

2D extremal black holes as solitons

Mariano Cadoni*

*Dipartimento di Scienze Fisiche, Università di Cagliari, Via Ospedale 72, I-09100 Cagliari, Italy
and INFN, Sezione di Cagliari, Cagliari, Italy*

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We discuss the relationship between two-dimensional (2D) dilaton gravity models and sine-Gordon-like field theories. We show that there is a one-to-one correspondence between the solutions of 2D dilaton gravity and the solutions of a (two fields) generalization of the sine-Gordon model. In particular, we find a connection between the soliton solutions of the generalized sine-Gordon model and extremal black hole solutions of 2D dilaton gravity. As a by-product of our calculations we find an easy way to generate cosmological solutions of 2D dilaton gravity. [S0556-2821(98)02518-1]

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I. INTRODUCTION

The connection between black holes and nonperturbative structures of string theory, such as Bogomol'nyi-Prasad-Sommerfield (BPS) solitons or D -branes, has been one of the main ingredients of the last developments in string theory [1,2]. Black hole thermodynamics seems to have a natural explanation in terms of microscopic string and membrane physics [2], opening new ways to address old (and new) fundamental problems of black hole physics.

On the other hand, the same fundamental problems of black hole physics have been analyzed in the recent literature using low-dimensional gravity models. In particular, two-dimensional (2D) dilaton gravity models have been used to tackle challenging questions such as the ultimate fate of black holes or the loss of quantum coherence in the black hole evaporation process [3]. Although no definitive answers to the abovementioned problems have been found, 2D dilaton gravity models still provide a useful and simple framework to describe 4D black hole physics.

If one wants to use the new ideas of string theory in the context of 2D dilaton gravity models, one has to investigate the role that solutions such as solitons play in these models. Moreover, for particular 2D dilaton gravity models we have a direct relationship between BPS solitons of the 4D string effective theory and solutions of the 2D model. For instance, the Jackiw-Teitelboim (JT) model [4] can be used to describe the S -wave sector of the extremal $D=4$, supersymmetric, black hole solutions of models with dilaton coupling $a = 1/\sqrt{3}$ [5,6].

In a recent paper [7], using the well-known correspondence between solutions of the sine-Gordon theory and constant curvature metrics, Gegenberg and Kunstatter found a relationship between black holes of JT dilaton gravity and solitons of the sine-Gordon field theory [8]. In this paper we explore the possibility to generalize this correspondence to generic 2D dilaton gravity models, whose solutions, in general, are not spacetimes of constant curvature. We find that the field equations for 2D dilaton gravity are equivalent to those derived from a (two fields) generalization of the sine-Gordon model. From this correspondence we derive a con-

nection between solitons of the generalized sine-Gordon model and *extremal* black hole solutions of 2D dilaton gravity. We also explain why in the JT model this correspondence holds for the generic black hole solutions and not only for the extremal one.

The structure of the paper is the following. In Sec. II we show that the field equations of 2D dilaton gravity can be reduced to those of a generalized sine-Gordon model. In Sec. III we derive the static solutions of the generalized sine-Gordon model. The conditions that have to be satisfied for these solutions to describe solitons, are also presented and implemented. In Sec. IV we discuss the relationship between the solitons and the black holes of the 2D dilaton gravity theory. In Sec. V we use topological arguments to classify the soliton solutions of the generalized sine-Gordon model. In Sec. VI we apply the general formulas that we have derived to some particular 2D dilaton gravity models. In Sec. VII we discuss a by-product of our calculations, namely, an easy way to generate cosmological solutions of 2D dilaton gravity models. Finally, in Sec. VIII we present our conclusions.

II. 2D DILATON GRAVITY AND GENERALIZED SINE-GORDON FIELD THEORY

Let us consider the generic two-dimensional dilaton gravity model. Using a Weyl rescaling of the metric and a reparametrization of the dilaton, one can write the most general action for the model in the form [9]

$$S[g_{\mu\nu}, \Phi] = \frac{1}{2\pi} \int d^2x \sqrt{-g} [\Phi R + \lambda^2 V(\Phi)], \quad (1)$$

where R is the curvature of the 2D spacetime, $V(\Phi)$ is an arbitrary function of the dilaton Φ , and the 2D metric $g_{\mu\nu}$ has signature $(-1,1)$. The field equations derived from the action (1) have the simple form [9]

