Generalized Xanthopoulos theorem in the low-energy limit of string theory

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(Received 4 May 1998; revised manuscript received 22 June 1998; published 19 October 1998)

It is proved that if $(g_{\mu\nu},F_{\mu\nu},\phi,\eta)$ is an exact solution of the Einstein-Maxwell-dilaton-axion equations, then $(g_{\mu\nu}+l_\mu l_\nu,F_{\mu\nu},\phi,\eta)$ is also an exact solution of those equations if and only if $(l_\mu l_\nu,0,0,0)$ satisfies the Einstein-Maxwell-dilaton-axion equations linearized about $(g_{\mu\nu},F_{\mu\nu},\phi,\eta)$, provided that the null vector field l_μ be simultaneously a principal null direction of the electromagnetic field $F_{\mu\nu}$ and orthogonal to the gradients of the dilaton ϕ and axion η fields. Furthermore, it is shown that the matter field equations with sources are invariant under changes of the metric $g_{\mu\nu}$ by $g_{\mu\nu}+l_\mu l_\nu$ if l_μ satisfies the above conditions. An application of these results in the study of perturbations of the solution corresponding to colliding plane waves is given. [S0556-2821(98)09120-6]

PACS number(s): 04.20.Jb, 04.40.Nr

I. INTRODUCTION

With the purpose of introducing a useful tool in the search for exact solutions of the Einstein vacuum equations and their corresponding physical interpretations, Xanthopoulos proposed and proved the now called Xanthopoulos theorem [1], which establishes that if the metric $g_{\mu\nu}$ is an exact solution of the Einstein vacuum equations and l_μ a null vector field, then $g_{\mu\nu}+l_\mu l_\nu$ is also an exact solution of those equations if and only if $l_\mu l_\nu$ satisfies the Einstein equations linearized around $g_{\mu\nu}$. Recently, Torres del Castillo [2] has proved that analogous results hold in the presence of different matter fields provided that the null vector l_μ be suitably aligned to those matter fields, in addition to that has been shown the invariance of the corresponding matter field equations under changes of the metric $g_{\mu\nu}$ by $g_{\mu\nu}+l_\mu l_\nu$. More specifically, it was proved that the Maxwell equations with sources, are invariant under those changes of the metric if l_μ is a principal null direction of the electromagnetic field $F_{\mu\nu}$, with an analogous result in the case when the matter field corresponds to a scalar field Φ, if l_μ is orthogonal to the gradient of this field; a generalized Xanthopoulos theorem is established in each case. From these results, it is easy to prove that in the more general Einstein-Maxwell-scalar equations (where all matter fields appear minimally coupled to gravity), if $(g_{\mu\nu},F_{\mu\nu},\Phi)$ is an exact solution of these equations, then $(g_{\mu\nu}+l_\mu l_\nu,F_{\mu\nu},\Phi)$ is also an exact solution of these equations if and only if $(l_\mu l_\nu,0,0)$ satisfies the Einstein-Maxwell-scalar equations linearized around $(g_{\mu\nu},F_{\mu\nu},\Phi)$, provided that l_μ satisfies both conditions; i.e., it is a principal null direction of $F_{\mu\nu}$ and simultaneously orthogonal to the gradient of the scalar field Φ. Although the presence of scalar fields may be an optional issue in ordinary gravity, the more recent unification theories predict the rigorous presence of scalar partners to the usual gravity tensor. These scalar fields appear, unlike the scalar fields appearing in ordinary Einstein gravity, nonminimally coupled to gravity and other matter fields. This unusual property yields the fact that the effect of the gravity on the matter fields is not only given through the covariant derivative compatible with the metric tensor in the matter field equations, but also through the presence, in those equations, of additional terms involving the metric tensor itself and other matter fields; besides, the matter fields do not contribute in the same form as sources of curvature of the space-time in the Einstein equations, as they do in ordinary gravity theory. These facts lead to a more close coupling between gravity and matter fields. Hence, the following natural step along these lines, is to answer under what conditions, in the presence of these scalar partners, we have analogous results to those presented above for ordinary gravity theories. In this manner, the aim of the present paper is to demonstrate that when the low-energy degrees of freedom most characteristic of the string theory, namely, dilaton and axion scalar fields are excited for interacting with electromagnetic fields in the stringy way, analogous results hold just requiring the same conditions of alignment of the null vector field l_μ to the matter fields required in ordinary gravity theories, which means that l_μ be a principal null direction of $F_{\mu\nu}$ and also orthogonal to the gradients of the dilaton and axion fields. These results are not obvious, since the nonminimal coupling of gravity to the matter fields through the metric tensor in the stringy way, is radically different from that of the usual Einstein gravity theory.

