Topology of event horizons

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The topology of event horizons is investigated. Considering the existence of the end point of the event horizon, the event horizon cannot be differentiable. Then there are new possibilities for the topology of the event horizon, excluded in smooth event horizons. The relation between the spatial topology of the event horizon and its end points is revealed. A toroidal event horizon is caused by two-dimensional end point sets. One-dimensional end point sets provide the coalescence of spherical event horizons. Moreover, these aspects can be removed by an appropriate time slicing. The result will be useful to discuss the stability and generality of the topology of the event horizon. [S0556-2821(98)07620-6]

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I. INTRODUCTION

The existence of an event horizon is one of the most characteristic concepts of general relativity. So many authors have studied the properties of the event horizon. Mathematically, the event horizon is defined as the boundary of the causal past of future null infinity [1]. Since the natural asymptotic structure of spacetimes is supposed to be asymptotically flat, where the topology of the future null infinity is $S^2 \times R$, we naively think that the ''spatial'' topology of the event horizon (TOEH[1]) will always be S^2.

Simple situations arise in general stationary spacetimes for which it can be shown that any event horizon must have a spherical topology [2,3]. The first work dealing with the topology of nonstationary black holes is due to Gannon [4]. With the physically reasonable condition of asymptotic flatness, it was proved that the topology of a smooth event horizon must be either a sphere or a torus (when the dominant energy condition is satisfied). Such an approach has recently been extended and generalized to give stronger theorems, with the assumptions of asymptotic flatness, global hyperbolicity, and a suitable energy condition. Friedmann, Schleich, and Witt proved the ''topological censorship'' theorem that any two causal curves extending from past to future null infinity are homotopy equivalent to each other [5]. Jacobson and Venkataramani [6] have established a theorem that strengthens a recent result due to Browdy and Galloway that the TOEH with a time slicing is a sphere if no new null generators enter the horizon at later times [7]. The theorem of Jacobson and Venkataramani limits the time for which a toroidal event horizon can persist.

Some of these works are based on the differentiability of the event horizon or the absence of an end point implying nondifferentiability. Considering the whole structure of the event horizon, however, the event horizon cannot always be differentiable. For example, even in the case of a spherically symmetric spacetime (for example, Oppenheimer-Snyder spacetime) the event horizon is not differentiable where the event horizon is formed. If the event horizon is not smooth, we cannot say that the event horizon should always be a sphere.

In fact, the existence of an event horizon whose spatial topology is not a single S^2 is reported in the numerical simulations of gravitational collapse. Shapiro and coworkers [8] numerically observed a toroidal event horizon in the collapse of toroidal matter. Seidel and coworkers have numerically shown the coalescence of two spherical event horizons [9]. This is because, as shown in the present article, an event horizon is not differentiable at the subset of the end points of the null geodesics generating the event horizon. In the present article, we are mainly concerned with such nondifferentiability at the end point. We are not concerned with nondifferentiability not related to the end points.

In a physically realistic gravitational collapse, it is believed that a spacetime is quasistationary far in the future. So, it may be natural to assume that the TOEH should be a sphere for a single asymptotic region. Then the problem of the TOEH is regarded as a topology changing process from a nonspherical surface to a sphere in a three-dimensional manifold (the event horizon). Therefore we will put theorems of topology change [11,12] into this problem.

In the next section, we prepare the theorems of the topology change of a spacetime, which is applied to the event horizon in Sec. III. The final section is devoted to the summary and discussions.

II. THE TOPOLOGY CHANGE OF A SPACETIME

Many works have been concerned with the topology change of a spacetime. Some of these are useful to discuss the TOEH, which is a three-dimensional null surface imbedded into a four-dimensional spacetime. Now we briefly present several theorems concerning the topology change of a spacetime.

A. The Poincaré-Hopf theorem

Our investigation is based on a well-known theorem regarding the relation between the topology of a manifold and

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[1]The TOEH means the topology of the spatial section of the EH throughout the present article. Of course, it may depend on a time slicing.

a vector field on it.[2] The following Poincaré-Hopf theorem (Milnor, 1965) is essential for our investigation.

Theorem II.1: Poincaré-Hopf. Let M be a compact n-dimensional ($n \geqslant 2$) C^r ($r \geqslant 1$) *manifold. X is any* C^{r-1} *vector field with at most a finite number of zeros, satisfying the following two conditions: (a) The zeros of X are contained in IntM. (b) X has outward directions at* ∂M*. Then the sum of the indices of X at all its zeros is equal to the Euler number* χ *of M:*

$$\chi(M) = \mathrm{ind}(X). \tag{1}$$

The index of the vector field X at a zero p is defined as follows. Let $X_a(x)$ be the components of X with respect to local coordinates $\{x^a\}$ in a neighborhood about p. Set $v_a(x) = X_a(x)/|X|$. If we evaluate v on a small sphere centered at $x(p)$, we can regard $v_a(S^{n-1})$ as a continuous mapping[3] from S^{n-1} into S^{n-1}. The mapping degree [13] of this map is called the index of X at the zero p. For example, if the map is homeomorphic, the mapping degree of the orientation preserving (reversing) map is $+1$ (-1). Figure 1 gives some examples of the zeros in two and three dimensions.

In the present article, we treat a three-dimensional manifold imbedded into a four-dimensional spacetime manifold as an EH. The three-dimensional manifold has two two-dimensional boundaries as an initial boundary and a final boundary (which is assumed to be a sphere in the next section). For such a manifold, we use the following modification of the Poincaré-Hopf theorem. Now we consider an odd-dimensional manifold with two boundaries, Σ_1 and Σ_2.

Theorem II.2: Sorkin 1986. Let M be a compact n-dimensional ($n > 2$ *is an odd number)* C^r ($r \geqslant 1$) *manifold with* $\Sigma_1 \cup \Sigma_2 = \partial M$ *and* $\Sigma_1 \cap \Sigma_2 = \phi$*. X is any* C^{r-1} *vector field with at most a finite number of zeros, satisfying the following two conditions: (a) The zeros of X are contained in IntM. (b) X has inward directions at* Σ_1 *and outward directions at* Σ_2*. Then the sum of the indices of X at all its zeros is related to the Euler numbers of* Σ_1 *and* Σ_2*:*

$$\chi(\Sigma_2) - \chi(\Sigma_1) = 2\,\mathrm{ind}(X). \tag{2}$$

A proof of this theorem is given in Sorkin's work [11].

[2]It should be noted that we never take the affine parametrization of a vector field so that the vector field is continuous even at the end point of the curve tangent to the vector field, since we deal with the end point as the zero of the vector field. If we chose affine parameters, the vector field is not unique at the crease set (see next section).

