On lattice computations of $K^+ \to \pi^+ \pi^0$ decay at $m_K = 2m_\pi$

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We use one-loop chiral perturbation theory to compare potential lattice computations of the $K^+ \to \pi^+ \pi^0$ decay amplitude at $m_K = 2m_\pi$ with the experimental value. We find that the combined one-loop effect due to this unphysical pion to kaon mass ratio and typical finite volume effects is still of order minus 20–30%, and appears to dominate the effects from quenching.
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I. INTRODUCTION

Recently, we have used chiral perturbation theory (ChPT) to one loop in order to investigate three systematic effects which affect lattice computations of the weak matrix element for $K^+ \to \pi^+ \pi^0$ decay: quenching, finite-volume effects, and the use of unphysical values of the quark masses and pion external momenta [1] (to which we will refer as I). (For an extension to the partially quenched case, see Ref. [2]; for a recent lattice computation, see [3]; for other references, see I.) We considered the case of a lattice computation (extrapolated to the continuum limit) with three degenerate light quarks, and final-state pions at rest. This is unphysical because SU(3) flavor is broken in the real world and because this choice of external momenta does not conserve energy.

Here, we extend our results to the case $m_K = 2m_\pi$, considering both the quenched and unquenched theories. This choice of masses conserves energy when all external mesons are at rest, and, since in reality $m_K \approx 3.6 m_\pi$, it is closer to the real-world meson masses. One might therefore expect the systematic errors for this choice of masses to be smaller than in the energy-nonconserving case with degenerate masses.

The choice $m_K = 2m_\pi$ was also advertized in Ref. [4], where it is used as one of the ingredients in an improvement program for lattice computations of nonleptonic kaon decays with Wilson fermions.

Here, we restrict ourselves to a summary of our results, and a brief discussion of differences with *I*; for other details we refer to the extensive explanations contained in I. We also discuss similar systematic effects for B_K. The notation is the same as that of I.

II. LATTICE METHOD

On the lattice, usually operators with zero spatial momentum, $O(t) = \Sigma_{\vec{x}} O(\vec{x}, t)$, are used. For $K^+ \to \pi^+ \pi^0$ decay on the lattice, one computes the time-correlation function

$$C(t_2, t_1) \equiv \langle 0 | \pi^+(t_2) \pi^0(t_2) O_4(t_1) K^-(0) | 0 \rangle \tag{2.1}$$

$$\to \frac{\langle 0 | \pi^+(0) \pi^0(0) | \pi^+ \pi^0 \rangle \langle K^+ | K^-(0) | 0 \rangle}{\langle \pi^+ \pi^0 | \pi^+ \pi^0 \rangle \langle K^+ | K^+ \rangle} \times \langle \pi^+ \pi^0 | O_4(0) | K^+ \rangle e^{-E_{2\pi}(t_2 - t_1)} e^{-m_K t_1}, \tag{2.2}$$

where O_4 is the $\Delta S = 1$, $\Delta I = 3/2$ part of the weak effective Hamiltonian [5]. Equation (2.2) shows the dominant term for large time separations, from which the desired K^+-decay matrix element $\langle \pi^+ \pi^0 | O_4 | K^+ \rangle$ can be extracted. This matrix element represents the decay process, confined to a finite lattice volume (we will restrict ourselves to a cubic volume with linear dimension L and periodic boundary conditions), in which a kaon, at rest, decays into a state consisting of two pions at rest with the lowest energy $E_{2\pi}$ [6]. The case for which $m_K = 2m_\pi$ is of special interest, since, in that case, the matrix element corresponds to an energy-conserving process (in the infinite-volume limit), in contrast to, for instance, the mass-degenerate case ($m_K = m_\pi$) in which energy is injected through the weak operator O_4.

The aim of this paper is to compare real-world physical quantities with those obtained from a hypothetical lattice computation in full or quenched QCD with $m_K = 2m_\pi$, using ChPT. Here, we will refer to the choice $m_K = 2m_\pi$, as ‘‘lattice masses.’’ Also, we will only consider the case with unbroken isospin, $m_u = m_d \neq m_s$.

