x,\infty)$ exponentially]. The third definition represents instead the "best" approximation on a finite lattice since $\xi^{(m,3)}(L,T)$ converges to $\xi^{(m)}(\infty)$ exponentially. This is indeed a general result that can be proved using the relation

$$\sum_{i=1-[(L+1)/2]}^{\lfloor L/2 \rfloor} i^2 f(i) = 2 \sum_{n=1}^{L-1} \frac{(-1)^n}{\hat{q}^2} [\hat{f}(q) - \hat{f}(0)] \mathcal{P}_L(q), \tag{2.33}$$

valid for every function f. Here $\hat{f}$ is the Fourier transform of f, $q = 2\pi n/L$ and

$$\mathcal{P}_L\left(\frac{2\pi n}{L}\right) = \begin{cases} 1 & \text{for } L \text{ even,} \\ \cos\dfrac{\pi n}{L} & \text{for } L \text{ odd.} \end{cases} \tag{2.34}$$

If $\hat{f}(q)$ is meromorphic (as a function of the *complex* variable q) in the strip $0 \leq \operatorname{Re} q \leq 2\pi$, periodic of period 2π, and with simple poles at $\bar{q}_i$, then we can evaluate this sum to obtain

$$-\hat{f}''(0) - \frac{L}{4} \sum_i \frac{R_i}{\sin^2(\bar{q}_i/2)\sin(L\bar{q}_i/2)} \mathcal{P}_L(\bar{q}_i), \tag{2.35}$$

where R_i is the residue of $\hat{f}(q)$ at $\bar{q}_i$. Thus the convergence rate is $L \exp(-L\bar{q}_I/2)$ where $\bar{q}_I = \min_i |\operatorname{Im}(\bar{q}_i)|$. For the specific case of the isovector correlation length one expects the nearest singularities (for $\beta \to \infty$ at least) to be at $q = \pm i\mu(L)$ where $\mu(L)$ is the mass gap. Thus we expect a convergence rate of $Le^{-\mu(L)L/2}$. A general Green function will not be in general a meromorphic function of q as cuts will appear as well. We expect however that the definition (2.31) will show the same exponential convergence rate.

Using Eq. (2.33) we can rewrite Eq. (2.31) as

$$[\xi^{(m,3)}(L,T)]^2$$

$$= \frac{1}{2\hat{G}(0;L,T)} \sum_{(n_1,n_2)\neq(0,0)} \frac{1}{\hat{q}^2} [\hat{G}(q;L,T) - \hat{G}(0;L,T)]$$

$$\times [(-1)^{n_1}\mathcal{P}_L(q_1)\delta_{q_2 0} + (-1)^{n_2}\mathcal{P}_T(q_2)\delta_{q_1 0}]. \tag{2.36}$$

Let us now give explicit expressions for the isovector correlation length: using the isovector two-point function (2.15) we get on a finite lattice:

$$[\xi_V^{(m,1)}(L,T)]^2$$

$$= \frac{1}{2}\left(\frac{w(p_{0x})}{\hat{p}_{0x}^2} + \frac{w(p_{0y})}{\hat{p}_{0y}^2}\right)\frac{1}{m_{L,T}^2}, \tag{2.37}$$

$$[\xi_V^{(m,2)}(L,T)]^2$$

$$= \frac{1}{8\pi^2}\left(\frac{w(p_{0x})L^2}{w(p_{0x})+m^2} + \frac{w(p_{0y})T^2}{w(p_{0y})+m^2}\right), \tag{2.38}$$

$$[\xi_V^{(m,3)}(L,T)]^2$$

$$= -\frac{1}{2}\sum_{n_1=1}^{L-1}(-1)^{n_1}\frac{w(q_1)}{\hat{q}_1^2}\frac{1}{w(q_1)+m_{L,T}^2}\mathcal{P}_L(q_1)$$

$$-\frac{1}{2}\sum_{n_2=1}^{T-1}(-1)^{n_2}\frac{w(q_2)}{\hat{q}_2^2}\frac{1}{w(q_2)+m_{L,T}^2}\mathcal{P}_T(q_2). \tag{2.39}$$

In infinite volume we have instead

$$\xi_V^{(m)}(\infty) = \frac{1}{m_\infty}. \tag{2.40}$$

For the mass gap $\mu_V(L)$ and the exponential correlation length $\xi_V^{(\text{exp})}(L)$ we must solve the equation

$$w[i\mu_V(L),0] + m_{L,\infty}^2 = 0. \tag{2.41}$$

An explicit solution can be obtained only for the simplest $w(p)$. For the Hamiltonians we have considered in this section we have

$$\mu_V(L) = \begin{cases} 2\operatorname{arcsinh}\left(\dfrac{m_{L,\infty}}{2}\right) & \text{for } H^{\text{std}} \text{ and } H^{\text{diag}}, \\[2ex] 2\operatorname{arcsinh}\left\{\dfrac{\sqrt{6}}{2}\left[1-\left(1-\dfrac{m_{L,\infty}^2}{3}\right)^{1/2}\right]^{1/2}\right\} & \text{for } H^{\text{Sym}}. \end{cases} \tag{2.42}$$

In our calculation we will only need the expression of $\mu_V(L)$ for $m_{L,\infty} \to 0$. In this limit we obtain

$$\mu_V(L) = m_{L,\infty}\left[1 + \frac{1}{2}\left(\alpha_1 + \alpha_2 - \frac{1}{12}\right)m_{L,\infty}^2 + O(m_{L,\infty}^4)\right]. \tag{2.43}$$

Isotensor observables are defined using the tensor two-point function (2.17). For the mass gap it is easy to verify that $\mu_T(L) = 2\mu_V(L)$.

III. N-VECTOR MODEL

A. The gap equation

In this section we want to discuss the corrections to FSS for the Hamiltonian (2.1). Let us consider a fixed value of β. Let m_∞ and $m_{L,T}$ be the mass parameters corresponding to β in infinite volume and in a box $L \times T$. It is immediate to obtain a relation between m_∞ and $m_{L,T}$. Indeed from the gap equation we obtain

$$\mathcal{I}_{L,T}(m_{L,T}^2) = \mathcal{I}_\infty(m_\infty^2). \tag{3.1}$$

Now let us consider the finite-size-scaling limit $m_\infty, m_{L,T} \to 0, L, T \to \infty$ with $m_{L,T}L \equiv z$ and $T/L \equiv \rho$ fixed. Using the results (B88) and (C6) we obtain

$$\frac{m_\infty^2}{m_{L,T}^2} = f_m(z;\rho)\left(1 + \frac{\Delta_{m,1}(z;\rho)}{L^2}\log L + \frac{\Delta_{m,2}(z;\rho)}{L^2}\right) \tag{3.2}$$

with corrections of order $O(\log^2 L/L^4)$, where

$$f_m(z;\rho) = \frac{32}{z^2}e^{-4\pi F_0(z;\rho)}, \tag{3.3}$$

$$\Delta_{m,1}(z;\rho) = \frac{1}{4}(12\alpha_1 + 16\alpha_2 - 1)(32e^{-4\pi F_0(z;\rho)} - z^2), \tag{3.4}$$

$$\Delta_{m,2}(z;\rho) = 16\pi(12\alpha_1 + 16\alpha_2 - 1)F_0(z;\rho)e^{-4\pi F_0(z;\rho)}$$
$$- 4(8\alpha_1 + 8\alpha_2 - 1)e^{-4\pi F_0(z;\rho)} - 4\pi[\mathcal{F}_1(z;\rho)$$
$$+ 32e^{-4\pi F_0(z;\rho)}\Lambda_1]. \tag{3.5}$$

The functions $F_0(z;\rho)$ and $\mathcal{F}_1(z;\rho)$ are defined in Appendix B, Eqs. (B54) and (B90). The function $f_m(z;\rho)$ is the FSS function for the ratio $m_\infty^2/m_{L,T}^2$. As expected, it is universal [it does not depend on the explicit form of the coupling $J(x)$] and depends on the modular parameter ρ. The corrections instead are *not* universal. However, the dependence on $J(x)$ is very simple: the only relevant quantities are α_1 and α_2 that are connected to the small-q^2 behavior of $w(q)$ and Λ_1 given by

$$\Lambda_1 = \int \frac{d^2p}{(2\pi)^2}\left[\frac{1}{w(p)^2} - \frac{1}{(\hat{p}^2)^2} + \frac{2}{(\hat{p}^2)^3}\right.$$
$$\left. \times\left(\alpha_1\sum_\mu \hat{p}_\mu^4 + \alpha_2(\hat{p}^2)^2\right)\right]. \tag{3.6}$$

The corrections behave in general as $\log L/L^2$, except when $12\alpha_1 + 16\alpha_2 - 1 = 0$. This cancellation happens for improved Hamiltonians for which $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$ and also for many other Hamiltonians that are not improved but nonetheless satisfy $12\alpha_1 + 16\alpha_2 - 1 = 0$. To understand the relevance of this combination, let us introduce polar coordinates $q_x = q\cos\theta$, $q_y = q\sin\theta$. Then

$$w(q) = q^2 + \frac{1}{16}(12\alpha_1 + 16\alpha_2 - 1)q^4$$
$$+ \frac{1}{48}(12\alpha_1 - 1)q^4\cos 4\theta + O(q^6). \tag{3.7}$$

Thus, if $(12\alpha_1 + 16\alpha_2 - 1) = 0$, one cancels the first rotationally invariant subleading operator, leaving a correction that is associated to a lattice operator that vanishes when averaged over the angle θ. This last property is the reason why this quantity does not couple to the leading correction. This fact is not unexpected. Indeed the leading correction to scaling is usually associated to a rotationally invariant operator (for a discussion for the two-point function in infinite volume see Ref. [66]).

For $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$ the expression for $\Delta_{m,2}(z;\rho)$ simplifies drastically, becoming

$$\Delta_{m,2}(z;\rho) = 4\pi(32e^{-4\pi F_0(z;\rho)} - z^2)\left(\frac{1}{96\pi} - \Lambda_1\right). \tag{3.8}$$

Thus for improved Hamiltonians there is the possibility of eliminating even the $1/L^2$ corrections choosing $J(x)$ so that

$$\Lambda_1 = \frac{1}{96\pi}. \tag{3.9}$$

Notice that this condition is global, that is it does not only fix the small-q behavior of $w(q)$, but it depends on the behavior of $w(q)$ over all the Brillouin zone.

A particular Hamiltonian satisfying Eq. (3.9) is

$$H^{\text{Sym2}} = -N\sum_{x\mu}\left[\left(\frac{4}{3} + 15a\right)\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+\mu}\right.$$
$$\left. - \left(\frac{1}{12} + 6a\right)\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+2\mu} + a\boldsymbol{\sigma}_x \cdot \boldsymbol{\sigma}_{x+3\mu}\right], \tag{3.10}$$

where $a = 0.00836553968(1)$. The function $w(q)$ is given by

$$w(q) = \hat{q}^2 + \frac{1}{12}\sum_\mu \hat{q}_\mu^4 + a\sum_\mu \hat{q}_\mu^6. \tag{3.11}$$

TABLE I. Values of $R_{\text{exact}}(L;z,\rho)$, $R_{\text{expan}}(L;z,\rho)$, $D_{\text{exact}}(L;z,\rho)$, $D_{\text{expan}}(L;z,\rho)$, and $\delta_2(L;z,\rho)$ for the standard Hamiltonian H^{std}, $\rho=1$ and $z=2$.

L	$R_{\text{exact}}(L;2,1)$	$R_{\text{expan}}(L;2,1)$	$D_{\text{exact}}(L;2,1)$	$D_{\text{expan}}(L;2,1)$	$\delta_2(L;2,1)$
4	0.23892847	0.24124682	0.0649479	0.0752812	-0.428
6	0.23332246	0.23379910	0.0399609	0.0420854	-0.345
8	0.23028713	0.23044079	0.0264318	0.0271167	-0.303
10	0.22856980	0.22863479	0.0187774	0.0190671	-0.282
12	0.22751364	0.22754600	0.0140699	0.0142141	-0.270
14	0.22681775	0.22683573	0.0109682	0.0110483	-0.262
16	0.22633412	0.22634493	0.0088126	0.0088607	-0.255
20	0.22572118	0.22572580	0.0060806	0.0061012	-0.247
32	0.22497047	0.22497124	0.0027345	0.0027380	-0.232
64	0.22453987	0.22453993	0.0008153	0.0008155	-0.218
128	0.22441005	0.22441006	0.0002367	0.0002367	-0.208
∞		0.22435696			-0.148

For this Hamiltonian the corrections to FSS behave as $\log L/L^4$. Of course one could improve further: using a Hamiltonian with $w(q)=q^2+O(q^8)$ satisfying Eq. (3.9) one could get rid also of the terms $\log L/L^4$. However, the cancellation of the terms $1/L^4$ will again require a global condition of the type (3.9).

It is interesting to understand our results in terms of perturbation theory. Within the Symanzik improvement program the conditions $\alpha_1=\frac{1}{12}$ and $\alpha_2=0$ are required for tree-level improvement: if the theory is tree-level improved, then the corrections behave as $1/L^2$ instead of $\log L/L^2$. In Ref. [45] it was shown that the condition (3.9) is necessary for improvement at one loop. The simplifying feature of the model is that, once the theory is one-loop improved, it is improved to all orders of perturbation theory. This explains why, if condition (3.9) is satisfied, corrections to scaling behave as $\log L/L^4$. As we shall discuss in the following section, for a generic model, for instance for a mixed $O(N)$-RP^{N-1} theory, we expect only the $1/L^2$ term to be cancelled so that the corrections to scaling would still behave as $1/(L^2\log L)$.

We have performed various checks of the expressions (3.3)–(3.5). First of all we have compared our results with previous work. For the strip $f_m(z;\infty)$ was computed by Lüscher [41]. In this case, as $\lim_{\rho\to\infty} M_{1,1}(z;\rho)=0$, using the explicit expression for $F_0(z;\rho)$, Eq. (B54), and Eq. (A12), we get

$$f_m(z;\infty)=\frac{32}{z^2}\exp\left[-\frac{2\pi}{z}-2\gamma_E+\log\frac{\pi^2}{2}-2G_0\left(\frac{z}{2\pi}\right)\right]$$

$$=\exp\left[-4\sum_{n=1}^{\infty}K_0(nz)\right], \tag{3.12}$$

which agrees with the result of Ref. [41].

We have furthermore performed a detailed numerical check for the standard Hamiltonian H^{std}. Given L, z, and ρ we have first computed $m_{L,T}^2=z^2/L^2$, then β from the finite-volume gap equation $\beta=\mathcal{I}_{L,T}(m_{L,T}^2)$ and finally m_∞^2 from β

$=\mathcal{I}_\infty(m_\infty^2)$: in this way we have obtained for each lattice size L and z the ratio $R_{\text{exact}}(L;z,\rho)\equiv m_\infty^2/m_{L,T}^2$. Then we computed

$$\delta_1(L;z,\rho)=\frac{L^4}{\log^2 L}[R_{\text{exact}}(L;z,\rho)-R_{\text{expan}}(L;z,\rho)], \tag{3.13}$$

where $R_{\text{expan}}(L;z,\rho)$ is the right-hand side of Eq. (3.2). In this way we have tried to verify that indeed $\delta_1(L;z,\rho)$ at fixed z goes to a constant for $L\to\infty$. Numerically we find that $\delta_1(L;z,\rho)$ varies slowly with L and that the behavior is compatible with the presence of $1/\log L$ and $1/\log^2 L$ corrections. A better check can be obtained if we include the term of order $\log^2 L/L^4$ that can be easily computed

$$\Delta_{m,3}(z;\rho)=(12\alpha_1+16\alpha_2-1)^2$$

$$\times\left(96e^{-8\pi F_0(z;\rho)}-4z^2e^{-4\pi F_0(z;\rho)}+\frac{z^4}{32}\right). \tag{3.14}$$

Then we consider

$$\delta_2(L;z,\rho)=\frac{L^4}{\log L}\left[R_{\text{exact}}(L;z,\rho)-R_{\text{expan}}(L;z,\rho)\right.$$

$$\left.-f_m(z;\rho)\Delta_{m,3}(z;\rho)\frac{\log^2 L}{L^4}\right]. \tag{3.15}$$

In this case we should be able to verify that

$$\delta_2(L;z,\rho)\approx\delta_{20}(z;\rho)+\frac{\delta_{21}(z;\rho)}{\log L} \tag{3.16}$$

for large values of L. The results for $\rho=1$ and $z=2$ (this value of z corresponds to the region where the corrections to FSS are larger) are shown in Table I where we also give the deviations from FSS, i.e., the quantity

$$D(L;z,\rho) = \left[\frac{R(L;z,\rho)}{f_m(z;\rho)} - 1 \right]. \qquad (3.17)$$

A plot of $\delta_2(L;2,1)$ versus $1/\log L$ shows, as expected, a linear behavior from which we can estimate $\delta_{20}(2;1) \approx -0.148$ and $\delta_{21}(2;1) \approx -0.291$.

Let us now consider the limits $z \to \infty$ and $z \to 0$. Asymptotic expansions of the FSS functions can be obtained using the expansions of $F_0(z;\rho)$ and $\mathcal{F}_1(z;\rho)$ reported in Appendixes B 2 a and B 2 b. For large z we have

$$f_m(z;\rho) = 1 - 2e^{-z}\sqrt{\frac{2\pi}{z}} - 2e^{-\rho z}\sqrt{\frac{2\pi}{\rho z}}$$
$$+ O(z^{-3/2}e^{-z}, z^{-3/2}e^{-\rho z}), \qquad (3.18)$$

$$\Delta_{m,1}(z;\rho) = -\frac{1}{2}(12\alpha_1 + 16\alpha_2 - 1)z^2$$
$$\times \left(e^{-z}\sqrt{\frac{2\pi}{z}} + e^{-\rho z}\sqrt{\frac{2\pi}{\rho z}} \right)$$
$$+ O(z^{1/2}e^{-z}, z^{1/2}e^{-\rho z}), \qquad (3.19)$$

$$\Delta_{m,2}(z;\rho) = \frac{\pi}{6}(12\alpha_1 + 12\alpha_2 - 1)z^2$$
$$\times \left(e^{-z}\sqrt{\frac{z}{2\pi}} + e^{-\rho z}\sqrt{\frac{\rho z}{2\pi}} \right)$$
$$+ \frac{z^2}{4}(12\alpha_1 + 16\alpha_2 - 1)$$
$$\times \log \frac{z^2}{32} \left(e^{-z}\sqrt{\frac{2\pi}{z}} + e^{-\rho z}\sqrt{\frac{2\pi}{\rho z}} \right)$$
$$+ O(z^{3/2}e^{-z}, z^{3/2}e^{-\rho z}). \qquad (3.20)$$

The result agrees with what is expected: for $z \to \infty$ the FSS function converges to 1 exponentially [67]. Also the corrections vanish in the same way and thus they are extremely tiny for large z.