[3]For the theorem in this statement, we only need a continuous vector field and the index of its zero defined by the continuous map v: $S^{n-1} \to S^{n-1}$. Nevertheless, if one wishes to relate the index and the Hesse matrix $H = \nabla_a v_b$, a C^2 manifold and a C^1 vector field will be required.

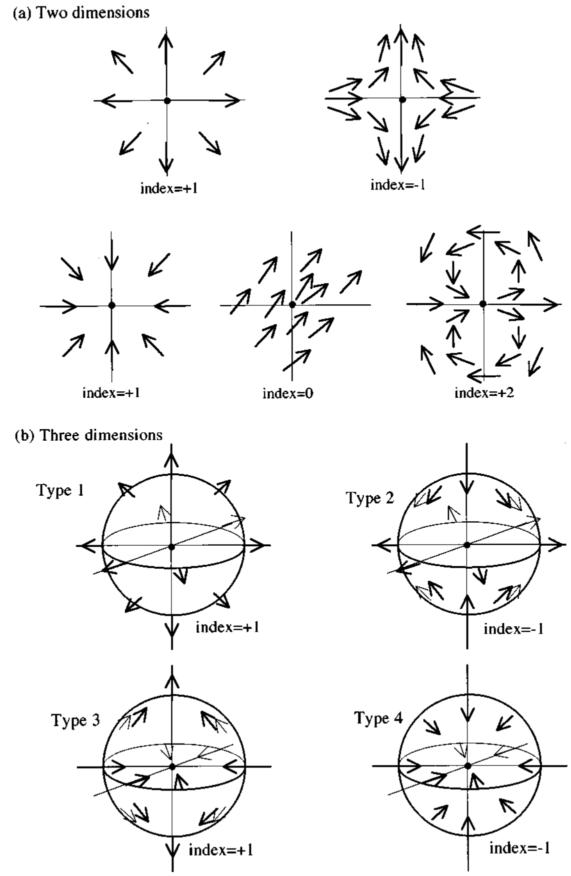

FIG. 1. (a) Two-dimensional zeros and a vector field around them. Five types of zeros are shown in this figure. (b) Three-dimensional zeros and a vector field around them. Only the zeros with |index| = 1 are shown. Other cases can easily be understood by analogy to 1(a).

B. Geroch's theorem

Geroch proved that there is no topology change of a spacetime without a closed timelike curve[4] [12].

Theorem II.3: Geroch 1967. Let M be a C^r ($r \geqslant 2$) *n-dimensional compact spacetime manifold whose boundary is the disjoint union of two compact spacelike* ($n-1$) *manifolds,* Σ_1 *and* Σ_2*. Suppose M is isochronous and has no closed timelike curve. Then* Σ_1 *and* Σ_2 *are* C^{r-1} *diffeomorphic, and M is topologically* $\Sigma_1 \times [0,1]$.

This theorem is not directly applicable to a null surface *H*, where a chronology is determined by null geodesics generated by a null vector field *K*. In this case, "isochronous" means that there is no zero of *K* in the interior of *H*. On the other hand, the closed timelike curve does not correspond to a closed null curve in a rigorous sense, since on a null surface an imprisoned null geodesic cannot be distorted, remaining null, so as to become a closed curve as stipulated by theorem II.3 [12]. Then we require a strongly causal condition [10] on a spacetime rather than the condition of no

[4]Originally he assumed a C^∞-differentiable spacetime. Nevertheless, his theorem is easily generalized to a C^r ($r \geqslant 2$) spacetime.

closed causal curve. The following modified version of Geroch's theorem arises.

Theorem II.4. Let H be a C^r $(r \geq 2)$ *n-dimensional compact null surface whose boundary is the disjoint union of two compact spacelike* $(n-1)$ *manifolds,* Σ_1 and Σ_2. *Suppose that there exists a* C^{r-1} *null vector field K which is nowhere zero in the interior of H and has inward and outward directions at* Σ_1 *and* Σ_2, *respectively, and H is imbedded into a strongly causal spacetime (M,g). Then* Σ_1 *and* Σ_2 *are* C^{r-1} *diffeomorphic, and H is topologically* $\Sigma_1 \times [0,1]$.

Proof. Let γ be a curve in H, beginning on Σ_1, and everywhere tangent to K. Suppose first that γ has no future end point both in the interior of H and its boundary Σ_2. Parametrizing γ by a continuous variable t with range zero to infinity, the infinite sequence $P_i = \gamma(i)$, $i = 1,2,3,...$, on the compact set H has a limit point P. Then for any positive number s, there must be a $t > s$ with $\gamma(t)$ in a sufficiently small open neighborhood $\mathcal{U}_P$ (since P is a limit point of P_i), and a $t' > s$ with $\gamma(t')$ not in $\mathcal{U}_P$ (since γ has no future end point). That is, γ must pass into and then out of the neighborhood $\mathcal{U}_P$ an infinite number of times. Since $\mathcal{U}_P$ can be regarded as the open neighborhood of $\gamma(t) \in \mathcal{U}_P$, this possibility is excluded by the hypothesis that H is imbedded into a strongly causal spacetime (M,g). Then such a curve γ must have a future end point on Σ_2, because there is no zero of K which is the future end point of γ in the interior of H, from the assumption of the theorem. Hence we can draw the curve γ through each point p of H from Σ_1 to Σ_2. By defining the appropriate parameter of each γ, a one parameter family of surfaces from Σ_1 to Σ_2 passing thorough every point of H is given [12]. Furthermore the C^{r-1} congruence K provides a one-to-one correspondence between any two surfaces of this family. Hence, Σ_1 and Σ_2 are C^{r-1} diffeomorphic and $H \sim \Sigma_1 \times [0,1]$. □

III. THE TOPOLOGY OF EVENT HORIZONS

Now, we apply the topology change theorems given in the previous section to EHs. Let (M,g) be a four-dimensional C^∞ spacetime whose topology is R^4. In the remainder of this article, the spacetime (M,g) is assumed to be strongly causal, and also weak cosmic censorship is assumed. Furthermore, for simplicity, the TOEH is assumed to be a smooth S^2 far in the future and the EH is not an eternal one (in other words, the EH begins somewhere in the spacetime, and it is open to the infinity in the future direction with a smooth S^2 section). We expect that those assumptions could be valid if we consider only one regular $(\sim R \times S^2)$ asymptotic region, namely the future null infinity $\mathcal{J}^+$, to define the EH, and the formation of a black hole. The following investigation, however, is easily extended to the case of different final TOEHs far in the future.