$$R = -\lambda^2 \frac{dV}{d\Phi}, \quad (2)$$

$$\nabla_\mu \nabla_\nu \Phi - \frac{\lambda^2}{2} g_{\mu\nu} V = 0. \quad (3)$$

*Email address: CADONI@CA.INFN.IT

By means of a suitable parametrization of the 2D metric one can map the solutions of Eqs. (2), (3) into solutions of a generalized sine-Gordon model in 2D Euclidean space. In fact, using the invariance of the theory under coordinate transformations we can write the 2D spacetime metric in the form

$$ds^2 = -\sin^2\left(\frac{u}{2}\right)dt^2 + \cos^2\left(\frac{u}{2}\right)dx^2, \quad (4)$$

where $u = u(x, t)$. Taking into account that the corresponding curvature tensor is

$$R = -\frac{2}{\sin u}(\partial_t^2 u + \partial_x^2 u), \quad (5)$$

one can show that the field equations (2), (3) are equivalent to the following set of equations in 2D Euclidean space:

$$(\partial_t^2 u + \partial_x^2 u) = \frac{\lambda^2}{2} \frac{dV}{d\Phi} \sin u, \quad (6)$$

$$(\partial_t^2 \Phi + \partial_x^2 \Phi) = \frac{\lambda^2}{2} V \cos u. \quad (7)$$

Notice that Eq. (2) is equivalent to Eq. (6), one of the three equations in Eq. (3) translates into Eq. (7), whereas the other two equations in Eq. (3) are integrability conditions for the system (6), (7).

Instead of considering 2D dilaton gravity in Minkowski space, one can also start from the Euclidean formulation of this theory. In this case, one can easily demonstrate the equivalence of the Euclidean field equations (2), (3) with the Minkowskian counterpart of Eqs. (6), (7). The corresponding equations are obtained performing the Wick rotation $t \rightarrow it$.

The field theory defined by the field equations (6), (7) can be considered as a (two field) generalization of the sine-Gordon model. Equation (6) reduces to the sine-Gordon equation for $V = \Phi$ [7] or, more generally, for constant configurations Φ_0 of the dilaton, with $V(\Phi_0) = 0$, $(dV/d\Phi)(\Phi_0) > 0$. The field equations (6), (7) can be also obtained extremizing an action, which in Minkowski space has the form

$$S = \frac{1}{2} \int d^2x \left(\Phi \square u - \frac{\lambda^2}{2} V \sin u \right), \quad (8)$$

where $\square = -\partial_t^2 + \partial_x^2$.

An unpleasant feature of the model (8) is that it describes a system of two scalar fields of opposite signature. This can be easily seen performing a field redefinition that diagonalizes the kinetic energy of the fields

$$\Phi = w + \phi, \quad u = w - \phi. \quad (9)$$

Up to surface terms, the action (8) becomes

$$S = -\frac{1}{2} \int d^2x \left(\eta^{\mu\nu} \partial_\mu w \partial_\nu w - \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right.$$

$$\left. + \frac{\lambda^2}{2} \sin(w - \phi) V(w + \phi) \right), \quad (10)$$

where $\eta^{\mu\nu} = (-1, 1)$. The scalar field ϕ has negative kinetic energy.

Let us conclude this section with some remarks on the correspondence that we have established between the dilaton gravity model (1) and the generalized sine-Gordon field theory (8). Although there is a one-to-one correspondence (up to spacetime diffeomorphisms of the dilaton gravity theory) between the solutions of the two theories, this does not mean that one can construct a metric solution that covers the whole 2D spacetime, once a solution of Eqs. (6), (7) is known. In general, this is not possible owing to the particular parametrization of the metric, given by Eq. (4), which allows the metric coefficients g_{tt} and g_{xx} to take values only in $[0, 1]$. However, one can take analytic continuations of the solutions. To this end, we can consider a parametrization of the metric obtained by replacing in Eq. (4) the trigonometric with the hyperbolic functions

$$ds^2 = -\sinh^2\left(\frac{u}{2}\right)dt^2 + \cosh^2\left(\frac{u}{2}\right)dx^2. \quad (11)$$

Starting from this expression for the metric, we can repeat the steps that led to the field equations (6), (7) and to the action (8). What we find now is an equivalence between the Minkowskian (Euclidean) dilaton gravity field equations (2), (3) and the generalized Minkowskian (Euclidean) sinh-Gordon field theory obtained by replacing in Eqs. (6)–(8) the trigonometric with the corresponding hyperbolic functions. In conclusion, in order to have a complete correspondence between 2D spacetime structures of the dilaton gravity theory and solutions of sine-Gordon-like field theories, we need both the sine- and sinh-Gordon models.