We start introducing the four-dimensional effective action, which appears as a bosonic part of a generic low-energy limit of string theory and as a result of the dimensional reduction of the Kaluza-Klein (KK) theory:

$$S=\int d^4x\sqrt{-g}\{R-2(\partial_\mu\phi)\partial^\mu\phi-\tfrac{1}{2}\zeta(\phi)(\partial_\mu\eta)\partial^\mu\eta +\xi(\phi)F_{\mu\nu}F^{\mu\nu}+\omega(\eta)F_{\mu\nu}\tilde{F}^{\mu\nu}+V(\phi,\eta)\}, \tag{1}$$

where R is the scalar curvature, $g=\det(g_{\mu\nu})$, $F_{\mu\nu}=2\nabla_{[\mu}A_{\nu]}$ is the electromagnetic field, $\tilde{F}^{\mu\nu}=1/(2\sqrt{-g})\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$ corresponds to the dual of $F_{\mu\nu}$. On the other hand, ϕ represents the dilaton (scalar) field and η the axion (pseudoscalar) field; the arbitrary functions $\zeta(\phi)$,

$\xi(\phi)$, and $\omega(\eta)$ are collectively known as the coupling functions (chosen arbitrary for generality). The presence of these functions makes that the dilaton and axion appear nonminimally coupled to the gravity and matter fields. $V(\phi,\eta)$ represents the dilaton-axion potential, which is a (well-behaved) function of the dilaton and the axion alone, and contains no derivatives of these fields. A specific choice of the coupling functions and the dilaton-axion potential corresponds to a particular gravity theory. For example, for the special choice $\zeta=e^{4\phi}$, $\xi=e^{-2\phi}$, and $\omega=\eta$, the action (1) reduces to the usual low-energy effective action for the heterotic string theory. V may be a Liouville-type dilaton potential, $\Lambda e^{b\phi}$, i.e., a cosmological constant term with dilaton coupling; V also may contain the possible mass terms for dilaton and axion fields, i.e., it can contain terms of the form $m_D\phi^2+m_A\eta^2$, where m_D and m_A are the masses of the dilaton and axion fields respectively, etc. The factors -2 and $-\frac{1}{2}$ appearing in the action (1) are introduced for future convenience. As we can see, S covers a large family of nontrivial actions for four-dimensional gravity appearing in the modern literature.

Variations of the action S with respect to the axion field, dilaton field, gauge field A_μ, and metric tensor $g_{\mu\nu}$ give respectively the following Einstein-Maxwell-dilaton-axion (EMDA) equations:

$$g^{\mu\nu}\nabla_\mu(\zeta\partial_\nu\eta)+\frac{d\omega}{d\eta}F_{\mu\nu}\tilde{F}^{\mu\nu}+\frac{\partial V}{\partial\eta}=\eta_s \quad \text{(axion)}, \tag{2}$$

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi+\frac{1}{4}\left[\frac{d\xi}{d\phi}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\frac{d\zeta}{d\phi}(\partial_\mu\eta)\partial^\mu\eta+\frac{\partial V}{\partial\phi}\right]=\phi_s \quad \text{(dilaton)}, \tag{3}$$

$$\nabla_\mu(\omega\tilde{F}^{\mu\nu}+\xi F^{\mu\nu})=J^\nu \quad \text{(Maxwell modified)}, \tag{4}$$

$$\nabla_\mu\tilde{F}^{\mu\nu}=0 \quad \text{(Bianchi identities)}, \tag{5}$$

$$R_{\mu\nu}=2(\partial_\mu\phi)\partial_\nu\phi+\frac{1}{2}\zeta(\partial_\mu\eta)\partial_\nu\eta-2\xi\left(F_{\mu\lambda}F_\nu{}^\lambda-\frac{1}{4}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}\right)-\frac{1}{2}g_{\mu\nu}V \quad \text{(Einstein)}, \tag{6}$$

where for generality, in Eqs. (2)–(4) we have included the scalar sources η_s and ϕ_s for the axion and dilaton fields, respectively, and a current density J^ν for the electromagnetic field.