III. ANALYTIC RESULTS FOR $m_K = 2m_\pi$ FROM ONE-LOOP ChPT

To $O(p^4)$ in (quenched) ChPT, the one-loop diagrams (a) to (d) of Fig. 1 in I, along with the relevant wave-function renormalizations, as well as the tree-level contributions of $O(p^4)$ weak operators [7], have to be evaluated in order to obtain the K^+-decay matrix element. In I, where the mass-degenerate case was considered, special care was taken to accommodate the kinematic situation in which energy is not conserved. In contrast, for $m_K = 2m_\pi$ and in infinite volume, standard Feynman diagram techniques can be used. In a finite volume, it was shown in I that (power-like) finite-volume corrections come exclusively from diagram (b) of Fig. 1, in which the final-state pions from the weak decay of the kaon rescatter. We have derived the finite-volume corrections for $m_K \neq m_\pi$, and, in particular, for $m_K = 2m_\pi$ in the same way.

The weak decay operator O_4 does not couple directly to the singlet meson η_0. However, for $m_u = m_d \neq m_s$, mixing between η_8 and η_0 occurs. As a result, the η two-point function inherents the ‘‘double pole’’ of the quenched theory [8], introducing dependence on the parameters δ and α through η loops [$\delta = m_0^2/(24\pi^2 f_\pi^2)$, with m_0 the singlet part of the η' mass], which is not present in the mass-degenerate case. It turns out that these contributions are finite (independent of the cutoff), and, therefore, unambiguously predicted in quenched ChPT.

The one-loop result for the unquenched case follows directly from Eqs. (43),(44) [re-expressing the $1/f_\pi^3$ in terms of the bare decay constant f using Eq. (7) of I] by substituting $m_K=2m_\pi$. In the quenched case (which in I was only considered for $m_K=m_\pi$), we obtain for $m_K=2m_\pi$, ignoring the contributions from $O(p^4)$ operators,

$$\langle\pi^+\pi^0|O_4|K^+\rangle^q=\frac{9i\alpha_{27}^q}{\sqrt{2}f_q^3}m_K^2\left[1+\frac{m_K^2}{(4\pi f_q)^2}\times\left(-2\log\frac{m_K^2}{\Lambda_q^2}+c_K+\frac{1}{4}G(m_\pi L)\right)+\delta c_\delta+\alpha c_\alpha\frac{m_K^2}{(4\pi f_q)^2}\right],\tag{3.1}$$

with

$$c_K=\frac{7}{12}\log 4+\frac{1}{12}\log 7-\frac{1}{2}\sqrt{3}\arctan\frac{\sqrt{3}}{5},$$

$$c_\delta=-\frac{5}{6}-2\log 4+\frac{17}{9}\log 7,$$

$$c_\alpha=\frac{14}{9}+2\log 4-\frac{431}{208}\log 7-\frac{2}{\sqrt{3}}\arctan\frac{\sqrt{3}}{5},\tag{3.2}$$

and

$$G(x)=\frac{17.827}{x}+\frac{10\pi^2}{x^3}.\tag{3.3}$$

The super/subscript q denotes quenched quantities: f_q is the bare decay constant of the quenched theory, etc. The first factor of Eq. (3.1) is the tree-level result; the factor in square brackets gives the one-loop correction factor.

Before substituting $m_K=2m_\pi$ in order to obtain Eq. (3.1), one finds nonanalytic contributions coming from poles at m_π^2, m_K^2 and $2m_K^2-m_\pi^2$ (which is the mass of a pure $s\bar{s}$ meson). These nonanalytic contributions have the values c_K, c_δ and c_α after substituting $m_K=2m_\pi$. The reason that we do not display the more general result here is that it is different depending on whether energy is conserved, or all external mesons are at rest. Only at $m_K=2m_\pi$ do both more general expressions agree with each other.

The finite-volume correction [term proportional to $G(m_\pi L)$ in Eq. (3.1)] has the same form in the full and quenched theories, since it comes only from the pion-pion rescattering diagram, which has no internal quark loops. Note that the function $G(x)$ is different from the one that appears in the mass-degenerate case, cf. Eq. (84) in I.

For $m_K\neq m_\pi$, $O(p^4)$ weak operators lead to contributions to the one-loop correction factor of the form $(Am_\pi^2+Bm_\pi m_K+Cm_K^2)/(4\pi f_q)^2$. In the mass-degenerate limit, this leaves a dependence on the linear combination $A+B+C$; whereas for $m_K=2m_\pi$, the linear combination is $A+2B+4C$. In Eq. (3.1), we have set $A+2B+4C=0$. Needless to say, for a comparison of complete $O(p^4)$ results between different masses, information will be needed about the values of the coefficients A, B and C.