III. SOLITON SOLUTIONS

It is well known that the dilaton gravity model (1) admits solutions that can be interpreted as 2D black holes [10, 11, 6]. On the other hand, one expects the generalized sine-Gordon theory (8) to have soliton solutions that, in view of the results of the previous section, should be related to 2D black hole solutions. For arbitrary potential V the existence of soliton solutions is not *a priori* evident. We will therefore begin our discussion by answering the question about the existence of solitons in the model (8).

Solitons are nonsingular field configurations that describe localized states of finite energy. Usually, necessary conditions for the existence of solitons can be found requiring the energy of the solution to be finite. Differently from the usual sine-Gordon model, in the case under consideration the energy functional is not positive definite. This is a consequence of the presence of a scalar field with negative kinetic energy. From the action (8) it follows for the energy functional

$$E(u, \Phi) = \frac{1}{2} \int_{-\infty}^{\infty} dx \left(\partial_t \Phi \partial_t u + \partial_x \Phi \partial_x u + \frac{\lambda^2}{2} V \sin u \right). \quad (12)$$

Let us focus on static solutions of the field equations. To single out soliton solutions we require $E \geq 0$ and, at $x \rightarrow \pm \infty$,

$$\partial_x u = 0, \quad \partial_x \Phi = 0. \quad (13)$$

Note that for static solutions, the field equations imply $\partial_x u \propto V$ and $\partial_x \Phi \propto \sin u$ [see Eq. (15) later]. Hence, conditions (13) imply also $V \sin u = 0$ at $x \rightarrow \pm \infty$. Conditions (13) are sufficient but not necessary for the finiteness of the energy. In fact, one can easily construct field configurations of finite energy for which $\partial_x u$ and/or $\partial_x \Phi$ are different than zero asymptotically. The existence of these configurations is related to the fact that the energy is not positive definite. It is not clear to us whether a soliton interpretation of these solutions is also possible.

The system of differential equations (6), (7) admits the constant-field solutions $u = n\pi$, $\Phi = \Phi_0$, with $V(\Phi_0) = 0$. One would like to identify some of these constant solutions as vacua of the theory. However, in the model under consideration one cannot define the vacuum in the usual way, just by looking for local minima of the potential energy. The problem is that our model contains a field with the wrong sign of the kinetic energy term. This fact makes the usual arguments about stability meaningless. On the other hand, because we are looking for soliton solutions, which tend asymptotically to some constant field configuration, we need to use a notion of vacuum of the model. The vacua of the model are defined as the zero energy, constant-field configurations Φ_0, u_0 that satisfy, additionally, the following conditions. For $\Phi = \Phi_0$ the field equations (6), (7) reduce to the usual sine-Gordon equation for u , whereas for $u = u_0$ they reduce to the equations of motion of a scalar field Φ with potential $V(\Phi)$. These conditions single out, as vacua of the model, the following constant values of the fields

$$u = 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots, \quad \Phi = \Phi_0,$$

$$V(\Phi_0) = 0, \quad \frac{dV}{d\Phi}(\Phi_0) > 0. \quad (14)$$

For static configurations the field equations (6), (7) can be integrated exactly. The first integral is

$$u' = \lambda a V, \quad \Phi' = \frac{\lambda}{2a} \sin u, \quad (15)$$

where $' = d/dx$ and a is an integration constant. A further integration gives the final form of the solutions

$$\lambda(x - x_0) = \pm \int \frac{d\Phi}{\sqrt{(K-c)[1 - a^2(K-c)]}}, \quad (16)$$

$$\sin \frac{u}{2} = \pm a \sqrt{K-c}, \quad (17)$$

where $K = K(\Phi) = \int^\Phi d\tau V(\tau)$ and c, x_0 are integration constants.

