Hierarchical quark mass matrices

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(Received 7 April 1998; published 7 October 1998)

I define a set of conditions that the most general hierarchical Yukawa mass matrices have to satisfy so that the leading rotations in the diagonalization matrix are a pair of (2,3) and (1,2) rotations. In addition to Fritzsch structures, examples of such hierarchical structures include also matrices with (1,3) elements of the same order or even much larger than the (1,2) elements. Such matrices can be obtained in the framework of a flavor theory. To leading order, the values of the angle in the (2,3) plane (s_{23}) and the angle in the (1,2) plane (s_{12}) do not depend on the order in which they are taken when diagonalizing. We find that any of the Cabibbo-Kobayashi-Maskawa matrix parametrizations that consist of at least one (1,2) and one (2,3) rotation may be suitable. In the particular case when the s_{13} diagonalization angles are sufficiently small compared to the product $s_{12}s_{23}$, two special CKM parametrizations emerge: the $R_{12}R_{23}R_{12}$ parametrization follows with s_{23} taken before the s_{12} rotation, and vice versa for the $R_{23}R_{12}R_{23}$ parametrization. [S0556-2821(98)01119-9]

PACS number(s): 11.30.Hv, 12.15.Ff, 12.15.Hh

I. INTRODUCTION

A hierarchical structure of the Yukawa matrix is the most widely used structure. It can follow, for example, from flavor theories with either Abelian or non-Abelian symmetries. In theories with Abelian symmetries the hierarchy is obtained by assigning different charges to different families [1]. Families that have a larger charge will have a higher power of the flavor symmetry breaking parameter and thus will have a smaller Yukawa coupling. In theories with non-Abelian symmetries, the hierarchy in the couplings is a reflection of the hierarchy in symmetry breaking scales [2]. Hierarchies can also be generated radiatively where the small numbers originate in the loop factors [3]. General hierarchical structures, but only texture zeroes, have been studied before [4,5,6]. Another very popular structure, which we do not consider here, is a democratic one [7] where the elements are all of order one and sufficiently close to each other so that only one eigenvalue is large. Other structures may combine hierarchy and democracy [8,9].

In this paper I give a set of conditions that define the most general hierarchical matrix, with the condition that leading rotations in the diagonalization matrix are a pair of s_{12} and s_{23} rotations. In what follows I will assume that there are no large accidental cancellations between the up and down mixing angles, so that the hierarchies in both up and down sectors are of the same order or smaller than the corresponding observed quark masses and mixings. If the up and down quark mass matrices are hierarchical, at least one of them, if not both,[1] must fall into the above category. In addition to the well known Fritzsch structures, the hierarchy conditions permit structures which may have a large (1,3) element.

Next, I discuss possible parametrizations of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [10,11] that emerge from hierarchical mass matrices. I will use some recently obtained *exact* results about diagonalizing 3×3 matrices [12], in order to control the corrections involving small terms [for example the rotation angles in the (1,3) plane, s_{13}]. The basic result is that any CKM parametrization that has at least one (2,3) rotation and one (1,2) rotation is practical. Which one of the parametrizations should be used will at the end depend on the flavor theory, i.e. the explicit structure of the Yukawa matrices. If the theory has a prediction, e.g. if some of the diagonalizing angles can be expressed in terms of quark masses, it might be obvious in one parametrization but not in another.

A particular example is the case when the s_{13} angles are small compared to the $s_{12}s_{23}$ product. Two parametrizations emerge as winners: the "$R_{12}R_{23}R_{12}$" parametrization [proposed by Dimopoulos, Hall and Raby [13] (see also [14,15,16]); it was recently proposed as "standard" by Fritzsch and Xing [17] and the "$R_{23}R_{12}R_{23}$" parametrization (the original Kobayashi-Maskawa parametrization [11]). It will depend on the underlying flavor theory that predicted the hierarchical structures which one of these two parametrizations should be used. If one has precise predictions for the s_{12} rotations in terms of quark masses one should use the first parametrization. Conversely, if one can predict more precisely the s_{23} rotations in terms of the quark masses, one should use the second parametrization.

In the next section we review some of the notation and results about diagonalizing quark mass matrices from reference [12]. In Sec. III, we define the hierarchical structures of Yukawa matrices and list some illustrative examples. Interesting structures emerge beyond the more familiar Fritzsch type ones. Then we turn to the question of which CKM parametrization is most practical to use for the hierarchical structures. First, in Sec. IV we show that the values of s_{23} and s_{12} do not depend on the order in which the rotations are taken when diagonalizing the mass matrices. Using this result we compare various CKM parametrizations for hierarchical structures in Sec. V. We present examples of predictions with particular CKM parametrizations and conclude in Sec. VI.