The outline of this paper is as follows. In Sec. II we introduce the basic framework and it is shown the invariance of the matter field equations (2)–(5) under changes of the metric. In Sec. III we consider the generalization of the Xanthopoulos theorem for the EMDA theory. In Sec. IV we apply the results of the previous sections to the study of the perturbations of the solution which represents colliding plane waves in the scheme of the EMDA theory. We then finish in Sec. V with some concluding remarks on our results.

II. INVARIANCE OF THE MATTER FIELD EQUATIONS

In this section we set up briefly the basic framework and notation that will be used in this paper. For more details we refer readers to [2].

If l_μ is a null vector field and we consider two geometries whose metrics are related by

$$g'_{\mu\nu}=g_{\mu\nu}+l_\mu l_\nu, \tag{7}$$

then it is easy to find that

$$g'^{\mu\nu}=g^{\mu\nu}-l^\mu l^\nu, \tag{8}$$

$$\Gamma'^\mu_{\nu\lambda}=\Gamma^\mu_{\nu\lambda}+C^\mu_{\nu\lambda}, \tag{9}$$

$$C^\mu_{\nu\lambda}=\frac{1}{2}[\nabla_\nu(l^\mu l_\lambda)+\nabla_\lambda(l^\mu l_\nu)-\nabla^\mu(l_\nu l_\lambda)+l^\mu l^\rho\nabla_\rho(l_\nu l_\lambda)], \tag{10}$$

$$l^\rho C^\mu_{\rho\nu}=\frac{1}{2}(l^\mu X_\nu+l_\nu X^\mu),\quad l_\rho C^\rho_{\mu\nu}=-\frac{1}{2}(l_\mu X_\nu+X_\mu l_\nu),$$

$$C^\mu_{\mu\lambda}=0,\quad l^\nu l^\lambda C^\mu_{\nu\lambda}=0,\quad g^{\nu\lambda}C^\mu_{\nu\lambda}=\nabla_\nu(l^\mu l^\nu)=X^\mu+\theta l^\mu, \tag{11}$$

$$R'_{\mu\nu}=R_{\mu\nu}+2\nabla_{[\rho}C^\rho_{\mu]\nu}+2C^\rho_{\mu[\nu}C^\lambda_{\lambda]\rho}, \tag{12}$$

where $\theta=\nabla^\mu l_\mu$, and $X_\mu=l^\rho\nabla_\rho l_\mu$ satisfies $l^\mu X_\mu=0$; ∇_μ is the covariant derivative with respect to the background metric $g_{\mu\nu}$, which raises and lowers the indices. Furthermore, it is not difficult to show that

$$g'=g. \tag{13}$$

We now consider the necessary and sufficient conditions of alignment of the vector l_μ with the matter fields, in order to establish our results. First, we take l_μ along a principal null direction of the electromagnetic field $F_{\mu\nu}$,

$$l^\mu F_{\mu\nu}=\lambda l_\nu, \tag{14}$$

where λ is some scalar function and simultaneously, orthogonal to the gradients of the dilation and axion fields, that is

$$l^\mu\nabla_\mu\phi=0, \tag{15}$$

$$l^\mu\nabla_\mu\eta=0. \tag{16}$$

In this manner, we can show, using Eqs. (8) and (14) and the nullness of l_μ that

$$(F_{\mu\nu}F^{\mu\nu})'\equiv g'^{\mu\alpha}g'^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}=g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}=F_{\mu\nu}F^{\mu\nu}, \tag{17}$$

which means that the quantity $F_{\mu\nu}F^{\mu\nu}$ appearing in the dilation equation (3) is invariant under changes of the metric considered. Similarly, from Eq. (13) we have that

$$(\tilde{F}^{\mu\nu})' \equiv \frac{1}{2\sqrt{-g'}}\,\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \frac{1}{2\sqrt{-g}}\,\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \tilde{F}^{\mu\nu},$$

and then

$$(F_{\mu\nu}\tilde{F}^{\mu\nu})' = F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{18}$$

