Interaction of Hawking radiation with static sources outside a Schwarzschild black hole

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We show that the response rate of (i) a static source interacting with Hawking radiation of a massless scalar field in Schwarzschild spacetime (with the Unruh vacuum) and that of (ii) a uniformly accelerated source with the same proper acceleration in Minkowski spacetime (with the Minkowski vacuum) are equal. We show that this equality will not hold if the Unruh vacuum is replaced by the Hartle-Hawking vacuum. It is verified that the source responds to the Hawking radiation near the horizon as if it were at rest in a thermal bath in Minkowski spacetime with the same temperature. It is also verified that the response rate in the Hartle-Hawking vacuum approaches that in Minkowski spacetime with the same temperature far away from the black hole. Finally, we compare our results with others in the literature. [S0556-2821(98)02922-1]

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I. INTRODUCTION

Recently we analyzed the emission and absorption of ''zero-energy particles'' by a static source interacting with Hawking radiation outside a Schwarzschild black hole [1]. It was found that the total response rate of a point-like static scalar source in the Unruh vacuum is given by

$$R_{\text{tot}} = \frac{q^2 a}{4\pi^2}, \qquad (1.1)$$

where q is the coupling constant between the source and the massless scalar field, and a is the proper acceleration of the source. The remarkable fact about this result is that Eq. (1.1) also corresponds to the total response rate of a uniformly accelerated source for a massless scalar field in Minkowski spacetime provided that the initial quantum state is the Minkowski vacuum. [In fact, according to inertial observers, Eq. (1.1) is associated with the emission rate of finite-energy Minkowski particles while according to coaccelerated observers it is associated with the emission and absorption of zero-energy Rindler particles [2].] Thus, an equality between the behavior of static sources in Schwarzschild spacetime (with the Unruh vacuum) and uniformly accelerated sources in Minkowski spacetime (with the Minkowski vacuum), concerning their emission and absorption rates, was found. Here we analyze in detail some related points that were not discussed in Ref. [1], provide some consistency checks of our results and demonstrate their compatibility with related results in the literature. The paper is organized as follows. In Sec. II, we review the general formalism for computing the response rate of classical sources for a massless scalar field in static spacetime. In Sec. III, we analyze the case of a static point-like source in the Rindler wedge—i.e. a uniformly accelerated point-like source in Minkowski spacetime—and

show that its total response rate (with the initial quantum state being the Minkowski vacuum) is given by Eq. (1.1). Next, in Sec. IV, we consider a static source immersed in a thermal bath in Minkowski spacetime and calculate its response rate for later use. In Sec. V, we consider a toy model for a static source outside a static black hole characterized by a simplified gravitational effective potential and calculate its response rate assuming that the initial quantum state is the Unruh vacuum. In Sec. VI, the response rate is calculated with the true Schwarzschild effective potential. The results found are compared with those obtained in Sec. III. In particular, it is shown that the total response rate here is also given by Eq. (1.1). This equivalence is our main result [1]. In Sec. VII, we calculate the response rate replacing the Unruh vacuum by the Hartle-Hawking one and show that Eq. (1.1) does not hold. In Sec. VIII, we discuss the case where the source approaches the horizon and the case where it is far away from the black hole using the method described in Refs. [3,4] and show that the results agree with the suitable limit of the one obtained in Sec. VI, i.e., Eq. (1.1). In Sec. IX, we discuss our results. We will use natural units $\hbar = c = G = k_B = 1$ throughout this paper.

II. GENERAL FORMALISM

It is well known that field theory quantized in globally hyperbolic spacetime possessing a global timelike Killing field admits a unique vacuum state and a corresponding unique ''particle interpretation'' (under certain technical conditions) [5,6]. This is so because the use of the time parameter corresponding to the Killing field allows us to distinguish, in a natural way, between positive and negative frequency modes. This is the case in globally hyperbolic static spacetime described by the metric

$$ds^2 = f(\mathbf{x})dt^2 - h_{ij}(\mathbf{x})dx^i dx^j \qquad (2.1)$$

(under certain technical conditions). We consider emission of particles in these spacetimes by classical static scalar sources $J(\mathbf{x})$ coupled to a massless scalar field. The response of a classical source in the vacuum is entirely due to spontaneous emission. If the source is static, then this vanishes (unless there are severe infrared divergences). However, if the static source is in a thermal bath, the absorption and induced emission also contribute to the response rate. Now, the static source interacts only with zero-energy modes and Planck's distribution formula diverges at zero energy. It will turn out that this makes the rates of absorption and induced emission nonzero. Thus, the static source responds with a finite probability to thermal baths for the cases we consider in this paper.

In order to avoid the appearance of intermediate indefinite results due to the divergence mentioned above, we will introduce oscillation as a regulator. Thus, we consider at this point a source of the form

$$j_{\omega_0}(t,\mathbf{x}) = \sqrt{2}J(\mathbf{x})\cos\omega_0 t \qquad (2.2)$$

and take the limit $\omega_0 \to 0$ at the end. The factor $\sqrt{2}$ has been introduced to keep the time average $\langle|j_{\omega_0}(t,\mathbf{x})|^2\rangle_t$ equal to $|J(\mathbf{x})|^2$. This makes $j_{\omega_0}(t,\mathbf{x})$ equivalent to $J(\mathbf{x})$ in the limit $\omega_0 \to 0$ because the response rate at the lowest order is proportional to the square of (the Fourier transform of) the source. We will be interested in the point source where

$$J(\mathbf{x}) = q\,\delta(\mathbf{x} - \mathbf{x}_0)/\sqrt{h} \qquad (2.3)$$

with q being the coupling constant, $\mathbf{x}_0$ being the position of the source and $h(\mathbf{x}) \equiv \det[h_{ij}(\mathbf{x})]$. With this definition, we have

$$\int_{\Sigma_t} d\Sigma\, J(\mathbf{x}) = q \qquad (2.4)$$

for any Cauchy surface Σ_t with constant t.