[1] There may be a simpler up or down matrix, i.e. with mixings between only two generations, which is diagonalized with only one rotation.

II. DIAGONALIZING QUARK MASS MATRICES

Following the notation of [12], we denote the Yukawa matrices as

$$u^c\mathbf{h}^u Q + d^c\mathbf{h}^d Q. \tag{1}$$

Each of the matrices $\mathbf{h}^{u,d}$ is diagonalized by a biunitary transformation

$$\mathbf{m} = \mathbf{S}^\dagger\mathbf{h}\mathbf{R}. \tag{2}$$

The matrices **S** and **R** diagonalize the following products of **h**:

$$\mathbf{m}^2 = \mathbf{S}^\dagger\mathbf{h}\mathbf{h}^\dagger\mathbf{S}, \quad \mathbf{m}^2 = \mathbf{R}^\dagger\mathbf{h}^\dagger\mathbf{h}\mathbf{R}. \tag{3}$$

The CKM matrix is given by

$$\mathbf{V} = \mathbf{R}^{u\dagger}\mathbf{R}^d. \tag{4}$$

Let us neglect phases for the moment,[2] we will discuss them later in the text. Then the matrix $\mathbf{h}^\dagger\mathbf{h}$ is of the form

$$\mathbf{h}^\dagger\mathbf{h} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}, \tag{5}$$

where we assume all the elements to be non-negative (i.e. any negative signs are absorbed with the phases which are discussed later). Since this matrix is Hermitian, the eigenvalues λ_i are real and non-negative. They can be found as the solution of the cubic equation in λ:

$$\begin{aligned}\det(\mathbf{h}^\dagger\mathbf{h} - \lambda\mathbf{1}) = & -\lambda^3 + \lambda^2(\lambda_{11}+\lambda_{22}+\lambda_{33}) \\ & -\lambda(\lambda_{11}\lambda_{22}+\lambda_{11}\lambda_{33}+\lambda_{22}\lambda_{33} \\ & -\lambda_{23}^2-\lambda_{13}^2-\lambda_{12}^2)+\lambda_{11}(\lambda_{22}\lambda_{33}-\lambda_{23}^2) \\ & -\lambda_{12}(\lambda_{12}\lambda_{33}-\lambda_{13}\lambda_{23}) \\ & +\lambda_{13}(\lambda_{12}\lambda_{23}-\lambda_{13}\lambda_{22}) = 0. \end{aligned} \tag{6}$$

The diagonalizing matrix **R** is a product of three plane rotations. It is completely arbitrary which three rotations we pick; the only requirement is that two successive rotations are not in the same plane, since they can be trivially combined into one rotation.

We can write the three rotation angles in terms of the eigenvalues λ_i and matrix elements λ_{ij}. Let us show the procedure for a choice of rotations

$$\mathbf{R} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}. \tag{7}$$

From Eq. (3)

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \mathbf{R}_{12}^T\mathbf{R}_{13}^T\mathbf{R}_{23}^T \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix} \times \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}. \tag{8}$$

We can rewrite the above equation

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix} \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12} = \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{9}$$

Now, comparing the off diagonal elements (2,3), (1,3) and (1,2) on both sides we can find the rotation angles

$$\begin{aligned} \frac{s_{23}}{c_{23}} &= \frac{(\lambda_3-\lambda_{11})\lambda_{23}+\lambda_{13}\lambda_{12}}{(\lambda_3-\lambda_{22})(\lambda_3-\lambda_{11})-\lambda_{12}^2}, \\ \frac{s_{13}}{c_{13}} &= \frac{\lambda_{12}s_{23}+\lambda_{13}c_{23}}{\lambda_3-\lambda_{11}}, \\ \frac{s_{12}}{c_{12}} &= \frac{\lambda_{12}c_{23}-\lambda_{13}s_{23}}{(\lambda_2-\lambda_{11})c_{13}+(\lambda_{12}s_{23}+\lambda_{13}c_{23})s_{13}}, \end{aligned} \tag{10}$$

where $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$.