IV. COMPARISONS

In this section, we will compare a hypothetical lattice computation, in a finite volume and with $m_K=2m_\pi$, of the $K^+\to\pi^+\pi^0$ matrix element and B_K with the real world. We will denote the lattice meson masses by $m_{K,latt}$ and $m_{\pi,latt}=\frac{1}{2}m_{K,latt}$, and the real-world masses by m_π and m_K, with $m_\pi=136$ MeV and $m_K=496$ MeV. We will use $f_{\pi,latt}=f_\pi=132$ MeV (f_π will only appear in one-loop corrections). The mass of the η is always determined from the tree-level relation $m_\eta^2=\frac{4}{3}m_K^2-\frac{1}{3}m_\pi^2$.

Little is known about the values of $O(p^4)$-operator coefficients. Therefore, as in I, we will ignore contributions from $O(p^4)$ operators, and use the values 770 MeV and 1 GeV for the cutoffs Λ and Λ_q of the full and quenched theories, respectively, and take the spread as an indication of the uncertainty from the lack of information about the $O(p^4)$ constants. In subsection IV A, we will consider the case of a full-QCD lattice computation, and in subsection IV B the quenched case.

A. Full QCD at $m_K=2m_\pi$

In the full theory, the $K^+\to\pi^+\pi^0$ matrix element for the real world (subscript *phys*) and on the lattice (subscript *latt*) can be related to one loop using Eqs. (43), (44), (7) of I (which was derived for arbitrary m_K and m_π). For the real world, one evaluates the result for $\langle\pi^+\pi^0|O_4|K^+\rangle$ at the physical values of m_K and m_π; whereas on the lattice, one chooses $m_{\pi,latt}=\frac{1}{2}m_{K,latt}$ and therefore $m_\eta^2=5m_{K,latt}^2/4$. We obtain

$$\langle\pi^+\pi^0|O_4|K^+\rangle_{phys}^f=X\,\frac{4}{3}\,\frac{m_K^2-m_\pi^2}{m_{K,latt}^2}\langle\pi^+\pi^0|O_4|K^+\rangle_{latt}^f,\tag{4.1}$$

where the superscript f denotes the full theory, and where

$$X=\frac{1+U_{phys}}{1+U_{latt}+(m_{K,latt}^2/4(4\pi f_\pi)^2)\,G(m_{K,latt}L/2)}\tag{4.2}$$

is the one-loop correction to the tree-level ''conversion factor'' $4(m_K^2-m_\pi^2)/(3m_{K,latt}^2)$ (U denotes the one-loop correction term [cf. Eq (87) of I] in infinite volume, i.e., with $G=0$). We will restrict ourselves to the case where the lattice kaon mass is the same as the physical kaon mass, i.e., $m_{K,latt}=m_K$, which will presumably be accessible in future lattice computations. It is straightforward to consider other examples.

TABLE I. The factor X for different values of $m_K L$ and different combinations of values of the cutoff Λ [super/subscripts (1) and (0.77) denote cutoff used in the numerator/denominator of Eq. (4.2)].

$m_K L$	$X^{(1)}_{(1)}$	$X^{(0.77)}_{(0.77)}$	$X^{(1)}_{(0.77)}$	$X^{(0.77)}_{(1)}$
6	0.71	0.72	0.80	0.64
8	0.74	0.77	0.85	0.67
∞	0.82	0.86	0.95	0.74

TABLE II. The factor Y for different values of $m_K L$ and different combinations of values of the cutoffs Λ and Λ_q [super/subscripts (1) and (0.77) denote cutoff used in the numerator/denominator of Eq. (4.7)].

	$Y^{(1)}_{(1)}$	$Y^{(0.77)}_{(0.77)}$	$Y^{(1)}_{(0.77)}$	$Y^{(0.77)}_{(1)}$
$m_K L=6$	0.71	0.69	0.76	0.65
$m_K L=8$	0.75	0.73	0.81	0.68
∞ volume	0.83	0.81	0.89	0.75

The tree-level conversion factor is unambiguous [9], and, for our example, equal to 1.23. We will therefore concentrate on the one-loop factor X. From Eq. (87) of I, $U_{phys}=0.0888$ and -0.0146 for $\Lambda=1$ GeV and 770 MeV, respectively (we will ignore the imaginary part of the matrix element, since it does not contribute to the magnitude of the amplitude to order p^4). For $m_{K,latt}=m_K$, we obtain $U_{latt}=0.328$ and 0.147 for $\Lambda=1$ GeV and 770 MeV, respectively.