Let us consider the coupling of our classical source $j_{\omega_0}(t,\mathbf{x})$ to a massless real scalar field Φ, which is described by the action

$$S = \int d^4x \sqrt{fh}\left(\frac{1}{2}\nabla^\mu\Phi\nabla_\mu\Phi + j_{\omega_0}\Phi\right).$$

Let

$$u_{\omega\mathbf{s}\lambda}(x) = \sqrt{\frac{\omega}{\pi}}\,U_{\omega\mathbf{s}\lambda}(\mathbf{x})\exp(-i\omega t) \qquad (2.5)$$

with frequency $\omega > 0$, and their complex conjugates $u_{\omega\mathbf{s}\lambda}(x)^*$ be solutions to $\Box u = 0$, where $\mathbf{s} = (s_1, \ldots, s_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_m)$ are sets of continuous and discrete quantum numbers, respectively, for the complete set of modes. We have assumed ω to be continuous because this is the case in the spacetimes we study, and adopted it as one of the mode labels. The factor $\sqrt{\omega/\pi}$ has been inserted for later convenience. We orthonormalize these solutions with respect to the Klein-Gordon inner product:

$$i\int_{\Sigma_t} d\Sigma\, n^\mu(u^*_{\omega\mathbf{s}\lambda}\nabla_\mu u_{\omega'\mathbf{s}'\lambda'} - \nabla_\mu u^*_{\omega\mathbf{s}\lambda}\cdot u_{\omega'\mathbf{s}'\lambda'})$$
$$= \delta(\omega - \omega')\delta(\mathbf{s} - \mathbf{s}')\delta_{\lambda\lambda'}, \qquad (2.6)$$

$$i\int_{\Sigma_t} d\Sigma\, n^\mu(u_{\omega\mathbf{s}\lambda}\nabla_\mu u_{\omega'\mathbf{s}'\lambda'} - \nabla_\mu u_{\omega\mathbf{s}\lambda}\cdot u_{\omega'\mathbf{s}'\lambda'})$$
$$= 0, \qquad (2.7)$$

where n^μ is the future-pointing unit normal to the volume element of a Cauchy surface Σ_t. The in-field Φ^{in} satisfying the free field equation $\Box\Phi^{\text{in}} = 0$ can now be expanded as

$$\Phi^{\text{in}}(x) = \sum_\lambda \int d\omega d^n\mathbf{s}[u_{\omega\mathbf{s}\lambda}(x)a^{\text{in}}_{\omega\mathbf{s}\lambda} + \text{H.c.}],$$

where $a^{\text{in}}_{\omega\mathbf{s}\lambda}$ and $a^{\text{in}\dagger}_{\omega\mathbf{s}\lambda}$ are annihilation and creation operators, respectively, which satisfy the usual commutation relations

$$[a^{\text{in}}_{\omega\mathbf{s}\lambda}, a^{\text{in}\dagger}_{\omega'\mathbf{s}'\lambda'}] = \delta(\omega - \omega')\delta(\mathbf{s} - \mathbf{s}')\delta_{\lambda\lambda'}. \qquad (2.8)$$

Let the initial quantum state be the in-vacuum state $|0\rangle$ defined by $a^{\text{in}}_{\omega\mathbf{s}\lambda}|0\rangle = 0$ for all ω, $\mathbf{s}$ and λ. The rate of spontaneous emission per total *proper* time $T\sqrt{f(\mathbf{x}_0)}$, where $T \equiv 2\pi\delta(0)$ is the total *coordinate* time [7], with fixed $\mathbf{s}$ and λ to lowest order is

$$R_{\text{sp}}(\omega_0;\mathbf{s},\lambda)d^n\mathbf{s} = T^{-1}f(\mathbf{x}_0)^{-1/2}$$
$$\times \int d\omega\left|\left\langle\omega\mathbf{s}\lambda\left|\int d^4x\sqrt{fh}\,j_{\omega_0}\Phi\right|0\right\rangle\right|^2 d^n\mathbf{s}. \qquad (2.9)$$

By performing the integration with respect to ω, we obtain

$$R_{\text{sp}}(\omega_0;\mathbf{s},\lambda)d^n\mathbf{s} = \frac{\omega_0}{\sqrt{f(\mathbf{x}_0)}}|\tilde{J}(\omega_0;\mathbf{s},\lambda)|^2 d^n\mathbf{s}, \qquad (2.10)$$

where

$$\tilde{J}(\omega_0;\mathbf{s},\lambda) = \int d^3\mathbf{x}\sqrt{f(\mathbf{x})h(\mathbf{x})}\,J(\mathbf{x})U_{\omega_0\mathbf{s}\lambda}(\mathbf{x}).$$

If the source is immersed in a thermal bath of inverse temperature β, the rates of absorption and *induced* emission are both given by $R_{\text{sp}}(\omega_0;\mathbf{s},\lambda)/[\exp(\beta\omega_0) - 1]$. Adding the absorption rate and the spontaneous and induced emission rates, we find that the *response* rate for modes with fixed $\mathbf{s}$ and λ is given by

$$R(\omega_0;\mathbf{s},\lambda) = \frac{\omega_0}{\sqrt{f(\mathbf{x}_0)}}|\tilde{J}(\omega_0;\mathbf{s},\lambda)|^2\coth(\beta\omega_0/2).$$

In the case of interest here, i.e., for static sources, we take the limit $\omega_0 \to 0$ as explained above, obtaining

$$R(0;\mathbf{s},\lambda) = \frac{2\beta^{-1}}{\sqrt{f(\mathbf{x}_0)}} |\tilde{J}(0;\mathbf{s},\lambda)|^2. \qquad (2.11)$$

This is the general expression for the response rate of a static point source in a thermal bath interacting with a massless scalar field. The total rate is obtained from Eq. (2.11) by integrating with respect to $\mathbf{s}$ and summing over λ. Note that although the particle content of a field theory depends in general on the Killing field with respect to which the vacuum is defined, the total response rate does not.

III. STATIC SOURCE IN THE RINDLER WEDGE

We first review the computation of the response rate of a static source in the Rindler wedge [8] (see Refs. [2,9]). This source corresponds to a uniformly accelerated source in Minkowski spacetime. The Rindler wedge is the portion of Minkowski spacetime limited by $z > |t|$, where (t,x,y,z) are the usual Minkowski coordinates. These are related to the Rindler coordinates (τ,x,y,ξ) by

$$t = a^{-1}e^{a\xi}\sinh a\tau, \quad z = a^{-1}e^{a\xi}\cosh a\tau.$$

In these coordinates, the line element of the Rindler wedge is written as

$$ds^2 = e^{2a\xi}(d\tau^2 - d\xi^2) - dx^2 - dy^2.$$

We consider a point-like source fixed in space with a time-dependent magnitude [see Eqs. (2.2) and (2.3)]

$$j_{\omega_0} = \sqrt{2}\, q\cos\omega_0\tau\, \delta(\xi)\delta(x)\delta(y) \qquad (3.1)$$

and take the limit $\omega_0 \to 0$ at the end. Note that j_{ω_0} describes a source with constant proper acceleration a.