III. HIERARCHICAL STRUCTURES OF YUKAWA MATRICES

In this paper I will define a *hierarchical Yukawa matrix* as any Yukawa matrix that has a hierarchy in the elements with the following conditions:

There is a hierarchy in the eigenvalues $\lambda_1 \ll \lambda_2 \ll \lambda_3$. The cubic equation in λ with coefficients in terms of the eigenvalues reduces in the leading order to

$$\begin{aligned} & (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) \\ & \quad \approx -\lambda^3+\lambda^2\lambda_3-\lambda\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_3 = 0. \end{aligned} \tag{11}$$

The hierarchy is such that the largest element is λ_{33}. In addition the second eigenvalue is to the leading order given in terms of the closest neighbors of the largest element, that is λ_2 is given in terms of λ_{22} and λ_{23}. Comparing the cubic equations (6) and (11) we see

$$\begin{aligned} \lambda_3 &\approx \lambda_{33}, \\ \lambda_2 &\approx \lambda_{22} - \frac{\lambda_{23}^2}{\lambda_3}, \\ \lambda_1 &\approx \lambda_{11} - \frac{\lambda_{12}^2}{\lambda_2} - \frac{\lambda_{13}(\lambda_{13}\lambda_{22}-2\lambda_{12}\lambda_{23})}{\lambda_2\lambda_3}. \end{aligned} \tag{12}$$

This is achieved with the following conditions:

[2]A complete diagonalization of the general case with complex phases was given in [12].

$$(\mathbf{h1}) \quad \lambda_{33} \qquad \gg \text{all other } \lambda_{ij};$$
$$(\mathbf{h2}) \quad (\lambda_{22}-\lambda_{23}^2/\lambda_{33}) \quad \gg \lambda_{11},\lambda_{12},\lambda_{13}^2/\lambda_{33}. \tag{13}$$

The (1,3) rotation needed to diagonalize such a matrix is much smaller than the (1,2) and (2,3) rotations. This requirement follows from the observed CKM values, assuming there are no accidental cancellations between the (1,3) rotations coming from diagonalizing up and down sector. Looking at the exact results (10), for $s_{13} \ll s_{23}$ the condition is

$$(\mathbf{h3}) \quad \lambda_{23} \gg \lambda_{13}. \tag{14}$$

Finally, for $s_{13} \ll s_{12}$ we need

$$(\mathbf{h4}) \quad \lambda_{13} \lesssim \lambda_{12} s_{23} \text{ or}$$
$$(\lambda_{13} \gg \lambda_{12} s_{23} \text{ and } \lambda_{22} \lesssim \lambda_{23}^2/\lambda_{33} \text{ or}$$
$$[(\lambda_{12} \gg \lambda_{13}\lambda_{22}/\lambda_{33} \text{ and } \lambda_{12} \gg \lambda_{13}\lambda_{23}/\lambda_{33})]). \tag{15}$$

Notice that the condition (**h4**) for a sufficiently small λ_{22} does not further constrain the element λ_{13} (see examples III and IV below). The angles are now to leading approximation

$$\frac{s_{23}}{c_{23}} \approx \frac{\lambda_{23}}{\lambda_3}$$
$$\frac{s_{13}}{c_{13}} \approx \frac{\lambda_{12}s_{23}+\lambda_{13}}{\lambda_3}$$
$$\frac{s_{12}}{c_{12}} \approx \frac{\lambda_{12}-\lambda_{13}s_{23}}{\lambda_2}. \tag{16}$$

Conditions (**h1**)–(**h4**) define the hierarchical structures. Notice that these are the conditions on elements of $\mathbf{h}^\dagger\mathbf{h}$, not on $\mathbf{h}$.[3] Conditions can be worked out for the elements of $\mathbf{h}$ itself. Here we just give the mass eigenvalues $m_i = \sqrt{\lambda_i}$ in terms of h_{ij}

$$m_3 \approx h_{33},$$
$$m_2 \approx \left| h_{22} - \frac{h_{23}h_{32}}{m_3} \right|,$$
$$m_1 \approx \left| h_{11} - \frac{h_{12}h_{21}}{m_2} - \frac{h_{13}(h_{31}h_{22}-h_{21}h_{32}) - h_{31}h_{12}h_{23}}{m_2 m_3} \right|. \tag{18}$$

In what follows we will only need the actual conditions (**h1**)–(**h4**).

Let us now show five illustrative examples of hierarchical structures, which have certain relations between diagonalizing angles and quark masses. For simplicity I assume that the structures are themselves Hermitian in the first four examples so that conditions (**h1**)–(**h4**) apply to elements of **h** itself.[4] The fifth example is an asymmetric matrix.

