

ζ -function regularization, the multiplicative anomaly, and finite temperature quantum field theory

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We discuss the role of the multiplicative anomaly for a complex scalar field at finite temperature and density. It is argued that physical considerations must be applied to determine which of the many possible expressions for the effective action obtained by the functional integral method is correct. This is done by first studying the non-relativistic field where the thermodynamic potential is well known. The relativistic case is also considered. We emphasize that the role of the multiplicative anomaly in finite temperature quantum field theory is not to lead to new physics, but rather to preserve the equality among the various expressions for the effective action. [S0556-2821(98)11320-6]

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I. INTRODUCTION

The Feynman functional integral (or path integral) is now the most widely used approach to quantum field theory at zero as well as finite temperature. Of course, it is perfectly possible to use more standard techniques based on operator methods with the Feynman path integral never appearing. What must be the case is that regardless of the approach adopted, all valid methods lead to the same physical consequences. Because of the lack of rigor in defining the Feynman functional integral, care must be exercised in accepting the results of the formal manipulations involved. In the end we can only accept the results of the functional integral approach if they are in agreement with other methods of calculation.

In a one-loop calculation of the effective action for a scalar field theory the result typically involves the determinant of a differential operator, a result which is infinite and must be defined through some regularization technique. If we use the background field method [1], then the one-loop effective action reads

$$\Gamma = S[\bar{\phi}] + \frac{1}{2} \ln \det(l^2 S_{,ij}[\bar{\phi}]) \quad (1.1)$$

where $\bar{\phi}$ denotes the background field, $S[\bar{\phi}]$ is the classical action functional, and

$$S_{,ij}[\bar{\phi}] = \frac{\delta^2 S[\bar{\phi}]}{\delta \bar{\phi}^i(x) \delta \bar{\phi}^j(x')}. \quad (1.2)$$

The second term of Eq. (1.1) contains the quantum corrections to the classical theory (at one-loop order) and arises from performing the functional integral over a Gaussian. l in Eq. (1.1) is a unit of length (the renormalization scale) introduced in order that the argument of the logarithm in Eq. (1.1) be dimensionless. In finite temperature field theory it is con-

venient to adopt the imaginary time formalism in which the path integral extends over all fields periodic in time with period $\beta = 1/T$ with T the temperature. We choose to work with real scalar fields.

The problem of evaluating the second term of Eq. (1.1) now arises. To be explicit we will consider the case of a single complex scalar field at finite charge density. Writing the complex field in terms of its real and imaginary parts we have [2-6]

$$\begin{aligned} S[\bar{\phi}] = & \int_0^\beta dt \int_\Sigma d\sigma_x \left(\frac{1}{2} (\dot{\bar{\phi}}_1 - ie\mu\bar{\phi}_2)^2 \right. \\ & + \frac{1}{2} (\dot{\bar{\phi}}_2 + ie\mu\bar{\phi}_1)^2 + \frac{1}{2} |\nabla\bar{\phi}_1|^2 + \frac{1}{2} |\nabla\bar{\phi}_2|^2 \\ & \left. + \frac{1}{2} m^2 (\bar{\phi}_1^2 + \bar{\phi}_2^2) + \frac{\lambda}{4!} (\bar{\phi}_1^2 + \bar{\phi}_2^2)^2 \right). \end{aligned} \quad (1.3)$$

Here Σ is the spatial part of the spacetime, e is the electronic charge and μ is the chemical potential. If we choose $\bar{\phi}_1 = \bar{\phi}$ and $\bar{\phi}_2 = 0$ as the background fields it is easy to show that

$$\frac{\delta^2 S[\bar{\phi}]}{\delta \bar{\phi}_1(x) \delta \bar{\phi}_1(x')} = \left(-\square_x + m^2 - e^2 \mu^2 + \frac{\lambda}{2} \bar{\phi}^2 \right) \delta(x, x'), \quad (1.4)$$

$$\frac{\delta^2 S[\bar{\phi}]}{\delta \bar{\phi}_2(x) \delta \bar{\phi}_2(x')} = \left(-\square_x + m^2 - e^2 \mu^2 + \frac{\lambda}{6} \bar{\phi}^2 \right) \delta(x, x'), \quad (1.5)$$

$$\frac{\delta^2 S[\bar{\phi}]}{\delta \bar{\phi}_1(x) \delta \bar{\phi}_2(x')} = - \frac{\delta^2 S[\bar{\phi}]}{\delta \bar{\phi}_2(x) \delta \bar{\phi}_1(x')} = 2ie\mu \frac{\partial}{\partial t} \delta(x, x'). \quad (1.6)$$

There are now several approaches one could take to evaluate $\ln \det(l^2 S_{,ij}[\bar{\phi}])$. To illustrate this we will simplify the prob-

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lem by taking $\lambda=0$, so that the theory is free. We will comment on the interacting case at the end of our paper.

The first way is to take the determinant over the 2×2 matrix first to obtain

$$\begin{aligned} \Gamma_A^{(1)} &= \frac{1}{2} \ln \det l^2 (l^2 S_{,ij}[\bar{\phi}]) \\ &= \frac{1}{2} \ln \det l^4 \left((-\square_x + m^2 - e^2 \mu^2)^2 - 4e^2 \mu^2 \frac{\partial^2}{\partial t^2} \right). \end{aligned} \quad (1.7)$$

The remaining determinant in this last expression is understood to be a functional one. At this stage we note that this step has been criticized recently by Dowker [7] who claims that it is incorrect to take the 2×2 determinant first. As we will show in the next section, this need not be an incorrect step. The relativistic case will be considered in Sec. III.

A second expression for $\Gamma^{(1)}$ is obtained by diagonalizing the functional matrix $S_{,ij}[\bar{\phi}]$ by a transformation of the fields to give

$$\begin{aligned} \Gamma_B^{(1)} &= -\frac{1}{2} \ln \det l^2 \left[-\square_x + m^2 - e^2 \mu^2 + 2ie\mu \frac{\partial}{\partial t} \right] \\ &\quad - \frac{1}{2} \ln \det l^2 \left[-\square_x + m^2 - e^2 \mu^2 - 2ie\mu \frac{\partial}{\partial t} \right]. \end{aligned} \quad (1.8)$$

This can also be understood to correspond to a particular factorization of Eq. (1.7).