In our investigation, the most important concept is the existence of the end points of null geodesics λ which lie completely in the EH and generate it. We call these the end points of the EH. To generate the EH the null geodesics λ are maximally extended to the future and past as long as they belong to the EH. Then the end point is the point where such null geodesics are about to come into the EH (or go out of the EH), though the null geodesic can continue to the outside or the inside of the EH through the end point in the sense of the whole spacetime. We consider a null vector field K on the EH which is tangent to the null geodesics λ. K is not affinely parametrized, but parametrized so as to be continuous even on the end point where the caustic of λ appears. Then the end points of λ are the zeros of K, which can become only past end points, since λ must reach to infinity in the future direction. Of course, using an affine parametrization, K becomes ill-defined at a subset of the set of the end points. We call such a subset the *crease set*. To be precise, we define the crease set by the set of the end points contained by two or more null generators of the EH. Thus the set of the end points consists of the crease set and end points contained by one null generator. As stated in Ref. [14], the crease set contains the interior of the set of the end points and the closure of the crease set contains the set of the end points.[5]

Moreover, the fact that the EH defined by $\dot{J}^-(\mathcal{J}^+)$ (the boundary of the causal past of the future null infinity) is an achronal boundary (the boundary of a future set) tells us that the EH is an imbedded C^{1-} submanifold without a boundary (see Ref. [1]). Introducing normal coordinates (x^1,x^2,x^3,x^4) in a neighborhood $\mathcal{U}_\alpha$ about p on the EH, the EH is immersed as $x^4 = F(x^1,x^2,x^3)$, where $\partial/\partial x^4$ is timelike. Since the EH is an achronal boundary, F is a Lipschitz function and one-to-one map ψ_α: $\mathcal{V}_\alpha \to R^3$, $\psi_\alpha(p) = x^i(p)$ is a homeomorphism, where $\mathcal{V}_\alpha$ is the intersection of $\mathcal{U}_\alpha$ and the EH [1]. Then the EH is an imbedded three-dimensional C^{1-} submanifold.

First we study the relation between the crease set and the differentiability of the EH. We see that the EH is not differentiable at the crease set.

Lemma III.1 Suppose that H is a three-dimensional null surface imbedded into the spacetime (M,g) by a function F as

$$H: \; x^4 = F(x^i, i = 1,2,3), \tag{3}$$

in a coordinate neighborhood $(\mathcal{U}_\alpha, \phi_\alpha)$, ϕ_α: $\mathcal{U}_\alpha \to R^4$, *where* $\partial/\partial x^4$ *is timelike. When H is generated by the set of null geodesics whose tangent vector field is K, we define the crease set by the set of the end points of the null geodesics contained by two or more null generators of H. Then, H and the imbedding function F are nondifferentiable at the crease set.*

Proof. If H is a C^r $(r \geq 1)$ null surface around p, we can define the tangent space T_p of H, which is spanned by one null vector and two independent spacelike vectors. On the contrary, a point in the crease set is contained by two or more null generators of H. Therefore, there exists two or more null vectors tangent to H at p, and there is no unique choice of a null vector defining T_p. This implies that H and the imbedding function F are not differentiable at the crease set.

In the present article, we deal only with this nondifferentiability. Then, we assume that the EH is C^r $(r \geq 2)$ differ-

[5]Though Ref. [14] deals with a Cauchy horizon, the same proof is available for an EH.

entiable (the inequality $r \geqslant 2$ is necessary for theorem II.4), except on the crease set of the EH and we assume that the set of the end points is compact. Thus we suppose that the EH is nondifferentiable only on a compact subset. Incidentally, in the case where future null infinity possesses pathological structure, the EH could be nowhere differentiable [15]. Nevertheless we have no concrete example of a physically reasonable spacetime with such a noncompact nondifferentiability. Similarly there might be the case that the EH is nondifferentiable elsewhere than at the end point of the EH. In spite of this possibility, the reason we consider only the nondifferentiability caused by the end points is that EHs possess at least one end point, as long as the EHs are not eternal. Most of the nondifferentiability which we can imagine would be concerned with the end point.

Next, we prepare a basic proposition. Suppose there is no past end point of a null geodesic generator of an EH between Σ_1 and Σ_2. Then, Geroch's theorem stresses the topology of a smooth EH does not change.

Proposition III.2. Let H be a compact subset of the EH of (M,g) whose boundaries are an initial spatial section Σ_1 *and a final spatial section* Σ_2, $\Sigma_1 \cap \Sigma_2 = 0$. Σ_2 *is assumed to be a smooth sphere far in the future. Suppose that H is* C^r ($r \geqslant 2$) *differentiable. Then the topology of* Σ_1 *is* S^2.

Proof. As proved in Ref. [14], if there is any end point of the null geodesic generator of the EH in the interior of *H*, *H* cannot be C^1 differentiable there. Using theorem II.4, it is concluded that Σ_1 is topologically S^2, since *H* is imbedded into a strongly causal spacetime (*M,g*). □

Now we discuss the possibilities of nonspherical topologies. From Sorkin's theorem there should be at least one zero of *K* in the interior of *H* provided that the Euler number of Σ_1 is different from that of $\Sigma_2 \sim S^2$. Such a zero can only be the past end point of the EH, since the null geodesic generator of the EH cannot have a future end point. With regard to this past end point and the crease set of the EH we state the following two propositions.

Proposition III.3. The crease set (consisting of the past end points) of the EH is an acausal set.

Proof. The crease set is obviously an achronal set, as the EH is a null surface (an achronal boundary). Suppose that the crease set includes a null segment l_p through an event *p*. By lemma III.1, the null segment l_p consists of the points at which the EH is not differentiable. The EH, however, is differentiable in the null direction tangent to l_p at *p*, since l_p is smoothly imbedded into the smooth spacetime (*M,g*). Then the section S_H of the EH on a spatial hypersurface through *p* is nondifferentiable at *p*, as shown in Fig. 2. Considering a sufficiently small neighborhood $\mathcal{U}_p$ about *p*, the local causal structure of $\mathcal{U}_p$ is similar to that of Minkowski spacetime, since (*M,g*) is smooth there. Therefore, when S_H is convex at *p*, the EH will be C^1 differentiable at q_v, which is on a adjacent future of the null segment l_p (see Fig. 2), because the EH is the outer side of the enveloping surface of the light cones standing along S_H in the neighborhood $\mathcal{U}_p$ about *p*. Nevertheless, from lemma III.1, also $q_v \in l_p$ cannot be smooth in this section. Also, if S_H is concave, q_c which is on a nearby future of l_p, will invade the inside of the EH and fails to be on the EH (see Fig. 2). Thus the crease set cannot