Let us now consider the perturbative limit (small z). For finite values of ρ, for $z \ll 1$ and $z \ll 1/\rho$, we find

$$f_m(z;\rho) = \frac{16\pi^2}{z^2} \eta(i\rho)^4 e^{-2\gamma_E}\exp\left(-\frac{4\pi}{\rho z^2} \right)$$
$$\times [1 - 4\pi z^2 F_{01}(\rho) + O(z^4)], \qquad (3.21)$$

$$\Delta_{m,1}(z;\rho) = -\frac{z^2}{4}(12\alpha_1 + 16\alpha_2 - 1) + O[e^{-4\pi/(\rho z^2)}], \qquad (3.22)$$

$$\Delta_{m,2}(z;\rho) = -4\pi(1 - 12\alpha_1)F_{10}(\rho) - \frac{4\pi\alpha_2}{\rho}$$
$$-4\pi z^2 \Big[(1 - 12\alpha_1)F_{11}(\rho) + 2\alpha_2 F_{00}(\rho)$$
$$+ \frac{\alpha_1}{8\pi} - \Lambda_1 \Big] + O(z^4). \qquad (3.23)$$

Here $\eta(\tau)$ is Dedekind's η function [68]

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \qquad (3.24)$$

and $F_{01}(\rho)$, $F_{10}(\rho)$, and $F_{11}(\rho)$ are defined in Appendix B: see Eqs. (B62)–(B64).

For the strip the previous expansions are not valid. In this case, for small z, we get

$$f_m(z;\infty) = \frac{16\pi^2}{z^2} e^{-2\gamma_E}e^{-2\pi/z}\left(1 + \frac{z^2}{4\pi^2}\zeta(3) + O(z^4) \right), \qquad (3.25)$$

$$\Delta_{m,1}(z;\infty) = -\frac{z^2}{4}(12\alpha_1 + 16\alpha_2 - 1) + O[\exp(-2\pi/z)], \qquad (3.26)$$

$$\Delta_{m,2}(z;\infty) = \frac{\pi^2}{18}(12\alpha_1 - 1) - \frac{\pi z}{4}(12\alpha_1 + 12\alpha_2 - 1)$$
$$+ O(z^2). \qquad (3.27)$$

These formulas can also be used when $\rho \gg 1$. Indeed they approximate the FSS functions for $1/\rho \ll z \ll 1$.

It is interesting to obtain these expansions within perturbation theory (PT). The idea [41,45] is to start from the perturbative prediction for $m_\infty(\beta)$,

$$m_\infty^2 = 32e^{4\pi\Lambda_0}e^{-4\pi\beta}[1 + O(\beta e^{-4\pi\beta})], \qquad (3.28)$$

and the perturbative expansion of $m_{L,T}(\beta)$,

$$z \equiv Lm_{L,T} = \sum_{n=1}^{\infty} \frac{m_n(L,T)}{\beta^n}. \qquad (3.29)$$

Then we use the last equation to express perturbatively β in terms of z and finally we substitute the result in Eq. (3.28). In this way we obtain the expansions (3.21)–(3.23) and the analogous ones on the strip. It must be noted that in this perturbative expansion the combination $(12\alpha_1 + 16\alpha_2 - 1)$ arises naturally: indeed it is the coefficient of the unique $\log L/L^2$ term that appears in the expansion. Thus $12\alpha_1 + 16\alpha_2 - 1 = 0$ is the improvement condition of the renormalized perturbative expansion.

To conclude the discussion we want to comment on the validity of PT: finite-volume PT is valid in the limit $\beta \to \infty$ at L fixed while the FSS limit we are interested in corresponds to $\beta \to \infty$, $L \to \infty$ at z fixed. Thus our perturbative derivation of the FSS scaling functions involves an *a priori* unjustified
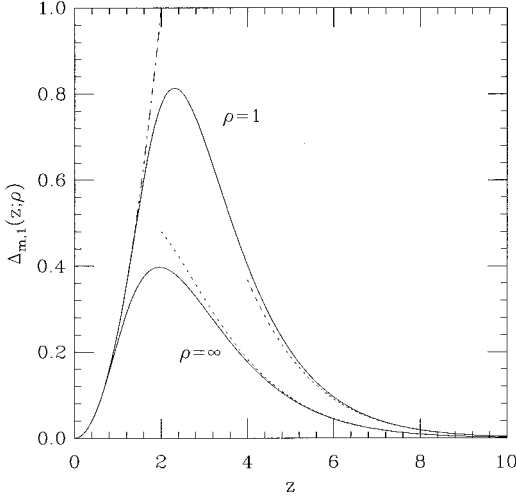
FIG. 1. $\Delta_{m,1}(z;\rho)$ for the standard Hamiltonian H^{std} for $\rho=1$ and $\rho=\infty$. The dashed lines correspond to the asymptotic expansions (3.19) and (3.22). $\Delta_{m,1}(z;\rho)$ for H^{diag} is obtained by multiplying the vertical scale by 4/3.
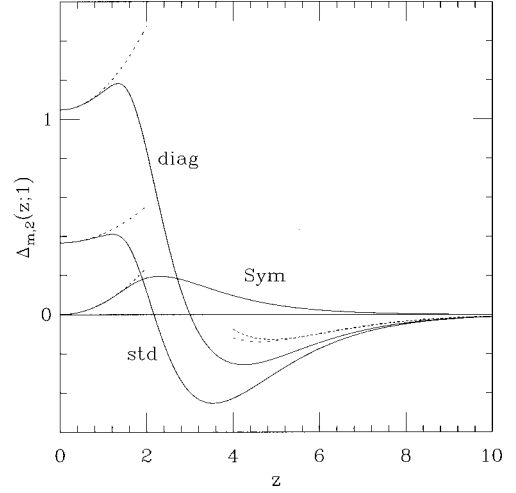

FIG. 2. $\Delta_{m,2}(z;1)$ for H^{std} (std), H^{diag} (diag), and H^{Sym} (Sym). The dashed lines are the asymptotic expansions (3.20) and (3.23).

extension of the validity of PT [51,52]. For the leading contribution this should be correct (naively because the result is L independent) but the situation is less clear for the corrections: in this case the explicit calculation shows that the extension is valid also for the $1/L^2$ corrections, but, as we shall see in the next section, this is a special feature of the large-N N-vector model: in general the corrections to FSS computed in PT need a ''resummation'' to correctly describe the FSS regime.

The functions $\Delta_{m,1}(z;\rho)$ and $\Delta_{m,2}(z;\rho)$ are reported in the Figs. 1, 2, and 3 for the torus with $\rho=1$ and for the strip $\rho=\infty$ for the various Hamiltonians we have introduced. From these plots one can immediately recognize a few basic facts: the region where the corrections to FSS are larger corresponds to $1\lesssim z\lesssim 4$ (the same has been found numerically in Monte Carlo simulations of H^{std} with $N=3$ [10]). In this region, for H^{std} and H^{diag} and small values of L, say $L=10$, the $\log L/L^2$ term gives a contribution which is 2–4 times larger than the $1/L^2$ term and the corrections are positive. For these two Hamiltonians the corrections become negative for large values of z [this can be easily checked from the asymptotic expansions (3.19) and (3.20)]. They are also negative for H^{std} in the small-z region on the strip or on the torus for large values of ρ. Numerically we find that H^{diag} is the Hamiltonian with the largest corrections, while H^{Sym} is the ''best'' one, as expected.

We have finally checked that our expansion (3.2) describes well the corrections to FSS even for small values of L. In Table II we give $D_{\text{exact}}(L;z,\rho)$ and $D_{\text{expan}}(L;z,\rho)$ for H^{diag} and H^{Sym} for $\rho=1$ and $z=2$. For the first Hamiltonian there is good agreement even at $L=4$, while for the latter there is a somewhat larger discrepancy, probably due to the larger spatial extent of the Symanzik Hamiltonian. We have also computed $D_{\text{exact}}(L;z,\rho)$ for the same values of ρ and z for H^{Sym2}: for $L=4$ (10) we get $D_{\text{exact}}(L;2,1)=0.0005743$

(0.00001853). The corrections are extremely small (at $L=4$ they are 100 times smaller than those present for H^{std}): improvement really works.

B. Observables

Let us now compute the FSS curves and the corresponding corrections for the observables we have introduced in Sec. II. We will first consider the quantities that are obtained from the isovector correlation function, then we will discuss isotensor observables.

1. Isovector sector

From Eq. (3.2) it is immediate to obtain the finite-size-scaling curves and their leading corrections for the various observables. The susceptibility χ_V does not require any additional calculation since

$$\frac{\chi_V(L,T)}{\chi_V(\infty)} = \frac{m_\infty^2}{m_{L,T}^2}. \tag{3.30}$$

For the second-moment correlation lengths, neglecting terms of order $\log^2 L/L^4$, we obtain

$$\left(\frac{\xi_V^{(m,1)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2 = \frac{m_\infty^2}{m_{L,T}^2}\left(1+\frac{2\pi^2}{L^2}\left(\alpha_1+\alpha_2\right)\frac{1+\rho^2}{\rho^2}\right), \tag{3.31}$$

$$\left(\frac{\xi_V^{(m,2)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2 = \frac{z^2}{2}A_1(z;\rho)\frac{m_\infty^2}{m_{L,T}^2}\left(1+\frac{\pi^2}{3L^2}\right.$$
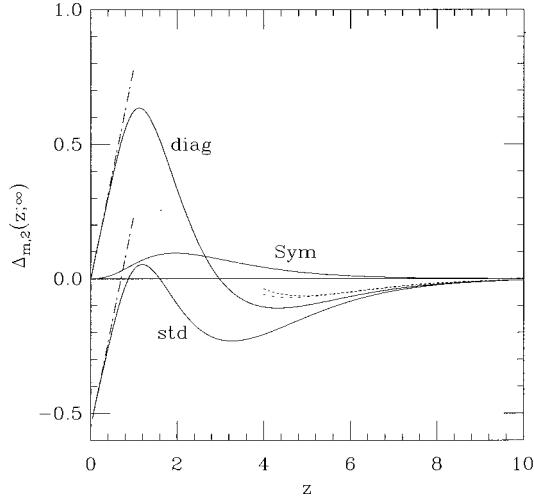$$\left.\times(12\alpha_1+12\alpha_2-1)\frac{A_2(z;\rho)}{A_1(z;\rho)}\right), \tag{3.32}$$

FIG. 3. $\Delta_{m,2}(z;\infty)$ for H^{std} (std), H^{diag} (diag), and H^{Sym} (Sym). The dashed lines are the asymptotic expansions (3.20) and (3.27).

TABLE II. Deviations from FSS for H^{diag} and H^{sym}: here $\rho = 1$ and $z = 2$.

	H^{diag}		H^{Sym}	
L	$D_{\text{exact}}(L;2,1)$	$D_{\text{expan}}(L;2,1)$	$D_{\text{exact}}(L;2,1)$	$D_{\text{expan}}(L;2,1)$
4	0.1363230	0.1435180	0.0039380	0.0116907
6	0.0736526	0.0752402	0.0035162	0.0051959
8	0.0463553	0.0468952	0.0023457	0.0029227
10	0.0320544	0.0322879	0.0016198	0.0018705
12	0.0235981	0.0237157	0.0011666	0.0012990
14	0.0181629	0.0182287	0.0008802	0.0009543
16	0.0144512	0.0144909	0.0006858	0.0007307
20	0.0098297	0.0098468	0.0004483	0.0004676

$$\left(\frac{\xi_V^{(m,3)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2 = \frac{m_\infty^2}{m_{L,T}^2}\left[1-\frac{z}{4}\left(\frac{1}{\sinh z/2}+\frac{\rho}{\sinh \rho z/2}\right)\right.$$
$$\left.+\frac{1}{L^2}\left(Q_{p(L)}(z)+\frac{1}{\rho^2}Q_{p(T)}(\rho z)\right)\right],$$
$$(3.33)$$

where $p(M)$ is the parity of M ($M=L,T$) and

$$A_1(z;\rho)=\frac{4\pi^2(1+\rho^2)+2\rho^2 z^2}{(4\pi^2+z^2)(4\pi^2+\rho^2 z^2)},\qquad(3.34)$$

$$A_2(z;\rho)=\frac{z^2}{(4\pi^2+z^2)^2}+\frac{\rho^2 z^2}{(4\pi^2+\rho^2 z^2)^2},\qquad(3.35)$$

$$Q_{\text{even}}(z)=(12\alpha_1+12\alpha_2-1)\frac{z^4}{192}\frac{\cosh z/2}{\sinh^2 z/2}$$
$$-(4\alpha_1+4\alpha_2-1)\frac{z^3}{32}\frac{1}{\sinh z/2},\qquad(3.36)$$

$$Q_{\text{odd}}(z)=(12\alpha_1+12\alpha_2-1)\frac{z^4}{192}\frac{\cosh z/2}{\sinh^2 z/2}$$
$$-(\alpha_1+\alpha_2)\frac{z^3}{8}\frac{1}{\sinh z/2}.\qquad(3.37)$$

Let us now consider the asymptotic limit $z\to\infty$. In the FSS limit it is easy to obtain

$$\left(\frac{\xi_V^{(m,1)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2_{\text{FSS}}=1-2e^{-z}\sqrt{\frac{2\pi}{z}}-2e^{-\rho z}\sqrt{\frac{2\pi}{\rho z}}$$
$$+O(z^{-3/2}e^{-z},z^{-3/2}e^{-\rho z}),\qquad(3.38)$$

$$\left(\frac{\xi_V^{(m,2)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2_{\text{FSS}}=1-\frac{2\pi^2(1+\rho^2)}{\rho^2 z^2}+O(z^{-4}),$$
$$(3.39)$$

$$\left(\frac{\xi_V^{(m,3)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2_{\text{FSS}}=1-\frac{z}{2}e^{-z/2}-\frac{\rho z}{2}e^{-\rho z/2}$$
$$+O(z^{-1/2}e^{-z},z^{-1/2}e^{-\rho z}).\qquad(3.40)$$

From these expansions one immediately sees that the FSS function for $\xi_V^{(m,2)}$ goes to 1 only as a power as $z\to\infty$. The approach is very slow and indeed it reaches 1 at the 1% level only for $z\approx 60$. This is extremely inconvenient for Monte Carlo applications: indeed in order to determine numerically the FSS curve one has to perform runs up to the value of z where the FSS curves becomes 1 within error bars: in this case runs with $z\approx 60$ are required, which means that simulations on very large lattices are needed. The origin of these power corrections can be identified in the definition that approximates the infinite volume $\xi_V^{(m)}(\infty)$ with corrections of order $1/L^2$: the $1/L^2$ terms give rise to the corrections of order $1/z^2$. The first definition should suffer from the same problem because also in this case $\xi_V^{(m,1)}(L,T)$ converges to $\xi_V^{(m)}(\infty)$ with corrections of order $1/L^2$. Instead Eq. (3.38) shows corrections of order $O(e^{-z}/\sqrt{z})$. This is a peculiarity due to the particular form of $G_V(x)$ [$G_V(x)$ is a free-field two-point function]. However, for different Green's functions terms of order $1/z^2$ are expected and indeed they are present for $\xi_T^{(m,1)}$, cf. Eq. (3.68). As expected the FSS function for $\xi_V^{(m,3)}$ converges to 1 with corrections of order $ze^{-z/2}$: in this case the FSS function is 1 at the 1% level already at $z\approx 15$.

The large-z behavior of the FSS functions can be easily computed not only in the large-N limit, but for all values of N. The basic observation is that $G_V(x;L,T)$ converges to $G_V(x;\infty)$ with corrections of order $L^p e^{-\mu_V(\infty)L}$. Therefore, in order to compute the large-z expansion, one can simply replace $G_V(x;L,T)$ with $G_V(x;\infty)$. The function $G_V(x;\infty)$ is well known in the critical limit. Indeed, if $\hat{G}_V(p;\infty)$ is the corresponding Fourier transform, then, in the limit $p\to 0$, $\xi_V^{(m)}(\infty)\to\infty$ with $p\xi_V^{(m)}(\infty)\equiv Q$ fixed, we have [69,66,70]

FIG. 4. $\Delta_{\xi_V,2}(z;1)$ for H^{std} for the three different definitions of second-moment correlation length.
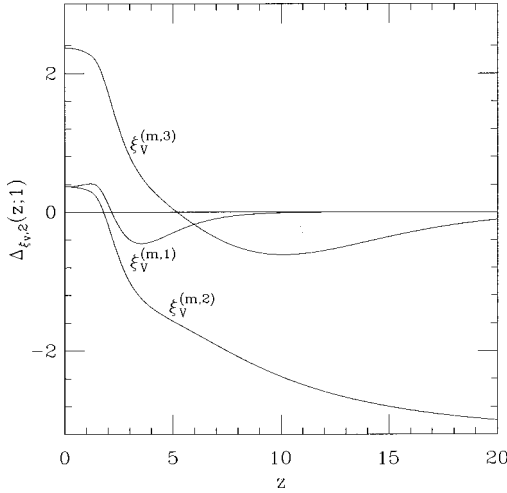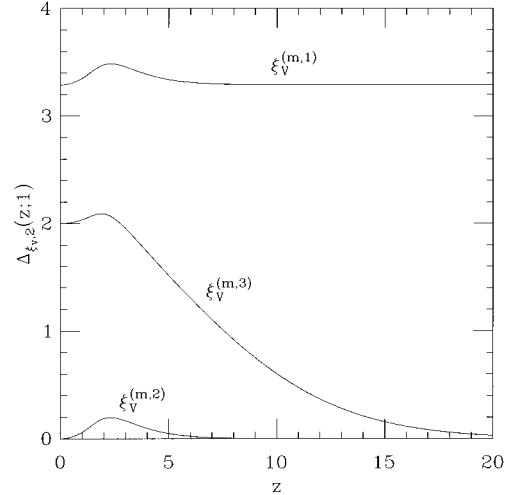

FIG. 5. $\Delta_{\xi_V,2}(z;1)$ for H^{Sym} for the three different definitions of second-moment correlation length.