A third way to evaluate the effective action is to integrate over one of the fields, say ϕ_1 , first, and then perform the remaining integral over ϕ_2 . This leads to a result

$$\begin{aligned} \Gamma_C^{(1)} &= -\frac{1}{2} \ln \det l^2 (-\square_x + m^2 - e^2 \mu^2) \\ &\quad - \frac{1}{2} \ln \det l^2 \left[(-\square_x + m^2 - e^2 \mu^2) - 4e^2 \mu^2 \frac{\partial^2}{\partial t^2} \right] \\ &\quad \times (-\square_x + m^2 - e^2 \mu^2)^{-1}. \end{aligned} \quad (1.9)$$

The inverse operator $(-\square_x + m^2 - e^2 \mu^2)^{-1}$ can be understood to be the Green function. This expression is equivalent to factoring out $(-\square_x + m^2 - e^2 \mu^2)$ from the determinant in Eq. (1.7).

At the formal level, by adopting the usual rules for the manipulation of finite dimensional matrices, all of the three expressions for the effective action are identical. However, as pointed out in Refs. [8, 9] based on the application of the Wodzicki residue in Ref. [10], the usual formal manipulations are vitiated by the presence of an anomaly, called the multiplicative anomaly. The essential feature is that this multiplicative anomaly leads to a difference between the three expressions for the effective action $\Gamma_{A,B,C}^{(1)}$. The question arises as to which, if any, of the three expressions is correct. Our viewpoint is that this issue cannot be settled by consid-

ering the formal Feynman functional integral without recourse to physics. We will justify this in the subsequent sections. In Sec. II we will consider the simpler non-relativistic theory where there is no question as to the correct thermodynamics. In Sec. III we will return to the relativistic field.

II. NON-RELATIVISTIC THEORY

The action functional for a non-relativistic Schrödinger field Ψ is (in the imaginary time formalism)

$$\begin{aligned} S[\Psi, \Psi^\dagger] &= \int_0^\beta dt \int_\Sigma d\sigma_x \left(\frac{1}{2} \left(\Psi^\dagger \frac{\partial \Psi}{\partial t} - \frac{\partial \Psi^\dagger}{\partial t} \Psi \right) \right. \\ &\quad \left. + \frac{1}{2m} |\nabla \Psi|^2 - \mu |\Psi|^2 \right). \end{aligned} \quad (2.1)$$

We will not include self-interactions here for simplicity. The theory defined by Eq. (2.1) is already complicated enough to demonstrate the discrepancy between different approaches. Rather than deal with the complex field Ψ we can decompose Ψ into its real and imaginary parts and define

$$\Psi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2). \quad (2.2)$$

This gives

$$\begin{aligned} S[\phi_1, \phi_2] &= \int_0^\beta dt \int_\Sigma d\sigma_x \left(\frac{i}{2} (\phi_1 \dot{\phi}_2 - \phi_2 \dot{\phi}_1) + \frac{1}{4m} |\nabla \phi_1|^2 \right. \\ &\quad \left. + \frac{1}{4m} |\nabla \phi_2|^2 - \frac{1}{2} \mu (\phi_1^2 + \phi_2^2) \right). \end{aligned} \quad (2.3)$$

If we perform the Feynman functional integral over the two real fields in Eq. (2.3) it is easy to see that the one-loop part of Γ is

$$\Gamma^{(1)} = \frac{1}{2} \ln \det l^2 \begin{pmatrix} -\frac{1}{2m} \nabla^2 - \mu & i \frac{\partial}{\partial t} \\ -i \frac{\partial}{\partial t} & \frac{1}{2m} \nabla^2 - \mu \end{pmatrix} \quad (2.4)$$

$$= \frac{1}{2} \ln \det l^4 \left[\left(-\frac{1}{2m} \nabla^2 - \mu \right)^2 - \frac{\partial^2}{\partial t^2} \right]. \quad (2.5)$$

In the second line above we have taken the determinant of the 2×2 matrix which we claim is a valid step and that Eq. (2.5) is the correct answer. (This will be justified below.)

In order to define Eq. (2.5) we make use of ζ -function regularization. Let $\{f_n(x)\}$ be a complete orthonormal set of solutions to

$$-\nabla^2 f_n(x) = \sigma_n f_n(x). \quad (2.6)$$

σ_n represent the eigenvalues of the Laplacian for $f_n(x)$ obeying whatever boundary conditions apply to the space Σ . Because the fields ϕ_1 and ϕ_2 are periodic in imaginary time, the eigenvalues of the operator in Eq. (2.5) are $(\sigma_n/2m - \mu)^2 + \omega_j^2$ where

$$\omega_j = \frac{2\pi j}{\beta} \quad (2.7)$$

with $j=0, \pm 1, \pm 2, \dots$. We have the formal result

$$\Gamma^{(1)} = \frac{1}{2} \sum_{j,n} \ln l^4 \left[\left(\frac{\sigma_n}{2m} - \mu \right)^2 + \omega_j^2 \right]. \quad (2.8)$$

To give meaning to Eq. (2.8) we introduce the generalized ζ-function [11–13] defined by

$$\zeta(s) = \sum_{j=-\infty}^{+\infty} \sum_n \left[\left(\frac{\sigma_n}{2m} - \mu \right)^2 + \omega_j^2 \right]^{-s}. \quad (2.9)$$

$\Gamma^{(1)}$ is then defined to be

$$\Gamma^{(1)} = -\frac{1}{2} \zeta'(0) + \frac{1}{2} \zeta(0) \ln l^4. \quad (2.10)$$

Knowledge of $\zeta(s)$ in a neighborhood of $s=0$ will therefore give us an expression for the one-loop effective action.

The ζ-function is considered in the Appendix. If we take $\mu=0$ in Eq. (A1) and also $E_n = \sigma_n/2m - \mu$ then we may deduce $\zeta(0)$ and $\zeta'(0)$ from Eq. (A7) once we have evaluated the energy ζ-function of the first term. From Eq. (A4) we have

$$E(\alpha) = \sum_n \left(\frac{\sigma_n}{2m} - \mu \right)^{1-\alpha} \quad (2.11)$$

in this case. If we now specialize Σ to be flat space with the large volume limit taken, then we have

$$\sigma_n \rightarrow k^2, \quad (2.12)$$

$$\sum_n \rightarrow V \int \frac{d^3k}{(2\pi)^3}. \quad (2.13)$$

It is now easy to show that

$$E(\alpha) = V \frac{\Gamma(\alpha - 5/2)}{\Gamma(\alpha - 1)} \left(\frac{m}{2\pi} \right)^{3/2} (-\mu)^{5/2-\alpha}. \quad (2.14)$$

Assuming that $E(2s)$ is analytic at $s=0$, referring to Eq. (A7) we find

$$\zeta(0) = 0, \quad (2.15)$$

$$\zeta'(0) = -\beta E(0) - 2 \sum_n \ln[1 - e^{-\beta(\sigma_n/2m - \mu)}]. \quad (2.16)$$