In order to show the invariance of Eq. (2), we write the relevant terms of the left-hand side of this equation for the metric $g'_{\mu\nu}$ and using Eqs. (8), (9), and (18) we have that

$$g'^{\mu\nu}\nabla'_{\mu}(\zeta\partial_{\nu}\eta) + \frac{d\omega}{d\eta}(F_{\mu\nu}\tilde{F}^{\mu\nu})' = g^{\mu\nu}\nabla_{\mu}(\zeta\partial_{\nu}\eta) - g^{\mu\nu}C^{\lambda}_{\mu\nu}(\zeta\partial_{\lambda}\eta) - l^{\mu}l^{\nu}\nabla_{\mu}(\zeta\partial_{\nu}\eta) + l^{\mu}l^{\nu}C^{\lambda}_{\mu\nu}(\zeta\partial_{\lambda}\eta) + \frac{d\omega}{d\eta}F_{\mu\nu}\tilde{F}^{\mu\nu}$$
$$= g^{\mu\nu}\nabla_{\mu}(\zeta\partial_{\nu}\eta) + \frac{d\omega}{d\eta}F_{\mu\nu}\tilde{F}^{\mu\nu} - \nabla_{\mu}(l^{\mu}l^{\nu}\zeta\partial_{\nu}\eta), \tag{19}$$

where the last expression has been obtained using Eqs. (11). The last term of Eq. (19) vanishes according to Eq. (16); therefore, the scalar source η_s for the axion field is the same in both geometries. The term $\partial V/\partial\eta$ appearing in Eq. (2) may be added to both sides of Eq. (19) (or it may be absorbed in the definition of η_s) and the conclusion remains valid.

Similarly, the relevant terms of the left-hand side of Eq. (3) satisfy

$$g'^{\mu\nu}\nabla'_{\mu}(\partial_{\nu}\phi) + \frac{1}{4}\frac{d\xi}{d\phi}(F_{\mu\nu}F^{\mu\nu})' - \frac{1}{8}\frac{d\zeta}{d\phi}g'^{\mu\nu}(\partial_{\mu}\eta)\partial_{\nu}\eta = g^{\mu\nu}\nabla_{\mu}(\partial_{\nu}\phi) + \frac{1}{4}\frac{d\xi}{d\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}\frac{d\zeta}{d\phi}g^{\mu\nu}(\partial_{\mu}\eta)\partial_{\nu}\eta$$
$$-\nabla_{\mu}(l^{\mu}l^{\nu}\partial_{\nu}\phi) + \frac{1}{8}\frac{d\zeta}{d\phi}(l^{\mu}\partial_{\mu}\eta)^2, \tag{20}$$

where Eq. (17) has been used; the two last terms of Eq. (20) vanish in accordance with Eqs. (15) and (16), respectively. In this manner, the scalar source ϕ_s for the dilaton field is the same in both geometries.

As is well known, the Bianchi identities (5) can be expressed in the form

$$\nabla_{\mu}\tilde{F}^{\mu\nu} = 0 = \epsilon^{\mu\nu\alpha\beta}\partial_{\mu}F_{\alpha\beta},$$

which means that these equations actually do not depend on the background metric. These equations are equivalent to those in Eqs. (11b) of Ref. [2].