$$\frac{\hat{G}_V(0;\infty)}{\hat{G}_V(p;\infty)}=D(Q). \tag{3.41}$$

The function $D(Q)$ can be expanded in the limit $Q\to 0$ in powers of Q^2:

$$D(Q)=\sum_{n=0}^{\infty} b_n Q^{2n}, \tag{3.42}$$

with $b_0=b_1=1$. This expansion converges up to the three-particle cut, i.e., for $|Q|<3s_m$ where s_m is defined by

$$s_m = \lim_{\beta\to\infty} \mu_V(\infty)\xi_V^{(m)}(\infty); \tag{3.43}$$

s_m is the ratio between the second-moment and the exponential correlation length. Moreover $D(Q)$ has a zero in correspondence to the one-particle poles, $Q=\pm is_m$. In the neighborhood of these points, we have

$$D(Q)=s_z\left(\frac{Q^2}{s_m^2}+1\right). \tag{3.44}$$

Using these results it is straightforward to compute the FSS scaling curves in terms of $y=L/\xi_V^{(m)}(\infty)$ in the limit $y\to\infty$. Disregarding terms of order $y^p e^{-y}$, $y^p e^{-\rho y}$ we obtain

$$\left(\frac{\xi_V^{(m,1)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2\bigg|_{\text{FSS}}=1+\frac{1}{2}\sum_{n=1}^{\infty} b_{n+1}\left[\left(\frac{2\pi}{y}\right)^{2n}+\left(\frac{2\pi}{\rho y}\right)^{2n}\right], \tag{3.45}$$

$$\left(\frac{\xi_V^{(m,2)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2\bigg|_{\text{FSS}}=\frac{1}{2}\left[\frac{\sum_{n=0}^{\infty}b_{n+1}(2\pi/y)^{2n}}{\sum_{n=0}^{\infty}b_n(2\pi/y)^{2n}}+(y\to\rho y)\right], \tag{3.46}$$

$$\left(\frac{\xi_V^{(m,3)}(L,T)}{\xi_V^{(m)}(\infty)}\right)^2\bigg|_{\text{FSS}}=1-\frac{y}{2s_z s_m}\left[e^{-s_m y/2}+\rho e^{-s_m \rho y/2}\right]. \tag{3.47}$$

In the large-N limit $b_n=0$ for $n\geq 2$, $s_m=s_z=1$, and $y=z$ in the FSS limit, so that one recovers our previous results, Eqs. (3.38)–(3.40). For generic values of N numerical estimates of the various constants are reported in Ref. [70]. The deviations from the large-N values are extremely small: for $N=3$ one finds from a strong-coupling analysis [70] $b_2=-1.2(2)\times 10^{-3}$, $s_m=0.9994(1)$, $s_z=1.0013(2)$, while a precise Monte Carlo simulation gives [71] $s_m=0.9992(6)$. Using Eqs. (3.45)–(3.47), it is evident that the first definition is always the most convenient one except for extremely large values of y ($y\gtrsim 20$ for $N=3$), where the deviations are extremely tiny. This is in agreement with the observation of Ref. [36]: they found numerically that, for $7\lesssim y\lesssim 10$, $\xi_V^{(m,3)}$ had finite-size corrections larger than $\xi_V^{(m,1)}$. Using their data we can check the large-y behavior of the FSS function of $\xi_V^{(m,3)}$. We find that the data of Ref. [36]—they belong to the range $7\lesssim y\lesssim 10$—are well described by the formula

$$\left(\frac{\xi_V^{(m,3)}(L,T)}{\xi_V^{(m,1)}(L,T)}\right)^2=1-\alpha\,\frac{L}{\xi_V^{(m,1)}(L,T)}\,\exp\left[-\frac{L}{2\xi_V^{(m,1)}(L,T)}\right], \tag{3.48}$$

where

$$\alpha=\begin{cases} 1.023\pm 0.012 & \text{for } N=3, \\ 1.001\pm 0.007 & \text{for } N=8, \end{cases} \tag{3.49}$$

in good agreement with our previous results.

Let us now consider the corrections to scaling. The term proportional to $\log L/L^2$ is identical in all cases to $\Delta_{m,1}(z;\rho)$, cf. Eq. (3.4). The contribution proportional to $1/L^2$ depend instead on the definition of ξ. In Figs. 4 and 5 we report the deviations from FSS for the three definitions for the standard and the Symanzik Hamiltonians [$\Delta_{\xi,1}$ and $\Delta_{\xi,2}$ are defined in analogy with Eq. (3.2)]. Notice that for $\xi_V^{(m,1)}$ and $\xi_V^{(m,3)}$ the corrections proportional to $1/L^2$ do not vanish even when $\alpha_1=\frac{1}{12}$, $\alpha_2=0$ and Eq. (3.9) are satisfied. This is expected since the second-moment correlation length is an off-shell

quantity. Therefore the definition of the correlation length must be improved, as well as the Hamiltonian. For instance, if one uses $\xi_V^{(m,1)}$ and the Symanzik Hamiltonian one does not see any improvement: this definition has large corrections to scaling, and the behavior is worse for the Symanzik Hamiltonian than for the standard one. In this case there is a simple remedy to the problem: modify the definition in such a way that $\xi(L,T) \approx \xi_\infty + O(L^{-4}, T^{-4}, L^{-2}T^{-2})$. Analogously one could proceed for $\xi_V^{(m,3)}$. The second definition is automatically improved but this is a peculiarity of the large-N limit.

Let us finally discuss the mass gap $\mu_V(L)$ and the exponential correlation length $\xi_V^{(\text{exp})} = 1/\mu_V(L)$. We have computed the FSS functions expressing them in terms of $\mu_V(L)$ itself, i.e., using as variable $x \equiv \mu_V(L)L$ instead of z. We get

$$\left(\frac{\mu_V(\infty)}{\mu_V(L)}\right)^2 = f_m(x;\infty)\left(1 + \frac{\Delta_{\mu,1}(x;\infty)}{L^2}\log L + \frac{\Delta_{\mu,2}(x;\infty)}{L^2}\right),$$

(3.50)

where $\Delta_{\mu,1}(x;\infty) = \Delta_{m,1}(x;\infty)$ and

$$\Delta_{\mu,2}(x;\infty) = \Delta_{m,2}(x;\infty)$$

$$+ \frac{\pi x^3}{6}(12\alpha_1 + 12\alpha_2 - 1)\frac{\partial F_0}{\partial x}(x;\infty)$$

$$+ \frac{8}{3}(12\alpha_1 + 12\alpha_2 - 1)e^{-4\pi F_0(x;\infty)}. \quad (3.51)$$

Thus only the $1/L^2$ term differs from the expansion of $m_\infty^2/m_{L,T}^2$. The asymptotic behavior for large x is analogous to Eq. (3.20) while for $x \to 0$ we have

$$\Delta_{\mu,2}(x;\infty) = \frac{\pi^2}{18}(12\alpha_1 - 1)$$

$$- \frac{\pi x}{3}(12\alpha_1 + 12\alpha_2 - 1) + O(x^2).$$

(3.52)

Notice that, since this quantity is defined on-shell, it is improved once the Hamiltonian is improved.

2. Isotensor sector

Let us now consider the isotensor observables. The calculation of the FSS function for the isotensor susceptibility χ_T is straightforward. We obtain

$$\frac{\chi_T(L,T)}{\chi_T(\infty)} = f_{\chi_T}(z;\rho)\left(1 + \frac{\Delta_{\chi_T,1}(z;\rho)}{L^2}\log L + \frac{\Delta_{\chi_T,2}(z;\rho)}{L^2}\right),$$

(3.53)

with corrections of order $\log^2 L/L^4$ where

$$f_{\chi_T}(z;\rho) = -\frac{64\pi}{z}\frac{\partial F_0}{\partial z}(z;\rho)e^{-4\pi F_0(z;\rho)}, \quad (3.54)$$

$$\Delta_{\chi_T,1}(z;\rho) = \frac{1}{4}(12\alpha_1 + 16\alpha_2 - 1)\left[64e^{-4\pi F_0(z;\rho)}\right.$$

$$\left. -z^2 + \frac{z}{2\pi}\left(\frac{\partial F_0}{\partial z}(z;\rho)\right)^{-1}\right], \quad (3.55)$$

$$\Delta_{\chi_T,2}(z;\rho) = 32\pi(12\alpha_1 + 16\alpha_2 - 1)F_0(z;\rho)$$

$$\times e^{-4\pi F_0(z;\rho)} - 4(28\alpha_1 + 32\alpha_2 - 3)$$

$$\times e^{-4\pi F_0(z;\rho)} - 4\pi\mathcal{F}_1(z;\rho)$$

$$+ \frac{\partial\mathcal{F}_1/\partial z(z;\rho)}{\partial F_0/\partial z(z;\rho)} - 256\pi\Lambda_1 e^{-4\pi F_0(z;\rho)}.$$

(3.56)

As before the $\log L/L^2$ corrections cancel if $12\alpha_1 + 16\alpha_2 - 1 = 0$. The function $\Delta_{\chi_T,2}(z;\rho)$ simplifies considerably if $\alpha_1 = \frac{1}{12}$ and $\alpha_2 = 0$. In this case

$$\Delta_{\chi_T,2}(z;\rho) = \left(\frac{1}{96\pi} - \Lambda_1\right)\left[256\pi e^{-4\pi F_0(z;\rho)} - 4\pi z^2\right.$$

$$\left. + 2z\left(\frac{\partial F_0}{\partial z}(z;\rho)\right)^{-1}\right]. \quad (3.57)$$

Therefore, if Eq. (3.9) is satisfied, χ_T has only corrections of order $\log L/L^4$. It is straightforward to compute the expansions of the various FSS functions in the limit $z \to \infty$. Using the results in Appendixes B 2 a and B 2 b we obtain

$$f_{\chi_T}(z;\rho) = 1 + \sqrt{2\pi z}e^{-z} + \sqrt{2\pi\rho z}e^{-\rho z}$$

$$+ O(z^{-1/2}e^{-z}, z^{-1/2}e^{-\rho z}), \quad (3.58)$$

$$\Delta_{\chi_T,1}(z;\rho) = \frac{z^2}{4}(12\alpha_1 + 16\alpha_2 - 1)(\sqrt{2\pi z}e^{-z}$$

$$+ \sqrt{2\pi\rho z}e^{-\rho z}) + O(z^{3/2}e^{-z}, z^{3/2}e^{-\rho z}), \quad (3.59)$$

$$\Delta_{\chi_T,2}(z;\rho) = -\frac{\pi z^4}{12}(12\alpha_1 + 12\alpha_2 - 1)$$

$$\times\left(\frac{e^{-z}}{\sqrt{2\pi z}} + \frac{\rho^2 e^{-\rho z}}{\sqrt{2\pi\rho z}}\right)$$

$$- \frac{z^2}{8}(12\alpha_1 + 16\alpha_2 - 1)\log\frac{z^2}{32}$$

$$\times(e^{-z}\sqrt{2\pi z} + e^{-\rho z}\sqrt{2\pi\rho z})$$

$$+ O(z^{5/2}e^{-z}, z^{5/2}e^{-\rho z}). \quad (3.60)$$

As expected, $f_{\chi_T}(z;\rho)$ behaves as $z^p e^{-z}$, but p differs from the value it assumes for other observables (see, e.g., the large-z behavior of χ_V). Indeed, while the exponential behavior is completely general, the power p depends on the observable.
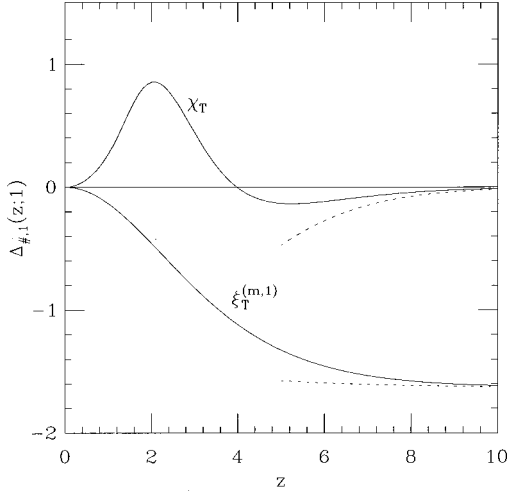
FIG. 6. $\Delta_{\chi_T,1}(z;1)$ and $\Delta_{\xi_T,1}(z;1)$ for H^{std}. The dashed lines are the large-z asymptotic expansions, Eqs. (3.59) and (3.69).

In Figs. 6 and 7 we report the graphs of $\Delta_{\chi_T,1}(z;\rho)$ and $\Delta_{\chi_T,2}(z;\rho)$ for $\rho=1$ and different Hamiltonians. The behavior is very similar to the behavior of $m^2_{L,T}/m^2_\infty$. The FSS corrections are quite small. The Symanzik Hamiltonian shows the best behavior, while the diagonal Hamiltonian is the one with the largest deviations from FSS.

Finally let us compute the FSS curve for the tensor correlation length. We will restrict the discussion to the standard action; the generalization to generic Hamiltonians is straightforward but the final expressions are cumbersome. Moreover we will restrict our discussion to $\xi_T^{(m,1)}$, that is to the definition used in numerical simulations. We obtain

$$\left(\frac{\xi_T^{(m,1)}(L,T)}{\xi_T^{(m)}(\infty)}\right)^2 = f_{\xi_T}(z;\rho)$$

$$\times\left(1+\frac{\Delta_{\xi_T,1}(z;\rho)}{L^2}\log L+\frac{\Delta_{\xi_T,2}(z;\rho)}{L^2}\right),$$

(3.61)

where

$$f_{\xi_T}(z;\rho)=192\Xi_1(z;\rho)e^{-4\pi F_0(z;\rho)},\qquad(3.62)$$

$$\Delta_{\xi_T,1}(z;\rho)=\Delta_{m,1}(z;\rho)+\frac{\Xi_2(z;\rho)}{\Xi_1(z;\rho)}$$

$$+8e^{-4\pi F_0(z;\rho)},\qquad(3.63)$$

$$\Delta_{\xi_T,2}(z;\rho)=\Delta_{m,2}(z;\rho)+\frac{\Xi_3(z;\rho)}{\Xi_1(z;\rho)}-4e^{-4\pi F_0(z;\rho)}$$

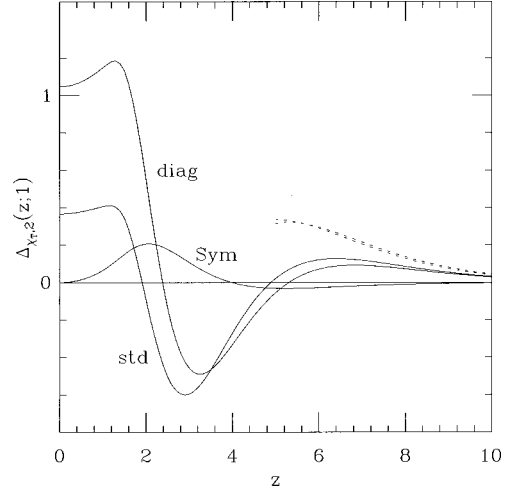$$+16\pi F_0(z;\rho)e^{-4\pi F_0(z;\rho)},\qquad(3.64)$$

and

FIG. 7. $\Delta_{\chi_T,2}(z;1)$ for H^{std}, H^{Sym}, and H^{diag}. The dashed lines are the large-z asymptotic expansions, Eq. (3.60).

$$\Xi_1(z;\rho)=\frac{1}{8\pi^2}\left[-1-\rho^2-\frac{1}{2z}\frac{\partial F_0}{\partial z}(z;\rho)\right.$$

$$\left.\times\left(\frac{1}{F_3(z;\rho)}+\frac{1}{F_3(\rho z;1/\rho)}\right)\right],\qquad(3.65)$$

$$\Xi_2(z;\rho)=\frac{1}{128\pi^3}\left[\frac{1}{F_3(z;\rho)}+\frac{1}{F_3(\rho z;1/\rho)}+\frac{1}{2z}\frac{\partial F_0}{\partial z}(z;\rho)\right.$$

$$\left.\times\left(\frac{1}{F_3^2(z;\rho)}+\frac{1}{\rho^2 F_3^2(\rho z;1/\rho)}\right)\right],\qquad(3.66)$$

$$\Xi_3(z;\rho)=\frac{1}{8\pi^2}\left[-\frac{1}{2z}\frac{\partial F_1}{\partial z}(z;\rho)\left(\frac{1}{F_3(z;\rho)}+\frac{1}{F_3(\rho z;1/\rho)}\right)\right.$$

$$-\frac{2\pi^2}{3}-\frac{\pi^2}{6z}\frac{\partial F_0}{\partial z}(z;\rho)\left(\frac{1}{F_3(z;\rho)}\right.$$

$$\left.+\frac{1}{\rho^2 F_3(\rho z;1/\rho)}\right)+\frac{1}{2z}\frac{\partial F_0}{\partial z}(z;\rho)\left(\frac{F_4(z;\rho)}{F_3^2(z;\rho)}\right.$$

$$\left.\left.+\frac{F_4(\rho z;1/\rho)}{\rho^2 F_3^2(\rho z;1/\rho)}+\frac{1}{16\pi}\frac{\log\rho}{\rho^2 F_3^2(\rho z;1/\rho)}\right)\right].$$

(3.67)

Using Eqs. (B57), (B58), (B106), and (B107) it is straightforward to compute the large-z behavior of the various FSS functions. We obtain

$$f_{\xi_T}(z;\rho)=1-\frac{\pi^2}{15z^2}\frac{1+\rho^2}{\rho^2}+\frac{2\pi^4}{63z^4}\frac{1+\rho^4}{\rho^4}+O(z^{-6}),$$

(3.68)

$$\Delta_{\xi_T,1}(z;\rho)=-\frac{\pi^2}{12}\frac{1+\rho^2}{\rho^2}+\frac{\pi^4}{180\rho^4 z^2}(3-2\rho^2+3\rho^4)$$

$$+O(z^{-4}),\qquad(3.69)$$

$$\Delta_{\xi_T,2}(z;\rho) = \frac{\pi^2}{24\rho^2}(1+\rho^2)\left(\log\frac{z^2}{32}+4\right)$$

$$-\frac{\pi^4}{360\rho^4 z^2}(3-2\rho^2+3\rho^4)\log\frac{z^2}{32}$$

$$-\frac{\pi^4}{90\rho^4 z^2}(1-\rho^2)^2 + O(\log z/z^4).$$

(3.70)

As expected $f_{\xi_T}(z;\rho)$ approaches one as $1/z^2$ and thus it reaches the asymptotic value for $z\to\infty$ within 1% only for $z\approx 9$. Notice moreover that $\Delta_{\xi_T,2}(z;\rho)$ diverges logarithmically as $z\to\infty$. This fact signals the nonuniformity of the expansion in z. This is not unexpected. Indeed, for each fixed z, we expect the expansion to be reliable only if $m_{L,T}\ll 1$, i.e., if the correlation length is much larger than a lattice spacing. Therefore we expect the expansion to be valid only if $z\ll L$. If $1\ll z\ll L$, $\log z/L^2$ is a small number and thus the expansion is completely under control.

