More on lattice BRST invariance

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In the gauge-fixing approach to (chiral) lattice gauge theories, the action in the U(1) case implicitly contains a *free* ghost term, in accordance with the continuum Abelian theory. On the lattice there is no BRST symmetry and, without fermions, the partition function is strictly positive. Recently, Neuberger pointed out, Phys. Rev. D **58**, 057502 (1998), that a different choice of the ghost term would lead to a BRST-invariant lattice model, which is ill defined nonperturbatively. We show that such a lattice model is inconsistent already in perturbation theory, and clearly different from the gauge-fixing approach. [S0556-2821(98)08719-0]

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A central difficulty in constructing chiral lattice gauge theories stems from the coupling between the fermions and the longitudinal degrees of freedom of the lattice gauge field. The physical reason for this coupling is the need to correctly reproduce the contribution to the anomaly for each fermion species, in the background of smooth gauge fields [1]. However, for generic lattice gauge fields, this (momentum-dependent) coupling is not small, and renders the fermion spectrum vectorlike instead of chiral in many proposals for lattice chiral gauge theories [2].

Gauge fixing is a natural way to try to control the dynamics of the longitudinal degrees of freedom [3–5]. In the gauge-fixing approach one transcribes to the lattice a chiral gauge theory, whose action in the continuum already contains Lorentz gauge-fixing and ghost terms. One then looks for a continuum limit governed by renormalized perturbation theory, requiring that this continuum limit indeed corresponds to the target gauge-fixed chiral gauge theory.

In the U(1) case, strong evidence for the existence of this continuum limit was found by us in a ''reduced'' model, where one restricts the U(1) gauge field to the trivial orbit. [In the future we plan to investigate the model with fully dynamical U(1) gauge fields.] The continuum limit corresponds to a continuous phase transition between a normal broken phase and an exotic broken phase where, in addition, rotation symmetry is broken by a vector condensate [4]. Analytical and numerical evidence for the existence and continuity of the phase transition is given in Refs. [6,7]. Evidence that the correct chiral spectrum is obtained in the continuum limit is given in Refs. [7,8]. As explained in Ref. [9] (which contains a less technical account of our work) the gauge-fixing approach does not contradict the Nielsen-Ninomiya theorem [10,1] as reformulated for interacting lattice theories in Ref. [11].

The lattice action of the gauge-fixing approach [Eq. (14) below] includes a *free* ghost term in the U(1) case, in accordance with the target continuum Abelian theory. Evidently, this exactly decoupled ghost sector does not affect any observable constructed out of the vector and fermion fields, hence it was dropped from the definition of the U(1) lattice action given in Refs. [5–8] (for a concise formulation see Ref. [8]). With the ghost action included, one can formulate Becchi-Rouet-Stora-Tyutin (BRST) transformations, but the lattice action is not BRST invariant. Following Ref. [3], one adds counterterms to the action in order to restore BRST invariance in the continuum limit. In particular, the continuous phase transition mentioned above corresponds to a vanishing photon mass.

Recently [12], Neuberger pointed out that a different choice of the ghost action exists, such that the sum of the gauge-fixing term of Refs. [5–8] and the new ghost term is BRST invariant. The U(1) lattice model defined using that BRST invariant action is in fact ill defined. This is a consequence of a general ''no-go'' theorem [13], also due to Neuberger, which asserts that the partition function itself, as well as (unnormalized) expectation values of gauge-invariant operators, vanish identically in a lattice model with exact BRST invariance. As a result, (normalized) expectation values of gauge-invariant operators always lead to ill-defined expressions of the form ''0/0.''

We will first describe Neuberger's observation in detail, and then explain why it is irrelevant for the gauge-fixing approach. The BRST-invariant U(1) model which was considered in Ref. [12] is defined by the path integral

$$Z=\int \mathcal{D}U\mathcal{D}\bar{c}\,\mathcal{D}c\,\exp(-S_{\mathrm{BRST}}(U;\bar{c},c)), \tag{1}$$

$$S_{\mathrm{BRST}}(U;\bar{c},c)=S_{\mathrm{gaugeinv}}(U)+S_{\mathrm{gaugefix}}(U)+S_{\mathrm{ghost}}(U;\bar{c},c). \tag{2}$$