We will describe the free massless scalar field theory using Rindler coordinates. For this purpose we look for positive-frequency solutions to $\Box u_{\omega k_x k_y} = 0$ with respect to the Killing field $\partial/\partial\tau$:

$$u_{\omega k_x k_y}(\tau,x,y,\xi) = \sqrt{\frac{\omega}{\pi}}\,\psi_{\omega k_\perp}(\xi)\frac{e^{ik_x x + ik_y y - i\omega\tau}}{2\pi}. \quad (3.2)$$

Here the $\psi_{\omega k_\perp}$ must satisfy

$$\left[-\frac{d^2}{d\xi^2} + k_\perp^2 e^{2a\xi}\right]\psi_{\omega k_\perp}(\xi) = \omega^2\psi_{\omega k_\perp}(\xi) \qquad (3.3)$$

and $k_\perp \equiv \sqrt{k_x^2 + k_y^2}$. We assume a Minkowski vacuum which corresponds to a thermal state of Rindler particles [10–12]. This is the Fulling-Davies-Unruh (FDU) thermal bath characterized by a temperature $\beta^{-1} = a/2\pi$. The term $k_\perp^2 e^{2a\xi}$ in Eq. (3.3) acts as an effective potential that is unbounded for the modes with nonvanishing transverse momentum. The solutions $\psi_{\omega k_\perp}(\xi)$ that tend to zero as $\xi \to +\infty$ are

$$\psi_{\omega k_\perp}(\xi) = C_\omega K_{i\omega/a}[(k_\perp/a)e^{a\xi}], \qquad (3.4)$$

where C_ω is a normalization constant. In order to determine it, we substitute normal modes (3.2) in the Klein-Gordon inner product (2.6), obtaining

$$(\omega + \omega')\int_{-\infty}^{+\infty} d\xi\, \psi_{\omega k_\perp}^*(\xi)\psi_{\omega' k_\perp}(\xi) = \frac{\pi}{\omega}\delta(\omega - \omega').$$
$$(3.5)$$

Next we use the wave equation (3.3) to turn the integral in Eq. (3.5) into a surface term:

$$\frac{1}{\omega - \omega'}\left[\psi_{\omega' k_\perp}\frac{d\psi_{\omega k_\perp}^*}{d\xi} - \psi_{\omega k_\perp}^*\frac{d\psi_{\omega' k_\perp}}{d\xi}\right]_{\xi \to -\infty} = \frac{\pi}{\omega}\delta(\omega - \omega').$$
$$(3.6)$$

Using Eq. (3.4) in Eq. (3.6) and noting that

$$\lim_{r \to +\infty} \frac{\sin(\omega \mp \omega')r}{\omega \mp \omega'} = \pi\delta(\omega \mp \omega'), \qquad (3.7)$$

we obtain the normalization constant (up to a phase)

$$C_\omega = \sqrt{\frac{\sinh(\pi\omega/a)}{\pi a\omega}}. \qquad (3.8)$$

From Eqs. (3.4) and (3.8) we obtain the normalized zero-energy modes

$$\psi_{0k_\perp}(\xi) = a^{-1}K_0[(k_\perp/a)e^{a\xi}]. \qquad (3.9)$$

In fact, one can directly normalize $\psi_{0k_\perp}$ referring only to solutions of Eq. (3.3) with $\omega = 0$. This method will be very useful in the Schwarzschild black-hole case, where the analogue of Eq. (3.4) cannot be explicitly obtained. One considers the form of the solution of Eq. (3.3) with arbitrary frequency for large and negative values of ξ. This has a simple form

$$\psi_{\omega k_\perp}(\xi) \approx -\frac{1}{\omega}\sin[\omega\xi + \alpha(\omega)] \quad (\xi < 0, |\xi| \gg 1),$$
$$(3.10)$$

where the normalization constant has been fixed to make Eq. (3.10) compatible with Eq. (3.6). In particular,

$$\psi_{0k_\perp}(\xi) \approx -\xi + \mathrm{const} \quad (\xi < 0, |\xi| \gg 1). \qquad (3.11)$$

By solving Eq. (3.3) with $\omega = 0$ and fitting the solution obtained to Eq. (3.11) for large and negative values of ξ, we recover Eq. (3.9).

In order to calculate the response rate $R^R(k_x,k_y)$ with fixed transverse momentum (k_x,k_y), we use the general expression (2.11), identifying $U_{0s\lambda}(\mathbf{x})$ with $\psi_{0k_\perp}(\xi)e^{ik_x x + ik_y y}/(2\pi)$ [see Eqs. (2.5), (3.2) and (3.9)]. Thus, we find

$$R^R(k_x,k_y)dk_x dk_y = \frac{q^2}{4\pi^3 a}[K_0(k_\perp/a)]^2 dk_x dk_y.$$
$$(3.12)$$

The total response rate is obtained by integrating Eq. (3.12) over the whole range of transverse momenta as

$$R_{\text{tot}}^R = \frac{q^2 a}{4\pi^2}. \qquad (3.13)$$

It is interesting to recall at this point that by a standard Cartesian-coordinate calculation (see, e.g., Refs. [7,13]), Eq. (3.13) can be shown to be identical to the emission rate of usual Minkowski particles. This result is interpreted as follows [2,9]: *The emission of a usual finite-energy particle from a uniformly accelerated source in Minkowski vacuum as described by inertial observers corresponds to either the emission or the absorption of a zero-energy Rindler particle to or from the FDU thermal bath as described by uniformly accelerated observers.* This is in agreement with Unruh and Wald's inertial interpretation of the excitation of an accelerated detector [14], and with the discussion of this problem in terms of classical radiation [15]. Although these zero-energy particles are conceptually well defined, they are not observable by accelerated observers [2]. This is compatible with the fact that observers coaccelerated with the source associate no emission of classical radiation with it [16,17].

We would like to call attention to the fact that we are implicitly assuming that the classical source is adiabatically switched on and off asymptotically. Thus, we are not concerned with the controversy as to whether or not there is radiation from uniformly accelerated sources which are eternally turned on. (See Ref. [18] and references therein for a comprehensive analysis of this issue and Ref. [1] for a brief discussion of its relation to our problem.)

IV. STATIC SOURCE IN MINKOWSKI SPACETIME

Before analyzing static sources in the spacetime with a black hole, it is useful for later purposes to work out the response of a static source in Minkowski spacetime using spherical coordinates and assuming a background thermal bath. The line element of Minkowski spacetime in spherical coordinates is

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (4.1)$$

In spherical coordinates we write our oscillating source in the form

$$j_{\omega_0}(x) = \frac{\sqrt{2}q\cos\omega_0 t}{\sqrt{h}}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0), \qquad (4.2)$$

where $\sqrt{h} = r^2\sin\theta$.