The functions $\Delta_{\xi_T,1}(z;\rho)$ and $\Delta_{\xi_T,2}(z;\rho)$ are reported for $\rho=1$ in Figs. 6 and 8. From these plots, comparing with the analogous graphs for other observables, one can immediately see that the corrections to FSS for $\xi_T^{(m,1)}$ are quite large. This is particularly evident in the large-z region, where the $\log L/L^2$ term goes to a constant (for the isovector correlation length this term vanishes exponentially), while the $1/L^2$ term diverges as $\log z$.

IV. MIXED $O(N)-\text{RP}^{N-1}$ MODEL

In this section we compute the FSS corrections for the Hamiltonian (2.18). As before we want to obtain a relation between m_∞^2 and $m_{L,T}^2$ at fixed β. Using now the gap equation (2.19) we have

$$\frac{4I_{L,T}(m_{L,T}^2)^2}{4I_{L,T}(m_{L,T}^2)+r[m_{L,T}^2 I_{L,T}(m_{L,T}^2)-1]}$$

$$= \frac{4I_\infty(m_\infty^2)^2}{4I_\infty(m_\infty^2)+r[m_\infty^2 I_\infty(m_\infty^2)-1]}.$$

(4.1)

We will now discuss the FSS limit in which $L,T\to\infty$, $\beta\to\infty$, $m_{L,T}$, $m_\infty\to 0$ with $m_{L,T}L\equiv z$ and $T/L\equiv\rho$ fixed. At leading order we can disregard the terms $m_\infty^2 I_\infty(m_\infty^2)$ and $m_{L,T}^2 I_{L,T}(m_{L,T}^2)$ in the denominators obtaining

$$\frac{I_{L,T}(m_{L,T}^2)^2}{4I_{L,T}(m_{L,T}^2)-r} = \frac{I_\infty(m_\infty^2)^2}{4I_\infty(m_\infty^2)-r},$$

(4.2)

which implies $I_{L,T}(m_{L,T}^2)=I_\infty(m_\infty^2)$. Thus, at leading order, the relation between $m_{L,T}$ and m_∞ is identical to the one we have discussed in the previous section. Consequently the FSS functions for the Hamiltonian (2.18) are identical to the
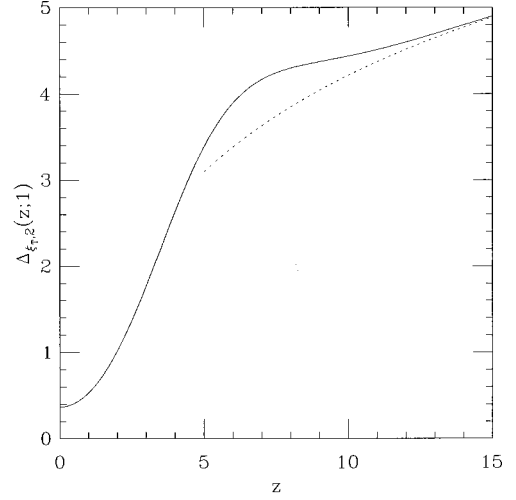

FIG. 8. $\Delta_{\xi_T,2}(z;1)$ for H^{std}. The dashed line is the large-z asymptotic expansion, Eq. (3.70).

FSS functions of models with Hamiltonian (2.1), as expected on the basis of universality.[4] The corrections will, however, be different. Writing

$$I_\infty(m_\infty^2)=I_{L,T}(m_{L,T}^2)+\delta m_{L,T}^2,$$

(4.3)

a simple computation gives

$$\delta = \frac{r}{2}\left(\frac{m_\infty^2}{m_{L,T}^2}-1\right)\frac{I_{L,T}(m_{L,T}^2)^2}{2I_{L,T}(m_{L,T}^2)-r}.$$

(4.4)

Solving for $m_\infty^2/m_{L,T}^2$ we get finally

$$\frac{m_\infty^2}{m_{L,T}^2}=f_m(z;\rho)\left(1+\frac{[\Delta_{m,1}(z;\rho)+\Delta_{m,1}^r(z;\rho)]}{L^2}\log L\right.$$

$$\left.+\frac{[\Delta_{m,2}(z;\rho)+\Delta_{m,2}^r(z;\rho)]}{L^2}+\frac{\widehat{\Delta_m^r}(L;z;\rho)}{L^2}\right),$$

(4.5)

where $f_m(z;\rho)$, $\Delta_{m,1}(z;\rho)$, and $\Delta_{m,2}(z;\rho)$ are defined in Eqs. (3.3)–(3.5), with $\alpha_1=\alpha_2=0$ and

$$\Delta_{m,1}^r(z;\rho)=\frac{r}{2}(z^2-32e^{-4\pi F_0(z;\rho)}),$$

(4.6)

---

[4]To be precise universality applies only to models with $r\neq 1$. For $r=1$ (RP^{N-1} σ model) there is an additional gauge invariance and therefore we do not expect its FSS functions to be identical to those of the N-vector model. However, under suitable assumptions (absence of vortices) that are verified in the large-N limit, one can prove that the FSS functions for the RP^{N-1} model with periodic boundary conditions are equal to the N-vector FSS functions with fluctuating periodic-antiperiodic boundary conditions [56]. In the large-N limit the antiperiodic contribution vanishes, hence RP^∞ has the same FSS functions of the N-vector model. For a discussion in one dimension, see Ref. [40].

TABLE III. Values of $R_{\text{exact}}(L;z,\rho)$, $R_{\text{expan}}(L;z,\rho)$, $D_{\text{exact}}(L;z,\rho)$, and $D_{\text{expan}}(L;z,\rho)$ for the RP$^\infty$ model ($r=1$), $\rho=1$ and $z=2$.

L	$R_{\text{exact}}(L;2,1)$	$R_{\text{expan}}(L;2,1)$	$D_{\text{exact}}(L;2,1)$	$D_{\text{expan}}(L;2,1)$
8	0.35709886	0.34256990	0.5916549	0.5268967
10	0.29398737	0.29002279	0.3103555	0.2926846
12	0.26809044	0.26656021	0.1949281	0.1881076
14	0.25463070	0.25391316	0.1349356	0.1317374
16	0.24665643	0.24627448	0.0993928	0.0976904
32	0.22943124	0.22941299	0.0226170	0.0225356
64	0.22561220	0.22561112	0.0055948	0.0055900
128	0.22467784	0.22467777	0.0014302	0.0014300
∞		0.22435696		

$$\Delta^r_{m,2}(z;\rho) = \frac{\pi r}{2}\left[2F_0(z;\rho)+r\right](z^2-32e^{-4\pi F_0(z;\rho)}),$$
(4.7)

$$\widehat{\Delta^r_m}(L;z;\rho) = \frac{\pi^2 r^3}{2}(z^2-32e^{-4\pi F_0(z;\rho)})$$
$$\times \frac{1}{\log L + 2\pi F_0(z;\rho) - \pi r}. \quad (4.8)$$

The result (4.5) is quite different than Eq. (3.2). Indeed, while before the corrections had a very simple dependence on $\log L$, now the corrections involve $\widehat{\Delta^r_m}(L;z;\rho)$ that is not a simple polynomial in $\log L$. Notice that, for large L at fixed z, $\widehat{\Delta^r_m}(L;z;\rho)$ behaves as $1/\log L$. Therefore, in the FSS limit, the corrections still behave as $\log L/L^2$.

We should make a second remark about $\widehat{\Delta^r_m}(L;z;\rho)$. It is easy to convince oneself from the asymptotic expansions, Eqs. (B57), (B59), and (B69), that $F_0(z;\rho)$ assumes any real value. Therefore, for each value of L, there is a value z_c such that the denominator in Eq. (4.8) vanishes, and therefore $\widehat{\Delta^r_m}(L;z;\rho)$ diverges. When $L\to\infty$, $z_c\to\infty$, more precisely, using Eq. (B57), we have $z_c\approx e^{-\pi r}\sqrt{32L}$. This singularity is a signal of the fact that the expansion is not uniform in z. For each z the expansion is valid only when $L\gg z$, i.e., for $m_{L,T}\ll 1$. In other words the expansion makes sense only when the correlation length is much larger than a lattice spacing.

The corrections are larger for the mixed model than for the vector model. For instance, $\Delta^r_{m,1}(z;\rho)/\Delta_{m,1}(z;\rho)=2r$ so that the logarithmic correction in the RP$^\infty$ model ($r=1$) is three times larger than the corresponding one in the N-vector model ($r=0$). For the values of L that are used in Monte Carlo simulations, say $8\leqslant L\leqslant 128$, however, all terms in Eq. (4.5) contribute to the FSS corrections. In Table III we report for $r=1$, $\rho=1$, and $z=2$ the same quantities reported in Table I. Comparing the two tables we see that the corrections to FSS for RP$^\infty$ are seven times larger than those for the N-vector model in the same range of values of L. Only for $L\approx 50$ ($L\approx 150$) the corrections are smaller than 1% (0.1%).
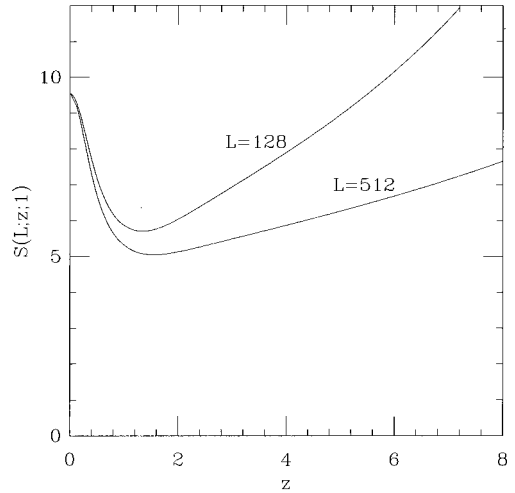

FIG. 9. $S(L;z;1)$ for $L=128$ and $L=512$. For $L\to\infty$, $S(L;z;1)$ converges to 3 for all $z\neq 0$.

To compare the corrections for the RP$^\infty$ and the N-vector model with the standard Hamiltonian for all values of z, in Fig. 9 we report

$$S(L;z;\rho) = \frac{D_{\text{expan}}(L;z,\rho)_{r=1}}{D_{\text{expan}}(L;z,\rho)_{r=0}}$$
(4.9)

for $L=128$, 512 and $\rho=1$ [$D(L;z,\rho)$ is defined in Eq. (3.17)]. For these values of L, corrections are 5–10 times larger in the RP$^\infty$ model.

It is interesting to understand the origin of $\widehat{\Delta^r_m}(L;z;\rho)$ in terms of perturbation theory. First of all, for z small (for simplicity we consider the strip case, analogous results are valid for generic ρ) we can expand, cf. Eq. (B69),

$$\widehat{\Delta^r_m}(L;z;\rho) = \frac{\pi^2 r^3}{2}\frac{z^3}{\pi+z\log L}$$
$$\times\left[1-\frac{\pi z(2\bar{F}_{00}-r)}{\pi+z\log L}+O(z^2)\right]. \quad (4.10)$$

This is not yet a perturbative expansion in z due to the presence of the term $z\log L$ in the denominators. This term cannot be expanded in z since we are considering an asymptotic expansion at fixed z with $L\to\infty$. However, if we ignore this problem and expand the denominators in powers of z, we obtain

$$\widehat{\Delta^r_m}(L;z;\rho)_{\text{pert}} = \frac{\pi r^3}{2}z^3\left[1-\frac{z}{\pi}\log L-z(2\bar{F}_{00}-r)\right.$$
$$\left.+O(z^2\log^2 L)\right]. \quad (4.11)$$

In general $\widehat{\Delta^r_m}(L;z;\rho)$ has a polynomial expansion in z with coefficients that are polynomials in $\log L$:
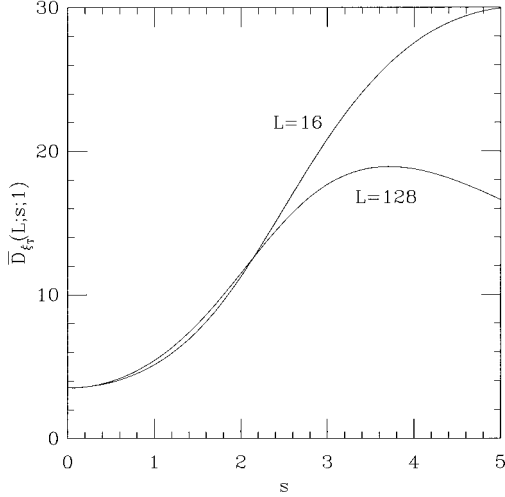
FIG. 10. $\bar{D}_{\xi_T}(L;s;1)$ for the RP$^\infty$ model for two different values of L: $L=16$ and $L=128$.

FIG. 11. $\bar{D}_{\chi_T}(L;s;1)$ for the RP$^\infty$ model for two different values of L: $L=16$ and $L=128$.

$$\widehat{\Delta_m^r}(L;z;\rho)_{\text{pert}} = z^3 \sum_{n=0}^{\infty} P_n(\log L) z^n, \qquad (4.12)$$

where $P_n(x)$ is a polynomial of degree n. This expansion is clearly incorrect in the FSS limit $L\to\infty$ at fixed z. However Eq. (4.12) correctly describes the theory in the limit $z\to 0$ at fixed L. This is the limit in which PT works correctly [51,52], and indeed the expansion (4.12) can be directly obtained with a perturbative calculation. Therefore our results show that in order to correctly compute the corrections to FSS one needs to resum the perturbative expansion. This reflects the fact that the perturbative limit $z\to 0$ with L large and fixed does not commute with the FSS limit $L\to\infty$ with z fixed and small. It should also be noticed that the infinite series of logarithms appearing in Eq. (4.12) resums to give corrections of order $1/(L^2\log L)$. This is a result that is far from obvious: in general one expects series of the form (4.12) to give powers of L, i.e., to resum to $L^{p(z)}$, where $p(z)$ is some function of z. For $N=\infty$ no power is generated, but we have no proof that this will be true for generic values of N. The only argument we have against the appearance of power corrections is based on a naive application of renormalization-group ideas. The corrections to FSS are due to the irrelevant operators of the theory. Since the N-vector model is asymptotically free, operators have canonical scaling dimensions with logarithmic corrections. Therefore we always expect a behavior of the form $(\log L)^p/L^2$.

Using the results of the previous section it is easy to obtain the FSS functions and their leading corrections for the various observables. For the isovector second-moment correlation lengths the expressions (3.31)–(3.33) still hold with $m_\infty^2/m_{L,T}^2$ given by Eq. (4.5). For the susceptibility χ_V we have

$$\frac{\chi_V(L,T)}{\chi_V(\infty)} = \frac{m_\infty^2}{m_{L,T}^2} \left[1 + \frac{r}{4z^2 L^2} (32e^{-4\pi F_0(z;\rho)} - z^2) \right.$$

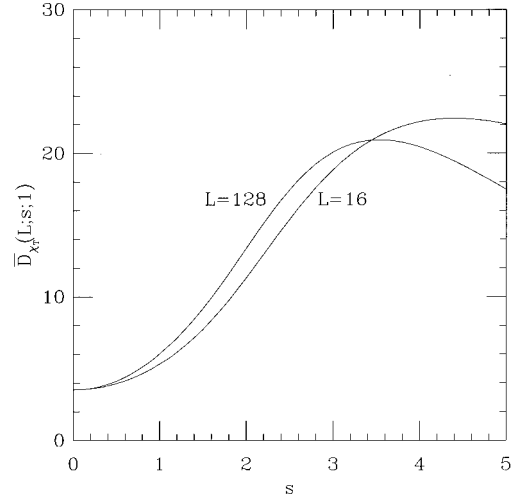$$\left. \times \frac{\log L + 2\pi F_0(z;\rho)}{\log L + 2\pi F_0(z;\rho) - \pi r} \right], \qquad (4.13)$$

while for χ_T we have

$$\frac{\chi_T(L,T)}{\chi_T(\infty)} = \left(\frac{\chi_T(L,T)}{\chi_T(\infty)} \right)_{r=0} \left[1 + \frac{\Delta_{m,1}^r(z;\rho)}{L^2}\log L + \frac{\Delta_{m,2}^r(z;\rho)}{L^2} \right.$$

$$+ \frac{\widehat{\Delta_m^r}(L;z;\rho)}{L^2} + \frac{r}{2z^2 L^2} (32e^{-4\pi F_0(z;\rho)} - z^2)$$

$$\left. \times \frac{\log L + 2\pi F_0(z;\rho)}{\log L + 2\pi F_0(z;\rho) - \pi r} \right]. \qquad (4.14)$$

Finally for $\xi_T^{(m,1)}$ we have

$$\left(\frac{\xi_T^{(m,1)}(L,T)}{\xi_T^{(m,1)}(\infty)} \right)^2 = \left(\frac{\xi_T^{(m,1)}(L,T)}{\xi_T^{(m,1)}(\infty)} \right)_{r=0}^2 \left[1 + \frac{\Delta_{m,1}^r(z;\rho)}{L^2}\log L \right.$$

$$\left. + \frac{\Delta_{m,2}^r(z;\rho)}{L^2} + \frac{\widehat{\Delta_m^r}(L;z;\rho)}{L^2} \right]. \qquad (4.15)$$

To conclude our discussion we come back again to the RP$^\infty$ model. In this case the FSS functions are usually reported in terms of $s = L/\xi_T^{(m,1)}(L,T)$. Indeed, because of the gauge symmetry, one cannot define observables in the isovector sector. For any observable $\mathcal{O}$, we define the FSS deviations

$$\bar{D}_{\mathcal{O}}(L;s;\rho) = L^2 \left[\frac{\mathcal{O}(L,T)}{\mathcal{O}(\infty)} \frac{1}{\bar{f}_{\mathcal{O}}(s;\rho)} - 1 \right], \qquad (4.16)$$

where $\bar{f}_{\mathcal{O}}(s;\rho)$ is the FSS function of $\mathcal{O}$ expressed in terms of s. In Figs. 10 and 11 we report $\bar{D}_{\xi_T}(L;z;1)$ and $\bar{D}_{\chi_T}(L;z;1)$ for $L=16$ and $L=128$. The corrections are extremely large if one compares them with the analogous results for the N-vector model. This is especially true in the large-s region. Moreover the corrections are positive. These results are in qualitative agreement with the results of Ref. [56].

V. CONCLUSIONS

In this paper we have investigated the corrections to FSS in the large-N limit for a vast class of models and we have studied their relation with the improvement program of Symanzik. In the large-N limit we find that the corrections behave as $f(z,L)\log L/L^2$ where $f(z,L)$ can be expanded in powers of $1/\log L$ and is such that $f(z,\infty)$ is finite for all values of z. Thus, for large values of L, corrections behave as $\log L/L^2$. Tree-level improved Hamiltonians have corrections to FSS behaving as $1/L^2$: the effect of the improvement is the cancellation of a logarithm. Subsequent perturbative improvement should give Hamiltonians with corrections of order $1/(L^2\log^l L)$ ($l=1$ for one-loop improvement and so on).