We therefore have, from Eq. (2.10),

$$\Gamma^{(1)} = \frac{1}{2} \beta E(0) + \sum_n \ln[1 - e^{-\beta(\sigma_n/2m - \mu)}]. \quad (2.17)$$

The first term in $\Gamma^{(1)}$ is the regularized zero-point energy. In the simple case we are considering here, $E(0)=0$, so that the zero-point energy makes no contribution. We are left with the second term of Eq. (2.17) which is in agreement with what is written down in standard statistical mechanics. We can therefore be certain that Eq. (2.5) and ζ-function regularization have led to results which agree with those found by other methods. If we use Eqs. (2.12) and (2.13) in Eq. (2.17) we find

$$\Gamma^{(1)} = -V \left(\frac{m}{2\pi\beta} \right)^{3/2} Li_{5/2}(e^{\beta\mu}) \quad (2.18)$$

where

$$Li_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p} \quad (2.19)$$

defines the polylogarithm. This result may now be used to discuss Bose-Einstein condensation of the ideal gas. (See Ref. [14] for example.)

In order to see what can go wrong, suppose that we return to Eq. (2.5). This time, by noting that

$$\begin{aligned} & \left(-\frac{1}{2m} \nabla^2 - \mu^2 \right)^2 - \frac{\partial^2}{\partial t^2} \\ &= \left(-\frac{1}{2m} \nabla^2 - \mu - \frac{\partial}{\partial t} \right) \left(-\frac{1}{2m} \nabla^2 - \mu + \frac{\partial}{\partial t} \right) \end{aligned} \quad (2.20)$$

we will write

$$\begin{aligned} \tilde{\Gamma}^{(1)} &= \frac{1}{2} \ln \det l^2 \left(-\frac{1}{2m} \nabla^2 - \mu - \frac{\partial}{\partial t} \right) \\ &+ \frac{1}{2} \ln \det l^2 \left(-\frac{1}{2m} \nabla^2 - \mu + \frac{\partial}{\partial t} \right). \end{aligned} \quad (2.21)$$

Formally Eqs. (2.21) and (2.5) are the same, since we have simply written the determinant of a product as a product of determinants. However, as pointed out in Refs. [8, 9] this may not be justified for differential operators. Because of the simplicity of the model, we can evaluate $\tilde{\Gamma}^{(1)}$ in Eq. (2.21) explicitly and see if it agrees with $\Gamma^{(1)}$, which we know to be correct.

We will define the generalized ζ-function

$$\tilde{\zeta}(s) = \sum_{j=-\infty}^{+\infty} \sum_n \left(\frac{\sigma_n}{2m} - \mu + i\omega_j \right)^{-s}. \quad (2.22)$$

Because the only difference between the two terms in Eq. (2.21) lies in the sign of t , and the sum over j in Eq. (2.22) goes from $-\infty$ to $+\infty$ it is easy to see that

$$\tilde{\Gamma}^{(1)} = -\tilde{\zeta}'(0) + \tilde{\zeta}(0) \ln l^2. \quad (2.23)$$

This is also the result which would be obtained by performing the Feynman functional integral over the complex field Ψ using Eq. (2.1). (See Ref. [15].)

From Eqs. (2.12) and (2.13) we have

$$\begin{aligned} \tilde{\zeta}(s) &= V \frac{\Gamma(s-3/2)}{\Gamma(s)} \left(\frac{m}{2\pi}\right)^{3/2} \\ &\times \sum_{j=-\infty}^{+\infty} (i\omega_j - \mu)^{3/2-s} \end{aligned} \quad (2.24)$$

after performing the integral over k . The sum in Eq. (2.24) was evaluated in the Appendix of [15]. Making use of this result we find

$$\begin{aligned} \tilde{\zeta}(s) &= V \frac{\Gamma(s-3/2)}{\Gamma(s)} \left(\frac{m}{2\pi}\right)^{3/2} (-\mu)^{3/2-s} \\ &+ V \left(\frac{m}{2\pi\beta}\right)^{3/2} \frac{\beta^s}{\Gamma(s)} Li_{5/2-s}(e^{\beta\mu}). \end{aligned} \quad (2.25)$$

Expansion about $s=0$ shows that

$$\tilde{\zeta}(0) = 0, \quad (2.26)$$

$$\tilde{\zeta}'(0) = \frac{4}{3} \sqrt{\pi} V \left(\frac{m}{2\pi}\right)^{3/2} (-\mu)^{3/2} + V \left(\frac{m}{2\pi\beta}\right)^{3/2} Li_{5/2}(e^{\beta\mu}). \quad (2.27)$$

We therefore find

$$\tilde{\Gamma}^{(1)} = -\frac{4}{3} \sqrt{\pi} V \left(-\frac{m\mu}{2\pi}\right)^{3/2} - V \left(\frac{m}{2\pi\beta}\right)^{3/2} Li_{5/2}(e^{\beta\mu}) \quad (2.28)$$

with the definition (2.21).

Comparison of Eq. (2.28) with the correct result (2.18) shows that

$$\Gamma^{(1)} - \tilde{\Gamma}^{(1)} = \frac{4}{3} \sqrt{\pi} V \left(-\frac{m\mu}{2\pi}\right)^{3/2}. \quad (2.29)$$

The results are different. It might be thought that the difference (2.29) is just a trivial constant which can lead to no physical consequences; however this is incorrect. The difference depends on the chemical potential μ and can affect thermodynamic quantities, such as the particle number or internal energy. Use of $\tilde{\Gamma}^{(1)}$ rather than $\Gamma^{(1)}$ will lead to results which are not in agreement with those found from statistical mechanics, and therefore not in agreement with observations of physical systems. In particular, the particle number follows from the normal Bose-Einstein distribution function as

$$N = \sum_n [e^{\beta(\sigma_n/2m - \mu)} - 1]^{-1} \quad (2.30)$$

$$= V \left(\frac{m}{2\pi\beta}\right)^{3/2} Li_{3/2}(e^{\beta\mu}). \quad (2.31)$$

We also must have (since Γ is related to the Helmholtz free energy)

$$N = -\frac{1}{\beta} \left(\frac{\partial \Gamma^{(1)}}{\partial \mu}\right)_{\beta, V}. \quad (2.32)$$

The result (2.18) used in Eq. (2.32) agrees with Eq. (2.31), but if we use the result (2.28) in Eq. (2.32) we do not get the correct particle number. The only conclusion to be drawn is that Eq. (2.28) which arose from Eq. (2.21) is not correct.