Let us write the positive-frequency solutions of the massless Klein-Gordon equation with respect to the Killing field $\partial/\partial t$ as

$$u_{\omega l m} = \sqrt{\frac{\omega}{\pi}}\frac{\psi_{\omega l}(r)}{r}Y_{lm}(\theta, \phi)e^{-i\omega t}, \qquad (4.3)$$

where $Y_{lm}(\theta, \phi)$ are spherical harmonics [19] with $l \geq 0$ and $-l \leq m \leq +l$. [The form (4.3) will also be adopted in the following sections.] Here $\psi_{\omega l}(r)$ is the solution of the ordinary differential equation

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}\right)\psi_{\omega l}(r) = \omega^2\psi_{\omega l}(r). \qquad (4.4)$$

The solutions of Eq. (4.4) which are finite at $r = 0$ are

$$\psi_{\omega l}(r) = C_{\omega l}rj_l(\omega r), \qquad (4.5)$$

where the $j_l(x)$ are the spherical Bessel functions. The normalization constants $C_{\omega l}$ are found by substituting Eq. (4.3) in the Klein-Gordon inner product (2.6) as

$$(\omega + \omega')\int_0^{+\infty}dr\,\psi_{\omega l}(r)^*\psi_{\omega' l}(r) = \frac{\pi}{\omega}\delta(\omega - \omega')$$

and using Eq. (4.4) to turn this integral into a surface term:

$$\frac{C_{\omega l}^*C_{\omega' l}}{\omega - \omega'}[-\omega'r^2 j_l(\omega r)j_{l+1}(\omega'r)$$
$$+ \omega r^2 j_l(\omega'r)j_{l+1}(\omega r)]_{r \to +\infty}$$
$$= \frac{\pi}{\omega}\delta(\omega - \omega'), \qquad (4.6)$$

where use has been made of the identity [see (10.1.22) of Ref. [19]]

$$\frac{d}{dr}j_l(\omega r) = \frac{l}{r}j_l(\omega r) - \omega j_{l+1}(\omega r).$$

In order to evaluate the left-hand side (LHS) of Eq. (4.6) we use [see (10.1.1) and (9.2.1) of Ref. [19]]

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x) \approx x^{-1}\sin(x - l\pi/2) \quad (x \gg 1) \qquad (4.7)$$

and Eq. (3.7). Hence, from Eq. (4.6) we obtain $C_{\omega l} = 1$ (up to a phase), and the normalized zero-energy mode is

$$\psi_{0l}(r) = rj_l(0) = r\delta_{l0}. \qquad (4.8)$$

In spherical coordinates, a general expression for the response rate with fixed angular momentum can be obtained by using Eq. (4.3) in Eq. (2.11):

$$R_{lm} = \frac{2q^2\sqrt{f(\mathbf{x}_0)}}{\beta r_0^2}|\psi_{0l}(r_0)|^2|Y_{lm}(\theta_0, \phi_0)|^2. \qquad (4.9)$$

Here ψ_{0l} is given by Eq. (4.8) and $f(\mathbf{x}_0) = 1$. Thus, we obtain

$$R_{lm}^M = 2q^2\beta^{-1}|Y_{lm}(\theta_0, \phi_0)|^2\delta_{l0} = \frac{q^2}{2\pi\beta}\delta_{l0}, \qquad (4.10)$$

where we have used $|Y_{00}(\theta,\phi)|^2=(4\pi)^{-1}$. Since this expression vanishes for every l except for $l=0$, the *total* response rate is

$$R_{\text{tot}}^M=\frac{q^2}{2\pi\beta}. \tag{4.11}$$

One can readily verify that the same response rate (4.11) is obtained by repeating the calculation in Cartesian coordinates [7,13]. [In this case the normalized positive-frequency modes are the standard ones: $u_{\mathbf{k}}(x)=e^{-ik_\mu x^\mu}/\sqrt{16\pi^3\omega}$.] This should clearly be the case since the vacuum is defined through the same timelike Killing field $\partial/\partial t$.

We note that one obtains Eq. (3.13) by substituting $\beta^{-1}=a/2\pi$ in Eq. (4.11). This shows that a uniformly accelerated source for a massless scalar field in Minkowski spacetime responds to the FDU thermal bath as if it were at rest in Minkowski spacetime with a background thermal bath provided that both thermal baths have the same temperature, as is well known (see, e.g., Ref. [13]). We will return to this point in Secs. VI and VII.

V. STATIC SOURCE OUTSIDE A TOY BLACK HOLE

In this section we will treat a static source in the spacetime of a black hole with an artificial gravitational effective potential simple enough to enable us to find the normal modes in terms of well-known functions. This will allow us to normalize them for every frequency ω and thus we will be able to take the limit $\omega\to0$ explicitly. We will consider a potential that reproduces the main features of the effective potential for the Schwarzschild black hole, and compare the results with those obtained in the next section where we treat the Schwarzschild case using the method outlined in Sec. III. This will provide a useful check for the latter method.

The Schwarzschild line element is

$$ds^2=f(r)dt^2-f(r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{5.1}$$

where $f(r)=1-2M/r$. The scalar source $j_{\omega_0}(x)$ is given by Eq. (4.2) with $\sqrt{h}=f^{-1/2}r^2\sin\theta$. The positive-frequency solutions $u_{\omega lm}$ of the massless scalar field equation can be written as in Eq. (4.3) where the $\psi_{\omega l}(r)$ here satisfy the differential equation

$$\left[-f(r)\frac{d}{dr}\left(f(r)\frac{d}{dr}\right)+V_{\text{eff}}(r)\right]\psi_{\omega l}(r)=\omega^2\psi_{\omega l}(r), \tag{5.2}$$

with

$$V_{\text{eff}}(r)=(1-2M/r)[2M/r^3+l(l+1)/r^2]. \tag{5.3}$$

The effective potential $V_{\text{eff}}(r)$ vanishes at the horizon and goes to zero like $1/r^2$ for large r. It is useful to introduce the dimensionless Wheeler tortoise coordinate $x\equiv y+\ln(y-1)$ where $y\equiv r/2M$. Then Eq. (5.2) can be rewritten as

$$\left[-\frac{d^2}{dx^2}+4M^2V_{\text{eff}}[r(x)]\right]\psi_{\omega l}=4M^2\omega^2\psi_{\omega l}. \tag{5.4}$$

There are two independent solutions to Eq. (5.4). One solution corresponds to the mode purely incoming from the past horizon H^- and the other to the mode purely incoming from the past null infinity $\mathcal{J}^-$. These modes are orthogonal to each other with respect to the Klein-Gordon inner product (2.6).