We have shown explicitly that the FSS limit and the perturbative limit commute in the calculations of the FSS functions but *not* for the calculation of the next-to-leading term. Corrections to FSS cannot be computed in perturbation theory unless an infinite series of logarithms is resummed.

Finally we have investigated if there is any sign of large correction in the RP^∞ model. We find that this model shows deviations from FSS that are much larger than those of the N-vector model. We find qualitative agreement with the results of Ref. [56].

ACKNOWLEDGMENTS

APPENDIX A: DEFINITIONS

1. Functions $G_k(\alpha)$, $H_k(\alpha)$, and $\mathcal{H}_k(\alpha)$

Let us define the following functions:

$$
G_k(\alpha) = \sum_{n=1}^{\infty} \left[(n^2+\alpha^2)^{k-1/2} - \sum_{m=0}^{k} \binom{k-1/2}{m} \alpha^{2m} n^{2k-2m-1} \right], \qquad (A1)
$$

$$
H_k(\alpha) = \sum_{n=1}^{\infty} \frac{1}{(n^2+\alpha^2)^{k+1/2}}, \qquad (A2)
$$

$$
\mathcal{H}_k(\alpha) = \sum_{n=1}^{\infty} \frac{1}{(1-4n^2)(n^2+\alpha^2)^{k+1/2}}. \qquad (A3)
$$

The first and the third ones are defined for integers $k \geq 0$ while the second one is defined for $k \geq 1$: for these values of k the sums converge for all values of α. The functions $G_k(\alpha)$ and $H_k(\alpha)$ are known in statistical mechanics under the name of remnant functions (see Appendix D of Ref. [46] and Ref. [60]).

We want now to discuss their behavior for $\alpha \to 0$ and $\alpha \to \infty$. We will focus on those values of k that appear in our final results, i.e., to $k=0,1$.

The expansion for $\alpha \to 0$ is trivial. We obtain

$$
G_0(\alpha) = \sum_{k=1}^{\infty} (-1)^k \binom{2k}{k} \zeta(2k+1) \left(\frac{\alpha}{2}\right)^{2k}, \qquad (A4)
$$

$$
G_1(\alpha) = 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{2k}{k} \zeta(2k+1) \left(\frac{\alpha}{2}\right)^{2k+2}, \qquad (A5)
$$

$$
H_1(\alpha) = \sum_{k=0}^{\infty} (-1)^k (2k+1) \binom{2k}{k} \zeta(2k+3) \left(\frac{\alpha}{2}\right)^{2k}, \qquad (A6)
$$

$$
\mathcal{H}_0(\alpha) = \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} \alpha^{2k} \\ \times \left[1 - 2\log 2 + \sum_{n=1}^{k} \frac{\zeta(2n+1)}{4^n} \right], \qquad (A7)
$$

$$
\mathcal{H}_1(\alpha) = 4 \sum_{k=0}^{\infty} (-1)^k (2k+1) \binom{2k}{k} \alpha^{2k} \\ \times \left[1 - 2\log 2 + \sum_{n=1}^{k+1} \frac{\zeta(2n+1)}{4^n} \right], \qquad (A8)
$$

where we have used

$$
\sum_{n=1}^{\infty} \frac{1}{(1-4n^2)n^{2q+1}} = 4^q(1-2\log 2) + 4^q \sum_{s=1}^{q} \frac{\zeta(2s+1)}{4^s}. \qquad (A9)
$$

All series converge for $|\alpha| < 1$.

We want now to derive the asymptotic expansions for $\alpha \to \infty$. In order to obtain them let us derive a different representation for the functions $G_k(\alpha)$ and $H_k(\alpha)$.

Let us first consider $G_0(\alpha)$. We rewrite it as

$$
G_0(\alpha) = \lim_{\epsilon \to 0+} \left[-\zeta(1+\epsilon) + \frac{1}{\Gamma(1/2+\epsilon/2)} \times \int_0^{\infty} \frac{dx}{x^{(1-\epsilon)/2}} e^{-x\alpha^2} \sum_{n=1}^{\infty} e^{-xn^2} \right]. \qquad (A10)
$$

Using the Poisson resummation formula [72] we obtain

$$G_0(\alpha) = \lim_{\epsilon \to 0+} \left\{ -\zeta(1+\epsilon) + \frac{\alpha^{-\epsilon}}{\Gamma(1/2+\epsilon/2)} \right.$$

$$\times \left[\frac{\sqrt{\pi}}{2} \Gamma\left(\frac{\epsilon}{2}\right) - \frac{1}{2\alpha} \Gamma\left(\frac{1+\epsilon}{2}\right) \right]$$

$$+ \frac{\sqrt{\pi}}{\Gamma(1/2+\epsilon/2)} \int_0^\infty \frac{dx}{x^{1-\epsilon/2}} e^{-x\alpha^2}$$

$$\left. \times \sum_{n=1}^\infty \exp\left(-\frac{\pi^2 n^2}{x}\right) \right\}. \tag{A11}$$

Taking the limit we get the representation [60]

$$G_0(\alpha) = -\log\frac{\alpha}{2} - \gamma_E - \frac{1}{2\alpha} + 2\sum_{n=1}^\infty K_0(2\pi n\alpha), \tag{A12}$$

where K_0 is a modified Bessel function [73]. The corresponding representations for $G_k(\alpha)$ and $H_k(\alpha)$, $k \geq 1$ can be obtained by integration and derivation of the previous relation. We obtain [60]

$$G_1(\alpha) = \frac{1}{12} + \frac{\alpha^2}{2}\left(-\log\frac{\alpha}{2} + \frac{1}{2} - \gamma_E - \frac{1}{\alpha}\right)$$

$$- \frac{\alpha}{\pi} \sum_{n=1}^\infty \frac{1}{n} K_1(2\pi n\alpha), \tag{A13}$$

$$H_1(\alpha) = \frac{1}{\alpha^2} - \frac{1}{2\alpha^3} + \frac{4\pi}{\alpha} \sum_{n=1}^\infty n K_1(2\pi n\alpha). \tag{A14}$$

This representation of the functions $G_k(\alpha)$ and $H_k(\alpha)$ allows an immediate derivation of the asymptotic expansion for large values of α since [73]

$$K_n(x) = \sqrt{\frac{\pi}{2x}} e^{-x}\left[1 + \frac{1}{2x}\left(n^2 - \frac{1}{4}\right) + O(x^{-2})\right]. \tag{A15}$$

Let us now consider the functions $\mathcal{H}_k(\alpha)$. We rewrite $\mathcal{H}_k(\alpha)$ as

$$\mathcal{H}_k(\alpha) = -\frac{1}{2}\frac{1}{\alpha^{2k+1}} + \frac{1}{2}\frac{1}{\Gamma(k+1/2)} \int_0^\infty dx\, x^{k-1/2} e^{-x\alpha^2} g(x), \tag{A16}$$

where

$$g(x) = \sum_{n=-\infty}^{+\infty} \frac{e^{-xn^2}}{1-4n^2}. \tag{A17}$$

Now in the integral appearing in Eq. (A16) the relevant region for large α^2 corresponds to small values of x. Therefore we need the small-x expansion of $g(x)$. First of all notice that

$$g(0) = 1 + 2\sum_{n=1}^{+\infty} \frac{1}{1-4n^2} = 0. \tag{A18}$$

Then one immediately verifies that $g(x)$ satisfies an equation of the form

$$g'(x) + \frac{1}{4} g(x) - \frac{1}{4} \sum_{n=-\infty}^{+\infty} e^{-xn^2} = 0, \tag{A19}$$

so that, using Eq. (A18), we get

$$g(x) = \frac{1}{4} e^{-x/4} \int_0^x dy\, e^{y/4} \sum_{n=-\infty}^{+\infty} e^{-yn^2}. \tag{A20}$$

Using the Poisson resummation formula [72] we can rewrite it as

$$g(x) = \frac{\sqrt{\pi}}{4} e^{-x/4} \int_0^x \frac{dy}{\sqrt{y}} e^{y/4} \sum_{n=-\infty}^{+\infty} e^{-\pi^2 n^2/y}. \tag{A21}$$

In the small-x region only the term with $n=0$ is relevant so that

$$g(x) \approx \frac{\sqrt{\pi}}{4} e^{-x/4} \int_0^x \frac{dy}{\sqrt{y}} e^{y/4}$$

$$\approx \frac{\sqrt{\pi x}}{2}\left(1 - \frac{x}{6} + \frac{x^2}{60} - \frac{x^3}{840} + \frac{x^4}{15120} + O(x^5)\right). \tag{A22}$$

We obtain eventually for large values of α

$$\mathcal{H}_k(\alpha) = -\frac{1}{2}\frac{1}{\alpha^{2k+1}} + \frac{\sqrt{\pi}}{4\Gamma(k+1/2)}\frac{1}{\alpha^{2k+2}}$$

$$\times \left[k! - \frac{(k+1)!}{6\alpha^2} + \frac{(k+2)!}{60\alpha^4} - \frac{(k+3)!}{840\alpha^6} + \frac{(k+4)!}{15120\alpha^8} \right]$$

$$+ O(\alpha^{-2k-12}). \tag{A23}$$

2. The functions $M_{pq}(z;\rho)$, $N_{pq}(\rho)$, $\mathcal{M}_{pq}(z;\rho)$, and $\mathcal{N}_{pq}(\rho)$

A second set of functions appear in our calculations. We define

$$M_{pq}(z;\rho)=(2\pi)^p\sum_{n=-\infty}^{\infty}\frac{1}{(4\pi^2n^2+z^2)^{p/2}}$$

$$\times\frac{1}{[\exp(\rho\sqrt{4\pi^2n^2+z^2})-1]^q},\quad(A24)$$

$$\mathcal{M}_{pq}(z;\rho)=(2\pi)^p\sum_{n=-\infty}^{\infty}\frac{1}{(1-4n^2)(4\pi^2n^2+z^2)^{p/2}}$$

$$\times\frac{1}{[\exp(\rho\sqrt{4\pi^2n^2+z^2})-1]^q},\quad(A25)$$

$$N_{pq}(\rho)=\sum_{n=1}^{\infty}\frac{1}{n^p}\frac{1}{[\exp(2\pi\rho n)-1]^q},\quad(A26)$$

$$\mathcal{N}_{pq}(\rho)=\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)n^p}\frac{1}{[\exp(2\pi\rho n)-1]^q},$$
$$(A27)$$

where q is a positive integer. We want to compute here the asymptotic expansion of $M_{pq}(z;\rho)$ and $\mathcal{M}_{pq}(z;\rho)$ for $z\to0$ and $z\to\infty$ at fixed ρ. The first expansion is straightforward and we get for $M_{pq}(z;\rho)$

$$M_{pq}(z;\rho)=\left(\frac{2\pi}{z}\right)^p\frac{1}{(\rho z)^q}\sum_{n_1,\ldots,n_q=0}^{\infty}\frac{B_{n_1}\cdots B_{n_q}}{n_1!\cdots n_q!}$$

$$\times(\rho z)^{n_1+\ldots+n_q}+2N_{pq}(\rho)$$

$$+2z^2\left[-\frac{p}{8\pi^2}N_{p+2,q}(\rho)\right.$$

$$\left.-\frac{q\rho}{4\pi}(N_{p+1,q}(\rho)+N_{p+1,q+1}(\rho))\right]$$

$$+O(z^4).\quad(A28)$$

For $\mathcal{M}_{pq}(z;\rho)$ the expansion is analogous with the substitution of N_{pq} with $\mathcal{N}_{pq}$.

For large z let us consider first $M_{pq}(z;\rho)$. We obtain

$$M_{pq}(z;\rho)=\sum_{n=-\infty}^{\infty}\frac{(2\pi)^p}{(4\pi^2n^2+z^2)^{p/2}}$$

$$\times\exp[-q\rho\sqrt{4\pi^2n^2+z^2}]+O(e^{-2pqz}z^{1/2-p}).$$
$$(A29)$$

This last sum can be evaluated using the Poisson resummation formula [72]. Define

$$\hat{f}(\omega)=\int_{-\infty}^{\infty}dt\,e^{i\omega t}\frac{(2\pi)^p}{(4\pi^2t^2+z^2)^{p/2}}\exp[-q\rho\sqrt{4\pi^2t^2+z^2}].$$
$$(A30)$$

Then

$$M_{pq}(z;\rho)\approx\sum_{n=-\infty}^{\infty}\hat{f}(2\pi n).\quad(A31)$$

For $z\to\infty$ we have

$$\hat{f}(\omega)=\frac{2\pi}{\sqrt{z}}(q\rho z)^{1-p}(\omega^2+4\pi^2q^2\rho^2)^{(2p-3)/4}$$

$$\times\exp\left[-\frac{z}{2\pi}\sqrt{\omega^2+4\pi^2q^2\rho^2}\right][1+O(z^{-1})].$$
$$(A32)$$

Therefore

$$M_{pq}(z;\rho)=\frac{1}{\sqrt{q\rho}}\left(\frac{z}{2\pi}\right)^{1/2-p}e^{-q\rho z}[1+O(z^{-1})].$$
$$(A33)$$

For $\mathcal{M}_{pq}(z;\rho)$ we will not need the explicit asymptotic behavior. It is, however, easy to convince oneself that, for large z, $\mathcal{M}_{pq}(z;\rho)$ goes to zero faster than $z^{-p}e^{-q\rho z}$.

To conclude this subsection let us derive a set of relations for the functions $N_{pq}(\rho)$. First of all let us notice that $N_{1,1}(\rho)$ can be related to Dedekind's η function [68]

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}).\quad(A34)$$

Indeed

$$N_{1,1}(\rho)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n}e^{-2\pi nk\rho}=-\log\left[\prod_{k=1}^{\infty}(1-e^{-2\pi k\rho})\right]$$

$$=-\frac{\pi\rho}{12}-\log\,\eta(i\rho).\quad(A35)$$

Following the same steps it is possible to prove that, for $p>0$,

$$N_{-p,1}(\rho)=\frac{(-1)^p}{(2\pi)^p}\frac{d^p}{d\rho^p}N_{p,1}(\rho).\quad(A36)$$

This relation, together with

$$\frac{d}{d\rho}N_{pq}(\rho)=-2\pi q[N_{p-1,q}(\rho)+N_{p-1,q+1}(\rho)],$$
$$(A37)$$

allows us to prove that all $N_{pq}(\rho)$ with $p\le-1$ can be expressed in terms of $N_{0,q'}(\rho)$. From Eqs. (A36) and (A37) we obtain the following relations we will use in the following:

$$N_{-1,1}(\rho)=N_{0,1}(\rho)+N_{0,2}(\rho)=\frac{1}{24}+\frac{1}{2\pi}\frac{d}{d\rho}\log\,\eta(i\rho),$$
$$(A38)$$

$$N_{-2,1}(\rho)+N_{-2,2}(\rho)=N_{-1,1}(\rho)+3N_{-1,2}(\rho)+2N_{-1,3}(\rho)$$

$$=-\frac{1}{4\pi^2}\frac{d^2}{d\rho^2}\log\,\eta(i\rho). \qquad (A39)$$

Finally, for $p\geqslant 0$, let us derive a relation between $N_{4p+3,1}(\rho)$ and $N_{4p+3,1}(1/\rho)$ that will allow us to compute explicitly $N_{4p+3,1}(1)$. Let us start from

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+\alpha^2}=\frac{\pi}{\alpha}+\frac{2\pi}{\alpha}\frac{1}{e^{2\pi\alpha}-1}. \qquad (A40)$$

It follows that

$$\sum_{m,n=1}^{\infty}\left(\frac{1}{n^{4p+2}}+\frac{1}{(\rho m)^{4p+2}}\right)\frac{1}{n^2+\rho^2 m^2}$$

$$=-\frac{1}{2}\,\zeta(4p+4)+\frac{\pi}{2\rho}\,\zeta(4p+3)+\frac{\pi}{\rho}\,N_{4p+3,1}(1/\rho)$$

$$+\frac{1}{\rho^{4p+2}}\left[-\frac{1}{2\rho^2}\,\zeta(4p+4)+\frac{\pi}{2\rho}\,\zeta(4p+3)\right.$$

$$\left.+\frac{\pi}{\rho}\,N_{4p+3,1}(\rho)\right]. \qquad (A41)$$

However, for integer $p\geqslant 0$, we can also compute the first sum as

$$\sum_{m,n=1}^{\infty}\frac{1}{(nm\rho)^{4p+2}}\frac{n^{4p+2}+(\rho m)^{4p+2}}{n^2+(\rho m)^2}$$

$$=\sum_{k=0}^{2p}(-1)^k\sum_{m,n=1}^{\infty}\frac{1}{(mn\rho)^{4p+2}}(\rho m)^{2k}n^{4p-2k}$$

$$=\sum_{k=0}^{2p}(-1)^k\frac{1}{\rho^{4p-2k+2}}\,\zeta(4p-2k+2)\zeta(2k+2). \qquad (A42)$$

Comparing Eqs. (A41) and (A42) we obtain a relation between $N_{4p+3,1}(\rho)$ and $N_{4p+3,1}(1/\rho)$. For $\rho=1$ we obtain

$$N_{4p+3,1}(1)=\frac{1}{2\pi}\,\zeta(4p+4)-\frac{1}{2}\,\zeta(4p+3)$$

$$+\frac{1}{2\pi}\sum_{k=0}^{2p}(-1)^k\zeta(4p-2k+2)\zeta(2k+2). \qquad (A43)$$

Particular cases are

$$N_{3,1}(1)=\frac{7\pi^3}{360}-\frac{1}{2}\,\zeta(3)\approx 0.001871373, \qquad (A44)$$

$$N_{7,1}(1)=\frac{19\pi^7}{113400}-\frac{1}{2}\,\zeta(7)\approx 0.001870964. \qquad (A45)$$

Analogous relations can be obtained starting from the more general sum

$$\sum_{m,n=1}^{\infty}\left(\frac{1}{n^{4p+2}}+\frac{1}{(\rho m)^{4p+2}}\right)\frac{1}{(n^2+\rho^2 m^2)^q}. \qquad (A46)$$

We leave the derivation to the reader. We will not need these relations here.

APPENDIX B: ASYMPTOTIC EXPANSIONS OF LATTICE SUMS

In this appendix we will study generic sums involving the lattice propagator for a Gaussian theory with arbitrary interactions. We will present an algorithmic procedure that allows us to derive systematically the expansion in powers of $1/L$ of sums of this type. The results will be expressed in terms of the functions we have introduced in Appendix A. The method presented in this appendix applies to a square lattice $L\times T$ with periodic boundary conditions but generalizes easily to other types of boundary conditions. It can also be used to study lattice sums in more than two dimensions. In Appendix B 1 we compute some preliminary one-dimensional sums; the general procedure is presented in Appendix B 2.