Equations (16), (17) do not give full account of the general static solution of the field equations. In fact we have two two-parameter families of solutions that are not contained in Eqs. (16), (17). The first family is obtained by taking a constant u field,

$$\lambda(x - x_0) = \pm \int \frac{d\Phi}{\sqrt{(-1)^n K - b}}, \quad u = n\pi. \quad (18)$$

The second family of solutions corresponds to a constant dilaton field. These solutions exist only if $V(\Phi)$ has at least one zero $\Phi = \Phi_0$. For $(dV/d\Phi)(\Phi_0) > 0$ the field equations reduce to those of the usual sine-Gordon model

$$u'' = \frac{\lambda^2}{2} \frac{dV}{d\Phi}(\Phi_0) \sin u. \quad (19)$$

Using Eqs. (12) and (15), one can calculate the energy of the solutions (16), (17). A straightforward calculation gives

$$E = \lambda \{K[\Phi(\infty)] - K[\Phi(-\infty)]\}. \quad (20)$$

Having found an explicit form of the solutions, conditions (13) translate into restrictions on the admissible form of the dilaton potential V and on the values of the integration constants parametrizing the general solution. Using Eq. (15) in Eq. (13), one gets

$$V[\Phi(\pm\infty)] = 0. \quad (21)$$

From Eq. (16) it follows (from now on we will set the physically irrelevant integration constant $a = 1$)

$$\Phi' = \lambda \sqrt{(K-c)(1-K+c)}. \quad (22)$$

Inserting Eq. (22) into Eq. (13), one finds the value of the integration constant c for which the solutions (16), (17) describe solitons

$$c = K[\Phi(\pm\infty)]. \quad (23)$$

It follows that the model (8) admits static soliton solutions, approaching for $x \rightarrow \pm \infty$ the constant field configurations (14) with $n = \pm 1$, if the equation $V(\Phi) = 0$ admits at least one solution $\Phi_0 = \Phi(\pm\infty)$ with $(dV/d\Phi)(\Phi_0) > 0$. The soliton solutions are given by Eqs. (16), (17) with $c = K(\Phi_0)$.

Note that Eqs. (13) are solved also by $c = K[\Phi(\pm\infty)] - 1$. However, it is evident from Eq. (17) that the corresponding solutions tend asymptotically to $u = \pm \pi$. They cannot be taken into consideration if one requires, as we do here, that the soliton solutions approach asymptotically to one of the vacua (14). The energy of the soliton, given by Eq. (20), is zero if $\Phi(\infty) = \Phi(-\infty) = \Phi_0$, whereas it is different from zero if the potential V has more than one zero and if $K[\Phi(\infty)] \neq K[\Phi(-\infty)]$.

IV. SOLITON SOLUTIONS AND BLACK HOLES

In the previous section we derived soliton solutions of the model (8). The purpose of this section is to discuss the rela-

tionship between these soliton solutions and the black hole solutions of the 2D dilaton gravity model (1).

The solutions (16), (17) can be used in a straightforward way to generate static solutions of the dilaton gravity theory. We just need to insert Eqs. (16), (17) into the expression (4) for the metric of the 2D spacetime. Defining a new spacelike coordinate r ,

$$\frac{dr}{dx} = \sqrt{(1-K+c)(K-c)}, \quad (24)$$

we can write the spacetime metric and the dilaton in the form

$$ds^2 = -(K-c)dt^2 + (K-c)^{-1}dr^2, \quad \Phi = \lambda r. \quad (25)$$

This is the general form for the static solution of 2D dilaton gravity [11]. The parameter c appearing in Eqs. (25) is related to the mass M of the solution $c = 2M/\lambda$. Under suitable conditions the solutions (25) can be interpreted as 2D black holes. Hence, every dilaton gravity model, whose dilaton potential $V(\Phi)$ has at least one zero $\Phi = \Phi_0$ with $(dV/d\Phi)(\Phi_0) > 0$, has a solution, with mass given by $M = \lambda K(\Phi_0)/2$, that can be realized as a soliton solution (16), (17). Conversely, given a soliton solution of the generalized sine-Gordon model (8), one can always construct a solution of the 2D dilaton gravity model (1), which has the form (25) with $c = K(\Phi_0)$.

If the solution of the 2D dilaton gravity model describes a black hole, the soliton solution can be put in correspondence with a 2D black hole. Furthermore, one can easily show that the solutions of 2D dilaton gravity model that can be realized as solitons of the model (8) describe spacetimes with no event horizons, which eventually can be interpreted as extremal black holes. In fact, assuming that the function $K(\Phi)$ is monotonic in the considered range of variation of Φ (this condition is necessary for the black hole interpretation) it follows that the function $F(\Phi) = K(\Phi) - K(\Phi_0)$ has only one zero, at $\Phi = \Phi_0$, which owing to condition (21) is actually a double zero, $(dF/d\Phi)(\Phi_0) = V(\Phi_0) = 0$. As a consequence the Killing vector associated with the solution (25) with $c = K(\Phi_0)$ cannot become spacelike anywhere, i.e., the spacetime has no event horizons.