Example I [19]:

$$\mathbf{h} = \begin{pmatrix} 0 & \lambda_{12} & 0 \\ \lambda_{12} & 0 & \lambda_{23} \\ 0 & \lambda_{23} & \lambda_{33} \end{pmatrix}, \tag{19}$$

where the hierarchy conditions (**h1**)–(**h4**) mean $\lambda_{33} \gg \lambda_{23} \gg \lambda_{23}^2/\lambda_{33} \gg \lambda_{12}$.

To leading order the eigenvalues and mixing angles are $\lambda_3 \approx \lambda_{33}$, $\lambda_2 \approx \lambda_{23}^2/\lambda_3$, $\lambda_1 \approx \lambda_{12}^2/\lambda_2$, $s_{23} \approx \lambda_{23}/\lambda_3$, $s_{13} \approx \lambda_{12}s_{23}/\lambda_3$ and $s_{12} \approx \lambda_{12}/\lambda_2$.[5] We get the predictive relations

$$s_{23} \approx \sqrt{\frac{\lambda_2}{\lambda_3}},$$
$$s_{12} \approx \sqrt{\frac{\lambda_1}{\lambda_2}},$$
$$s_{13} \approx s_{23}s_{12}\frac{\lambda_2}{\lambda_3} \approx \sqrt{\frac{\lambda_1}{\lambda_3}}\frac{\lambda_2}{\lambda_3}. \tag{20}$$

Example II:

$$\mathbf{h} = \begin{pmatrix} 0 & \lambda_{12} & 0 \\ \lambda_{12} & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{23} & \lambda_{33} \end{pmatrix}, \tag{21}$$

where in addition I will assume

$$\lambda_{22} \simeq \lambda_{23}. \tag{22}$$

Then the hierarchy conditions (**h1**)–(**h4**) give $\lambda_{33} \gg \lambda_{22}$, $\lambda_{23} \gg \lambda_{12}$.

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[3]An example of a Yukawa matrix **h** with a somewhat unusual structure, but for which $\mathbf{h}^\dagger\mathbf{h}$ still satisfies conditions (**h1**)–(**h4**) was given in [18]

$$\mathbf{h} = \begin{pmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{pmatrix}, \tag{17}$$

with $a_i \gg b_j \gg c_k$. I thank K. S. Babu for bringing this example to my attention.

[4]The only difference is that some eigenvalues may be negative, which can be simply corrected by a sign redefinition of the fields.

[5]For this case exact relations have been obtained in [20,21].

To leading order the eigenvalues and mixing angles are $\lambda_3 \approx \lambda_{33}$, $\lambda_2 \approx \lambda_{22}$, $\lambda_1 \approx \lambda_{12}^2/\lambda_2$, $s_{23} \approx \lambda_{23}/\lambda_3$, $s_{13} \approx \lambda_{12} s_{23}/\lambda_3$ and $s_{12} \approx \lambda_{12}/\lambda_2$. Predictive relations are

$$s_{12} \approx \sqrt{\frac{\lambda_1}{\lambda_2}},$$

$$s_{23} \simeq \frac{\lambda_2}{\lambda_3},$$

$$s_{13} \approx s_{23} s_{12} \frac{\lambda_2}{\lambda_3} \approx \sqrt{\frac{\lambda_1}{\lambda_2}} \left(\frac{\lambda_2}{\lambda_3}\right)^2. \tag{23}$$

The second relation follows because I assumed $\lambda_{22} \simeq \lambda_{23}$.

A note about phases: in this example all phases cannot be completely eliminated by redefinitions of the fields. One phase will be included in the diagonalization matrix. (See for example [22]. For general treatment of phases see for example [23,12].)

Example III:

$$\mathbf{h} = \begin{pmatrix} 0 & 0 & \lambda_{13} \\ 0 & 0 & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}, \tag{24}$$

Then the hierarchy conditions (**h1**)–(**h4**) imply $\lambda_{33} \gg \lambda_{23} \gg \lambda_{13}$. The structure in this example appears in [24,25,26]. A similar structure appears in [5], with a nonzero λ_{22}, but small (of the order of $\lambda_{23}^2/\lambda_{33}$), so that the results and predictions are same as here.

To leading order the eigenvalues and mixing angles are $\lambda_3 \approx \lambda_{33}$, $\lambda_2 \approx \lambda_{23}^2/\lambda_3$, $\lambda_1 \approx 0$, $s_{23} \approx \lambda_{23}/\lambda_3$, $s_{13} \approx \lambda_{13}/\lambda_3$ and $s_{12} \approx -\lambda_{13}/\lambda_{23}$. It is interesting that here even though the structure has a (1,3) element, but vanishing (1,2) element, still the (1,2) rotation is bigger than the (1,3) rotation, as in the previous examples.