1. One-dimensional sums

a. The Euler-Mac Laurin formula

The basic tool we will use is the Euler-Mac Laurin formula [74]. In its general form it is given by

$$\sum_{k=n}^{m-1}f(k)=\int_n^m dx\,f(x)-\frac{1}{2}\,[f(m)-f(n)]$$

$$+\sum_{k=1}^{N}\frac{B_{2k}}{(2k)!}\,[f^{(2k-1)}(m)-f^{(2k-1)}(n)]$$

$$+\frac{1}{(2N+1)!}\int_n^m dx\,f^{(2N+1)}(x)B_{2N+1}(x-\lfloor x\rfloor), \qquad (B1)$$

where B_k are the Bernoulli numbers and $B_k(x)$ are the Bernoulli polynomials defined by [73,74]

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}. \qquad (B2)$$

In the following we will be interested in sums of the form

$$\frac{1}{L}\sum_{n=0}^{\alpha L-1}f(\rho), \qquad (B3)$$

where $p=2\pi n/L$. We will try to compute the asymptotic expansion of the sum (B3) for $L\to\infty$ with α fixed. It is easy to obtain from Eq. (B1) the following formula:

$$\frac{1}{L}\sum_{n=0}^{\alpha L-1} f(p)=\int_0^{2\pi\alpha}\frac{dp}{2\pi}f(p)-\frac{1}{2L}[f(2\pi\alpha)-f(0)]$$

$$+\frac{1}{2\pi}\sum_{k=1}^{N}\frac{B_{2k}}{(2k)!}\left(\frac{2\pi}{L}\right)^{2k}$$

$$\times[f^{(2k-1)}(2\pi\alpha)-f^{(2k-1)}(0)]$$

$$+\frac{1}{(2N+1)!}\left(\frac{2\pi}{L}\right)^{2N+1}\int_0^{2\pi\alpha}\frac{dp}{2\pi}f^{(2N+1)}$$

$$\times(p)\hat{B}_{2N+1}(p),\qquad(B4)$$

with

$$\hat{B}_n(p)=B_n\left(\frac{Lp}{2\pi}-\left\lfloor\frac{Lp}{2\pi}\right\rfloor\right).\qquad(B5)$$

Thus, as long as the last integral in Eq. (B4) is finite, i.e., $f^{(2N+1)}(p)$ is integrable in the interval $[0,2\pi\alpha]$, the previous formula gives the asymptotic expansion of the sum (B3) in powers of $1/L$ up to order $1/L^{2N}$. An important case corresponds to $\alpha=1$ and $f(p)$ periodic of period 2π, i.e., $f(p+2\pi)=f(p)$. In this case all the $1/L$ corrections vanish and we obtain

$$\frac{1}{L}\sum_{n=0}^{L-1}f(p)=\int_0^{2\pi}\frac{dp}{2\pi}f(p)+O(L^{-2N-1}).\qquad(B6)$$

It is easy to prove that a similar result holds for generic d-dimensional sums. If $f(p_1,\ldots,p_d)$ is a periodic function in all variables of period 2π and $\partial_1^{n_1}\cdots\partial_d^{n_d}f(p)$ is integrable in $[0,2\pi]^d$ for all $n_1,\ldots,n_d$ such that $n_1+\cdots+n_d\le 2N+1$, then, for $L\to\infty$ with $L_1/L,L_2/L,\ldots,L_d/L$ fixed, we have

$$\frac{1}{L_1\cdots L_d}\sum_{n_1=0}^{L_1-1}\sum_{n_2=0}^{L_2-1}\cdots\sum_{n_d=0}^{L_d-1}f(p_1,\ldots,p_d)$$

$$=\int_{[0,2\pi]^d}\frac{d^dp}{(2\pi)^d}f(p)+O(L^{-2N-1}),\qquad(B7)$$

where on the left-hand side, $p_i=2\pi n_i/L_i$.

b. Asymptotic expansions of Σn^{-p}

Here we will discuss the asymptotic expansion of sums of the form

$$\sum_{n=1}^{L}\frac{1}{n^p},\qquad(B8)$$

for $L\to\infty$. When p is a negative integer it is easy to perform the summation exactly. Indeed $(q=-p)$

$$\sum_{n=1}^{L}n^q=\frac{d^q}{d\alpha^q}\left[\frac{1-e^{\alpha(L+1)}}{1-e^{\alpha}}\right]\Bigg|_{\alpha=0}.\qquad(B9)$$

The simplest cases are

$$\sum_{n=1}^{L}n=\frac{1}{2}L(L+1),\qquad(B10)$$

$$\sum_{n=1}^{L}n^2=\frac{1}{6}L(L+1)(2L+1),\qquad(B11)$$

$$\sum_{n=1}^{L}n^3=\frac{1}{4}L^2(L+1)^2.\qquad(B12)$$

Let us now consider the sum (B8) with $p>1$. Rewriting it as

$$\sum_{n=1}^{L}\frac{1}{n^p}=\zeta(p)-\sum_{n=L+1}^{\infty}\frac{1}{n^p},\qquad(B13)$$

where $\zeta(n)$ is Riemann zeta function, and using the Euler-Mac Laurin formula for the second sum, we get the asymptotic expansion

$$\sum_{n=1}^{L}\frac{1}{n^p}=\zeta(p)-\frac{1}{\Gamma(p)}\sum_{n=0}^{\infty}\frac{B_n}{n!}\Gamma(n+p-1)L^{1-n-p}.\qquad(B14)$$

Finally, taking in the previous formula the limit $p\to 1$, we have the asymptotic expansion

$$\sum_{n=1}^{L}\frac{1}{n}=\log L+\gamma_E+\frac{1}{2L}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n}\frac{1}{L^{2n}},\qquad(B15)$$

where $\gamma_E\approx 0.577215665$ is the Euler constant.

c. Asymptotic expansions of $\Sigma(n^2+\alpha^2)^{k+1/2}$

In this section we consider sums of the form

$$\sum_{n=1}^{L-1}(n^2+\alpha^2)^{k+1/2}.\qquad(B16)$$

Again we want to compute the asymptotic expansion for $L\to\infty$ with α fixed. Let us first consider the case $k\ge -1$. In this case we rewrite the sum as

$$\sum_{n=1}^{L-1}\left[(n^2+\alpha^2)^{k+1/2}-\sum_{m=0}^{k+1}\binom{k+1/2}{m}\alpha^{2m}n^{2k-2m+1}\right]$$

$$+\sum_{m=0}^{k+1}\left[\binom{k+1/2}{m}\alpha^{2m}\sum_{n=1}^{L-1}n^{2k-2m+1}\right].\qquad(B17)$$

We have already explained how to compute the last sums in the previous subsection. We will now discuss the first sum that we rewrite as

$$G_{k+1}(\alpha)-\sum_{n=L}^{\infty}\left[(n^2+\alpha^2)^{k+1/2}\right.$$

$$\left.-\sum_{m=0}^{k+1}\binom{k+1/2}{m}\alpha^{2m}n^{2k-2m+1}\right],\qquad(B18)$$

where $G_k(\alpha)$ is defined in Eq. (A1). The last term appearing in Eq. (B18) can be easily computed using the Euler-Maclaurin formula (B1).

In the following we will need the previous sums for $k= -1,0$. Explicitly we have

$$\sum_{n=1}^{L-1} \sqrt{n^2+\alpha^2} = \frac{1}{2}L(L-1)+G_1(\alpha)+\frac{\alpha^2}{2}(\log L+\gamma_E)-\frac{\alpha^2}{4L}$$
$$+O(L^{-2}), \qquad (B19)$$

$$\sum_{n=1}^{L-1} \frac{1}{\sqrt{n^2+\alpha^2}} = \log L+\gamma_E+G_0(\alpha)-\frac{1}{2L}-\frac{1}{12L^2}$$
$$+\frac{\alpha^2}{4}\left(\frac{1}{L^2}+\frac{1}{L^3}\right)+O(L^{-4}). \qquad (B20)$$

For $k<-1$ the computation is straightforward as no subtraction is needed in this case. For $k=1$ we have

$$\sum_{n=1}^{L-1} \frac{1}{(n^2+\alpha^2)^{3/2}} = H_1(\alpha)-\frac{1}{2L^2}-\frac{1}{2L^3}+O(L^{-4}), \qquad (B21)$$

where $H_1(\alpha)$ is defined in Eq. (A3).

d. Computation of $\Sigma(\hat{p}^2+\alpha^2)^{-q}$

In this section we will compute exactly sums of the form

$$\sum_{n=0}^{L-1} \frac{1}{(\hat{p}^2+\alpha^2)^q}, \qquad (B22)$$

for integer values of q. As usual, $\hat{p}=2\sin(p/2)$.

If q is negative the summation is trivial as (here $k\geq 1$)

$$\sum_{n=0}^{L-1} \hat{p}^{2k} = \binom{2k}{k}L. \qquad (B23)$$

Let us now discuss the case $q\geq 1$. Consider first $q=1$. Then

$$\sum_{n=0}^{L-1} \frac{1}{\hat{p}^2+\alpha^2} = \frac{1}{2}\sum_{n=0}^{L-1} \frac{1}{\omega-\cos(2\pi n/L)}, \qquad (B24)$$

where $\omega=1+\alpha^2/2$. Then notice that

$$\lim_{R\to\infty} \int_{D_R} dz\, \frac{\cot\pi z}{\omega-\cos(2\pi z/L)} = 0, \qquad (B25)$$

where D_R is the rectangle in the complex z plane bounded by the lines $z=-\frac{1}{2}$, $z=L-\frac{1}{2}$, $z=\pm iR$. Using the residue theorem we get

$$\sum_{n=0}^{L-1} \frac{1}{\omega-\cos(2\pi n/L)} = \frac{L}{\sqrt{\omega^2-1}}\coth\left(\frac{L}{2}\,\text{arcch}\,\omega\right). \qquad (B26)$$

We thus end up with

$$\sum_{n=0}^{L-1} \frac{1}{\hat{p}^2+\alpha^2} = \frac{L}{\alpha\sqrt{4+\alpha^2}}\coth\left[L\,\text{arcsh}\left(\frac{\alpha}{2}\right)\right]. \qquad (B27)$$

Higher values of q can be handled by taking derivatives with respect to α^2 of the previous formula.

e. Asymptotic expansion of $\Sigma(\hat{p}^2+m^2)^{k+1/2}$

Let us now consider sums of the form

$$\sum_{n=0}^{L-1} (\hat{p}^2+m^2)^{k+1/2}, \qquad (B28)$$

where $\hat{p}=2\sin(p/2)$, $p=2\pi n/L$. We want to study these sums in the finite-size-scaling limit, i.e., for $L\to\infty$, $m\to 0$, with $mL\equiv z$ fixed. To compute these asymptotic expansions we proceed in the following way.

Assuming L even (the final result will not depend on this assumption) we rewrite

$$\sum_{n=0}^{L-1} (\hat{p}^2+m^2)^{k+1/2} = 2\sum_{n=1}^{L/2-1} (\hat{p}^2+m^2)^{k+1/2}$$
$$+(4+m^2)^{k+1/2}+m^{2k+1}. \qquad (B29)$$

Then let us consider the expansion of $(\hat{p}^2+z^2/L^2)^{k+1/2}$ in powers of $1/L^2$: it can be written in the form

$$\left(\hat{p}^2+\frac{z^2}{L^2}\right)^{k+1/2} = \left(p^2+\frac{z^2}{L^2}\right)^{k+1/2}$$
$$\times\sum_{h=0}^{\infty} \frac{a_h(n^2,z^2)}{L^{4h}}\frac{1}{(p^2+z^2/L^2)h}, \qquad (B30)$$

where $a_h(n^2,z^2)$ is a polynomial in n^2 and z^2. Let us indicate with $R_{k,q}(p,z;L)$ the sum of the first q terms in Eq. (B30). Then we rewrite

$$\sum_{n=0}^{L-1} (\hat{p}^2+m^2)^{k+1/2}$$
$$= 2\sum_{n=1}^{L/2-1} \left[\left(\hat{p}^2+\frac{z^2}{L^2}\right)^{k+1/2}-R_{k,q}(p,z;L)\right]$$
$$+2\sum_{n=1}^{L/2-1} R_{k,q}(p,z;L)+(4+m^2)^{k+1/2}+m^{2k+1}. \qquad (B31)$$

We must then choose q. To fix its value we must decide the order in $1/L$ to which we want to compute the expansion. Then we fix q so that we can use the Euler-Maclaurin formula for the first sum. It is trivial to reduce the second sum to a sum of terms of the form studied in the previous paragraph.

We will now illustrate the method by computing the asymptotic expansion of

$$\sum_{n=0}^{L-1} \frac{1}{\sqrt{\hat{p}^2+m^2}}, \tag{B32}$$

including terms of order $1/L^2$. Since

$$\frac{1}{\sqrt{\hat{p}^2+z^2/L^2}} = \frac{1}{\sqrt{p^2+z^2/L^2}}\left(1+\frac{1}{24}\frac{p^4}{p^2+z^2/L^2}+O(L^{-4})\right), \tag{B33}$$

we rewrite

$$\sum_{n=0}^{L-1} \frac{1}{\sqrt{\hat{p}^2+m^2}}$$

$$= 2\sum_{n=0}^{L/2-1}\left[\frac{1}{\sqrt{\hat{p}^2+m^2}}-\frac{1}{\sqrt{p^2+m^2}}-\frac{1}{24}\frac{p^4}{(p^2+m^2)^{3/2}}\right]$$

$$+\frac{1}{m}+2\sum_{n=1}^{L/2-1}\frac{1}{\sqrt{p^2+m^2}}$$

$$+\frac{1}{12}\sum_{n=1}^{L/2-1}\frac{p^4}{(p^2+m^2)^{3/2}}+\frac{1}{\sqrt{4+m^2}}. \tag{B34}$$

The first sum can be computed up to order $1/L^2$ using the Euler-Maclaurin formula. We obtain

$$\sum_{n=0}^{L/2-1}\left[\frac{1}{\sqrt{\hat{p}^2+m^2}}-\frac{1}{\sqrt{p^2+m^2}}-\frac{1}{24}\frac{p^4}{(p^2+m^2)^{3/2}}\right]$$

$$=L\left[-\frac{1}{2\pi}\log\frac{\pi}{4}-\frac{\pi}{96}-\frac{z^2}{16\pi L^2}\right.$$

$$\left.\times\left(-\frac{1}{12}-\frac{1}{2}\log\frac{\pi}{4}+\frac{2}{\pi^2}\right)\right]$$

$$-\frac{1}{4}+\frac{1}{2\pi}+\frac{\pi}{48}-\frac{z^2}{2L^2}\left(-\frac{1}{16}+\frac{1}{2\pi^3}+\frac{1}{16\pi}\right)$$

$$+\frac{\pi}{6L}\left(\frac{1}{\pi^2}-\frac{1}{24}\right)+O(L^{-3}). \tag{B35}$$

The two remaining sums can be computed using the results of the previous subsection. We obtain finally

$$\sum_{n=0}^{L-1}\frac{1}{\sqrt{\hat{p}^2+m^2}}=\frac{L}{\pi}\left[\log L+\gamma_E-\log\frac{\pi}{2}+G_0(z/2\pi)\right]$$

$$+\frac{L}{z}+\frac{\pi}{6L}\left(G_1(z/2\pi)-\frac{1}{12}\right)$$

$$+\frac{z^4}{96\pi^3 L}H_1(z/2\pi)-\frac{z^2}{16\pi L}$$

$$\times\left(\log L+\gamma_E-\log\frac{\pi}{2}+\frac{4}{3}G_0(z/2\pi)-\frac{1}{6}\right)$$

$$+O(L^{-3}). \tag{B36}$$

We will also need the expansion of Eq. (B29) for $k=0$ up to $O(L^{-3})$. Using the same method we obtain

$$\sum_{n=0}^{L-1}\sqrt{\hat{p}^2+m^2}=\frac{4L}{\pi}-\frac{\pi}{3L}+\frac{4\pi}{L}G_1(z/2\pi)+\frac{z}{L}$$

$$+\frac{z^2}{2\pi L}\left(\log L+\gamma_E-\log\frac{\pi}{2}\right)+O(L^{-3}). \tag{B37}$$

2. Two-dimensional sums

In this section we present our procedure to expand in powers of $1/L$ general sums with Gaussian propagators in the FSS limit. A general theorem for massless propagators was proved in Ref. [75]. Here we will improve their result showing that only even powers of $1/L$ appear in the expansion and providing an algorithmic method to compute the various coefficients.

a. Asymptotic expansions of $\Sigma\hat{p}_x^{2h}\hat{p}_y^{2k}(\hat{p}^2+m^2)^{-q}$

In this section we present a general procedure to derive asymptotic expansions of sums of the form

$$\frac{1}{LT}\sum_{n_x,n_y}\frac{\hat{p}_x^{2h}\hat{p}_y^{2k}}{(\hat{p}^2+m^2)^q}, \tag{B38}$$

where $p_x=2\pi n_x/L$, $p_y=2\pi n_y/T$, the sum extends over $0\leq n_x<L$, $0\leq n_y<T$, in the finite-size-scaling limit, i.e., for $L,T\to\infty$, $m^2\to0$ with $T/L\equiv\rho$ and $mL\equiv z$ fixed.