The previous statement seems to contradict the results of Ref. [7] for the JT model. In Ref. [7] it has been shown that every black solution of the JT theory can be put in correspondence with a soliton solution of the sine Gordon model. Later in this paper we will discuss this point in detail and we will argue that this behavior is due to a peculiar feature of the black hole solutions of the JT model.

As we have already noted in Sec. II the correspondence between black holes and soliton solutions is limited to a region of the 2D spacetime. Using the coordinate r defined in Eq. (24) to parametrize the 2D spacetime and taking into account Eq. (4), one can easily realize that this region is defined by $0 \leq (K(\lambda r) - 2M/\lambda) \leq 1$. From the discussion in Sec. II it is also evident that the spacetime region $(K(\lambda r) - 2M/\lambda) \geq 1$ can be put in correspondence with a generalized sinh-Gordon model.

V. TOPOLOGICAL ANALYSIS

The static configurations (16), (17) describe time-independent soliton solutions of the model (8). One can generate propagating soliton solutions performing in Eqs. (16), (17) the transformation

$$x \rightarrow \gamma(x \pm vt), \quad (26)$$

where $\gamma = 1/\sqrt{1-v^2}$ and v is the velocity of propagation of the soliton. Apart from the solutions (16), (17) there are no other time-independent soliton solutions of the theory (8). Being described by nonlinear equations the behavior of the many-soliton system is always time dependent. The admissible number of solitons can be determined in the usual way, using topological properties of the symmetry group of the model (8). Conditions (13) imply that every soliton solution of the field equations tends asymptotically to one of vacuum configuration (14). Therefore the number of admissible solitons is determined by the number of ways in which the points $x = \pm \infty$ (the zero sphere) can be mapped into the manifold of constant-field configurations (14). We have, therefore, a one-to-one correspondence between solitons and elements of the homotopy group $\pi_0(G/H)$, where G and H are, respectively, the group of internal symmetries of the model and the residual symmetry of the vacua (14). G and H depend on the form of the dilaton potential V . For generic V the model (8) has an invariance group $G = Z \times Z_2$, where Z and Z_2 are, respectively, the infinite discrete group of translations and the finite group of inversions of the field u :

$$u \rightarrow u + 2\pi n, \quad u \rightarrow -u. \quad (27)$$

Note that the effect of the transformation Z_2 on the action (8) is to flip its sign. The vacua (14) have a residual Z_2 symmetry, so that the homotopy group is

$$\pi_0\left(\frac{Z \times Z_2}{Z_2}\right) = \pi_0(Z) = Z. \quad (28)$$

The same result holds for the usual sine-Gordon model. For particular choices of the dilaton potential V we can have $Z \times Z_2 \subset G$ and $\pi(G/H) \neq Z$. In the next section we will give examples of this kind of behavior.

It is well known that field theories that admit soliton solutions have conserved currents, corresponding to conserved topological charges. For the model under consideration we can define two independent topological currents

$$J_{(u)}^\nu = \epsilon^{\nu\mu} \partial_\mu u, \quad J_{(\Phi)}^\nu = \epsilon^{\nu\mu} \partial_\mu \Phi. \quad (29)$$

The associated topological charges are

$$N_{(u)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx J_{(u)}^0 = \frac{1}{2\pi} [u(\infty) - u(-\infty)],$$

$$N_{(\Phi)} = \frac{1}{2} \int_{-\infty}^{\infty} dx J_{(\Phi)}^0 = \frac{1}{2} [\Phi(\infty) - \Phi(-\infty)]. \quad (30)$$

VI. SOME EXAMPLES

In this section we apply the general formulation described in the previous sections to some particular 2D dilaton gravity models. We will consider three examples: the JT theory defined by a dilaton potential $V(\Phi)=2\Phi$, a model with a degree-three polynomial potential $V(\Phi)=4\Phi(\Phi^2-1)$, and a $\sinh \Phi$ model with potential $V(\Phi)=\sinh 2\Phi$.