This leads us to the role of the multiplicative anomaly. As discussed in [8,9] on general grounds we would not expect Eqs. (2.21) and (2.5) to agree, since $\det(A_1 A_2)$ will not be the same as $(\det A_1)(\det A_2)$ if A_1 and A_2 are differential operators. The difference between $\det(A_1 A_2)$ and $(\det A_1)(\det A_2)$ is called the multiplicative anomaly. Our interpretation of the multiplicative anomaly is not that it is irrelevant to the physics as suggested by Evans [16] and Dowker [7], but rather that it is crucial for obtaining the correct physics. (It should be noted that the authors of Refs. [8, 9] have also responded [17,18] to the criticisms in Refs. [7, 16].) When we factored the differential operator in Eq. (2.5) to obtain Eq. (2.21) we should have included an additional term as discussed in [8,9]. The role of this term, which is the multiplicative anomaly, is to ensure that no matter how we handle Eq. (2.5) we end up with the correct result (2.18). Thus the multiplicative anomaly precisely cancels the first term of Eq. (2.28) resulting in the same answer for the effective action. This does not lead to any new physics in the case we have considered, but nevertheless does illustrate the importance of the multiplicative anomaly.

III. RELATIVISTIC FIELD

As we have already explained, the Feynman functional integral method should lead to a result for the effective action which is in agreement with other methods. Accordingly, we will first use more standard methods to determine what the effective action should be, and then see what different functional integral expressions lead to.

A. Thermodynamics and the charge

The effective action is related to the Helmholtz free energy, and can therefore be used in a standard way using thermodynamical relations to derive various physical results. Conversely, we can use known results for physical quantities and the thermodynamical relations to tell us the effective action.

Of particular importance to us is the total charge Q which is given by

$$Q = -\frac{1}{\beta} \frac{\partial \Gamma}{\partial \mu} \Big|_{\beta, V}. \quad (3.1)$$

Because a complex scalar field contains particles (of charge e) and antiparticles (of charge $-e$) with the Bose-Einstein distribution function, we must have

$$Q = e \sum_n \{ [e^{\beta(E_n - e\mu)} - 1]^{-1} - [e^{\beta(E_n + e\mu)} - 1]^{-1} \}. \quad (3.2)$$

Here E_n are the energy eigenvalues. We have

$$E_n = (\sigma_n + m^2)^{1/2} \quad (3.3)$$

with σ_n the eigenvalues of $-\nabla^2$. If we now use Eq. (3.2) in Eq. (3.1) and integrate with respect to μ keeping β and V fixed we find

$$\Gamma = \sum_n \{ \ln[1 - e^{-\beta(E_n - e\mu)}] + \ln[1 - e^{-\beta(E_n + e\mu)}] \} + \Gamma_1 \quad (3.4)$$

where Γ_1 is independent of μ . We can fix Γ_1 by setting $\mu = 0$ and demanding that Γ be the result for an uncharged complex scalar field. This fixes Γ_1 to be

$$\Gamma_1 = \sum_n \beta E_n. \quad (3.5)$$

(Γ_1 must of course be regularized). Γ_1 has the interpretation of the zero-point energy contribution to Γ . We end up with the expression

$$\Gamma = \sum_n \{ \beta E_n + \ln[1 - e^{-\beta(E_n - e\mu)}] + \ln[1 - e^{-\beta(E_n + e\mu)}] \} \quad (3.6)$$

which is equivalent to the result found in Ref. [3]. It can be expressed as

$$\Gamma = \Gamma_+ + \Gamma_- \quad (3.7)$$

where

$$\Gamma_{\pm} = \sum_n \left\{ \frac{1}{2} \beta (E_n \mp e\mu) + \ln[1 - e^{-\beta(E_n \mp e\mu)}] \right\}. \quad (3.8)$$

Γ_+ has the interpretation of the particle contribution and Γ_- the antiparticle contribution. Apart from the different relativistic expression for E_n , each of the terms Γ_+ and Γ_- has the same form as we found in the non-relativistic case in Eq. (2.17).

We will call Eq. (3.6) the correct expression for the effective action. It leads, via Eq. (3.1), to the standard result (3.2) for the charge. In addition, the internal energy

$$U = \left. \frac{\partial \Gamma}{\partial \beta} \right|_{\beta, \mu, V} \quad (3.9)$$

takes the familiar form

$$U = \sum_n E_n \{ 1 + [e^{\beta(E_n - e\mu)} - 1]^{-1} + [e^{\beta(E_n + e\mu)} - 1]^{-1} \}. \quad (3.10)$$

In the zero temperature limit, the net contribution to U is the zero-point energy contribution, which has its origin in the first term of Eq. (3.6).

B. Partition function and canonical quantization

The standard expression for the partition function is

$$Z = \text{tr} e^{-\beta(H - \mu Q)} \quad (3.11)$$

with H the Hamiltonian operator and Q the charge operator. The Hamiltonian operator may be expressed as

$$H = \sum_n H_n \quad (3.12)$$

with

$$H_n = E_n \left(a_n^\dagger a_n + \frac{1}{2} + b_n^\dagger b_n + \frac{1}{2} \right). \quad (3.13)$$

a_n, a_n^\dagger are the annihilation and creation operators for particles and b_n, b_n^\dagger are those for antiparticles. They satisfy the standard commutation relations for bosons. The charge operator is

$$Q = \sum_n Q_n \quad (3.14)$$

where

$$Q_n = e(a_n^\dagger a_n - b_n^\dagger b_n). \quad (3.15)$$

It is worth noting that the charge operator has been normal ordered. It is the normal ordered expression which leads to Eq. (3.2).

Because $[H_n, H_{n'}] = 0 = [Q_n, Q_{n'}]$ we can write

$$Z = \prod_n Z_n \quad (3.16)$$

where

$$Z_n = \text{tr} e^{-\beta(H_n - \mu Q_n)}. \quad (3.17)$$

This is a standard manipulation. Finally we can compute Z_n by noting that $a_n^\dagger a_n$ is the particle number operator and $b_n^\dagger b_n$ is the antiparticle number operator. Hence we have

$$\begin{aligned} Z_n &= \sum_{n'=0}^{\infty} \sum_{n''=0}^{\infty} e^{-\beta E_n - \beta(E_n - e\mu)n' - \beta(E_n + e\mu)n''} \\ &= e^{-\beta E_n} [1 - e^{-\beta(E_n - e\mu)}]^{-1} [1 - e^{-\beta(E_n + e\mu)}]^{-1}. \end{aligned} \quad (3.18)$$

Because the effective action Γ is related to the partition function Z by

$$\Gamma = -\ln Z, \quad (3.19)$$

we arrive at

$$\Gamma = \sum_n \{ \beta E_n + \ln[1 - e^{-\beta(E_n - e\mu)}] + \ln[1 - e^{-\beta(E_n + e\mu)}] \}. \quad (3.20)$$

This is consistent with the previous result (3.6).