We will focus in this and the next sections on the Unruh vacuum [11] where there is a thermal flux of temperature $\beta^{-1}=1/8\pi M$ coming out from H^-. This is basically the Hawking radiation [20] which leads to evaporation of black holes formed by gravitational collapse.

At this point we will replace the Schwarzschild effective potential $V_{\text{eff}}[r(x)]$ by a simpler potential

$$V_{\text{sim}}(x)=\frac{l(l+1)}{4M^2x^2}\Theta(x-1), \tag{5.5}$$

where $\Theta(x)$ is the step function. For $l\neq0$, this potential possesses the main features of the true potential (5.3): V_{sim} vanishes at the horizon and goes to zero like $1/r^2$ for large r.

For purely incoming waves from the past horizon, H^-, we write

$$\psi_{\omega l}(x)=\begin{cases}A_{\omega l}(e^{2iM\omega x}+\mathcal{R}_{\omega l}e^{-2iM\omega x}) & (x\leq1), \\ 2i^{l+1}A_{\omega l}\mathcal{T}_{\omega l}M\omega x h_l^{(1)}(2M\omega x) & (x>1), \end{cases} \tag{5.6}$$

where $A_{\omega l}$ is the normalization constant, and $|\mathcal{R}_{\omega l}|^2$ and $|\mathcal{T}_{\omega l}|^2$ are the reflection and transmission coefficients, respectively. Note that [see (8.451.3 and 8.451.4) of Ref. [21] and (10.1.1) of Ref. [19]]

$$h_l^{(1)}(x)=j_l(x)+in_l(x) \tag{5.7}$$

$$=\sqrt{\frac{\pi}{2x}}H_{l+1/2}^{(1)}(x) \tag{5.8}$$

$$=(-i)^{l+1}\frac{e^{ix}}{x}[1+\mathcal{O}(x^{-1})] \quad (|x|\gg1). \tag{5.9}$$

Using the continuity of the modes and their derivatives at $x=1$ we find the scattering coefficients:

$$\mathcal{T}_{\omega l}=\frac{e^{2iM\omega}}{i^{l+1}M\omega}[(1-i/2M\omega)h_l^{(1)}(2M\omega)-ih_l^{(1)'}(2M\omega)]^{-1} \tag{5.10}$$

and

$$\mathcal{R}_{\omega l}=e^{4iM\omega}\frac{[(1+i/2M\omega)h_l^{(1)}(2M\omega)+ih_l^{(1)'}(2M\omega)]}{[(1-i/2M\omega)h_l^{(1)}(2M\omega)-ih_l^{(1)'}(2M\omega)]}, \tag{5.11}$$

where primes indicate derivatives with respect to the argument. From these equations, we obtain the usual probability conservation

$$|\mathcal{T}_{\omega l}|^2 + |\mathcal{R}_{\omega l}|^2 = 1, \tag{5.12}$$

where we have used

$$h_l^{(1)}(x) h_l^{(1)\prime}(x)^* - h_l^{(1)}(x)^* h_l^{(1)\prime}(x) = -2i/x^2,$$

derived from [see (10.1.21)–(10.1.31) of Ref. [19]]

$$j_l'(x) n_l(x) - j_l(x) n_l'(x) = -1/x^2.$$

The normalization constant $A_{\omega l}$ is obtained, as usual, by substituting Eq. (4.3) in Eq. (2.6) and using Eq. (5.4) with $V_{\text{sim}}(x)$ in place of $V_{\text{eff}}[r(x)]$ to turn the integral into a surface term

$$\frac{1}{\omega-\omega'}\left[\psi_{\omega'l}(x)\frac{d}{dx}\psi_{\omega l}^*(x) - \psi_{\omega l}^*(x)\frac{d}{dx}\psi_{\omega'l}(x)\right]_{x\to-\infty}^{x\to+\infty}$$

$$= -\frac{2\pi M}{\omega}\delta(\omega-\omega'), \tag{5.13}$$

where $\psi_{\omega l}(x)$ is given by Eq. (5.6). Using the large $|x|$ behavior of $\psi_{\omega l}(x)$ obtained by substituting Eq. (5.9) in Eq. (5.6), we obtain (up to a phase), from Eq. (5.13),

$$A_{\omega l} = (2\omega)^{-1}. \tag{5.14}$$

Substituting the scattering coefficients (5.10)–(5.11) and the normalization constant (5.14) in Eq. (5.6), we obtain

$$\psi_{0l}(x) = \begin{cases} 2M(l^{-1}+1-x) & (x\leqslant 1), \\ 2Ml^{-1}x^{-l} & (x>1), \end{cases} \tag{5.15}$$

where we have used

$$j_l(x) = c_l x^l + \mathcal{O}(x^{l+2}),$$

$$n_l(x) = -d_l x^{-l-1} - e_l x^{-l+1} + \mathcal{O}(x^{-l+3}) \tag{5.16}$$

with $c_l = 1/(2l+1)!!$, $d_l = (2l-1)!!$, and $e_l = (2l-3)!!/2$. Note here that $\psi_{0l}(x) = -2Mx + \text{const}$ for $x<0$.

In order to calculate the response rate, we use Eq. (4.9) with $\beta^{-1} = 1/8\pi M$. If the source is at $x_0 \leqslant 1$ ($r_0/2M \leqslant 1.567$) we obtain

$$R_{lm}^{\text{T}}(x_0) = \frac{q^2 M}{\pi}\frac{f(r_0)^{1/2}}{r_0^2}\{l^{-1} - y(r_0)f(r_0) - \ln[y(r_0)f(r_0)]\}^2$$

$$\times |Y_{lm}(\theta_0,\phi_0)|^2, \quad x_0 \leqslant 1, \tag{5.17}$$

where $y(r) = r/2M$, while if the source is at $x_0 > 1$ ($r_0/2M > 1.567$) we obtain

$$R_{lm}^{\text{T}}(x_0) = \frac{q^2 M}{\pi}\frac{f(r_0)^{1/2}}{r_0^2 l^2}\{y(r_0) + \ln[y(r_0)f(r_0)]\}^{-2l}$$