A. The JT model

The JT model is characterized by the potential $V(\Phi)=2\Phi$, which has a zero at $\Phi=0$ with $dV/d\Phi=2$. According to the discussion of the previous sections, we expect the corresponding generalized sine-Gordon model to have soliton solutions, described by Eqs. (16), (17) with $c=K(0)=0$. With $K=\Phi^2$ and $c=0$ we can perform explicitly the integration in Eq. (16), the final form of the soliton solution is

$$u = \pm 4 \arctan \exp \lambda(x-x_0), \quad \Phi^{-1} = \cosh \lambda(x-x_0). \quad (31)$$

The previous solution has zero energy. Depending on the sign in Eqs. (31), we have solitons and antisolitons. The corresponding solution of the JT model is the extremal black hole solution (the ground state of the model) with $M=0$,

$$ds^2 = -\Phi^2 dt^2 + \Phi^{-2} dr^2, \quad \Phi = \lambda r. \quad (32)$$

The symmetry group of the model is $G=Z_2^{(\Phi)} \times Z_2^{(u)} \times Z^{(u)}$, where $Z_2^{(\Phi)}$ is the transformation $\Phi \rightarrow -\Phi$ and $Z_2^{(u)} \times Z^{(u)}$ is the symmetry group of the sine-Gordon model given by Eq. (27). The vacua $\Phi=0$, $u=2n\pi$ have a residual symmetry $Z_2^{(\Phi)} \times Z_2^{(u)}$, so that for the homotopy group we have

$$\pi_0 \left(\frac{G}{H} \right) = Z. \quad (33)$$

The model admits soliton solutions with topological charges (30) given by $N_{(\Phi)}=0$, $N_{(u)}=\pm n$, $n=0,1,\dots$, where the plus sign refers to soliton and the minus sign to antisoliton solutions. In particular the time-independent solitons (31) have topological charge $N_{(\Phi)}=0$, $N_{(u)}=\pm 1$, respectively, for the soliton and the antisoliton.

That the soliton solutions (31) can be put in correspondence with solutions of the JT theory has previously been demonstrated by Gegenberg and Kunstatler [7]. They have also shown that the soliton solution (31) can be put in correspondence with every ($M=0$ or $M>0$) black hole solution of the JT theory, not only with the $M=0$ extremal solution (32). This seems in contradiction with our result of Sec. IV, stating that only 2D spacetimes with no event horizons are in correspondence with soliton solutions of the theory (8). The two results are not in contradiction because of a well-known feature of the JT model, namely, the fact that all the static solutions of the JT theory are different parametrization of the same manifold, 2D anti-de Sitter spacetime (see, for example, Ref. [6], and references therein). There is always a coordinate transformation relating the $M=0$ solution (32) with the generic $M>0$ black hole

solution. Using these coordinate transformations one can always map the soliton (31) into a generic static solution of the JT model. This feature is a peculiarity of the JT model that is not expected to hold for a generic 2D dilaton gravity model.

B. Degree-three polynomial potential

Let us now consider a model with $V=4\Phi(\Phi^2-1)$. The model has the same symmetry group as the JT model, namely, $G=Z_2^{(\Phi)} \times Z_2^{(u)} \times Z^{(u)}$. Differently than the JT case, now the vacua $\Phi=\pm 1$, $u=2n\pi$ break the $Z_2^{(\Phi)}$ symmetry, the residual symmetry group being now $H=Z_2^{(u)}$. Therefore, the model will admit a richer spectrum of soliton solutions. The homotopy group is $\pi_0(G/H)=Z^2 \times Z$, so that in this case we can have a different than zero topological charge N_Φ . The soliton solutions can be classified using a topological charge vector $X=(N_\Phi, N_u)$ with entries N_Φ, N_u given by Eq. (30). The topologically trivial solutions, characterized by $X=(0,0)$, are given by the vacua $\Phi=\pm 1$, $u=2n\pi$. The soliton solutions labeled by $X=(0, \pm n)$ are usual sine-Gordon solitons, i.e they have a constant, $\Phi=\pm 1$, dilaton, whereas u is the solution of the sine-Gordon equation

$$-\partial_t^2 u + \partial_x^2 u = 4\lambda^2 \sin u. \quad (34)$$

The solitons characterized by $X=(\pm 1, 0)$ are *kinks*. They have a constant u field, $u=2n\pi$, whereas Φ is solution of the equation

$$-\partial_t^2 \Phi + \partial_x^2 \Phi = 2\lambda^2 \Phi(\Phi^2-1), \quad (35)$$

which admits the *kink* solution

$$\Phi = \pm \tanh \lambda(x-x_0). \quad (36)$$

Finally, we have the soliton solutions with $X=(\pm 1, \pm n)$. The solitons with $n=1$ are given by Eqs. (16), (17) with $K=\Phi^4-2\Phi^2$ and $c=K(\pm 1)=-1$. With these positions Eqs. (16), (17) can be easily integrated. Here, we do not write down the resulting expression because it is rather cumbersome and not essential for our purposes. This soliton solution has zero energy and is in correspondence with an extremal black hole solution of the 2D dilaton gravity model

$$ds^2 = -(\Phi^2-1)^2 dt^2 + (\Phi^2-1)^{-2} dr^2, \quad \Phi = \lambda r. \quad (37)$$