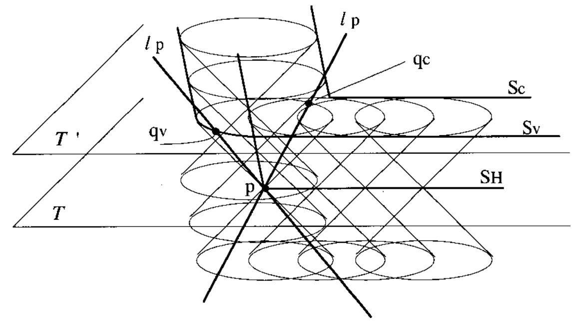

FIG. 2. The neighborhood of *p* is sliced by two spatial hypersurfaces $\mathcal{T}$ and $\mathcal{T}'$. S_H is on the lower hypersurface $\mathcal{T}$. l_p passes through *p*. In the convex (concave) case, the EH is given by the enveloping surface S_v (S_c). q_v (q_c) is a point on l_p at the future of *p*. S_v is C^1 differentiable at q_v. q_c is inside S_c.

contain either convex and concave null segments. Moreover if two disconnected segments could be connected by a null geodesic, a future end point of the null geodesic generator would exist. Hence the crease set is an acausal set. □

Proposition III.4. The crease set (consisting of past end points) of the EH of (M,g) is arcwise connected. Moreover, the tubular neighborhood [17] of the crease set is topologically a 3-disk D^3.

Proof. Consider all the null geodesics $\lambda_{p_e}(\tau)$ emanating from the crease set $\{p_e\}$ tangent to the null vector field *K*. Since the crease set is the set of zeros of *K*, p_e corresponds to $\lambda_{p_e}(-\infty)$. From proposition III.3, the spacelike section $\mathcal{S}$ of the EH very close to the crease set $\{p_e\}$ is determined by a map ϕ^K, with a large negative parameter $\Delta\tau$ of the null geodesic λ_{p_e}:

$$\phi^K\colon\ \{q \in \mathcal{S}\} \rightarrow \{p_e\}, \tag{4}$$

$$\text{s.t. } \lambda_{p_e}(-\infty) = p_e, \quad \lambda_{p_e}(\Delta\tau) = q. \tag{5}$$

With a sufficiently large negative $\Delta\tau(\rightarrow -\infty)$, *K* has inward directions to *H* at $\mathcal{S}$, where *H* is the subset of the EH bounded by $\mathcal{S}$ and the final spatial section Σ_2 which is far in the future and is a smooth sphere from the assumption. By this construction, the entire crease set is wrapped by $\mathcal{S}$, and $\mathcal{S}$ is compact because of the assumption that the set of end points is compact. *H* and the crease set are on the opposite side of $\mathcal{S}$. Therefore there is no end point in the interior of *H*. Since *H* is $C^r(r \geqslant 2)$ differentiable except on the crease set and compact from the assumption, proposition III.2 implies that $\mathcal{S}$ is homeomorphic to $\Sigma_2 \sim S^2$ and *H* is topologically $S^2 \times [0,1]$. If there were two or more connected components of the crease set, one would need the same number of spheres to wrap it with $\mathcal{S}$ being sufficiently close to the crease set. However, since $\mathcal{S}$ is homeomorphic to a single S^2, the crease set should be arcwise connected. In other words, the tubular neighborhood of the crease set is topologically a 3-disk D^3, because the EH and the crease set are imbedded into (*M,g*). □

The set of past end points is also arcwise connected, since the crease set is contained by it, and the closure of the crease set contains it [14].

Now we give theorems and corollaries regarding the topology of the spatial section of the EH on a time slicing. First we consider the case where the EH has a simple structure.

Theorem III.5. Let S_H be the section of an EH determined by a spacelike hypersurface. If the EH is C^r ($r \geqslant 1$) differentiable on S_H, it is topologically $\emptyset$ or S^2.

Proof. From lemma III.1, there is no intersection between S_H and the crease set. Since the EH is assumed not to be eternal, there exists at least one end point of the EH in the past of S_H as long as $S_H \neq \emptyset$. Therefore proposition III.4 implies there is no end point of the EH in the future of S_H. By the assumption that the EH is C^r ($r \geqslant 2$) differentiable except on the crease set and proposition III.2, it is concluded that S_H is topologically S^2. □

On the other hand, we obtain the following theorem about the change of the TOEH with the aid of Sorkin's theorem. Now, we introduce the dimension of the crease set. Considering an open subset of the crease set, if the subset is a n-dimensional topological submanifold, we state that the crease set is n dimensional, or the dimension of the crease set is n, in the open subset. Since an EH is an imbedded C^{1-} submanifold of a spacetime, the crease set where the EH is nondifferentiable has three-dimensional measure zero [14]. The crease set is zero, one, or two dimensional.

Theorem III.6. Consider a smooth time slicing $\mathcal{T} = \mathcal{T}(T)$ defined by a smooth function $T(p)$:

$$\mathcal{T}(T) = \{p \in M | T(p) = T = \text{const.}, \quad T \in [T_1, T_2]\}, \quad g(\partial_T, \partial_T) < 0. \quad (6)$$

Let H be the subset of the EH cut by $\mathcal{T}(T_1)$ and $\mathcal{T}(T_2)$ whose boundaries are the initial spatial section $\Sigma_1 \subset \mathcal{T}(T_1)$ and the final spatial section $\Sigma_2 \subset \mathcal{T}(T_2)$, and let K be the null vector field generating the EH. Suppose that Σ_2 is a sphere. If, in the time slicing $\mathcal{T}$, the TOEH changes ($\rightarrow \Sigma_1$ is not homeomorphic to Σ_2) then there must be a crease set (which is the zeros of K) in H and H is not smooth on Σ_1. When the time slice touches the one-dimensional segment of the crease set, two disconnected components of the EH coalesce, and the two-dimensional segment of the crease set, the genus of the EH decreases. Moreover the one-dimensional boundary of the crease set can work as a one-dimensional segment of the crease set in an appropriate time slicing. All the changes of the TOEH can be expressed by a combination of these processes taking account of a small deformation of the time slicing.