$$\times |Y_{lm}(\theta_0,\phi_0)|^2, \quad x_0 > 1, \tag{5.18}$$

where $l \neq 0$. Note here that for $x_0 \gg 1$ we have

$$R_{lm}^T(x_0) \approx (q^2/4\pi M) l^{-2} x_0^{-2l-2} |Y_{lm}|^2. \tag{5.19}$$

VI. STATIC SOURCE OUTSIDE THE SCHWARZSCHILD BLACK HOLE WITH THE UNRUH VACUUM

The fact that must be considered first in solving the full Schwarzschild case is that very little is known about the solutions of the wave equation (5.4) of nonzero frequency ω with potential (5.3). (See Ref. [22] for some known properties of these solutions.) Thus we use the method outlined in Sec. III [see the paragraph below Eq. (3.9)] in order to normalize the zero-energy modes which are the only relevant ones here [see Eq. (4.9)]. We will consider the Unruh vacuum as in the previous section. Thus, we need to consider only the modes incoming from H^-. Close to and far away from the horizon we can write

$$\psi_{\omega l}(x) \approx \begin{cases} A_{\omega l}(e^{2iM\omega x} + \mathcal{R}_{\omega l}e^{-2iM\omega x}) & (x<0,\ |x|\gg 1), \\ 2i^{l+1}A_{\omega l}\mathcal{T}_{\omega l}M\omega x h_l^{(1)}(2M\omega x) & (x\gg 1). \end{cases} \tag{6.1}$$

Zero-frequency modes coming from H^- are totally reflected back by the potential toward the horizon [see Eq. (5.15) with $x\gg 1$]. This implies that for $M\omega \ll 1$ the behavior of $\psi_{\omega l}(x)$ close to the horizon determines the inner product in Eq. (2.6). Taking this fact into account and disregarding the black-hole potential close to the horizon, we find the normalized solution of Eq. (5.4) in this region:

$$\psi_{\omega l}(x) \approx -\omega^{-1}\sin[2M\omega x + \alpha(\omega)] \quad (x<0,\ |x|\gg 1) \tag{6.2}$$

up to a phase. In the limit $\omega \to 0$, we find

$$\psi_{0l}(x) \approx -2Mx + \text{const} \quad (x<0,\ |x|\gg 1). \tag{6.3}$$

This agrees with the behavior of $\psi_{0l}(x)$ for large and negative x found in the previous section by normalizing the modes of nonzero ω and then taking the limit $\omega \to 0$ [see Eq. (5.15)].

We note that for $\omega = 0$ Eq. (5.4) can be reduced to the Legendre equation. The general solution is [3]

$$\psi_{0l}(y) = C_1 y Q_l(2y-1) + C_2 y P_l(2y-1), \tag{6.4}$$

where $y = r/2M$, and $P_l(z)$ and $Q_l(z)$ are the Legendre functions of the first and second kinds with the branch cut $(-\infty, 1]$ for $Q_l(z)$. By recalling that for $\omega \to 0$ the solution we seek must be totally reflected back to the horizon, and that $P_l(z) \sim z^l$ and $Q_l(z) \sim z^{-l-1}$ for large z, we conclude that we must let $C_2 = 0$. We find the normalization constant C_1 by comparing Eq. (6.4) close to the horizon with Eq. (6.3). For this purpose, note that [see (8.834.2) and (8.831.3) of Ref. [21]]

$$Q_l(z) = \frac{P_l(z)}{2}\ln\frac{z+1}{z-1} - W_{l-1}(z)$$

where

$$W_{l-1}(z) = \sum_{k=1}^{l} k^{-1} P_{k-1}(z) P_{l-k}(z).$$

Thus from Eq. (6.4), we have

$$\psi_{0l}(x) \approx C_1 \left(\frac{1}{2} - \frac{x}{2} - \sum_{m=1}^{l} \frac{1}{m} \right) \quad (x<0, \ |x| \gg 1).$$
(6.5)

By comparing Eq. (6.5) with Eq. (6.3) we obtain $C_1 = 4M$. Thus

$$\psi_{0l}(x) = 4M y Q_l(2y-1).$$
(6.6)

The response rate to quanta of given angular momentum is readily obtained by substituting Eq. (6.6) in Eq. (4.9):

$$R_{lm}^{S-U} = \frac{q^2}{\pi M} f(r_0)^{1/2} |Q_l(z_0)|^2 |Y_{lm}(\theta_0, \varphi_0)|^2,$$

where $z_0 \equiv r_0/M - 1$. Note that for $x_0 \gg 1$ we have

$$R_{lm}^{S-U}(x_0) \approx (q^2/4\pi M) [(l!)^2/(2l+1)!]^2 (x_0)^{-2l-2} |Y_{lm}|^2.$$

Comparison of this equation and Eq. (5.19) shows that the rate obtained with the toy black hole does model the exact response rate for moderate l provided that the source is set at large x_0. In order to obtain the total response rate R_{tot}^{S-U}, we sum over l and m. For this purpose we use

$$\sum_{m=-l}^{l} |Y_{lm}(\theta, \varphi)|^2 = \frac{2l+1}{4\pi}$$

and

$$\sum_{l=0}^{\infty} (2l+1) [Q_l(z)]^2 = \frac{1}{z^2-1}.$$

The latter expression can be obtained by squaring the formula $\Sigma_{l=0}^{+\infty} P_l(t) Q_l(z) = (z-t)^{-1}$, and integrating from -1 to 1 with respect to t. In this way we obtain

$$R_{\text{tot}}^{S-U} = \frac{q^2 a(r_0)}{4\pi^2},$$
(6.7)

where $a(r_0) = M f(r_0)^{-1/2}/r_0^2$ is the proper acceleration of the static source. Note that Eq. (6.7) is identical to Eq. (3.13) as a function of proper acceleration. This is our main result: *The emission and absorption of zero-energy particles by a static source outside a Schwarzschild black hole with the initial quantum state being the Unruh vacuum is exactly the same as if the source were static in the Rindler wedge with the initial quantum state being the Minkowski vacuum.* Note that close to the horizon Eq. (6.7) can be written as a function of the proper temperature (see Ref. [23]), $\beta^{-1} = f^{-1/2}/8\pi M$, as

$$R_{\text{tot}}^{S-U} \approx \frac{q^2}{2\pi\beta} \quad (x<0, \ |x| \gg 1),$$
(6.8)

which is the same as Eq. (4.11).