The previous metric is extremal because the generic solution of the 2D dilaton gravity model has an horizon for $\Phi^2=1+\sqrt{1+c}$. Solution (37) is the $c=-1$ solution and it is, therefore, the extremal one.

C. The $\sinh \Phi$ model

As a last example, we consider a model with $V(\Phi)=\sinh 2\Phi$. The vacua, the symmetry group G , and the group of residual symmetry are the same as those of the JT theory. As a consequence, also the homotopy group is the same, $\pi_0(G/H)=Z$. The soliton solutions with topological charge $N_u=\pm 1$ can be easily obtained using Eqs. (16), (17) with $K(\Phi)=(\cosh 2\Phi)/2$ and $c=K(0)=1/2$. These soliton solutions have zero energy and can be written in the form

$$\frac{u}{2} = \frac{\pi}{2} + \arctan[\sqrt{2} \sinh \lambda(x-x_0)], \quad (38)$$

$$\Phi = \text{Artanh}[\sqrt{2} \cosh \lambda(x-x_0)]^{-1}. \quad (39)$$

The corresponding black hole solution of the 2D dilaton gravity model is

$$ds^2 = -\sinh^2 \Phi dt^2 + \sinh^{-2} \Phi dr^2, \quad \Phi = \lambda r. \quad (40)$$

Again, this is an extremal solution because the horizon of the general solution is located at $\sinh \Phi = \sqrt{c-1/2}$.

VII. COSMOLOGICAL SOLUTIONS OF 2D DILATON GRAVITY

An interesting by-product of our discussion is an easy way to generate general cosmological solutions of 2D dilaton gravity. Although cosmological solutions have already been found for particular 2D dilaton gravity models [12,13], until now the solution for the generic model has not been derived.

Cosmological solutions of the model (1) can be easily found using the parametrization of the 2D metric given by Eq. (4) and the equivalence between the field equations (2), (3) in Minkowski space and the field equations (6), (7) in Euclidean space. The point is that, because Eqs. (6), (7) are written in Euclidean space, they maintain their form if we interchange $x \leftrightarrow t$. Cosmological solutions of 2D dilaton gravity can be, therefore, generated from the static solutions just by interchanging $x \leftrightarrow t$. From Eqs. (16), (17) the general form of the cosmological solutions follows:

$$ds^2 = -\sin^2\left(\frac{u(t)}{2}\right) dt^2 + \cos^2\left(\frac{u(t)}{2}\right) dx^2,$$

$$\sin\left(\frac{u}{2}\right) = \pm \sqrt{K-c},$$

$$\lambda(t-t_0) = \pm \int \frac{d\Phi}{\sqrt{(K-c)[1-(K-c)]}}.$$

Introducing the cosmological time $d\tau = \sin(u/2)dt$, the solutions take the form

$$ds^2 = -d\tau^2 + \cos^2\left(\frac{u}{2}\right) dx^2,$$

$$\sin\left(\frac{u}{2}\right) = \pm \sqrt{K-c},$$

$$\lambda(\tau - \tau_0) = \pm \int \frac{d\Phi}{\sqrt{1-(K-c)}}. \quad (41)$$

Let us give two examples of the application of Eqs. (41). Consider first the cosmological solutions of the JT model. With $K = \Phi^2$, we can perform the integration in Eq. (41) and obtain

$$ds^2 = -d\tau^2 + A^2 \sin^2 \lambda(\tau - \tau_0) dx^2,$$

$$\Phi = A \cos \lambda(\tau - \tau_0), \quad (42)$$

where $A = \sqrt{1+c}$. This form of the solution has already been found using a different method in Ref. [13].