This model contains vector and ghost fields, but no matter fields. The gauge-invariant term in the action represents the standard plaquette action, which in the classical continuum limit reduces to $\frac{1}{4}\int d^4x\,\Sigma_{\mu\nu}F_{\mu\nu}^2$. The gauge-fixing term has the general form

$$S_{\text{gaugefix}}(U)=\frac{1}{2\xi}\sum_x \mathcal{G}_x(U)^2, \tag{3}$$

where $\xi>0$ is the gauge-fixing parameter. $\mathcal{G}_x(U)$, which we will call the gauge condition, is a real local functional of the lattice link variables $U_{x\mu}=\exp(igA_{x\mu})$, which is continuously differentiable over the (compact) space of U(1) lattice gauge-field configurations. The general form of the ghost term is

$$S_{\text{ghost}}(U;\bar{c},c)=\sum_{xy} \bar{c}_x\Omega_{xy}(U)c_y, \tag{4}$$

where c and $\bar{c}$ are ghost and antighost fields. The ghost operator is

$$\Omega_{xy}(U)=\sum_\mu \frac{\delta\mathcal{G}_x(U)}{\delta A_{y\mu}}\Delta^+_{y\mu}, \tag{5}$$

where $\Delta^+_{y\mu}$ is the forward lattice derivative, defined as $\Delta^+_{x\mu}f=f_{x+\hat{\mu}}-f_x$ for any function f_x. The model in Eq. (1) has an exact BRST invariance if the same $\mathcal{G}_x(U)$ enters both the gauge-fixing and ghost terms. As mentioned above, in this case it was proved by Neuberger that the partition function (1) itself, as well as (unnormalized) expectation values of gauge invariant operators, vanish [13].

In Ref. [12] Neuberger showed that a BRST invariant action exists whose gauge-fixing term (3) coincides with the one defined in Refs. [5–8] up to a trivial constant $\mathcal{V}M$ where $\mathcal{V}$ is the lattice volume. The gauge-fixing term advocated in Refs. [5–8] has the form

$$S^{\text{L}}_{\text{gaugefix}}(U)=\tilde{\kappa}\left\{\sum_{xyz} \Box_{xy}(U)\Box_{yz}(U)-\sum_x B_x^2(U)\right\},$$

$$\tilde{\kappa}=\frac{1}{2\xi g^2}, \tag{6}$$

where

$$B_x(U)=\frac{1}{4}\sum_\mu (V_{x-\hat{\mu},\mu}+V_{x\mu})^2, \tag{7}$$

$$V_{x\mu}=\text{Im}\,U_{x\mu}=\sin(gA_{x\mu}), \tag{8}$$

and $\Box_{xy}(U)=\Sigma_\mu(\delta_{x+\hat{\mu},y}U_{x\mu}+\delta_{x-\hat{\mu},y}U^\dagger_{y\mu})-8\,\delta_{x,y}$ is the covariant nearest-neighbor lattice Laplacian. In the classical continuum limit $S^{\text{L}}_{\text{gaugefix}}(U)$ reduces to the Lorentz gauge-fixing action, $(1/2\xi)\int d^4x(\Sigma_\mu\partial_\mu A_\mu)^2$. The other properties of $S^{\text{L}}_{\text{gaugefix}}(U)$ are summarized later in this paper. Now, one can write

$$S^{\text{L}}_{\text{gaugefix}}(U)=\frac{1}{2\xi}\sum_x \mathcal{S}_x(U). \tag{9}$$

The BRST invariant action is defined by picking [12]

$$\mathcal{G}_x(U)=\sqrt{\mathcal{S}_x(U)+M}, \tag{10}$$

where M is a constant chosen such that $M>-\min\{\mathcal{S}_x(U)\}$. Note that the range of the functional $\mathcal{S}_x(U)$ over the entire lattice configuration space is a bounded closed interval, hence $\min\{\mathcal{S}_x(U)\}$ is necessarily finite. (In fact, $\min\{\mathcal{S}_x(U)\}=0$ [5].) As a special case of Neuberger's theorem, the partition function (1) vanishes if the functional (10) is used in its definition.

The gauge-fixing approach evades this inconsistency by *not* having BRST symmetry on the lattice. In the absence of fermions, the Boltzmann weight of the gauge-fixing approach in the U(1) case is strictly positive (see below), which implies that the ''0/0'' problem does not occur. Moreover, we wish to demonstrate that *perturbative* consistency already excludes the ghost action constructed in Ref. [12].