Proof. From proposition III.2 and theorem III.5, there is a crease set in H and H is not smooth at Σ_1. To treat the topology change of the spatial section of the EH by the time slicing, we take account of the crease set and apply theorem II.2 to the EH and K. Nevertheless, as discussed above, the zeros of the vector field K are not isolated and the EH is not differentiable, as demanded in theorem II.2. First of all, we must regularize H and K so that theorem II.2 can be applied to this case. To make H smooth, we define a smoothing map π in an atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}$. Introducing normal coordinates (x^1, x^2, x^3, x^4) in a neighborhood $\mathcal{U}_\alpha$ about $q \in H$, since the EH is an achronal boundary, H is imbedded by a function $x^4 = F(x^i, i = 1,2,3)$ which is Lipschitz continuous, where $\partial / \partial x^4$ is timelike (see Ref. [1]). Here, we set $x^4(p) = T(p) - T(q)$ in $\mathcal{U}_\alpha$ about q. Since M is a metric space, there is a partition of unit f_α for the atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}$, $\phi_\alpha : \mathcal{U}_\alpha \rightarrow R^4$ [16]. Then a smoothed function of the Lipschitz function $T(p \in H)$ (which is restricted on the nondifferentiable submanifold H for the smooth function $T(p)$ to become nondifferentiable) with a smoothing scale ϵ is given by

$$\tilde{T}(p \in H) = \Sigma_\alpha \int_{\mathcal{U}_\alpha} f_\alpha T(r = \phi_\alpha^{-1}(x^1, x^2, x^3, x^4)) W(p, r) \delta(x^4 - F(x^1, x^2, x^3)) dx^1 dx^2 dx^3 dx^4$$

$$= \Sigma_\alpha \int_{\mathcal{U}_\alpha} f_\alpha (F(x^1, x^2, x^3) + T(q)) W(p, r) \delta(x^4 - F(x^1, x^2, x^3)) dx^1 dx^2 dx^3 dx^4,$$

$$W(p, r) = 0, \quad p \notin \mathcal{U}_\alpha,$$

$$W(p, r) = w(|p - r|), \quad p \in \mathcal{U}_\alpha,$$

$$|p - r| = \sqrt{(x_p^1 - x_r^1)^2 + (x_p^2 - x_r^2)^2 + (x_p^3 - x_r^3)^2 + (x_p^4 - x_r^4)^2} \text{ in } \mathcal{U}_\alpha \text{ about } q,$$

$$w(x) \leqslant \infty, \quad w(x \geqslant \epsilon) \ll 1, \quad \int w(x) = 1,$$

where w is an appropriate window function with a smoothing scale ϵ. The support of W is a sphere with radii $\sim\epsilon$ and $w(|x|,\epsilon\to 0)=\delta^4(x)$. Of course, $\epsilon=0$ gives the original function $T(p\in H)=\tilde{T}(p\in H)$. Taking a sufficiently small nonvanishing ϵ, a new imbedded submanifold $\tilde{H}$, with a smooth function $\widetilde{x^4}(p\in\tilde{H})=\tilde{T}(p)-\tilde{T}(q)=:\tilde{F}(x^1,x^2,x^3)$ in $\mathcal{U}_\alpha$ about q, can become homeomorphic to H since $\tilde{H}$ can be homeomorphic to H in each $\mathcal{U}_\alpha$. Then $\tilde{H}$ is a smooth modification of H. From this smoothing procedure, we define a smoothing map π (homeomorphism) by

$$\pi:H\to\tilde{H}, \tag{7}$$

$$\phi_\alpha^{-1}(x^1,x^2,x^3,x^4=F)\to\phi_\alpha^{-1}(x^1,x^2,x^3,\widetilde{x^4}=\tilde{F}). \tag{8}$$

Of course, this map depends on the atlas $\{\mathcal{U}_\alpha,\phi_\alpha\}$ introduced. This smoothing map induces the correspondences

$$\lambda\to\tilde{\lambda},\quad \Sigma_{1,2}\to\widetilde{\Sigma_{1,2}}, \tag{9}$$

$$\mathcal{T}\to\tilde{\mathcal{T}},\quad \pi^*:K\to\tilde{K}, \tag{10}$$

where $\tilde{K}$ is the tangent vector field of curves $\tilde{\lambda}$ generating $\tilde{H}$. Now $\tilde{K}$ is not always null. Hereafter we call also the image of the crease set by the smoothing map π a crease set for $\tilde{\lambda}$, though the generators $\tilde{\lambda}$ are not null.

Furthermore, we need to modify $\tilde{K}$ by the transformed time slicing $\tilde{\mathcal{T}}$ so that the crease set for $\tilde{\lambda}$ becomes a zero-dimensional set, that is, the set of isolated zeros (where this set will no longer always be arcwise connected), keeping its time-direction in terms of $\tilde{\mathcal{T}}$ except on the end points. To make the zeros isolated, it is sufficient for a modified vector field $\bar{K}$ to be given on the crease set for $\tilde{\lambda}$ so as to generate this set. Therefore, the direction of $\bar{K}$ is determined on the crease set for $\tilde{\lambda}$ so that $\bar{K}$ is tangent to it. On the crease set for $\tilde{\lambda}$, $\bar{K}$ should be directed to the future in the sense of the time slicing $\tilde{\mathcal{T}}$ so that the direction of $\bar{K}$ at $\widetilde{\Sigma_1}$ and $\widetilde{\Sigma_2}$ is appropriate. At the branch, joint and boundary of the crease set (whether the crease set is open or closed), however, we should be careful. $\bar{K}$ will fail to be tangent to the crease set at the branch and the joint. There, $\bar{K}$ is determined to be zero. Also at the boundary points of a one-dimensional crease set, $\bar{K}$ is set to be zero. When the crease set is two-dimensional, the direction of $\bar{K}$ is still not fixed. In particular, at the boundary of the crease set for $\tilde{\lambda}$, we should be careful that $\bar{K}$ is tangent also to the non-zero-dimensional boundary of the crease set for $\tilde{\lambda}$ [see Fig. 3(b)], otherwise this non-zero-dimensional boundary becomes nonisolated zeros of $\bar{K}$. On the other hand, in the interior of the crease set for $\tilde{\lambda}$, we can determine the remaining arbitrariness of the direction of $\bar{K}$ by the projection of the time vector $\nabla^a\tilde{T}$ onto this crease set, so that $\bar{K}$ is continuous and future directed in the sense of $\tilde{\mathcal{T}}$ in there. The magnitude of $\bar{K}$ on the crease set except for the branch and joint of the crease set for $\tilde{\lambda}$ is determined by the projection of $\nabla^a\tilde{T}$ onto this direction. By this determination, the zeros of $\bar{K}$ remain only at points where the crease set or the boundary of it is tangent to the time slicing $\tilde{\mathcal{T}}$.

Now we should notice the discontinuity of $\bar{K}$ between the branch, joint or boundary of the crease set for $\tilde{\lambda}$ and their environs on the crease set. Since $\bar{K}$ satisfies the following two conditions, however, we can get continuous $\bar{K}$ on the crease set by the following modification.