In the case of the Maxwell modified equations (4) we have

$$g'^{\alpha\beta}g'^{\mu\nu}\nabla'_{\alpha}(\xi F_{\beta\mu}) + (\tilde{F}^{\mu\nu})'\partial_{\mu}\omega$$
$$= g'^{\mu\nu}[\xi g'^{\alpha\beta}\nabla'_{\alpha}F_{\beta\mu} + F_{\beta\mu}g'^{\alpha\beta}\partial_{\alpha}\xi] + \tilde{F}^{\mu\nu}\partial_{\mu}\omega, \tag{21}$$

but from Ref. [2]

$$g'^{\alpha\beta}\nabla'_{\alpha}F_{\beta\mu} = (g^{\alpha\beta}\nabla_{\alpha}F_{\beta\rho})g^{\rho\lambda}g'_{\lambda\mu}, \tag{22}$$

and using Eqs. (8) and (14) it is not difficult to show that

$$F_{\beta\mu}g'^{\alpha\beta}\partial_{\alpha}\xi = g^{\alpha\beta}F_{\beta\rho}(\partial_{\alpha}\xi)g^{\rho\lambda}g'_{\lambda\mu}, \tag{23}$$

and then from Eqs. (21)–(23) we have

$$g'^{\alpha\beta}g'^{\mu\nu}\nabla'_{\alpha}(\xi F_{\beta\mu}) + (\tilde{F}^{\mu\nu})'\partial_{\mu}\omega$$
$$= g^{\alpha\beta}g^{\mu\nu}\nabla_{\alpha}(\xi F_{\beta\mu}) + \nabla_{\mu}(\omega\tilde{F}^{\mu\nu}), \tag{24}$$

it means that the current density J^{ν} is the same in both geometries. Note that, although the Maxwell modified equations involve the dilaton and axion fields, only the alignment condition (14) has been required in order to obtain the relation (24). The results (19), (20), and (24) show that, in particular, the fields $F_{\mu\nu}$, ϕ, and η satisfy the source-free field equations with the metric $g'_{\mu\nu}$ if and only if they do with the metric $g_{\mu\nu}$.

III. GENERALIZATION OF THE XANTHOPOULOS THEOREM

As it has been suggested previously, the Xanthopoulos theorem for the EMDA theory reads the following.

Let $(g_{\mu\nu}, F_{\mu\nu}, \phi, \eta)$ be an exact solution of the source-free EMDA equations [i.e., Eqs. (2)–(6) with $J^{\nu}=0=\eta_s=\phi_s$], and l_{μ} a null vector field satisfying the alignment conditions (14)–(16), then $(g_{\mu\nu}+l_{\mu}l_{\nu}, F_{\mu\nu}, \phi, \eta)$ is also an exact solution of those equations if and only if $(l_{\mu}l_{\nu}, 0,0,0)$ satisfies the EMDA equations linearized about $(g_{\mu\nu}, F_{\mu\nu}, \phi, \eta)$, which will be proved now.

Linearized EMDA equations are obtained from Eqs. (2)–(6) taking first-order perturbations of the metric ($h_{\mu\nu}$) without perturbing the matter fields:

$$\nabla^\alpha\nabla_\alpha h_{\mu\nu}-2\nabla^\alpha\nabla_{(\mu}h_{\nu)\alpha}+\nabla_\mu\nabla_\nu(g^{\alpha\beta}h_{\alpha\beta})$$
$$=-4\xi h^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}-\xi h_{\mu\nu}F_{\alpha\beta}F_{\rho\gamma}g^{\alpha\rho}g^{\beta\gamma}$$
$$+2\xi g_{\mu\nu}h^{\alpha\beta}F_{\alpha\gamma}F_{\beta\rho}g^{\gamma\rho}+Vh_{\mu\nu}, \quad (25)$$

$$\nabla_\mu\xi(F_{\alpha\nu}h^{\mu\alpha}+F^{\mu\rho}h_{\rho\nu}-\tfrac{1}{2}F^\mu{}_\nu g^{\alpha\beta}h_{\alpha\beta})=0, \quad (26)$$

$$\nabla_\mu(h^{\mu\alpha}\nabla_\alpha\phi)-\tfrac{1}{2}(\nabla^\lambda\phi)\nabla_\lambda(g_{\mu\alpha}h^{\mu\alpha})+\frac{1}{2}\frac{d\xi}{d\phi}F_{\mu\rho}F_\alpha{}^\rho h^{\mu\alpha}$$
$$-\frac{1}{8}\frac{d\zeta}{d\phi}h^{\mu\alpha}(\partial_\mu\eta)(\partial_\alpha\eta)=0, \quad (27)$$