Thus we can have some faith in the expression (3.20) which has been arrived at using standard methods which do not involve functional integrals or ζ -function regularization, as the correct result for the effective action.

C. Functional integral approach

We now refer back to the result for $\Gamma^{(1)}$ found from the functional integral in Eq. (1.7). The determinant can be defined using ζ -function regularization. We have

$$\Gamma_A^{(1)} = -\frac{1}{2} \zeta'_A(0) + \frac{1}{2} \zeta_A(0) \ln l^4 \quad (3.21)$$

where

$$\zeta_A(s) = \sum_{j=-\infty}^{+\infty} \sum_n [(\omega_j^2 + \sigma_n + m^2 - e^2 \mu^2)^2 + 4e^2 \mu^2 \omega_j^2]^{-s}. \quad (3.22)$$

(Recall that σ_n are the eigenvalues of $-\nabla^2$.)

Alternatively we could factor the fourth order differential operator in Eq. (1.7) into a product of two second order operators, and define $\Gamma_B^{(1)}$ as in Eq. (1.8). If we define

$$\zeta_B(s) = \sum_{j=-\infty}^{\infty} \sum_n [(\omega_j + ie\mu)^2 + \sigma_n + m^2]^{-s} \quad (3.23)$$

then it is easy to see that

$$\Gamma_B^{(1)} = -\zeta'_B(0) + \zeta_B(0) \ln l^2 \quad (3.24)$$

is the ζ -function regularized expression. [Both terms in Eq. (1.8) involve the same ζ -function. Because the sum on j in Eq. (3.23) extends from $-\infty$ to $+\infty$ the sign of μ in each term is not relevant and each of the two terms gives an identical contribution]. This result, $\Gamma_B^{(1)}$, is the one used in [5,6], and is equivalent to that used in [2-4,19].

The third result we discussed in Sec. I was more complicated and was given in Eq. (1.9). This time we can define two ζ -functions

$$\zeta_C^{(1)}(s) = \sum_{j=-\infty}^{+\infty} \sum_n (\omega_j^2 + \sigma_n + m^2 - e^2 \mu^2)^{-s}, \quad (3.25)$$

$$\zeta_C^{(2)}(s) = \sum_{j=-\infty}^{+\infty} \sum_n \left[(\omega_j^2 + \sigma_n + m^2 - e^2 \mu^2) + \frac{4e^2 \mu^2 \omega_j^2}{(\omega_j^2 + \sigma_n + m^2 - e^2 \mu^2)} \right]^{-s}. \quad (3.26)$$

The regularized result for Eq. (1.9) reads

$$\Gamma_C^{(1)} = -\frac{1}{2} \zeta_C^{(1)'}(0) + \frac{1}{2} \zeta_C^{(1)}(0) \ln l^2 - \frac{1}{2} \zeta_C^{(2)'}(0) + \frac{1}{2} \zeta_C^{(2)}(0) \ln l^2. \quad (3.27)$$

The results for $\Gamma_B^{(1)}$ and $\Gamma_C^{(1)}$ may be regarded as different ways of factoring Eq. (1.7), although $\Gamma_C^{(1)}$ has the functional integral interpretation we have mentioned. If we manipulate the formal unregularized expressions for $\Gamma_{A,B,C}^{(1)}$ then they are all identical. However the ζ -function regularized results do not share this equality, as pointed out in [9]. The only way to decide which, if any, of the expressions for $\Gamma^{(1)}$ is correct is by comparison with a result which does not have any ambiguity. We have found in Secs. III A and III B that the standard result (3.20) holds. (Of course we have only written down three possible expressions for Γ , and there are many other ways to evaluate the functional integral.)

We turn first to $\Gamma_B^{(1)}$ because it is the easiest to evaluate, and as we shall show leads to a result in agreement with Eq. (3.20). We will first show this formally using the result of the Appendix, specifically Eq. (A7). To do this we need to know the behavior of the energy ζ -function $E(2s)$ near $s=0$. We have

$$E(2s) = \sum_n E_n^{1-2s} = \sum_n (\sigma_n + m^2)^{1/2-2s}. \quad (3.28)$$

Although it is possible to proceed generally, without knowing the eigenvalues σ_n of $-\nabla^2$ explicitly, we will for simplicity specialize to flat space in the infinite volume limit. In this case (3.28) becomes

$$E(2s) = V \int \frac{d^3k}{(2\pi)^3} (k^2 + m^2)^{1/2-2s} = \frac{V}{8\pi\sqrt{\pi}} \frac{\Gamma(s-2)}{\Gamma(s-1/2)} (m^2)^{2-s}. \quad (3.29)$$

The first term of Eq. (A7) therefore involves

$$\frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} E(2s) = \frac{\beta V}{16\pi^2} \frac{(m^2)^{2-s}}{(s-1)(s-2)} \quad (3.30)$$

$$= \frac{\beta V}{32\pi^2} m^4 \left[1 + \frac{3}{2} s - s \ln m^2 + \dots \right] \quad (3.31)$$

when expanded about $s=0$. We can use Eq. (A7) to conclude that

$$\zeta_B(0) = \frac{\beta V}{32\pi^2} m^4,$$

$$\begin{aligned} \zeta'_B(0) &= \frac{\beta V}{32\pi^2} m^4 \left(\frac{3}{2} - \ln m^2 \right) \\ &\quad - \sum_n \{ \ln[1 - e^{-\beta(E_n - e\mu)}] \\ &\quad + \ln[1 - e^{-\beta(E_n + e\mu)}] \}. \end{aligned}$$

Thus

$$\begin{aligned} \Gamma_B^{(1)} &= \sum_n \{ \ln[1 - e^{-\beta(E_n - e\mu)}] \\ &\quad + \ln[1 - e^{-\beta(E_n + e\mu)}] \} - \frac{\beta V}{32\pi^2} m^4 \left(\frac{3}{2} - \ln(m^2 l^2) \right). \end{aligned} \quad (3.32)$$

The last term in Eq. (3.32), which is independent of the chemical potential μ , contains the contribution of the zero-point energy after regularization.