Perturbation theory is an expansion around a classical vacuum, i.e., a translationally invariant global minimum of $S_{\text{gaugefix}}(U)$ on the trivial orbit. We consider in the following a gauge condition $\mathcal{G}_x(U)$ with a strictly positive range, i.e., $\mathcal{G}_x(U)>0$ and which is translationally covariant, i.e., $\mathcal{G}_x(U_{y\mu})=\mathcal{G}_{x-z}(U_{y-z,\mu})$. An example is the gauge condition (10). We will prove now that for such a gauge condition the Faddeev-Popov operator is identically zero, i.e., $\Omega_{xy}=0$, on a classical vacuum.

The proof is very simple. Let $U^0_{z\mu}=\exp(igA^0_{z\mu})=U^0_\mu$ be a translationally invariant saddle point of $S_{\text{gaugefix}}(U)$. Then

$$\mathcal{G}_x(U^0_\mu)\Omega_{xy}(U^0_\mu)=\frac{1}{2}\sum_\nu \left.\frac{\delta\mathcal{G}_x^2}{\delta A_{y\nu}}\right|_{U=U^0_\mu}\Delta^+_{y\nu}=0. \tag{11}$$

The first equality follows from Eq. (5). The second equality follows because, by Eq. (3), a translationally invariant U^0_μ is a saddle point of $S_{\text{gaugefix}}(U)$ if and only if it is a saddle point of $\mathcal{G}_x^2(U)$ for any x. Notice now that $\mathcal{G}_x(U^0_\mu)\neq 0$ by assumption. Dividing both sides of Eq. (11) by $\mathcal{G}_x(U^0_\mu)$, we obtain $\Omega_{xy}(U^0_\mu)=0$.

The conclusion is that perturbation theory is undefined if $\mathcal{G}_x(U)$ is a strictly positive functional, since the tree-level ghost operator $\Omega(U^0_\mu)$ vanishes identically. We note that the gauge condition, Eq. (10), is completely determined by the requirement that the gauge-fixing term, Eq. (3), of the BRST-invariant action should coincide (up to the constant $\mathcal{V}M$) with $S^{\text{L}}_{\text{gaugefix}}(U)$. Hence, this also proves that $S^{\text{L}}_{\text{gaugefix}}(U)$ cannot be the gauge-fixing term of any BRST invariant action that has the correct classical continuum limit. (Recall that, for the Lorentz gauge, the quadratic part of the continuum ghost action is $\bar{c}\,\Box\,c$, and not zero, in Abelian as well as in non-Abelian theories.)

As was shown in Ref. [14], if one is interested *only* in perturbation theory, one can employ the BRST construction just as in the continuum. Of course, one has to make sure that the gauge-fixing and ghost terms both have the correct classical continuum limit. In view of the above result, this implies that one must use an *indefinite-sign* functional for $\mathcal{G}_x(U)$. We conclude this section with an example of this. Consider the lattice discretization $\mathcal{G}^{\text{L,naive}}_x(U)$ of the Lorentz gauge condition $\Sigma_\mu\partial_\mu A_\mu$, with

$$\mathcal{G}_x^{\mathrm{L,naive}}(U)=\frac{1}{g}\sum_\mu \Delta^-_{x\mu}V_\mu, \tag{12}$$

where $\Delta^-_{x\mu}$ designates the backward lattice derivative, and V_μ is defined in Eq. (8). One expects that the equation $\Sigma_\mu\Delta^-_{x\mu}[\sin(gA_\mu-\Delta^+_\mu\theta)]=gv_x$ can be solved for sufficiently small A_μ and v. Therefore the range of $\mathcal{G}_x^{\mathrm{L,naive}}(U)$ contains an open neighborhood of zero and $\mathcal{G}_x^{\mathrm{L,naive}}(U)$ is an indefinite-sign functional. Equation (11) is now fulfilled on a classical vacuum because $\mathcal{G}_x^{\mathrm{L,naive}}(U^0_\mu)=0$ for all x. Since furthermore the gauge-fixing action

$$S^{\mathrm{L,naive}}_{\mathrm{gaugefix}}(U)=\frac{1}{2\xi}\sum_x (\mathcal{G}_x^{\mathrm{L,naive}}(U))^2, \tag{13}$$

and the Faddeev-Popov operator have the correct classical continuum limit, $\mathcal{G}_x^{\mathrm{L,naive}}(U)$ is a consistent gauge condition at the level of perturbation theory.