Let $\mathcal{K}$ be a vector field on a subset $\mathcal{S}$ of H (which is the crease set) and $C\subset\mathcal{S}$ be a center of $\mathcal{S}$ which is the branch, joint or boundary of the crease set.

Condition III.7. The vector field $\mathcal{K}$ is not past directed in the sense of $\tilde{\mathcal{T}}$. $\mathcal{K}$ is continuous except on C, and also continuous on C. It is discontinuous between C and its environs.

The reason is sketched as follows. The Tubular neighborhood theorem (see Munkres [17]) guarantees the existence of a sufficiently small finite envelope $\mathcal{E}$ of the center on the subset $\mathcal{S}$ and a continuous retraction $r:\mathcal{E}\mapsto C$, $\mathcal{E}$ can be regarded as the set of curves from each p_e on $\partial\mathcal{E}$ to $r(p_e)$ on C not intersecting each other except on C [17]. It can be covered by a finite number of local coordinate neighborhoods $\{\mathcal{U}_\alpha,\phi_\alpha\}$ and in each coordinate neighborhood a positive definite distance along a curve can be defined as $(p,q)=\int_p^q\sqrt{dx^\alpha dx^\beta\delta_{\alpha\beta}}$ in terms of the normal coordinate $\{x^1,x^2,x^3,x^4\}$. Since each point p in the envelope $\mathcal{E}$ determines a unique curve passing p, which starts from p_e on $\partial\mathcal{E}$ and ends at p_s on C, along this curve $\mathcal{K}_{new}|_p=\mathcal{K}|_{p_e}+(\mathcal{K}_{p_c}-\mathcal{K}_{p_e})(p_e,p)/(p_e,p_c)$ gives a continuation of $\mathcal{K}$ between $\mathcal{E}$ and C, in the coordinate neighborhood. The averaginge of $\mathcal{K}_{new}$ over each coordinate neighborhood with the partition of unit f_α gives a continuous $\mathcal{K}$ in the envelope. By this continuation, the location and index of $\bar{K}$'s zeros do not change.

Here it is noted that the case in which a finite part of the crease set or the boundary of it lies on the time slicing $\tilde{\mathcal{T}}$ is possible and we cannot determine the direction of $\bar{K}$ there. Since such a situation is unstable under the small deformation of the time slicing, however, we omit this possibility, as mentioned in the remark appearing after this proof. Hence $\bar{K}$ is determined on the crease set for $\tilde{\lambda}$ (see, for example, Fig. 3) and it is with the set of some isolated zeros. Of course, at this step, $\bar{K}$ on the crease set for $\tilde{\lambda}$, and $\tilde{K}$ except on the crease set for $\tilde{\lambda}$ is discontinuous. Then, we modify $\tilde{K}$ around the crease set for $\tilde{\lambda}$ along $\bar{K}$, and make the modified $\tilde{K}$ into $\bar{K}$, except on the crease set for $\tilde{\lambda}$, without changing the characters of the $\bar{K}$'s zeros, so that $\bar{K}$ becomes a continuous vector field on $\tilde{H}$. One may be afraid that an extra zero of $\bar{K}$ appears as a result of this continuation. Nevertheless it is guaranteed by the existence of the foliation by the time slice $\mathcal{T}$ or $\tilde{\mathcal{T}}$ that there exists a desirable modification of $\tilde{K}$ around the crease set for $\tilde{\lambda}$, since $\bar{K}$ and $\tilde{K}$ satisfy the condition III.7. Thus we get $\bar{K}$ and its integral curves $\bar{\lambda}$ on the whole of $\tilde{H}$. From this construction of $\bar{K}$, there are some isolated zeros of $\bar{K}$ only on the crease set for $\tilde{\lambda}$, and $\bar{K}$ is everywhere future

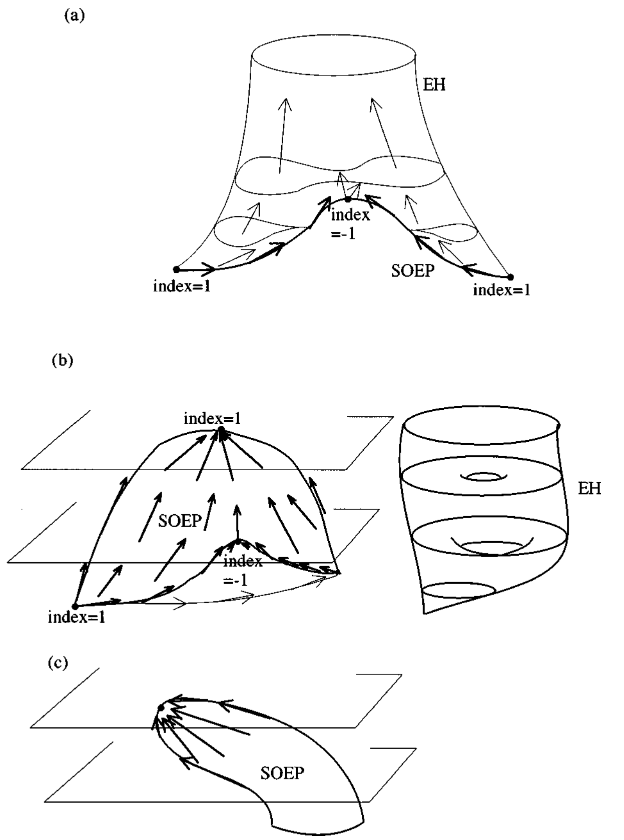

FIG. 3. (a) and (b) are the one-dimensional and two-dimensional crease set, respectively. In (b), we draw the entire EH separately. (c) is the case in which the edge of the crease set is hit from the future. By these vector fields K, the crease sets are generated. The zeros of K and their indices are indicated.

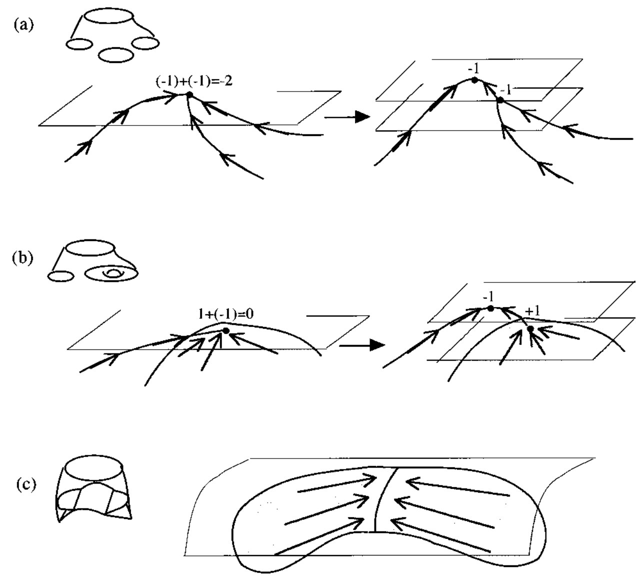

FIG. 4. (a) and (b) are examples of the branching crease set in an accidental time slicing. They are understood by a small deformation of the time slicing. On the other hand, (c) is the case in which the time slicing is partially tangent to the crease set. The two-dimensional crease set behaves as a one-dimensional crease set.

directed in the sense of the time slicing $\tilde{\mathcal{T}}$ (though they will be spacelike somewhere). Of course, $\tilde{\lambda}$ will have both future and past end points.