We now turn to the high temperature expansion of $\Gamma_B^{(1)}$. Haber and Weldon [3] obtained a result which ignored the zero-point energy, and we will show (following [5]) how to obtain their result directly from the generalized ζ -function. Separating off the $j=0$ term in Eq. (3.23) we have

$$\zeta_B(s) = \tilde{\zeta}_B(s) + F_+(s) + F_-(s) \quad (3.33)$$

where

$$\tilde{\zeta}_B(s) = \sum_n (\sigma_n + m^2 - e^2 \mu^2)^{-s}, \quad (3.34)$$

$$F_{\pm}(s) = \sum_{j=1}^{\infty} \sum_n [(\omega_j \pm i e \mu)^2 + \sigma_n + m^2]^{-s}. \quad (3.35)$$

Specializing to flat space, replacing $\sigma_n \rightarrow k^2$ and $\sum_n \rightarrow V \int [d^3 k / (2\pi)^3]$ we find

$$\begin{aligned} \tilde{\zeta}_B(s) &= \frac{V}{8\pi\sqrt{\pi}} \frac{\Gamma(s-3/2)}{\Gamma(s)} \\ &\quad \times (m^2 - e^2 \mu^2)^{3/2-s}, \end{aligned} \quad (3.36)$$

$$\begin{aligned} F_{\pm}(s) &= \frac{V}{8\pi\sqrt{\pi}} \frac{\Gamma(s-3/2)}{\Gamma(s)} \left(\frac{2\pi}{\beta} \right)^{3-2s} \\ &\quad \times \sum_{j=1}^{\infty} [(j \pm i e \bar{\mu})^2 + \bar{m}^2]^{3/2-s}, \end{aligned} \quad (3.37)$$

with $\bar{\mu} = \beta\mu/2\pi$ and $\bar{m} = \beta m/2\pi$. The binomial expansion may be used to evaluate the leading terms of $F_{\pm}(s)$ in the high temperature limit. After a bit of calculation we find

$$F_+(0) + F_-(0) = \frac{\beta V}{32\pi^2} m^4, \quad (3.38)$$

(which is an exact result), and

$$\begin{aligned} \frac{d}{ds} (F_+(s) + F_-(s))|_{s=0} &= \frac{4\pi^2 V}{3\beta^3} \left(\frac{1}{60} + \frac{\beta^2 e^2 \mu^2}{8\pi^2} - \frac{\beta^4 e^4 \mu^4}{32\pi^2} \right) \\ &\quad + \frac{V}{2\beta} m^2 \left(-\frac{1}{6} + \frac{\beta^2 e^2 \mu^2}{4\pi^2} \right) \\ &\quad + \frac{\beta V}{32\pi^2} m^4 \left(2\gamma + 2 \ln \frac{\beta}{4\pi} \right) + \dots, \end{aligned} \quad (3.39)$$

where γ is the Euler-Mascheroni constant. The expansion for $\Gamma_B^{(1)}$ becomes

$$\begin{aligned} \Gamma_B^{(1)} &= -\frac{V}{6\pi} (m^2 - e^2 \mu^2)^{3/2} - \frac{\pi^2 V}{45\beta^3} + \frac{V}{12\beta} (m^2 - 2e^2 \mu^2) \\ &\quad - \frac{\beta V}{16\pi^2} m^4 \left(\gamma + \ln \frac{\beta}{4\pi l} \right) \\ &\quad + \frac{\beta V}{24\pi^2} e^2 \mu^2 (e^2 \mu^2 - 3m^2) + \dots. \end{aligned} \quad (3.40)$$

Removing the zero-point energy term using Eq. (3.32) shows complete agreement between this result and that of Haber and Weldon [3]. We can safely conclude that the correct physics is contained in the expression $\Gamma_B^{(1)}$.

We now turn to the other two expressions $\Gamma_A^{(1)}$ and $\Gamma_C^{(1)}$. We have not found such an elegant way to analyze the ζ -functions in these two cases as that presented in the Appendix. Instead we will content ourselves with the high temperature limit only and compare with Eq. (3.40). We will show that different results are obtained in these two cases.

We take $\zeta_A(s)$ in Eq. (3.22) and expand in powers of μ , keeping terms up to order μ^4 . If we define

$$G(z, k) = \sum_{j=-\infty}^{+\infty} \sum_n \omega_j^{2k} (\omega_j^2 + \sigma_n + m^2 - e^2 \mu^2)^{-z} \quad (3.41)$$

it is easy to show that

$$\begin{aligned} \zeta_A(s) &= G(2s, 0) - 4e^2 \mu^2 s G(2s+2, 1) \\ &\quad + 8e^4 \mu^4 s(s+1) G(2s+4, 2) + \dots. \end{aligned} \quad (3.42)$$

Taking the case of flat space in the large box limit we find

$$\begin{aligned} G(z, k) &= \frac{V}{(4\pi)^{3/2}} \frac{\Gamma(z-3/2)}{\Gamma(z)} \\ &\quad \times \sum_{j=-\infty}^{+\infty} \omega_j^{2k} (\omega_j^2 + m^2 - e^2 \mu^2)^{3/2-z}. \end{aligned} \quad (3.43)$$

For $k \geq 1$, we can expand $G(z, k)$ to find (noting that the $j=0$ term in the sum makes no contribution for $k \geq 1$)

$$\begin{aligned}
 G(z,k) &= \frac{V}{4\pi\sqrt{\pi}} \frac{\Gamma(z-3/2)}{\Gamma(z)} \left\{ \left(\frac{2\pi}{\beta} \right)^{3+2k-2z} \right. \\
 &\quad \times \zeta_R(2z-3-2k) + \left(\frac{3}{2}-z \right) (m^2 - e^2\mu^2) \\
 &\quad \times \left(\frac{2\pi}{\beta} \right)^{1+2k-2z} \zeta_R(2z-1-2k) \\
 &\quad + \frac{1}{2} \left(\frac{3}{2}-z \right) (m^2 - e^2\mu^2)^2 \left(\frac{2\pi}{\beta} \right)^{2k-1-2z} \\
 &\quad \left. \times \zeta_R(2z+1-2k) + \dots \right\}. \tag{3.44}
 \end{aligned}$$

Here $\zeta_R(\alpha)$ denotes the Riemann ζ -function. For $k=0$ the $j=0$ term does make a contribution and we find

$$\begin{aligned}
 G(z,0) &= \frac{V}{8\pi\sqrt{\pi}} \frac{\Gamma(z-3/2)}{\Gamma(z)} (m^2 - e^2\mu^2)^{3/2-z} \\
 &\quad + \frac{V}{4\pi\sqrt{\pi}} \frac{\Gamma(z-3/2)}{\Gamma(z)} \left\{ \left(\frac{2\pi}{\beta} \right)^{3-2z} \right. \\
 &\quad \times \zeta_R(2z-3) + \left(\frac{3}{2}-z \right) (m^2 - e^2\mu^2) \left(\frac{2\pi}{\beta} \right)^{1-2z} \\
 &\quad \times \zeta_R(2z-1) + \frac{1}{2} \left(\frac{3}{2}-z \right) \left(\frac{1}{2}-z \right) \left(\frac{2\pi}{\beta} \right)^{-1-2z} \\
 &\quad \left. \times (m^2 - e^2\mu^2) \zeta_R(2z+1) + \dots \right\}. \tag{3.45}
 \end{aligned}$$