On the lattice, when L is such that $m_K L=6$ or 8, the relative one-loop contribution from the finite volume correction $[m_K^2/4(4\pi f_\pi)^2]G(m_K L/2)$ is 0.215 or 0.134, respectively. Since, as explained before, we have omitted *different* linear combinations of $O(p^4)$ coefficients in the real-world and lattice cases, respectively, we will vary the values of the cutoff in U_{phys} and U_{latt} independently.

We list in Table I the values of X for four combinations of the values 1 GeV or 770 MeV for the cutoff, and for volumes such that $m_K L=6$, $m_K L=8$ and $m_K L=\infty$. We take the spread of the factor X due to changes in the values of the cutoff as a systematic error, which, from Table I, is around 15–20%.

The one-loop expression for B_K in the full theory is given by Eq. (36) of I. For physical masses, $B_K^{f,phys}/B^f=1.72$ or 1.42 for $\Lambda=1$ GeV or 770 MeV, respectively; whereas for $m_{K,latt}=2m_{\pi,latt}=m_K$, $B_K^{f,latt}/B^f=1.73$ or 1.44, for the same values of the cutoff. If we compare at the same value of the cutoff, we see that the real-world and lattice values differ by about 1%, in contrast to the K^+-decay matrix element. However, if we compare at different values of the cutoff, again as an estimate of the error introduced by ignoring $O(p^4)$ coefficients, we see that the real-world and lattice values may differ by as much as 20%.

Following I, we can examine the ratio

$$\mathcal{R}=[f_K\langle\pi^+\pi^0|O_4|K^+\rangle/\langle\overline{K}^0|O'|K^0\rangle]^f_{latt}, \tag{4.3}$$

which is independent of the $O(p^2)$-operator coefficient α_{27}. The tree-level value for the above ratio is $9i/8\sqrt{2}$. At one loop, we find, for $m_{K,latt}=2m_{\pi,latt}=m_K$, corrections of -53%, -61% and -74% for $\Lambda=1$ GeV and $m_\pi L=6$, 8 and ∞, respectively (for $\Lambda=770$ MeV, the corrections are -29%, -37% and -50%).

B. Quenched QCD at $m_K=2m_\pi$

We now compare real-world quantities and quantities as would be obtained from a quenched lattice computation with $m_K=2m_\pi$. We get, from Eq. (3.1),

$$\langle\pi^+\pi^0|O_4|K^+\rangle^q_{unphys}=\frac{9i\alpha^q_{27}}{\sqrt{2}f_q^3}m^2_{K,latt}\left(1+U^q_{latt}+\frac{m^2_{K,latt}}{4(4\pi f_q)^2}G(m_{K,latt}L/2)\right), \tag{4.4}$$

with

$$U^q_{latt}=\frac{m_{K,latt}}{(4\pi f_q)^2}\left(-2\log\frac{m^2_{K,latt}}{\Lambda_q^2}+0.6820\right)+0.0697\delta+0.0603\alpha\frac{m^2_{K,latt}}{(4\pi f_q)^2}. \tag{4.5}$$

In analogy with Eq. (4.1), we relate the real-world K^+ decay matrix element to that of the quenched lattice theory, obtaining

$$\langle\pi^+\pi^0|O_4|K^+\rangle^f_{phys}=Y\frac{\alpha_{27}}{\alpha^q_{27}}\left(\frac{f_q}{f}\right)^3\times\frac{4}{3}\frac{m_K^2-m_\pi^2}{m^2_{K,latt}}\langle\pi^+\pi^0|O_4|K^+\rangle^q_{latt}, \tag{4.6}$$

where

$$Y=\frac{1+U_{phys}}{1+U^q_{latt}+[m^2_{K,latt}/4(4\pi f_q)^2]\,G(m_{K,latt}L/2)} \tag{4.7}$$

is again the one-loop correction to the tree-level conversion factor.