In the case of a model with an exponential potential $V = \exp \Phi$, Eqs. (41) give

$$ds^2 = -d\tau^2 + A^2 \left[\tanh^2 \frac{\lambda A}{2} (\tau - \tau_0) \right] dx^2, \quad (43)$$

$$\Phi = -2 \ln \cosh \frac{\lambda A}{2} (\tau - \tau_0) + \text{const}, \quad (44)$$

where A is an integration constant given as in Eq. (42).

VIII. CONCLUSIONS

In this paper we have shown that one can use solutions of a generalized sine-Gordon model to describe the classical dynamics of 2D dilaton gravity. As a consequence, we have found for a broad class of extremal 2D black holes an underlying solitonic solutions that, in principle, can be used to describe their classical behavior. Our results seem to indicate that there is a deep connection between extremal configurations of 2D black hole and solitonic states. However, for 2D dilaton gravity the situation seems rather different than that for supergravity theories in $D \geq 4$. In the latter case the condition that the solutions preserve at least $N=1$ supersymmetry implies that they are BPS states, i.e., that they saturate at least one Bogomol'nyi bound. In this way the extremal black hole solutions have a natural interpretation as BPS solitons. In the case of 2D dilaton gravity the absence of bounds related with symmetries of the model (such as the Bogomol'nyi bound) makes the notion of extremality model dependent. The simplest way to obtain these bounds would be to consider a supersymmetric extension of our model. Previous investigations of the supersymmetric sine-Gordon model indicate, however, that in this model the BPS states are not solitons [14]. It would be of interest to find out if this apply also to the supersymmetric extension of the model (8).

Apart from these difficulties, our approach represents just a first step in the analysis of the role that solitons play in 2D dilaton gravity. It leaves many open questions. We have seen that there is a sine-Gordon-like theory underlying the classical behavior of 2D extremal black holes. However, the most interesting questions in the black hole physics appear at the semiclassical (or full quantum) level. The natural development of our approach would be to use the solitons to describe the semiclassical and the quantum dynamics of 2D extremal black holes. For instance, one could try to use soliton physics to describe near extremal 2D black holes, to investigate the black hole evaporation process and to give a microscopic interpretation of the entropy of the hole.

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- [1] M. J. Duff, R. R. Khuri, and J. X. Lu, *Phys. Rep.* **259**, 213 (1995); C. M. Hull and P. K. Townsend, *Nucl. Phys.* **B438**, 109 (1995); E. Witten, *ibid.* **B443**, 85 (1995); D. Youm, “Black holes and solitons in string theory,” hep-th/9710046.
- [2] A. Strominger and C. Vafa, *Phys. Lett. B* **379**, 99 (1996); G. T. Horowitz and A. Strominger, *Phys. Rev. Lett.* **77**, 2368 (1996); G. T. Horowitz, J. Maldacena, and A. Strominger, *Phys. Lett. B* **383**, 151 (1996).
- [3] See, for instance, A. Strominger, lectures presented at the *1994 Les Houches Summer School*, hep-th/9501071.
- [4] R. Jackiw, in *Quantum Theory of Gravity*, edited by S. M. Christensen (Hilger, Bristol, 1984); C. Teitelboim, *ibid.*
- [5] M. Cadoni, “Dualities, compositeness and spacetime structure of 4D extreme stringy black holes,” hep-th/9711093.
- [6] M. Cadoni and S. Mignemi, *Phys. Rev. D* **51**, 4319 (1995).
- [7] J. Gegenberg and G. Kunstatter, *Phys. Lett. B* **413**, 274 (1997).
- [8] See, for instance, *Solitons*, edited by R. Bullough and P. J. Caudrey (Springer-Verlag, Berlin, 1980).
- [9] T. Banks and M. O’Loughlin, *Nucl. Phys.* **B362**, 649 (1991); R. B. Mann, *Phys. Rev. D* **47**, 4438 (1993).
- [10] C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, *Phys. Rev. D* **45**, 1005 (1992); M. Cadoni and S. Mignemi, *Phys. Lett. B* **358**, 217 (1995).
- [11] D. Louis-Martinez and G. Kunstatter, *Phys. Rev. D* **49**, 5227 (1994); M. Cadoni, *ibid.* **53**, 4413 (1996).
- [12] A. A. Tseytlin and C. Vafa, *Nucl. Phys.* **B372**, 443 (1992).
- [13] M. Cadoni and M. Cavaglia, *Phys. Rev. D* **50**, 6435 (1994).
- [14] A. Rebhan and P. van Nieuwenhuizen, *Nucl. Phys.* **B508**, 449 (1997).