Now we discuss the gauge-fixing approach in some more detail. Specifically, we will consider the lattice transcription of a Lorentz gauge-fixed U(1) theory, where the continuum theory consists of free photons only. Due to the presence of a quadratic covariant gauge-fixing term we expect to get all four polarizations as free, uncoupled states in the continuum limit of the lattice model. [We emphasize that the question here is not the practicality of working with a gauge-fixed U(1) lattice theory, but, rather, its existence.] The lattice model is now defined by the action

$$S(U;\bar{c},c)=S_{\mathrm{gaugeinv}}(U)+S^{\mathrm{L}}_{\mathrm{gaugefix}}(U)+S^{\mathrm{L}}_{\mathrm{ghost}}(\bar{c},c)+S_{\mathrm{counterterm}}(U). \tag{14}$$

The gauge-invariant term is again the plaquette action. $S^{\mathrm{L}}_{\mathrm{gaugefix}}(U)$ is the lattice discretization of the Lorentz gauge-fixing action introduced in Eq. (6). The *free* ghost action is

$$S^{\mathrm{L}}_{\mathrm{ghost}}(\bar{c},c)=\sum_{xy}\bar{c}_x\{-\Box_{xy}+\mu^2\,\delta_{xy}\}c_y, \tag{15}$$

where for definiteness we have chosen $\Box_{xy}$ as the nearest-neighbor free lattice Laplacian. We have added an infinitesimal mass term ($0<\mu^2\ll 1$) to avoid the trivial finite-volume zero mode. One can safely set $\mu=0$ after the infinite volume limit is taken. (Alternatively, one could, e.g., choose antiperiodic boundary conditions.) It is evident from Eqs. (14) and (15) that the Boltzmann weight of the gauge-fixing approach is strictly positive in the U(1) case.

Given the U(1) action (14), one can formulate lattice BRST transformations, but obviously, $S(U;\bar{c},c)$ is not BRST invariant. Following the procedure proposed and outlined in Ref. [3] (see in particular Sec. 6 of the last paper), one adds counterterms to the action, in order to restore BRST invariance in the continuum limit. In perturbation theory, this means that the continuum limit of any correlation function should obey the relevant *continuum* BRST identity. Because the ghosts are free, it is possible to impose BRST invariance without ghost counterterms, since all connected ghost correlation functions agree with the continuum ones in the continuum limit already. As we already mentioned in the introduction, the decoupled ghost sector cancels out from the expectation value of any operator constructed from the gauge (and/or matter) fields, hence it was dropped in Refs. [5–8]. [The U(1) continuum action is BRST invariant also with massive photon and ghost fields, provided their masses are equal (see for instance, Ref. [15]). On the lattice, one can impose the BRST identities of the massive theory in the continuum limit, sending $\mu\to 0$ in the end. Yet another possibility is to use the action *without* the free ghost term, in which case it is strictly speaking more appropriate to talk about recovering Ward identities rather than BRST invariance in the continuum limit.]

The gauge-fixing action density (cf. Eq. (9)) can be written as $\mathcal{S}_x(U)=\mathcal{S}_x^{(1)}(U)+\mathcal{S}_x^{(2)}(U)$, where $\mathcal{S}_x^{(1)}(U)=(\mathcal{G}_x^{\mathrm{L,naive}}(U))^2$, cf. Eq. (12). Thus, $\mathcal{S}_x^{(1)}(U)$ corresponds to the naive lattice transcription of the continuum $(\Sigma_\mu\partial_\mu A_\mu)^2$ discussed previously. While perturbation theory is self-consistent in this case, it may be unreliable in view of the proliferation of lattice Gribov copies of the $U_{x\mu}=1$ classical vacuum for the gauge condition $\mathcal{G}_x^{\mathrm{L,naive}}(U)$, each of which is a global minimum of $S^{\mathrm{L,naive}}_{\mathrm{gaugefix}}(U)$ [4] (see also Ref. [16]). In particular, the existence of the continuous phase transition where we want to take the continuum limit is *a priori* not guaranteed. This is remedied by the addition of $\mathcal{S}_x^{(2)}(U)$. The latter contains only irrelevant operators, and has a unique absolute minimum at $U_{x\mu}=1$. (That irrelevant terms can have a profound effect on the continuum limit should not come as a surprise, as the example of the Wilson term for lattice Wilson fermions shows.) We now summarize the key properties of the action $S(U;\bar{c},c)$, Eq. (14).