$$\nabla_\mu(h^{\mu\alpha}\zeta\nabla_\alpha\eta)-\tfrac{1}{2}\zeta(\nabla^\lambda\eta)\nabla_\lambda(g_{\mu\alpha}h^{\mu\alpha})$$
$$+\frac{1}{2}\frac{d\omega}{d\eta}F_{\mu\nu}\tilde{F}^{\mu\nu}(g_{\lambda\rho}h^{\lambda\rho})=0, \quad (28)$$

where $h_{\mu\nu}=l_\mu l_\nu$. We first assume that $h_{\mu\nu}$ satisfies the linearized equations (25)–(28) with l_μ satisfying the conditions (14)–(16); using these conditions and the nullness of l_μ it is straightforward to show that Eqs. (26)–(28) are satisfied identically. However, since $h_{\mu\nu}=l_\mu l_\nu$ and l_μ satisfies the condition (14), from Eq. (25) we have that

$$\nabla^\alpha\nabla_\alpha(l_\mu l_\nu)-2\nabla^\alpha\nabla_{(\mu}(l_{\nu)}l_\alpha)$$
$$=-(4\xi\lambda^2+\xi F_{\alpha\beta}F^{\alpha\beta}-V)l_\mu l_\nu, \quad (29)$$

if we contract Eq. (29) with $l^\nu l^\nu$, then it is easy to find that

$$X^\mu X_\mu=0, \quad (30)$$

if the contraction is with l^μ and we taking into account that $X_\mu=\psi l_\mu$ for some scalar function ψ [2], then the result is

$$(\nabla_\mu l_\nu)(\nabla^\mu l^\nu)=-\psi(\theta+\psi)-\dot{\theta}-\dot{\psi}, \quad (31)$$

where the overdot denotes the directional derivative along l^μ. On the other hand, using the fact that $l^\mu l^\nu R_{\mu\nu}=0$, which is a consequence of Eq. (6), and Eqs. (14)–(16), we can find that

$$(\nabla_\mu l_\nu)(\nabla^\nu l^\mu)=\theta\psi-\dot{\theta}+\dot{\psi}. \quad (32)$$

From expressions (30)–(32) we can find that [see Eq. (12)]

$$2\nabla_{[\rho}C^\rho_{\mu]\nu}+2C^\rho_{\mu[\nu}C^\lambda_{\lambda]\rho}$$
$$=-\frac{1}{2}[\nabla^\alpha\nabla_\alpha(l_\mu l_\nu)-2\nabla^\alpha\nabla_{(\mu}(l_{\nu)}l_\alpha)], \quad (33)$$

where the relation $X_\mu=\psi l_\mu$ has been used once more; the right-hand side of Eq. (33) is essentially the left-hand side of Eq. (29). Hence, we can finally find from Eqs. (6)–(8), (12), (29), and (33) that

$$R'_{\mu\nu}=2(\partial_\mu\phi)\partial_\nu\phi+\frac{1}{2}\zeta(\partial_\mu\eta)\partial_\nu\eta$$
$$-2\xi\left[g'^{\alpha\rho}F_{\mu\rho}F_{\nu\alpha}-\frac{1}{4}g'_{\mu\nu}g'^{\rho\lambda}g'^{\alpha\beta}F_{\rho\alpha}F_{\lambda\beta}\right]$$
$$-\frac{1}{2}g'_{\mu\nu}V. \quad (34)$$

Equation (34) shows then that the metric $g'_{\mu\nu}$ satisfies the Einstein equation (6) and, as it has been shown in the previous section, the electromagnetic, dilaton and axion fields satisfy the corresponding matter field equations in the metric $g'_{\mu\nu}$ [see paragraph after Eq. (24)]. With this, we demonstrate a part of the generalized Xanthopoulos theorem. Conversely, it is not difficult to prove that if $(g_{\mu\nu}+l_\mu l_\nu,F_{\mu\nu},\phi,\eta)$ is an exact solution of the EMDA equations then $h_{\mu\nu}=l_\mu l_\nu$ satisfies the linearized equations (25)–(28), which completes the proof.