VII. STATIC SOURCE OUTSIDE A SCHWARZSCHILD BLACK HOLE WITH THE HARTLE-HAWKING VACUUM

In this section we will calculate the response rate of the static source when the initial state is taken to be the Hartle-Hawking vacuum [24]. (This will show that the Unruh vacuum state is essential for the above mentioned equality.) In this state thermal fluxes come in from $\mathcal{J}^-$ as well as from H^-. The contribution of the flux from H^- to the response rate has already been calculated in the previous section. Thus, we consider here the modes incoming from $\mathcal{J}^-$. Close to and far away from the horizon these modes can be written as

$$\psi_{\omega l}(x) \approx \begin{cases} A_{\omega l} \mathcal{T}_{\omega l} e^{-2iM\omega x} & (x<0, \ |x| \gg 1), \\ A_{\omega l} [2(-i)^{l+1} M\omega x h_l^{(1)}(2M\omega x)^* + 2i^{l+1} \mathcal{R}_{\omega l} M\omega x h_l^{(1)}(2M\omega x)] & (x \gg 1), \end{cases}$$
(7.1)

where the normalization constant can be determined by the procedure used in Sec. V with the same result $A_{\omega l} = (2\omega)^{-1}$.

Recall that Eq. (6.4) gives the general expression for the zero-frequency solution $\psi_{0l}(x)$ of Eq. (5.4). Because zero-frequency modes must be totally reflected by the black-hole potential toward $\mathcal{J}^+$, we conclude that in this case $C_1 = 0$. [Note that $P_l(1) = 1$ while $Q_l(z) \approx -\log\sqrt{z-1}$ for $z \approx 1$ and that $P_l(z)$ and $Q_l(z)$ behave like z^l and z^{-l-1}, respectively, for $z \gg 1$.] Thus

$$\psi_{0l}[x(y)] = C_2 y P_l(2y-1),$$
(7.2)

where $x(y) = y + \ln(y-1)$. In order to determine the normal-

ization constant C_2 we first note that for large x Eq. (7.2) can be written as [see (8.820), (8.837.2) and (8.339.2) in Ref. [21]]

$$\psi_{0l}(x) \approx C_2 x F(-l, l+1; 1; 1-x) \quad (x \gg 1)$$

$$\approx C_2 \frac{(2l+1)!}{(l!)^2} x^{l+1} \quad (x \gg 1).$$
(7.3)

Now we find from Eq. (7.1) the following expression for $1 \ll x \ll 1/M\omega$ [see Eqs. (5.7) and (5.16)]:

$$\psi_{\omega l}(x) \approx \frac{2^{2l+1}l!M^{l+1}\omega^l}{(2l+1)!}x^{l+1} \quad (1 \ll x \ll 1/M\omega),$$
(7.4)

where we have set $\mathcal{R}_{\omega l} \approx (-1)^{l+1}$ for $\omega \ll 1$ so that $\psi_{\omega l}(x)$ behaves like x^{l+1} in the specified range of x. By comparing Eqs. (7.3) and (7.4), we obtain

$$C_2 = \frac{2^{2l+1}(l!)^3 M^{l+1}\omega^l}{[(2l+1)!]^2}\bigg|_{\omega\to 0}$$

$$= 2M\delta_{l0}.$$
(7.5)

Thus the only non-vanishing zero-energy mode $\psi_{0l}(x)$ has vanishing angular momentum.

Using Eq. (4.9) with Eqs. (7.2) and (7.5), we write the total contribution to the response rate due to the modes incoming from $\mathcal{J}^-$ in terms of the proper acceleration as

$$R_{\mathcal{J}^-} = \frac{q^2}{16\pi^2 r_0^2 a(r_0)}.$$
(7.6)

Therefore the total response rate of our scalar source in the Hartle-Hawking vacuum is given by the sum of Eqs. (7.6) and (6.7):

$$R_{\text{tot}}^{S-HH} = \frac{q^2 a}{4\pi^2} + \frac{q^2}{16\pi^2 r_0^2 a}.$$
(7.7)

Clearly Eq. (7.7) differs from Eq. (3.13) by the second term on the RHS. Thus, the equality found for the Unruh vacuum does not hold for the Hartle-Hawking vacuum.

Notice that near the horizon, the first term dominates. Hence, the response rate is approximately the same as in the Unruh vacuum and can physically be attributed to the acceleration of the source. For large x, the second term in Eq. (7.7) dominates and can be written as a function of the proper temperature as

$$R_{\text{tot}}^{S-HH} \approx \frac{q^2}{2\pi\beta} \quad (x \gg 1),$$
(7.8)

which agrees with Eq. (4.11). This is consistent with the fact that far away from the black hole the Hartle-Hawking vacuum is identical with a thermal bath in Minkowski spacetime. Note that if the equality were to hold for the Hartle-Hawking vacuum, the response rate would have to vanish in the limit $x \to \infty$. Thus, for large x the equality breaks down in the Hartle-Hawking vacuum because the source responds to a thermal bath of ''real particles'' as was found in Sec. IV. It is interesting to note that the second term in Eq. (7.7) would be absent if we considered the massless limit of a massive scalar field. Thus, the equality would be recovered. (There are no modes incoming from $\mathcal{J}^-$ that satisfy $\omega < m$, where m is the mass of the scalar field. This is why the contribution from these modes vanishes if we take the limit $\omega \to 0$ before the limit $m \to 0$.)

VIII. CONSISTENCY WITH THE LITERATURE

Before showing that our results are in agreement with those of Refs. [3,4] by Candelas and Sciama, Candelas and Deutsch (CSD), we recall (i) that our response rate is the *sum* of absorption and emission rates of zero-energy modes, and (ii) that the absorption and emission rates of zero-energy modes are equal. Thus our result would be twice the absorption rate obtained by CSD in the zero-energy limit *if we used the same source*. However, this is not the case. The classical source equivalent to the detector in Refs. [3,4] is proportional to $\exp(i\omega_0 t)$. This is replaced in our case by $\sqrt{2}\cos(\omega_0 t) = \exp(i\omega_0 t)/\sqrt{2} + \exp(-i\omega_0 t)/\sqrt{2}$. Since the second term on the RHS does not contribute to the absorption rate in the computation, our source is effectively $1/\sqrt{2}$ of the one in Refs. [3,4] as far as absorption is concerned. Eventually, when we square the amplitude to obtain the probability, we end up with an *absorption* rate which is 1/2 of the one obtained by CSD in the $\omega_0 \to 0$ limit. Hence, our results will be compatible with CSD if our total response (absorption + emission) rate is equal to the absorption rate of Refs. [3,4] in the $\omega_0 \to 0$ limit.