First of all let us notice that rewriting $\hat{p}_y^{2k}=[(\hat{p}^2+m^2)-\hat{p}_x^2-m^2]^k$ we can limit ourselves to consider sums with $k=0$, i.e., sums of the form

$$\frac{1}{LT}\sum_{n_x,n_y}\frac{\hat{p}_x^{2h}}{(\hat{p}+m^2)^q}. \tag{B39}$$

The summation over n_y can be performed exactly using the results of Appendix B 1 d. It is easy to see that the result will be a sum of terms of the form

$$\frac{1}{L}\sum_{n_x=0}^{L-1}(\hat{p}_x^2+m^2)^{a/2}(4+\hat{p}_x^2+m^2)^{b/2}$$

$$\times\left\{\exp\left[2T\,\text{arcsh}\left(\frac{1}{2}\sqrt{\hat{p}_x^2+m^2}\right)\right]-1\right\}^{-c}, \tag{B40}$$

for integers a, b, and $c\geq0$. If c is strictly positive it is simple to obtain an asymptotic expansion in powers of $1/L^2$. Indeed arcsh $x=0$ if and only if $x=0$. Therefore, for $L,T\to\infty$ the terms that contribute are those for which $\hat{p}\approx0$. Thus rewriting the previous sum as

$$\frac{1}{L}\sum_{n_x=\lfloor L/2\rfloor}^{1-\lfloor(L+1)/2\rfloor}g\left(\frac{2\pi n}{L},\frac{z^2}{L^2},\rho L\right), \tag{B41}$$

we expand the function g in powers of L at n, z, ρ fixed,

$$g\left(\frac{2\pi n}{L}, \frac{z^2}{L^2}, \rho L\right) = L^\alpha \sum_{m=0}^\infty \frac{1}{L^{2m}} \hat{g}_m(n,z^2,\rho). \quad (B42)$$

The expansion of Eq. (B40) is simply given by

$$L^{\alpha-1} \sum_{m=0}^\infty \frac{1}{L^{2m}} \left[\sum_{n=-\infty}^{+\infty} \hat{g}_m(n,z^2,\rho) \right]. \quad (B43)$$

Let us now consider the case $c=0$. If also $b=0$ we have discussed the asymptotic expansion in Appendix B 1 e. Suppose now $b\neq 0$. Then define

$$R_{kl} = 2^k \sum_{n=0}^l \binom{k/2}{n} \left(\frac{\hat{p}_x^2 + m^2}{4}\right)^n, \quad (B44)$$

and rewrite

$$\frac{1}{L} \sum_{n_x=0}^{L-1} (\hat{p}_x^2 + m^2)^{a/2} (4 + \hat{p}_x^2 + m^2)^{b/2}$$

$$= \frac{1}{L} \sum_{n_x=0}^{L-1} (\hat{p}_x^2 + m^2)^{a/2} [(4 + \hat{p}_x^2 + m^2)^{b/2} - R_{bl}]$$

$$+ \frac{1}{L} \sum_{n_x=0}^{L-1} (\hat{p}_x^2 + m^2)^{a/2} R_{bl}. \quad (B45)$$

Then choose l so that one can apply the Euler-Maclaurin formula to the first sum: as the function is periodic of period 2π, as we observed at the end of Appendix B 1 a [see formula (B7)] we can simply replace the sum with the corresponding integral. We thus obtain

$$\frac{1}{L} \sum_{n_x=0}^{L-1} (\hat{p}_x^2 + m^2)^{a/2} (4 + \hat{p}_x^2 + m^2)^{b/2}$$

$$= \int_0^{2\pi} \frac{dp}{2\pi} (\hat{p}_x^2 + m^2)^{a/2} [(4 + \hat{p}_x^2 + m^2)^{b/2} - R_{bl}]$$

$$+ \frac{1}{L} \sum_{n_x=0}^{L-1} (\hat{p}_x^2 + m^2)^{a/2} R_{bl}. \quad (B46)$$

The computation of the remaining sums was discussed in Appendix B 1 e.

To illustrate the method let us consider a specific case, the sum

$$I_{L,T}(m^2) \equiv \frac{1}{LT} \sum_{n_x, n_y} \frac{1}{\hat{p}^2 + m^2}. \quad (B47)$$

Using Eq. (B27) we can perform the summation over n_y obtaining

$$\frac{1}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2 + m^2} \sqrt{4 + \hat{p}_x^2 + m^2}}$$

$$+ \frac{2}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2 + m^2} \sqrt{4 + \hat{p}_x^2 + m^2}}$$

$$\times \left\{ \exp\left[2T \operatorname{arcsh}\left(\frac{1}{2} \sqrt{\hat{p}_x^2 + m^2}\right) \right] - 1 \right\}^{-1}. \quad (B48)$$

The asymptotic expansion of the second sum is immediately computed: we get

$$\frac{1}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2 + m^2} \sqrt{4 + \hat{p}_x^2 + m^2}} \left\{ \exp\left[2T \operatorname{arcsh}\left(\frac{1}{2} \sqrt{\hat{p}_x^2 + m^2}\right) \right] - 1 \right\}^{-1}$$

$$= \frac{1}{4\pi} M_{1,1}(z;\rho) + \frac{\pi}{24L^2} \left[\left(\frac{z}{2\pi}\right)^4 M_{3,1}(z;\rho) - 2\left(\frac{z}{2\pi}\right)^2 M_{1,1}(z;\rho) - 2M_{-1,1}(z;\rho) \right]$$

$$+ \frac{\pi^2 \rho}{12L^2} \left[2M_{-2,1}(z;\rho) - 2\left(\frac{z}{2\pi}\right)^2 M_{0,1}(z;\rho) + \left(\frac{z}{2\pi}\right)^4 M_{2,1}(z;\rho) \right.$$

$$\left. + 2M_{-2,2}(z;\rho) - 2\left(\frac{z}{2\pi}\right)^2 M_{0,2}(z;\rho) + \left(\frac{z}{2\pi}\right)^4 M_{2,2}(z;\rho) \right] + O(L^{-4}). \quad (B49)$$

Let us now consider the first sum. We want to compute its asymptotic expansion including terms of order $1/L^2$. We rewrite it as

$$\frac{1}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2+m^2}} \frac{1}{\sqrt{4+\hat{p}_x^2+m^2}}$$

$$= \frac{1}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2+m^2}} \left[\frac{1}{\sqrt{4+\hat{p}_x^2+m^2}} - \frac{1}{2} \right.$$

$$\left. + \frac{1}{16} (\hat{p}_x^2+m^2) \right] + \frac{1}{2L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2+m^2}}$$

$$- \frac{1}{16L} \sum_{n_x=0}^{L-1} \sqrt{\hat{p}_x^2+m^2}. \tag{B50}$$

The last two sums were discussed in Appendix B 1 e. The first one, up to terms of $O(L^{-4})$, can be replaced by the corresponding integral. Expanding the integrand in powers of m^2 we get

$$\frac{1}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2+m^2}} \left[\frac{1}{\sqrt{4+\hat{p}_x^2+m^2}} - \frac{1}{2} + \frac{1}{16}(\hat{p}_x^2+m^2) \right]$$

$$= \frac{1}{4\pi} (1-\log 2) + \frac{z^2}{64\pi L^2} (1+2\log 2)$$

$$+ O(L^{-4}\log L). \tag{B51}$$

It follows that

$$\frac{1}{L} \sum_{n_x=0}^{L-1} \frac{1}{\sqrt{\hat{p}_x^2+m^2}} \frac{1}{\sqrt{4+\hat{p}_x^2+m^2}}$$

$$= \frac{1}{2z} + \frac{1}{2\pi} \left[\log L + \gamma_E - \log \pi + \frac{1}{2}\log 2 + G_0\left(\frac{z}{2\pi}\right) \right]$$

$$+ \frac{\pi}{6L^2} \left[\frac{1}{12} - G_1\left(\frac{z}{2\pi}\right) \right] - \frac{z}{16L^2} + \frac{z^4}{192\pi^3 L^2} H_1\left(\frac{z}{2\pi}\right)$$

$$- \frac{z^2}{16\pi L^2} \left[\log L + \gamma_E - \log \pi + \frac{1}{2}\log 2 \right.$$

$$\left. + \frac{2}{3} G_0\left(\frac{z}{2\pi}\right) - \frac{1}{3} \right] + O(L^{-4}\log L). \tag{B52}$$

Using Eq. (B49) and the previous expression we obtain the following result:

$$I_{L,T}(m^2) = \frac{1}{2\pi} \log L + F_0(z;\rho) - \frac{z^2}{16\pi L^2} \log L + \frac{1}{L^2} F_1(z;\rho)$$

$$+ O(L^{-4}\log L), \tag{B53}$$

where

$$F_0(z;\rho) = \frac{1}{2z} + \frac{1}{2\pi} \left[\gamma_E - \frac{1}{2}\log\frac{\pi^2}{2} + G_0\left(\frac{z}{2\pi}\right) \right]$$

$$+ \frac{1}{2\pi} M_{1,1}(z;\rho) \tag{B54}$$

and

$$F_1(z;\rho) = \frac{\pi}{6} \left[\frac{1}{12} - G_1\left(\frac{z}{2\pi}\right) \right] - \frac{z}{16} + \frac{z^4}{192\pi^3} H_1\left(\frac{z}{2\pi}\right) - \frac{z^2}{16\pi} \left[\gamma_E - \frac{1}{2}\log\frac{\pi^2}{2} + \frac{2}{3} G_0\left(\frac{z}{2\pi}\right) - \frac{1}{3} \right] + \frac{\pi}{12} \left[\left(\frac{z}{2\pi}\right)^4 M_{3,1}(z;\rho) \right.$$

$$\left. -2\left(\frac{z}{2\pi}\right)^2 M_{1,1}(z;\rho) - 2M_{-1,1}(z;\rho) \right] + \frac{\pi^2\rho}{6} \left[2M_{-2,1}(z;\rho) - 2\left(\frac{z}{2\pi}\right)^2 M_{0,1}(z;\rho) + \left(\frac{z}{2\pi}\right)^4 M_{2,1}(z;\rho) \right.$$

$$\left. + 2M_{-2,2}(z;\rho) - 2\left(\frac{z}{2\pi}\right)^2 M_{0,2}(z;\rho) + \left(\frac{z}{2\pi}\right)^4 M_{2,2}(z;\rho) \right]. \tag{B55}$$

In addition to this sum we will also need

$$\frac{1}{LT}\sum_{n_x,n_y}\frac{\Sigma_\mu \hat{p}_\mu^4}{(\hat{p}^2+m^2)^2}=1-\frac{1}{\pi}+\frac{1}{L^2}\left\{\frac{\pi}{6}-2\pi G_1\left(\frac{z}{2\pi}\right)-\frac{3z}{4}+\frac{z^4}{16\pi^3}H_1\left(\frac{z}{2\pi}\right)\right.$$

$$-\frac{3z^2}{4\pi}\left[\log L-\log\pi+\frac{1}{2}\log 2+\gamma_E-\frac{1}{6}+\frac{2}{3}G_0\left(\frac{z}{2\pi}\right)\right]\right\}$$

$$-\frac{\pi}{L^2}\left[2M_{-1,1}(z;\rho)+2\left(\frac{z}{2\pi}\right)^2 M_{1,1}(z;\rho)-\left(\frac{z}{2\pi}\right)^4 M_{3,1}(z;\rho)\right]$$

$$+\frac{2\pi^2\rho}{L^2}\left[2M_{-2,1}(z;\rho)-2\left(\frac{z}{2\pi}\right)^2 M_{0,1}(z;\rho)+\left(\frac{z}{2\pi}\right)^4 M_{2,1}(z;\rho)\right]$$

$$+2M_{-2,2}(z;\rho)-2\left(\frac{z}{2\pi}\right)^2 M_{0,2}(z;\rho)+\left(\frac{z}{2\pi}\right)^4 M_{2,2}(z;\rho)+O(L^{-4}\log L)$$

$$=1-\frac{1}{\pi}-\frac{3z^2}{4\pi L^2}\log L+\frac{1}{L^2}\left(12F_1(z;\rho)-\frac{z^2}{8\pi}\right)+O(L^{-4}\log L). \tag{B56}$$

Finally we want to report the asymptotic expansions of $F_0(z;\rho)$ and $F_1(z;\rho)$ for $z\to 0$ and $z\to +\infty$. They are obtained using the asymptotic expansions of the functions $G_k(z)$, $H_1(z)$, and $M_{pq}(z;\rho)$ reported in Appendixes A 1 and A 2. For $z\to +\infty$, we obtain

$$F_0(z;\rho)=-\frac{1}{4\pi}\log\frac{z^2}{32}+\frac{e^{-z}}{\sqrt{2\pi z}}[1+O(z^{-1})]$$

$$+\frac{e^{-\rho z}}{\sqrt{2\pi\rho z}}[1+O(z^{-1})], \tag{B57}$$

$$F_1(z;\rho)=\frac{z^2}{32\pi}\left(\log\frac{z^2}{32}+1\right)+\frac{z^3}{24}\frac{e^{-z}}{\sqrt{2\pi z}}[1+O(z^{-1})]$$

$$+\frac{\rho z^3}{24}\frac{e^{-\rho z}}{\sqrt{2\pi\rho z}}[1+O(z^{-1})]. \tag{B58}$$

Let us now consider the perturbative limit. If $\rho\neq\infty$, we obtain for $z\ll 1$, $z\ll 1/\rho$

$$F_0(z;\rho)=\frac{1}{\rho z^2}+F_{00}(\rho)+z^2 F_{01}(\rho)+O(z^4), \tag{B59}$$

$$F_1(z;\rho)=F_{10}(\rho)+z^2 F_{11}(\rho)+O(z^4), \tag{B60}$$

where

$$F_{00}(\rho)=\frac{1}{2\pi}\left(\gamma_E-\log\pi+\frac{1}{2}\log 2\right)-\frac{1}{\pi}\log\eta(i\rho), \tag{B61}$$

$$F_{01}(\rho)=-\frac{1}{16\pi^3}\zeta(3)-\frac{\rho^3}{720}-\frac{1}{8\pi^3}N_{3,1}(\rho)$$

$$-\frac{\rho}{4\pi^2}[N_{2,1}(\rho)+N_{2,2}(\rho)], \tag{B62}$$

$$F_{10}(\rho)=\frac{\pi}{72}-\frac{1}{12\rho}-\frac{\pi}{3}N_{-1,1}(\rho)$$

$$+\frac{2\pi^2\rho}{3}[N_{-2,1}(\rho)+N_{-2,2}(\rho)], \tag{B63}$$

$$F_{11}(\rho)=-\frac{1}{16\pi}\left(\gamma_E-\log\pi+\frac{1}{2}\log 2-\frac{1}{3}\right)-\frac{\rho}{288}$$

$$+\frac{1}{8\pi}\log\eta(i\rho)+\frac{\rho}{12}[N_{0,1}(\rho)+N_{0,2}(\rho)]$$

$$-\frac{\pi\rho^2}{6}[N_{-1,1}(\rho)+3N_{-1,2}(\rho)+2N_{-1,3}(\rho)]$$

$$=-\frac{1}{8}F_{00}(\rho)-\frac{\rho}{4\pi}F_{10}(\rho). \tag{B64}$$

In the last formula we have used Eqs. (A38) and (A39) that also show that $F_{10}(\rho)$ can be expressed in terms of derivatives of $\log\eta(i\rho)$. For $\rho=1$ we obtain the following numerical values:

$$F_{00}(1)\approx 0.04876563317014130, \tag{B65}$$

$$F_{01}(1)\approx -0.00386694659073721, \tag{B66}$$

$$F_{10}(1)\approx -0.02924119479519021, \tag{B67}$$

$$F_{11}(1)\approx -0.00376876379948390. \tag{B68}$$

For the strip ($\rho=\infty$) the previous expansions are not valid. In this case we write

$$F_0(z;\infty)=\frac{1}{2z}+\bar{F}_{00}+z^2\bar{F}_{01}+O(z^4), \qquad (B69)$$

$$F_1(z;\infty)=\frac{\pi}{72}-\frac{z}{16}+z^2\bar{F}_{11}+O(z^4), \qquad (B70)$$

with

$$\bar{F}_{00}=\frac{1}{2\pi}\left(\gamma_E-\log\pi+\frac{1}{2}\log 2\right), \qquad (B71)$$

$$\bar{F}_{01}=-\frac{1}{16\pi^3}\zeta(3), \qquad (B72)$$

$$\bar{F}_{11}=-\frac{1}{16\pi}\left(\gamma_E-\log\pi+\frac{1}{2}\log 2-\frac{1}{3}\right). \qquad (B73)$$

Finally let us comment on the duality property of the functions $F_0(z;\rho)$ and $F_1(z;\rho)$. The sum $I_{L,T}(m^2)$ is clearly symmetric in L,T and thus it is a function $\Phi(mL,T/L,L)$ such that

$$\Phi(mL,T/L,L)=\Phi(mT,L/T,T), \qquad (B74)$$

i.e., $\Phi(z,\rho,L)=\Phi(\rho z,1/\rho,\rho L)$. This implies for the functions $F_0(z;\rho)$ and $F_1(z;\rho)$ the following relations:

$$F_0(z;\rho)=\frac{1}{2\pi}\log\rho+F_0(\rho z;1/\rho), \qquad (B75)$$

$$F_1(z;\rho)=-\frac{z^2}{16\pi}\log\rho+\frac{1}{\rho^2}F_1(\rho z;1/\rho). \qquad (B76)$$

These equations provide a nontrivial check for the correctness of our asymptotic expansions and moreover imply the following relations on the expansion coefficients for $z\to 0$:

$$F_{00}(\rho)=\frac{1}{2\pi}\log\rho+F_{00}(1/\rho), \qquad (B77)$$

$$F_{01}(\rho)=\rho^2 F_{01}(1/\rho), \qquad (B78)$$

$$F_{10}(\rho)=\frac{1}{\rho^2}F_{10}(1/\rho), \qquad (B79)$$

$$F_{11}(\rho)=-\frac{1}{16\pi}\log\rho+F_{11}(1/\rho). \qquad (B80)$$

The duality relation for $F_{00}(\rho)$, $F_{10}(\rho)$, and $F_{11}(\rho)$ can be obtained directly from the inversion property of Dedekind's η function [68]

$$\eta(-1/\tau)^2=-i\tau\eta(\tau)^2, \qquad (B81)$$

where, in our case, we would identify $\tau=i\rho$. To prove directly Eq. (B78) one should use the relation obtained comparing Eq. (A41) with Eq. (A42) for $p=0$.

b. Asymptotic expansion of $\Sigma(w(p)+m^2)^{-1}$

We want to compute here the asymptotic expansion, including terms of order $O(L^{-2})$, of the sum

$$\mathcal{I}_{L,T}(m^2)=\frac{1}{LT}\sum_{n_x,n_y}\frac{1}{w(p)+m^2} \qquad (B82)$$

in the FSS limit. Generic sums of the type (B38) can be easily computed with the same technique. We assume that, for $-\pi\leqslant p_i\leqslant\pi$, $w(p)$ vanishes only for $p=0$ and that in a neighborhood of the origin $w(p)$ has an expansion of the form (2.5). Then we rewrite

$$\mathcal{I}_{L,T}(m^2)=\frac{1}{LT}\sum_p\left[\frac{1}{w(p)+m^2}-\frac{1}{\hat{p}^2+m^2}\right.$$
$$\left.+\frac{\alpha_1\Sigma_\mu\hat{p}_\mu^4+\alpha_2(\hat{p}^2)^2}{(\hat{p}^2+m^2)^2}\right]+\frac{1}{LT}\sum_p\frac{1}{\hat{p}^2+m^2}$$
$$-\frac{1}{LT}\sum_p\frac{\alpha_1\Sigma_\mu\hat{p}_\mu^4+\alpha_2(\hat{p}^2)^2}{(\hat{p}^2+m^2)^2}. \qquad (B83)$$

Since we want to compute $\mathcal{I}_{L,T}(m^2)$ up to $O(L^{-2})$ we can substitute the first sum with the corresponding integral [cf. Eq. (B7)]. Then, expanding the integrand in powers of m^2, we obtain

$$\mathcal{I}_{L,T}(m^2)=\Lambda_0+\alpha_1\left(1-\frac{1}{\pi}\right)+\alpha_2-\frac{z^2}{L^2}\Lambda_1+I_{L,T}(m^2)$$
$$-\frac{1}{LT}\sum_p\frac{\Sigma_\mu\alpha_1\hat{p}_\mu^4+\alpha_2(\hat{p}^2)^2}{(\hat{p}^2+m^2)^2}, \qquad (B84)$$

where we have introduced

$$\Lambda_0=\int\frac{d^2p}{(2\pi)^2}\left(\frac{1}{w(p)}-\frac{1}{\hat{p}^2}\right), \qquad (B85)$$

$$\Lambda_1=\int\frac{d^2p}{(2\pi)^2}\left[\frac{1}{w(p)^2}-\frac{1}{(\hat{p}^2)^2}\right.$$
$$\left.+\frac{2}{(\hat{p}^2)^3}\left(\alpha_1\sum_\mu\hat{p}_\mu^4+\alpha_2(\hat{p}^2)^2\right)\right], \qquad (B86)$$

and we have used the result (see Appendix C of Ref. [39])

$$\int\frac{d^2p}{(2\pi)^2}\frac{\Sigma_\mu\hat{p}_\mu^4}{(\hat{p}^2)^2}=1-\frac{1}{\pi}. \qquad (B87)$$

We obtain eventually, using Eqs. (B56) and (B53),

TABLE IV. Values of Λ_0 and Λ_1 for various Hamiltonians.