(1) $S(U;\bar{c},c)$ is not invariant under BRST transformations. Moreover, there does not exist a BRST invariant lattice action with the correct classical continuum limit, whose gauge-fixing term coincides with $S^{\mathrm{L}}_{\mathrm{gaugefix}}(U)$.
(2) $S^{\mathrm{L}}_{\mathrm{gaugefix}}(U)$ has a unique absolute minimum at $U_{x\mu}=1$ [5].
(3) $S(U;\bar{c},c)$ has the correct classical continuum limit.

The second property ensures that the Euclidean functional integration is dominated by the unique global maximum of the Boltzmann weight. The third property implies that kinetic terms exist for all polarizations of the gauge field as well as the ghost fields. Therefore, perturbation theory is well-defined and renormalizable. This is at the heart of the good agreement between one-loop perturbation theory and nonperturbative numerical results found in the reduced model [6,8].

As explained above, in order to recover BRST invariance, we have introduced in Eq. (14) a finite number of counterterms that correspond to all relevant and marginal operators which are allowed by the exact lattice symmetries [3]. The only dimension-two counterterm is the photon mass term

$$S_{\mathrm{mass}}(U)=-2\kappa\sum_{x\mu}\mathrm{Re}\,U_{x\mu}. \tag{16}$$

So far, this is the only counterterm that we have studied in detail [5–8]. The mass counterterm is crucial because the

continuum limit mentioned in the introduction corresponds to a vanishing photon mass. This is achieved by tuning κ in Eq. (16) to its critical value. A brief discussion of nonderivative dimension-four counterterms is given in Ref. [5]. (In the future we plan to investigate the role of other counterterms in more detail.)

In conclusion unitarity and Lorentz invariance are consistency requirements for the continuum limit of any lattice gauge theory. In the gauge-fixing approach, the restoration of Lorentz invariance is expected to occur in a fairly standard fashion. As for unitarity, or exact decoupling of unphysical states, this may be achieved by imposing BRST invariance in the continuum limit. In fact, some of the counterterms needed to restore BRST invariance are also needed for the restoration of Lorentz symmetry [3].

For this program to succeed, BRST invariance needs not necessarily be present at finite lattice spacing. This observation plays a key role in the gauge-fixing approach. In view of Neuberger's theorem [13], *not* having BRST invariance is essential for the very existence of the lattice theory, and, hence, also for the existence of the continuous phase transition where one can make contact with the target gauge-fixed continuum theory. [In a *chiral* lattice gauge theory, BRST (or gauge) invariance is broken anyway by the fermion action. Sometimes the hope is expressed that this would be enough to avoid the consequences of Neuberger's theorem. We believe that one should first formulate gauge-fixed lattice theories without matter fields. If, before the introduction of matter fields, a gauge-fixed lattice model is ill defined due to exact BRST invariance, we see little reason why the attempt to incorporate chiral fermions should improve the situation.]

As discussed in this paper, in the Abelian case it is appropriate to choose a free, decoupled, lattice ghost action. [Note that we could have chosen a ghost action for the Abelian case which is not free on the lattice (but only in the classical continuum limit), but there is no reason to do so, since there is no BRST invariance on the lattice anyway.] Now, all properties of the gauge-fixing term (6) listed above generalize to the non-Abelian case [5]. But in the non-Abelian case we must also include a ghost-gauge field *interaction* term in the lattice action [3,4], because this interaction is present in the target gauge-fixed continuum theory. [Note that a non-Abelian ghost action *à-la* Eq. (5) will again not have the correct classical continuum limit, and therefore will not be a candidate for the lattice ghost action.]

In the non-Abelian case, the measure defined using the Faddeev-Popov determinant (rather than its absolute value [17]) is no longer positive. Therefore, a possibility that one should worry about is that Neuberger's theorem still applies in the continuum limit: approximate cancellations associated with ''smooth'' continuum Gribov copies might take place, and lead to the vanishing of the partition function in the continuum limit, even if such cancellations do not occur at finite lattice spacing. Also the (related, but separate) issue of enforcing BRST invariance nonperturbatively is highly nontrivial. These questions have to be addressed before the gauge-fixing approach can be successfully extended to non-Abelian theories.

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