These results are sufficient to show that

$$\zeta_A(0) = \frac{\beta V}{32\pi^2} m^4, \tag{3.46}$$

$$\begin{aligned}
 \zeta'_A(0) &= \frac{V}{3\pi} (m^2 - e^2\mu^2)^{3/2} + \frac{2\pi^2 V}{45\beta^3} \\
 &\quad - \frac{V}{6\beta} (m^2 - 2e^2\mu^2) + \frac{\beta V}{8\pi^2} m^4 \left(\gamma + \ln \frac{\beta}{4\pi} \right) \\
 &\quad + \frac{\beta V}{8\pi^2} e^2\mu^2 \left(m^2 - \frac{1}{3} e^2\mu^2 \right) + \dots. \tag{3.47}
 \end{aligned}$$

Used in Eq. (3.21) we find

$$\begin{aligned}
 \Gamma_A^{(1)} &= -\frac{V}{6\pi} (m^2 - e^2\mu^2)^{3/2} - \frac{\pi^2 V}{45\beta^3} \\
 &\quad + \frac{V}{12\beta} (m^2 - 2e^2\mu^2) - \frac{\beta V}{16\pi^2} m^4 \left(\gamma + \ln \frac{\beta}{4\pi l} \right) \\
 &\quad + \frac{\beta V}{48\pi^2} e^2\mu^2 (e^2\mu^2 - 3m^2) + \dots. \tag{3.48}
 \end{aligned}$$

Comparison with Eq. (3.40) shows that all terms are the same apart from the last one. In fact the difference between $\Gamma_A^{(1)}$ and $\Gamma_B^{(1)}$ corresponds precisely to the multiplicative anomaly computed in [9]. We will return to this at the end of the section.

Finally we will examine the high temperature limit of $\Gamma_C^{(1)}$. Again a straightforward binomial expansion can be used. Leaving out the details, we find

$$\zeta_C^{(1)}(0) = \frac{\beta V}{32\pi^2} (m^2 - e^2\mu^2)^2,$$

$$\begin{aligned}
 \zeta_C^{(1)'}(0) &= \frac{V}{3\pi} (m^2 - e^2\mu^2)^{3/2} + \frac{\pi^2 V}{45\beta^3} - \frac{V}{12\beta} (m^2 - e^2\mu^2) \\
 &\quad + \frac{\beta V}{16\pi^2} (m^2 - e^2\mu^2)^2 \left(\gamma + \ln \frac{\beta}{4\pi} \right) + \dots,
 \end{aligned}$$

$$\zeta_C^{(2)}(0) = \frac{\beta V}{16\pi^2} e^2\mu^2 (2m^2 - e^2\mu^2),$$

$$\begin{aligned}
 \zeta_C^{(2)'}(0) &= \zeta_C^{(1)'}(0) + \frac{V}{6\beta} e^2\mu^2 + \frac{\beta V}{4\pi^2} e^2\mu^2 (m^2 - e^2\mu^2) \\
 &\quad \times \left(\gamma + \frac{1}{2} + \ln \frac{\beta}{4\pi} \right) + \frac{\beta V}{8\pi^2} e^4\mu^4 \left(\gamma + \frac{11}{12} \right. \\
 &\quad \left. + \ln \frac{\beta}{4\pi} \right) + \dots.
 \end{aligned}$$

These results lead to

$$\begin{aligned}
 \Gamma_C^{(1)} &= -\frac{V}{6\pi} (m^2 - e^2\mu^2)^{3/2} - \frac{\pi^2 V}{45\beta^3} + \frac{V}{12\beta} (m^2 - 2e^2\mu^2) \\
 &\quad - \frac{\beta V}{16\pi^2} m^4 \left(\gamma + \ln \frac{\beta}{4\pi l} \right) \\
 &\quad - \frac{\beta V}{16\pi^2} e^2\mu^2 \left(m^2 - \frac{1}{12} e^2\mu^2 \right) + \dots. \tag{3.49}
 \end{aligned}$$

Again it is only the final term which differs from the result (3.40).

We are in the situation that we have three different ways of evaluating the formal expression for the effective action. Only one of these expressions, $\Gamma_B^{(1)}$ corresponds to a result found using canonical methods. The difference between the three results for the effective action cannot be due to the fact that we have not renormalized Γ . The only renormalization ambiguity resides in our choice of the renormalization length l . Rescaling l in any of the expressions we have found only alters the effective action by a term proportional to $\beta V m^4$, which does not involve μ . The only way to decide which of the expressions we have found is correct is by comparison with physical results as we have done. (Alternatively a more careful definition of the functional integral might settle the issue, but it must lead to $\Gamma_B^{(1)}$ if results of standard statistical mechanics are to be correct). To emphasize this point as clearly as possible, we can compute the charge in the high

temperature limit using Eq. (3.1). Because the μ -dependence in $\Gamma_A^{(1)}$, $\Gamma_B^{(1)}$ and $\Gamma_C^{(1)}$ are all different, we will obtain three different results for the charge. Only one of these results can correspond to that found from a direct high temperature expansion of Eq. (3.2), and this comes from $\Gamma_B^{(1)}$.

At this stage we return to the multiplicative anomaly. Although we claim that it does not lead to any new physics in the example presented, we do not agree with the authors of Refs. [7, 16] that it is of no importance. Rather we support the view of Refs. [8, 9, 17, 18] that it is extremely important. Having settled on the correct expression for the effective action $\Gamma_B^{(1)}$ in Eq. (1.8), the multiplicative anomaly is what ensures that if we combine the two operators to obtain Eq. (1.7) the correct result will be obtained. The reason is that it is necessary to add on the anomaly term when the operators are combined as found in [8,9] and this leads to the same result as found from $\Gamma_B^{(1)}$. A similar comment applies to $\Gamma_C^{(1)}$ with a different anomaly because of the different factorization. The multiplicative anomaly guarantees that formal manipulations of different factorizations of the effective action all lead to the same physics.