	Λ_0	Λ_1
H^{diag}	0.0322658881033520480	0.00371978402668476
H^{Sym}	-0.0471699346329274140	0.00811339924292905
H^{Sym2}	-0.0564354728047190420	0.00331572798724030

$$\mathcal{I}_{L,T}(m^2) = \frac{1}{2\pi}\log L + \mathcal{F}_0(z;\rho) - \frac{z^2}{16\pi L^2}$$

$$\times (1 - 12\alpha_1 - 16\alpha_2)\log L + \frac{1}{L^2}\,\mathcal{F}_1(z;\rho), \tag{B88}$$

where the neglected terms are of order $O(L^{-4}\log L)$ and

$$\mathcal{F}_0(z;\rho) = F_0(z;\rho) + \Lambda_0, \tag{B89}$$

$$\mathcal{F}_1(z;\rho) = (1-12\alpha_1)F_1(z;\rho) + z^2\left(\frac{\alpha_1}{8\pi} - \Lambda_1\right)$$

$$+ 2\alpha_2 z^2 F_0(z;\rho) + \frac{\alpha_2}{2}z^3\frac{\partial F_0}{\partial z}(z;\rho); \tag{B90}$$

$F_0(z;\rho)$ and $F_1(z;\rho)$ are defined in Eqs. (B54) and (B55). Explicit values of Λ_0 and Λ_1 for the Hamiltonians that were introduced in the text are reported in Table IV.

Using the expansions of $F_0(z;\rho)$ and $F_1(z;\rho)$ (see previous section) we can easily obtain the asymptotic expansions of $\mathcal{F}_1(z;\rho)$. For large z we obtain

$$\mathcal{F}_1(z;\rho) = -(12\alpha_1 + 16\alpha_2 - 1)\frac{z^2}{32\pi}\log\frac{z^2}{32} - \frac{z^2}{32\pi}$$

$$\times (8\alpha_1 + 8\alpha_2 - 1) - z^2\Lambda_1 - \frac{1}{24}$$

$$\times (12\alpha_1 + 12\alpha_2 - 1)z^3\left(\frac{e^{-z}}{\sqrt{2\pi z}} + \frac{\rho e^{-\rho z}}{\sqrt{2\pi\rho z}}\right)$$

$$+ O(z^{3/2}e^{-z}, z^{3/2}e^{-\rho z}). \tag{B91}$$

For finite ρ and $z \ll 1$, $z \ll 1/\rho$, neglecting terms of order z^4, we have

$$\mathcal{F}_1(z;\rho) = (1-12\alpha_1)F_{10}(\rho) + \frac{\alpha_2}{\rho} + z^2$$

$$\times\left[(1-12\alpha_1)F_{11}(\rho) + 2\alpha_2 F_{00}(\rho) + \frac{\alpha_1}{8\pi} - \Lambda_1\right], \tag{B92}$$

while on the strip, for $z \to 0$, we obtain

$$\mathcal{F}_1(z;\infty) = \frac{\pi}{72}(1-12\alpha_1) + \frac{z}{16}(12\alpha_1 + 12\alpha_2 - 1) + z^2$$

$$\times\left[(1-12\alpha_1)\bar{F}_{11} + 2\alpha_2\bar{F}_{00} + \frac{\alpha_1}{8\pi} - \Lambda_1\right] + O(z^4). \tag{B93}$$

c. Sum for the tensor correlation length

In this section we describe the computation, in the FSS limit, of

$$I_{2,LT}(m^2) = \frac{1}{LT}\sum_{n_x,n_y}\frac{1}{[\widehat{(p_x - p_0)}^2 + \hat{p}_y^2 + m^2][\hat{p}_x^2 + \hat{p}_y^2 + m^2]}, \tag{B94}$$

where $p_0 = 2\pi/L$ and, as before, $p_x = 2\pi n_x/L$ and $p_y = 2\pi n_y/T$.

First of all we rewrite Eq. (B94) as

$$I_{2,LT}(m^2) = \frac{2}{LT}\sum_{n_x,n_y}\frac{1}{\widehat{(p_x - p_0)}^2 - \hat{p}_x^2}\frac{1}{\hat{p}_x^2 + \hat{p}_y^2 + m^2}; \tag{B95}$$

then we sum over p_y to get

$$\frac{2}{L}\sum_{n=0}^{L-1}\frac{1}{\widehat{(p-p_0)}^2 - \hat{p}^2}\frac{1}{\sqrt{\hat{p}^2 + m^2}\sqrt{4 + \hat{p}^2 + m^2}}$$

$$\times\left\{1 + \frac{2}{\exp[2T\,\text{arcsh}(\frac{1}{2}\sqrt{\hat{p}^2 + m^2})] - 1}\right\}, \tag{B96}$$

where we have simplified the notation using p instead of p_x.

The contribution due to the second term in curly brackets is obtained by simply expanding in powers of $1/L^2$ [see the discussion of Eq. (B40) for $c \neq 0$]. The remaining term requires more care. Assuming L even, we rewrite

$$\frac{2}{L}\sum_{n=0}^{L-1}\frac{1}{\widehat{(p-p_0)}^2 - \hat{p}^2}\frac{1}{\sqrt{\hat{p}^2 + m^2}\sqrt{4 + \hat{p}^2 + m^2}}$$

$$= \frac{2}{L\hat{p}_0^2}\frac{1}{\sqrt{4+m^2}}\left(\frac{1}{m} - \frac{1}{\sqrt{8+m^2}}\right)$$

$$+ \frac{4}{L}\sum_{n=1}^{L/2-1}\frac{1-\hat{p}^2/2}{(\hat{p}_0^2 - 4\sin^2 p)}\frac{1}{\sqrt{\hat{p}^2 + m^2}\sqrt{4 + \hat{p}^2 + m^2}}. \tag{B97}$$

Consider now the last sum and notice that, for $L \to \infty$, $m \to 0$, beside the singularity at $p = 0$, there is an additional singularity at $p = \pi$. Using the fact that

$$\sum_{n=1}^{L/2-1}\frac{1}{\hat{p}_0^2 - 4\sin^2 p} = -\frac{1}{\hat{p}_0^2}, \tag{B98}$$

and keeping only those contributions that do not vanish for $L \to \infty$ we rewrite Eq. (B97) as

$$\frac{2}{z\hat{p}_0^2}\frac{1}{\sqrt{4+m^2}} + \frac{1}{2\sqrt{2}}\frac{1}{L\hat{p}_0^2} + \frac{4}{L}\sum_{n=1}^{L/2-1}\frac{1}{\hat{p}_0^2 - 4\sin^2 p}$$

$$\times\left(\frac{1}{4\sqrt{2}} + \frac{1-\hat{p}^2/2}{\sqrt{\hat{p}^2+m^2}\sqrt{4+\hat{p}^2+m^2}}\right). \qquad \text{(B99)}$$

In this way we have removed the singularity for $p=\pi$. The remaining part of the calculation follows the lines we have presented for Eq. (B40) when $c=0$. We subtract to the sum the first two terms of the asymptotic expansion in $1/L^2$ and then replace the sum with the integral. Explicitly, if we define

$$R(p,p_0,m^2) = \frac{1}{2(p_0^2 - 4p^2)}\left[\frac{1}{\sqrt{p^2+m^2}}\right.$$

$$\left.\times\left(1 + \frac{p_0^2}{12} - \frac{p^2}{4} - \frac{m^2}{6} + \frac{1}{24}\frac{m^4}{p^2+m^2}\right) + \frac{1}{2\sqrt{2}}\right],$$

$$\text{(B100)}$$

we obtain

$$\frac{2}{z\hat{p}_0^2}\frac{1}{\sqrt{4+m^2}} + \frac{1}{2\sqrt{2}}\frac{1}{L\hat{p}_0^2} + 4\int_0^\pi \frac{dp}{2\pi}$$

$$\times\left[-\frac{1}{4\sin^2 p}\left(\frac{1}{4\sqrt{2}} + \frac{1-\hat{p}^2/2}{\hat{p}\sqrt{4+\hat{p}^2}}\right) - R(p,0,0)\right]$$

$$+ \frac{4}{L}\sum_{n=1}^{L/2-1}R(p,p_0,m^2) + O(1/L). \qquad \text{(B101)}$$

The last sum can be dealt with following the strategy of Appendix B 1 c. We finally obtain

$$\frac{2}{L}\sum_{n=0}^{L-1}\frac{1}{(\widehat{p-p_0})^2 - \hat{p}^2}\frac{1}{\sqrt{\hat{p}^2+m^2}\sqrt{4+\hat{p}^2+m^2}}$$

$$= \frac{1}{z}\frac{L^2}{4\pi^2}\left(1 + \frac{\pi^2}{3L^2}\right) + \frac{1}{16\pi}\left(\log L + \gamma_E - \log \pi\right.$$

$$+ \frac{1}{2}\log 2 - \frac{2}{3}\right) + \frac{L^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)\sqrt{4\pi^2 n^2 + z^2}}$$

$$+ \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left\{\frac{1}{(1-4n^2)\sqrt{4\pi^2 n^2 + z^2}}\left[\frac{\pi^2}{3} - \pi^2 n^2 - \frac{z^2}{6}\right.\right.$$

$$\left.+ \frac{1}{24}\frac{z^4}{4\pi^2 n^2 + z^2}\right] - \frac{\pi}{8n}\right\}. \qquad \text{(B102)}$$

Collecting everything together and introducing the functions $\mathcal{H}_k(\alpha)$ and $\mathcal{M}_{pq}(z;\rho)$ defined in Appendixes A 1 and A 2 we have

$$I_{2,LT}(m^2) = L^2 F_3(z;\rho) + \frac{1}{16\pi}\log L + F_4(z;\rho) + O(\log L/L^2),$$

$$\text{(B103)}$$

where

$$F_3(z;\rho) = \frac{1}{4\pi^2 z} + \frac{1}{4\pi^3}\left[\mathcal{H}_0\left(\frac{z}{2\pi}\right) + \mathcal{M}_{1,1}(z;\rho)\right]$$

$$\text{(B104)}$$

and

$$F_4(z;\rho) = \frac{1}{12z} + \frac{1}{16\pi}\left(\gamma_E - \log \pi + \frac{1}{2}\log 2 - \frac{2}{3}\right) - \frac{z}{32\pi^2}$$

$$+ \frac{1}{48\pi^3}(\pi^2 - 2z^2)\mathcal{H}_0\left(\frac{z}{2\pi}\right) + \frac{1}{16\pi}G_0\left(\frac{z}{2\pi}\right)$$

$$+ \frac{1}{24\pi}\left(\frac{z}{2\pi}\right)^4\mathcal{H}_1\left(\frac{z}{2\pi}\right) + \frac{1}{24\pi}[\mathcal{M}_{1,1}(z;\rho)$$

$$+ M_{1,1}(z;\rho)] - \frac{1}{24\pi}\left[2\mathcal{M}_{-1,1}(z;\rho)\right.$$

$$+ 2\left(\frac{z}{2\pi}\right)^2 \mathcal{M}_{1,1}(z;\rho) - \left(\frac{z}{2\pi}\right)^4 \mathcal{M}_{3,1}(z;\rho)\right]$$

$$+ \frac{\rho}{12}\left[2\mathcal{M}_{-2,1}(z;\rho) - 2\left(\frac{z}{2\pi}\right)^2 \mathcal{M}_{0,1}(z;\rho)\right.$$

$$+ \left(\frac{z}{2\pi}\right)^4 \mathcal{M}_{2,1}(z;\rho) + 2\mathcal{M}_{-2,2}(z;\rho)$$

$$\left.- 2\left(\frac{z}{2\pi}\right)^2 \mathcal{M}_{0,2}(z;\rho) + \left(\frac{z}{2\pi}\right)^4 \mathcal{M}_{2,2}(z;\rho)\right].$$

$$\text{(B105)}$$

To conclude this section we give the asymptotic expansions of $F_3(z;\rho)$ and $F_4(z;\rho)$ for large and small values of z. The necessary formulas for the derivations are reported in Appendixes A 1 and A 2. For large z we have

$$F_3(z;\rho) = \frac{1}{4\pi z^2}\left[1 - \frac{2\pi^2}{3z^2} + \frac{8\pi^4}{15z^4} - \frac{16\pi^6}{35z^6} + O(z^{-8})\right],$$

$$\text{(B106)}$$

$$F_4(z;\rho) = -\frac{1}{32\pi}\left(\log\frac{z^2}{32} + 2\right) + \frac{\pi}{48z^2} - \frac{\pi^3}{360z^4} - \frac{\pi^5}{126z^6}$$

$$+ O(z^{-8}). \qquad \text{(B107)}$$

For finite ρ and $z \ll 1$, $z \ll 1/\rho$ we have

$$F_3(z;\rho) = \frac{1}{2\pi^2\rho z^2} + F_{30}(\rho) + O(z^2), \qquad \text{(B108)}$$

$$F_4(z;\rho) = \frac{1}{6\rho z^2} + F_{40}(\rho) + O(z^2),$$

(B109)

where

$$F_{30}(\rho) = \frac{1}{4\pi^3}(1 - 2\log 2) + \frac{\rho}{24\pi^2} + \frac{1}{2\pi^3}\mathcal{N}_{1,1}(\rho),$$

(B110)

$$F_{40}(\rho) = \frac{1}{16\pi}(\gamma_E - \log \pi) - \frac{1}{96\pi}(2 + \log 2) + \frac{\rho}{72}$$

$$- \frac{1}{24\pi^2\rho} + \frac{1}{12\pi}[N_{1,1}(\rho) + \mathcal{N}_{1,1}(\rho) - 2\mathcal{N}_{-1,1}(\rho)]$$

$$+ \frac{\rho}{3}[\mathcal{N}_{-2,1}(\rho) + \mathcal{N}_{-2,2}(\rho)].$$

(B111)

On the strip, for small z, we have

$$F_3(z;\infty) = \frac{1}{4\pi^2 z} + \frac{1}{4\pi^3}(1 - 2\log 2) + O(z^2),$$

(B112)

$$F_4(z;\infty) = \frac{1}{12z} + \frac{1}{16\pi}(\gamma_E - \log \pi) - \frac{1}{96\pi}(2$$

$$+ \log 2) - \frac{z}{32\pi^2} + O(z^2).$$

(B113)

APPENDIX C: ASYMPTOTIC EXPANSION OF LATTICE INTEGRALS

In this section we want to discuss the asymptotic expansion for $m_0^2 \to 0$ of the integrals

$$I_\infty(m_0^2) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{\hat{p}^2 + m_0^2},$$

(C1)

$$\mathcal{I}_\infty(m_0^2) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{w(p) + m_0^2}.$$

(C2)

More general integrals can be discussed following the same method and using the results of Appendix C of Ref. [39]. The expansion of $I_\infty(m_0^2)$ is easily obtained from its expression in terms of elliptic integrals [73]

$$I_\infty(m_0^2) = \frac{2}{\pi} \frac{1}{4 + m_0^2} K\left(\frac{4}{4 + m_0^2}\right) = -\frac{1}{4\pi}\log\frac{m_0^2}{32}$$

$$+ \frac{m_0^2}{32\pi}\left(\log\frac{m_0^2}{32} + 1\right) + O(m_0^4\log m_0^2). \quad \text{(C3)}$$

To obtain the expansion of $\mathcal{I}_\infty(m_0^2)$ let us proceed as in Appendix B 2 b. We rewrite

$$\mathcal{I}_\infty(m_0^2) = \int \frac{dp}{(2\pi)^2}\left[\frac{1}{w(p) + m^2} - \frac{1}{\hat{p}^2 + m^2}\right.$$

$$\left.+ \frac{\alpha_1\Sigma_\mu\hat{p}_\mu^4 + \alpha_2(\hat{p}^2)^2}{(\hat{p}^2 + m^2)^2}\right] + \int \frac{dp}{(2\pi)^2}\frac{1}{\hat{p}^2 + m^2}$$

$$- \int \frac{dp}{(2\pi)^2}\frac{\alpha_1\Sigma_\mu\hat{p}_\mu^4 + \alpha_2(\hat{p}^2)^2}{(\hat{p}^2 + m^2)^2}.$$

(C4)

If we want to compute the expansion neglecting terms of order $O(m_0^4\log m_0^2)$ we can expand the first integral in powers of m_0^2. Then, using (see Appendix C of Ref. [39])

$$\int \frac{dp}{(2\pi)^2}\frac{\Sigma_\mu\hat{p}_\mu^4}{(\hat{p}^2 + m_0^2)^2} = 1 - \frac{1}{\pi} + \frac{m_0^2}{8\pi}\left(3\log\frac{m_0^2}{32} + 2\right)$$

$$+ O(m_0^4\log m_0^2), \quad \text{(C5)}$$

we obtain

$$\mathcal{I}_\infty(m_0^2) = -\frac{1}{4\pi}\log\frac{m_0^2}{32} + \Lambda_0 + \frac{m_0^2}{32\pi}(1 - 12\alpha_1$$

$$- 16\alpha_2)\log\frac{m_0^2}{32} + \frac{m_0^2}{32\pi}(1 - 8\alpha_1 - 8\alpha_2) - m_0^2\Lambda_1,$$

(C6)

where Λ_0 and Λ_1 are defined in Eqs. (B85) and (B86).

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