To show that this is indeed the case it is convenient to interpret our results (with $q^2 = 1$) in terms of the two-point function as follows. Let us consider a static world line and let x_τ be the spacetime point which corresponds to proper time τ measured along the world line of the scalar source. Then the rate (per proper time) we have computed in Schwarzschild spacetime with the field in the *Unruh vacuum* initial state, $|0\rangle_U$, is

$$R_{\text{tot}}^{S-U} = \int d\tau \; {}_U\langle 0|\phi(x_\tau)\phi(x_0)|0\rangle_U,$$

where ${}_U\langle 0|\phi(x)\phi(y)|0\rangle_U$ is the two-point function of the massless scalar field ϕ. The behavior of this quantity near and far away from the horizon can be found using the results in CSD as follows.

The quantity considered by CSD is

$$\Pi(\omega|r) = \int_{-\infty}^{+\infty} dt \exp(-i\omega t) \; {}_U\langle 0|\phi(x(t))\phi(x(0))|0\rangle_U,$$

where t is the *coordinate* time along a static world line. Since $d\tau = (1 - 2M/r)^{1/2}dt$, our response rate R_{tot}^{S-U} and $\Pi(0|r)$ should be related by

$$R_{\text{tot}}^{S-U} = (1 - 2M/r)^{1/2}\Pi(0|r).$$

Now, according to CSD, for $r \approx 2M$ one has

$$(1 - 2M/r)^{1/2}\Pi(\omega|r)$$

$$\approx \frac{\omega}{2\pi(1 - 2M/r)^{1/2}[\exp(2\pi\omega/\kappa) - 1]},$$

where $\kappa = 1/4M$. Taking the limit $\omega \to 0$ for $r \approx 2M$, one finds

$$(1-2M/r)^{1/2}\Pi(0|r) \approx \frac{1}{16\pi^2 M(1-2M/r)^{1/2}}$$

$$\approx \frac{1}{4\pi^2}\frac{M}{r^2}(1-2M/r)^{-1/2}.$$

This is in agreement with our formula (6.7) with $q^2=1$.

The calculation for large values of r is a little more involved. The formula given by CSD for this case is

$$\Pi(\omega|r) \approx \frac{1}{r^2}\frac{\Sigma_{l=0}^{+\infty}(2l+1)|B_l(\omega)|^2}{8\pi\omega[\exp(2\pi\omega/\kappa)-1]} - \frac{\omega}{2\pi}\theta(-\omega).$$

[The factor $(1-2M/r)^{1/2}$ is irrelevant now.] Taking the $\omega \to 0$ limit, one has

$$\Pi(0|r) \approx \frac{1}{64\pi^2 M r^2}\sum_{l=0}^{+\infty}(2l+1)\lim_{\omega\to 0}\frac{|B_l(\omega)|^2}{\omega^2}, \quad (8.1)$$

where the $|B_l(\omega)|^2$ are the transmission coefficients for the modes incoming from H^-. In our notation where the positive-frequency modes are given by Eq. (4.3), the $B_l(\omega)$ are defined by

$$\psi_{\omega l}(r) \approx \begin{cases} C_{\omega l}[e^{2iM\omega x}+A_l(\omega)e^{-2iM\omega x}] & (r\approx 2M), \\ 2i^{l+1}C_{\omega l}B_l(\omega)M\omega x h_l^{(1)}(2M\omega x) & (r\gg 2M), \end{cases}$$
$$(8.2)$$

where $C_{\omega l}=(2\omega)^{-1}$ [see Eq. (5.14)]. Note that for $1\ll x \ll 1/M\omega$ we have, from Eq. (8.2),

$$\psi_{\omega l}(r) \approx B_l(\omega)i^{l+1}Mx[j_l(2M\omega x)+in_l(2M\omega x)] \quad (x\gg 1)$$

$$\approx \frac{B_l(\omega)}{\omega}\frac{i^{l+1}(2l)!\,\omega^{-l}}{2^{2l+1}l!M^l}x^{-l} \quad (1\ll x\ll 1/M\omega).$$
$$(8.3)$$

In order to determine $B_l(\omega)$ for small ω, we recall that the normalized zero-frequency solution of Eq. (5.4) can be written as [see Eq. (6.6)]

$$\psi_{0l}(r)=4MyQ_l(2y-1)$$

$$=4My\frac{\Gamma(l+1)\Gamma(1/2)}{2^{l+1}\Gamma(l+3/2)}$$

$$\times\frac{1}{(2y-1)^{l+1}}F\left(\frac{l+2}{2},\frac{l+1}{2};\frac{2l+3}{2};(2y-1)^{-2}\right)$$

$$\approx \frac{2M(l!)^2}{(2l+1)!}x^{-l} \quad (x\gg 1), \quad (8.4)$$

where $y=r/2M$. Thus, comparing Eqs. (8.3) and (8.4) we have

$$\left.\frac{B_l(\omega)}{\omega}\right|_{\omega\to 0}=\left.\frac{2^{2l+2}(l!)^3M^{l+1}\omega^l}{(2l)!(2l+1)!}\right|_{\omega\to 0}$$

$$=4M\delta_{l0}. \quad (8.5)$$

Finally, substituting Eq. (8.5) in Eq. (8.1), we recover the large r limit of our formula (6.7) with $q^2=1$. Thus, our results are in agreement with CSD for $r\approx 2M$ and for large r.

IX. DISCUSSIONS

We have shown that there is an equality between the response of a static source in Schwarzschild spacetime (with the Unruh vacuum) and that of a uniformly accelerated source in Minkowski spacetime (with the Minkowski vacuum) provided that the proper acceleration is the same. This result was quite unexpected since all classical formulations of the equivalence principle are valid only locally while quantum states are defined globally. What could have been naturally expected is an equivalence in the response of the type mentioned above only close to and far away from the horizon (see, e.g., Refs. [25,26]) rather than everywhere. (After the completion of this work, a similar calculation was performed for the electromagnetic field but *no* equality was obtained in this case [27]. Thus, the equality seems to hold only for the massless scalar field.) We have also verified that close and far away from the horizon the source responds to Hawking radiation as if it were at rest in a thermal bath in Minkowski spacetime characterized by the same proper temperature in the Unruh and Hartle-Hawking vacua, respectively. Clearly, Hawking radiation was crucial in obtaining non-vanishing rates: had we chosen the Boulware vacuum [28], we would have obtained vanishing response rates. It was also shown that the equality derived for the Unruh vacuum does not hold for the Hartle-Hawking vacuum.

The procedure used in Schwarzschild spacetime to normalize the massless Klein-Gordon scalar field in the zero-frequency limit was checked by comparing it with the one performed for a toy black hole where the normal modes can be written explicitly for every frequency. Finally, our results were compared with the literature and shown to be in agreement with it.

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