Again, we will restrict ourselves to the case $m_{K,latt}=2m_{\pi,latt}=m_K$, and take $f_q=f_\pi$. First, we estimate the importance of the δ- and α-dependent terms in U^q_{latt}, cf. Eq. (4.5). The δ- and α-independent part of U^q_{latt} is equal to 0.69 and 0.59 for $\Lambda_q=1$ GeV and $\Lambda_q=770$ MeV, respectively. The value for δ is estimated to be less than around 0.2 [10]. α is only poorly known, but is unlikely to be larger than one in magnitude [10]. Therefore, the total contribution of the δ and α terms is less than about ten percent of the δ- and α-independent contribution. We will omit the δ and α terms in Table II. Table II gives the values of Y for the four combinations of the values 1 GeV or 770 MeV for the cutoffs, and for volumes such that $m_K L=6$, $m_K L=8$ and $m_K L=\infty$. These values of Y deviate substantially from the tree-level value $Y=1$, like in the examples with $m_K=m_\pi$ considered in I. We see that the spread in the values of the factor Y is 15% or less.

For the quenched one-loop expression for B_K, we refer to Eq. (37) of I. For $m_{K,latt}=2m_{\pi,latt}=m_K$,

$$\frac{B_K^{q,latt}}{B^q}=1-0.14\delta+\begin{cases}0.691+0.050\alpha, & \Lambda_q=1\ \mathrm{GeV},\\ 0.358+0.085\alpha, & \Lambda_q=770\ \mathrm{MeV}.\end{cases} \tag{4.8}$$

The δ and α terms are again relatively small. If we compare with the values of $B_K^{f,phys}/B^f$ (see previous subsection), we see that again real-world and quenched lattice values differ only by a few percent if we compare at the same value of the cutoffs. At different values of the cutoffs, they may again differ by as much as 20%.

In the quenched case, the ratio $\mathcal{R}$ defined in Eq. (4.3) is, for $m_{K,latt}=2m_{\pi,latt}=m_K$ and $f_q=f_\pi$,

$$\begin{aligned}\mathcal{R}_q&=\left(f_K\frac{\langle\pi^+\pi^0|O_4|K^+\rangle}{\langle\overline{K}^0|O'|K^0\rangle}\right)^q_{latt}\\ &=\frac{9i}{8\sqrt{2}}\left(1+0.056\delta+0.0224G(m_KL/2)\right.\\ &\quad\left.+\begin{cases}-0.380-0.058\alpha, & \Lambda_q=1\ \mathrm{GeV}\\ -0.140-0.093\alpha, & \Lambda_q=770\ \mathrm{MeV}\end{cases}\right).\end{aligned} \tag{4.9}$$

We see that the quenched values of $\mathcal{R}$ are closer to the tree-level value than the unquenched values, for the same choice of meson masses and volumes.

V. CONCLUSION

Our main results are the ratios X and Y in Eqs. (4.2) and (4.7), which give the one-loop ChPT correction to the conversion factor between a lattice computation of the $K^+\to\pi^+\pi^0$ amplitude with $m_K=2m_\pi$ and the experimental value, for unquenched and quenched QCD, respectively. These ratios give an estimate of the systematic effect due to finite volume, unphysical quark masses, and quenching (Y) for this matrix element. (The tree-level conversion factor corrects only for unphysical masses.) For more discussion on the reliability of such estimates, and other systematic errors, see I.

In Tables I and II, we give numerical examples, illustrating these results for a lattice kaon mass equal to the experimental value, $m_K=496$ MeV. The four different values on each line represent four combinations of cutoffs we chose in evaluating the ratios X and Y, and we take the spread as an indication of the uncertainties introduced by the lack of information on $O(p^4)$ constants. We should emphasize that ChPT does not give us any information about the ratio $(\alpha_{27}f_q^3)/(\alpha_{27}^q f^3)$ in Eq. (4.6).

From the Tables, we conclude that even in the ''more physical'' case $m_K=2m_\pi$, the one-loop systematic effect may still be as large as minus 20–30%. Comparing Tables I and II, we see that quenching seems to have only a minor effect. The large deviations from one of the ratios X and Y are caused by the sensitivity of the matrix element to the ratio m_π/m_K and finite-volume effects.

This situation is different from that of B_K. Here, if we compare values of B_K between the real world and the lattice at the same value of the cutoff, the difference is only a few percent, both quenched and unquenched. However, comparing at different values, again in order to get an idea of the effect of the $O(p^4)$ constants, indicates that also here systematic effects can be as large as 20%.

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