Now we apply theorem II.2 to $\tilde{H}$ with the modified vector field K, whose boundaries are $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2 \sim S^2$. Since $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are on $\mathcal{T}(T_1)$ and $\mathcal{T}(T_2)$, respectively, K has inward directions at $\tilde{\Sigma}_1$ and outward directions at $\tilde{\Sigma}_2$.

From the construction above, we see that the type of the zero of K depends on the dimension of the crease set. In particular, for the zero most in the future, the one-dimensional crease set provides the zero of the second type in Fig. 1(b) corresponding to index$=-1$ and the two-dimensional crease set gives that of the third type in Fig. 1(b) with index$=+1$ (see Fig. 3). Following theorem II.2, the Euler number changes at the zero by an amount $2\times$index. Therefore if there is a one- (two)-dimensional crease set, the time slicing $\mathcal{T}$ gives the topology change of the EH from two spheres (a torus) to a sphere. When H contains the whole of the crease set, it will, according to theorem II.2, present all changes of the TOEH from the formation of the EH to a sphere far in the future, as shown in Fig. 3. To complete the discussion, we also consider uninteresting cases provided by a certain timeslicing. When the edge of the crease set is hit by the time slicing from the future, according to the construction above, it gives a zero with its index being zero [Fig. 3(c)], and there is no topological change of the EH. Also for a zero simply caused by the joint of the crease set, the index of this zero vanishes and it does not relate to the TOEH. □

This result is partially suggested in Shapiro, Teukolsky, and Winicour. [8] The following remark shows that we can also treat special situations where above discussion fails by a small deformation of the time slicing.

Remark. One may face special situations. The possibility of branching end points should be noted. If the crease set possesses a branching point, a special time slicing can make the branching point into an isolated zero, though such a time slicing loses this aspect under a small deformation of the time slicing. The index of this branching end point is hard to determine in a direct consideration. The situation, however, is regarded as the degeneration of the two distinguished zeros of K in $\tilde{H}$. Some examples are displayed in Fig. 4. Imagine a slightly slanted time slicing, and it will decompose the branching point into two distinguished zeros (of course, there are the possibilities of the degeneration of three or more zeros). The first case is the branch of the one-dimensional crease set[6] [Fig. 4(a)], where the branching point is the degeneration of two zeros of K with their index being -1, since they are the results of the one-dimensional crease set. Then the index of the branching point is -2 and, for example, three spheres coalesce there. The next case is a one-dimensional branch from the two-dimensional crease set [Fig. 4(b)]. This branching point is the degeneration of the

[6]We can also treat the branching points of the two-dimensional crease set in the same manner.

zeros of $\bar{K}$ from the one-dimensional crease set (index $=-1$) and the two-dimensional crease set (index$=+1$). This decomposition reveals that, though the index of this point vanishes, the TOEH changes at this point, for example, from a sphere and a torus to a sphere. Of course, the Euler number does not change in this process. Furthermore, these topology changing processes are stable under a small deformation of the time slicing. Finally, there is the case in which a time slicing is partially tangent to the crease set or its boundary. For instance, an accidental time slicing can hit, not a single point in the crease set, but a curve in the crease set from the future, as shown in Fig. 4(c). For such a time slicing, the contribution of the two-dimensional crease set to the index is not -1 but 1. This situation, however, is unstable under a small deformation of the timeslicing, and we omit such a case in the following.

A certain time slicing gives further changes of the Euler number.

Corollary III.8. An appropriate deformation of a time slicing turns a process in which the TOEH changes from n ($n=1,2,3,...$) *spheres to a sphere into a process in which the TOEH changes from m* ($m\neq n$) *spheres to a sphere. Also an appropriate deformation of a time slicing turns a process in which the TOEH changes from a surface with* genus$=n$ ($n=1,2,3,...$) *to a sphere into a process in which the TOEH changes from a surface with* genus$=m$ ($m\neq n$) *to a sphere.*

Proof. From theorem III.6, when the TOEH changes from $n\times S^2$ to a single S^2 in a time slicing, there should be a one-dimensional crease set (in which there may be some branches). Since the crease set is an acausal set (proposition III.3), there is another appropriate time slicing hitting the crease set at m different points simultaneously [Fig. 5(b)]. On this time slicing, the Euler number changes by $-2\times m$, and $m+1$ spheres coalesce. Using the same logic, the EH of a surface with genus$=n$ can be regarded as the EH of a surface with genus$=m$ by an appropriate change of its time slicing [see Fig. 5(c)].

As shown in corollary III.8, the TOEH depends strongly on the time slicing. Nevertheless, theorem III.6 tells us that there is a difference between the coalescence of n spheres, where the Euler number decreases by the one-dimensional crease set, and the EH of a surface with genus$=n$, where the Euler number increases by the two-dimensional crease set.

Finally we see that, in a sense, the TOEH is a transient term.

Corollary III.9. All the changes of the TOEH are reduced to the trivial creation of an EH which is topologically S^2.