IV. PURELY INCOMING PERTURBATIONS AS AN EXACT SOLUTION

As we will see in this section, our previous results will allow us to answer a question which remained open at the end of Ref. [3]. In that reference it was demonstrated the existence of purely incoming perturbations in the study of the perturbations of the space-time which represents plane waves bound to collision in the scheme of EMDA theory. Although such purely incoming perturbations correspond to a solution of the linearized EMDA equations, we demonstrate that the corresponding purely gravitational perturbations correspond also to an *exact* solution of the EMDA equations.

The purely gravitational perturbations can be obtained from Eqs. (63) in Ref. [3] when the electromagnetic field perturbations vanish setting $\partial_z^3 F=0$, since the dilaton and axion perturbations were to be vanishing. On the other hand, the corresponding first-order metric perturbations can be obtained from Eqs. (56) of Ref. [3], remembering that in this case $D\psi_G=0$,

$$h_{\mu\nu}=hl^{\rm NP}_\mu l^{\rm NP}_\nu, \quad (35)$$

where h is a real scalar function whose explicit form is not important; $l^{\rm NP}_\mu$ is the usual (real) null vector of the null tetrad of the Newman-Penrose (NP) formalism used in that reference, which is not necessarily our null vector l_μ used in the present approach. If we choose $l_\mu=\sqrt{h}l^{\rm NP}_\mu$, then from Eq. (35) $h_{\mu\nu}=l_\mu l_\nu$, then the metric perturbations take the form required by our present approach. Moreover, from Eq. (28) of Ref. [3] and from the expression for the electromagnetic field $F_{\mu\nu}$ in the NP formalism we find that

$$l^\mu F_{\mu\nu}=l^\mu[2\phi_2 l^{\rm NP}_{[\mu}m_{\nu]}+{\rm c.c.}]=0, \quad (36)$$

which corresponds to the alignment condition (14) with $\lambda=0$. Furthermore, from Ref. [3] it is easy to show that the gradients of the dilaton and axion fields are given respectively by

$$\nabla_\mu \phi = l_\mu^{\rm NP} \Delta \phi = \frac{1}{\sqrt{h}}\, l_\mu \Delta \phi, \tag{37}$$

$$\nabla_\mu \eta = l_\mu^{\rm NP} \Delta \eta = \frac{1}{\sqrt{h}}\, l_\mu \Delta \eta, \tag{38}$$

since in the regions prior to the collision, $\Delta\phi$ and $\Delta\eta$ are the only nonvanishing derivatives of these fields. Clearly, the expressions (37) and (38) satisfy the conditions (15) and (16) because of the nullness of l_μ. Hence, the purely gravitational perturbations (35) also correspond, in accordance with our previous results, to an exact solution of the EMDA equations. In this manner, the existence of such field perturbations is not only at level of linearized EMDA equations, but at level of the complete EMDA equations themselves.

These results for the EMDA theory contain as a particular case, analogous results for the Einstein-Maxwell-Dilaton theory (absence of the axion field), where only the conditions (14) and (15) are involved. In this manner, our present results also answer the question which remained open at the end of Ref. [4].

V. CONCLUDING REMARKS

Our present results represent, like those originally presented by Xanthopoulos in Ref. [1], a tool for studying the structure of solutions in the framework of the EMDA theory. However, the requirement of simultaneous alignment of l_μ to the matter fields, i.e., conditions (14)–(16), seems to limit the possible solutions of the EMDA theory which will satisfy those necessary conditions. For example, unlike the case of the colliding plane waves solution described in Sec. IV, in the case of the solutions which represent black holes with dilaton and axion fields [5,6], if one takes l_μ along the principal null direction of the electromagnetic field, i.e., satisfying condition (14), then l_μ does not satisfy necessarily the remainder conditions (15) and (16). In this manner, the rigorous presence of the dilaton and axion fields establishes a significant difference between the EMDA solutions and their analogues in the ordinary Einstein theory where, for example, just the alignment of l_μ to the electromagnetic field is required (see [2] and references cited therein). However, some possible applications of our results will be the subject of future investigations.

ACKNOWLEDGMENTS

This work was supported in part by CONACYT (México) and the Sistema Nacional de Investigadores (México). The authors wish to acknowledge Dr. G. F. Torres del Castillo for useful comments on the manuscript and suggestions.

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