IV. DISCUSSION AND CONCLUSIONS

We have argued in the preceding sections that to determine the validity of the inequivalent but formally identical expressions for the effective action obtained from the functional integral, physical considerations are of greater importance than mathematical ones. The functional integral method is merely one way of calculation, and obviously all valid methods of finding the effective action must lead to the same physical conclusions. We studied both the relativistic and non-relativistic scalar fields. In the non-relativistic case two results for the effective action, which are equivalent at the formal level, were evaluated, and the one which agreed with the results of standard thermodynamics was identified. In the relativistic case we evaluated three possible expressions for the effective action. The correct expression was identified both from looking at the charge, and from a direct evaluation of the partition function using canonical methods. Once the correct expression had been found we emphasized the crucial role of the multiplicative anomaly in maintaining equality between the correct expression and other formally equivalent expressions.

We have restricted our attention to non-interacting scalar field theories here. The role of interactions complicates the details, but does not lead to any differences of substance. At zero temperature the presence of a quartic self-interaction for the scalar field leads to a term in the multiplicative anomaly proportional to the interaction as found in [8]. However in this case the anomaly term is of no physical significance, in contrast to the view taken in [8], because the effective action (or potential) has not been renormalized. Once a renormalization condition has been imposed the anomaly is absorbed by the counterterms and the usual effective potential is obtained. At finite temperature the situation is slightly more complicated [9], but the approach we have outlined above settles the issue in favor of the standard expression for the effective action as used in [2–6] for example.

It is also worth commenting on the theory used in two recent criticisms [7,16] of the work in [8,9]. (See also Refs. [17, 18].) The model consists of two non-interacting scalar fields of different masses. The effective action can be expressed in two ways:

$$\Gamma_1 = -\frac{1}{2} \ln \det l^4(-\square_x + m_1^2)(-\square_x + m_2^2), \quad (4.1)$$

$$\Gamma_2 = -\frac{1}{2} \ln \det l^2(-\square_x + m_1^2) - \frac{1}{2} \ln \det l^2(-\square_x + m_2^2). \quad (4.2)$$

These expressions are formally equal. As shown in [8] if ζ -function regularization is used there is a multiplicative anomaly present so that an explicit evaluation leads to $\Gamma_1 \neq \Gamma_2$. However this theory is too simple to settle any issues about the role of the multiplicative anomaly. The difference between the two expressions Γ_1 and Γ_2 is proportional to $\beta V(m_1^2 - m_2^2)^2$. This is a constant term in the effective potential which has no physical significance. As for the self-interacting theory at zero temperature, the anomaly is unimportant once a renormalized result is considered by adopting a renormalization condition, in this case on the vacuum energy.

In conclusion, we support the authors of [8,9] that the multiplicative anomaly is important, but we do not agree that it contains any new physical consequences in the finite temperature field theory examples considered here. Rather the multiplicative anomaly is needed to explain the equality of formally identical expressions for the effective action arising from the functional integral. In this respect the multiplicative anomaly plays a vital and important role in the functional integral approach to quantum field theory.

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APPENDIX: THE GENERALIZED ζ-FUNCTION

We have the general definition

$$\zeta(s) = \sum_{j=-\infty}^{+\infty} \sum_n [(\omega_j + i\mu)^2 + E_n^2]^{-s}. \quad (A1)$$

This will be an analytic function of s in some region of the complex plane and the objective is to analytically continue it to a neighborhood of $s=0$ and find $\zeta(0)$ and $\zeta'(0)$. There are many ways to do this, and we will outline one way here.

The order of summations is irrelevant in the region of the complex s -plane where Eq. (A1) converges. We will perform the sum over j first. By making use of the summation formula

$$\sum_{j=-\infty}^{+\infty} f(j) = \int_{-\infty}^{+\infty} f(j) dj + \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dz (e^{-2\pi iz} - 1)^{-1} \times [f(z) + f(-z)] \quad (A2)$$

we obtain

$$\begin{aligned} \zeta(s) = & \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_n E_n^{1-2s} \\ & + \sum_n \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dz (e^{-2\pi iz} - 1)^{-1} \left\{ \left[\left(\frac{2\pi z}{\beta} + i\mu \right)^2 \right. \right. \\ & \left. \left. + E_n^2 \right]^{-s} + \left[\left(\frac{2\pi z}{\beta} - i\mu \right)^2 + E_n^2 \right]^{-s} \right\}. \end{aligned} \quad (\text{A3})$$

In arriving at the first term we have made use of the definition of the Γ -function. If we define an energy ζ -function by

$$E(\alpha) = \sum_n E_n^{1-\alpha} \quad (\text{A4})$$

which will be analytic for $\Re(\alpha)$ large enough, we can write

$$\begin{aligned} \zeta(s) = & \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} E(2s) \\ & + \sum_n \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dz (e^{-2\pi iz} - 1)^{-1} \\ & \times \left\{ \left[\left(\frac{2\pi z}{\beta} + i\mu \right)^2 + E_n^2 \right]^{-s} \right. \\ & \left. + \left[\left(\frac{2\pi z}{\beta} - i\mu \right)^2 + E_n^2 \right]^{-s} \right\}. \end{aligned} \quad (\text{A5})$$

In order to obtain the analytic continuation of $\zeta(s)$ to $s=0$ we must modify Eq. (A5) since the second term of Eq. (A5) diverges at $s=0$ as it stands. The integrand has branch

points at $2\pi z/\beta = i(E_n \pm \mu)$ in the upper half plane. By taking branch cuts along the imaginary axis, and deforming the contour around the branch cuts it is straightforward to show that

$$\begin{aligned} \zeta(s) = & \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} E(2s) + 2 \sin(\pi s) \\ & \times \sum_n \left\{ \int_{\beta(E_n - \mu)/2\pi}^{\infty} dx (e^{2\pi x} - 1)^{-1} \right. \\ & \times \left[\left(\frac{2\pi}{\beta} x + \mu \right)^2 - E_n^2 \right]^{-s} \\ & + \int_{\beta(E_n + \mu)/2\pi}^{\infty} dx (e^{2\pi x} - 1)^{-1} \\ & \left. \times \left[\left(\frac{2\pi}{\beta} x - \mu \right)^2 - E_n^2 \right]^{-s} \right\}. \end{aligned} \quad (\text{A6})$$

(An equivalent analytic continuation was found earlier by Ford [20]). By expanding the second term in Eq. (A6) about $s=0$ it is easy to show that

$$\begin{aligned} \zeta(s) = & \frac{\beta}{2\sqrt{\pi}} \frac{\Gamma(s-1/2)}{\Gamma(s)} E(2s) - s \sum_n \{ \ln(1 - e^{-\beta(E_n - \mu)}) \\ & + \ln(1 - e^{-\beta(E_n + \mu)}) \} + \dots \end{aligned} \quad (\text{A7})$$

where terms of order s^2 and higher have been dropped. The details of expanding the first term depend on the spectrum E_n . Another way of obtaining Eq. (A7) is given in the Appendix of [21].

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