Proof. We choose a point p_c on the boundary of the crease set. Since the Tubular neighborhood of the crease set is topologically a 3-disk D^3 from proposition III.4, by an appropriate distance function $l(p)=(p,p_c)$ along the crease set, we can slice the crease set by $l(p)=$const, and sections by this slicing do not intersect each other. Moreover, because the crease set is an acausal set, such a slicing of the crease set can be extended into the spacetime concerned as a time slicing, so that p_c becomes most in the past of the crease set. In this time slicing, since the crease set is sliced without the degeneration of the section, the zeros of $\bar{K}$ appear only on the boundary of the crease set. Then, $\tilde{H}$ has only one significant zero p_c of $\bar{K}$ [type 1 in Fig. 1(b)], which corresponds to the point where the EH is formed, and meaningless zeros [with the index 0, for example, see Fig. 3(c)] on the edge of the crease set. The index of p_c is $+1$, and a spherical EH is formed there. □

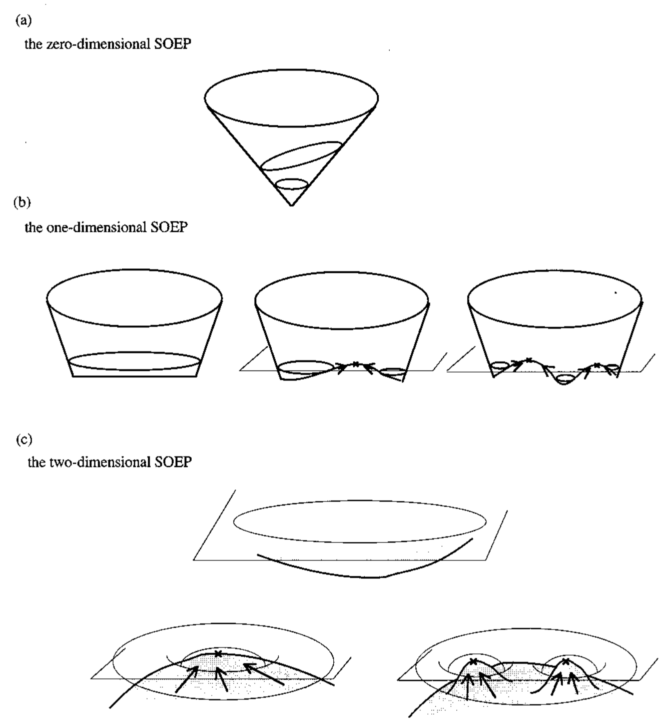

FIG. 5. EHs with zero-, one-, and two-dimensional crease sets are shown. We see that the one-dimensional crease set becomes a coalescence of an arbitrary number of spherical EHs. For the two-dimensional crease set, only sections of the EH and the crease set are drawn. It can become an EH with an arbitrary number of handles. It is also possible to change the EH into a trivial creation of a spherical EH.

Thus we see that the change of the TOEH is determined by the topology of the crease set and its time slicing. For example, we can imagine the graph of the crease set as Fig. 6. To determine the TOEH we must only give the order to each vertex of the graph by a time slicing. The graph in Fig.

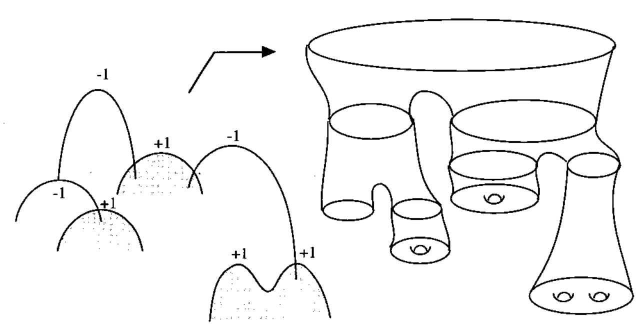

FIG. 6. An example of the graph of the crease set is drawn. Determining the order of the vertices, we see the TOEH from the index of each zero.

6 may be rather complex. Nevertheless, considering a small scale inhomogeneity, for example the scale of a single particle, the EH may admit such a complex crease set. It will be smoothed out in macroscopic physics.

IV. SUMMARY AND DISCUSSION

We have studied the spatial topology of the EH (TOEH), partially considering the nondifferentiability of the EH. We have found that the coalescence of EHs is related to a one-dimensional crease set and a toroidal EH is related to a two-dimensional crease set. In a sense, this is a generalization of the result of Shapiro, Teukolsky, and Winicour [8]. Furthermore these changes of the TOEH can be removed by an appropriate time slicing, since the crease set of an EH is a connected acausal set. We see that the TOEH depends strongly on the time slicing. The dimension of the crease set, however, plays an important role for the TOEH and, of course, is invariant under the change of the time slicing.

Based on these results, a question arises, what controls the dimension of the crease set. One may expect that something like an energy condition restricts the variety of the crease set. Nevertheless it is hopeless since, in fact, cases with each nontrivial TOEH—the coalescence of EHs (the one-dimensional crease set) and a toroidal EH (the two-dimensional crease set)—are reported in numerical simulations with energy conditions satisfied [8,9]. Are these generic in real gravitational collapses? It is probable that the gravitational collapse in which the EH is a single sphere in any time slicing is not generic, since the zero-dimensional crease set reflects the higher symmetry of a system than that of the one- or two-dimensional crease set. On balance, the symmetry of a system will control it. For example, it is possible to discuss the stability and generality of such a symmetry. We will show the stability of a spherical EH under linear perturbation and the structural stability of the crease set [18]. These discussions would tell something about how the structure of the crease set is determined dynamically.

In the present article, we have assumed some conditions about the structure of spacetime. Can other weaker conditions take the place of them? First, the strongly causal condition may be too strong. That is because this condition is needed only on the EH. For example, global hyperbolicity implies strong causality on the EH, because global hyperbolicity excludes a closed causal curve and a past imprisoned causal curve, and there should be no future imprisoned null curve on the EH. Next, we required that the TOEH is smooth S^2 far in the future. This, however, is not crucial. Since the present investigation is based on the topology change theory, the same discussion is possible for other final TOEHs. Next, the $C^r(r\geqslant 2)$ differentiability of the EH is supposed except on the compact crease set while it might be able to be violated in realistic situations. It is not clear whether this differentiability can be implied by other physically reasonable conditions. The nondifferentiability, however, is overwhelmingly easier to occur on the end point than not on the end point. Every noneternal EH possesses a point where the EH is not differentiable as a past end point and we do not have any simple example where the EH is nondifferentiable except at the end point. On the other hand, the case in which the EH is differentiable only on compact subsets (i.e., the crease set is not compact) might be excluded by a realistic requirement about the asymptotic structure of spacetime, as a nowhere differentiable spacetime [15] is excluded by asymptotic flatness. It would be worth to clarify such properties about the differentiability of the EH.

Incidentally, some of the statements in this article may be equivalent to results of previous works [2–7]. Nevertheless the condition required here is quite different from that appearing in their works (for example, energy conditions have never been assumed here). The present results may be considered as the extension of those in the previous works.

Finally we are reminded of an essential question. How can we see the spatial topology of the EH? Some of the previous works, for example ''topological censorship'' [5], stress that it is impossible. On the contrary, we expect phenomena depending strongly on the existence of the EH as the boundary condition of fields, for instance the quasi-normal mode of gravitational waves [19] or Hawking radiation [20], reflects the TOEH. For example, with regard to Hawking radiation, we would like to construct a toy model for the change of the TOEH, something like the Rindler spacetime for the Schwarzschild spacetime. This is our future problem.

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