Predictive relations are

$$s_{23} \approx \sqrt{\frac{\lambda_2}{\lambda_3}},$$

$$s_{13} \approx s_{23} s_{12}. \tag{25}$$

Example IV:

$$\mathbf{h} = \begin{pmatrix} 0 & \lambda_{12} & \lambda_{13} \\ \lambda_{12} & 0 & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} \end{pmatrix}, \tag{26}$$

where in addition I will assume

$$\lambda_{12} \simeq \frac{\lambda_{13}\lambda_{23}}{\lambda_{33}}, \tag{27}$$

that is λ_{12} still *smaller* then λ_{13}. The hierarchy conditions (**h1**)–(**h4**) imply $\lambda_{33} \gg \lambda_{23} \gg \lambda_{13}$ and $\lambda_{23}^2/\lambda_{33} \gg \lambda_{12}$. A similar structure appears in [24] with various relative sizes of λ_{13} and λ_{12}.

To leading order the eigenvalues and mixing angles are $\lambda_3 \approx \lambda_{33}$, $\lambda_2 \approx \lambda_{23}^2/\lambda_3$, $\lambda_1 \simeq \lambda_{12}^2/\lambda_2 \simeq \lambda_{13}^2/\lambda_3$, $s_{23} \approx \lambda_{23}/\lambda_3$, $s_{13} \approx \lambda_{13}/\lambda_3$ and $s_{12} \simeq \lambda_{12}/\lambda_2$. Still the (1,2) rotation is bigger than the (1,3) rotation, as in the previous examples.

Predictive relations are

$$s_{23} \approx \sqrt{\frac{\lambda_2}{\lambda_3}},$$

$$s_{12} \simeq \sqrt{\frac{\lambda_1}{\lambda_2}},$$

$$s_{13} \simeq s_{23} s_{12} \simeq \sqrt{\frac{\lambda_1}{\lambda_3}}. \tag{28}$$

It is interesting to note that the matrices in examples III and IV can be obtained in a U(2) flavor theory in a similar manner to Ref. [16]. Before the flavor symmetry is broken the only allowed term is λ_{33} and it is of order one. Other elements get generated from higher dimensional operators when the flavor symmetry is broken down. Which of the elements get created depends now on the flavon content on the theory. For example, one doublet can create λ_{23} when U(2) is broken first to U(1) and then λ_{13} and λ_{12} get created with another doublet and an antisymmetric singlet when U(1) is broken to nothing at a lower scale.

Example V [27]:

$$\mathbf{h} = \begin{pmatrix} 0 & c & 0 \\ c' & 0 & b \\ 0 & b' & a \end{pmatrix}, \tag{29}$$

with $a \gg b, b'$ and $(bb'/a) \gg c, c'$. This is an asymmetric matrix and we must diagonalize $\mathbf{h}^\dagger \mathbf{h}$. To leading order the eigenvalues of $\mathbf{h}$ and mixing angles are

$$m_3 = \sqrt{\lambda_3} \approx a,$$

$$m_2 = \sqrt{\lambda_2} \approx \frac{bb'}{a},$$

$$m_1 = \sqrt{\lambda_1} \approx \frac{cc'}{m_2},$$

$$s_{23} \approx \frac{b'}{a},$$

$$s_{13} \approx \frac{c'b}{a^2},$$

$$s_{12} \approx -\frac{c'}{\left(\frac{bb'}{a}\right)}, \tag{30}$$

with one relation

$$s_{12} \approx -s_{13}s_{23}\frac{m_3^2}{m_2^2}. \tag{31}$$

IV. IS IT IMPORTANT TO DO THE (2,3) ROTATION BEFORE THE (1,2) ROTATION?

As we saw in the previous section, diagonalization of hierarchical structures is done to leading order with only (2,3) and (1,2) rotations with the diagonalizing matrix (7)

$$\mathbf{R} \approx \mathbf{R}_{23}\mathbf{R}_{12}. \tag{32}$$

It is interesting, if not surprising, that diagonalization of the hierarchical structures can be done to leading order in the reverse order of rotations, that is first the (1,2) rotation, and then the (2,3) rotation. To show this let us consider the exact results for the diagonalizing unitary transformation

$$\mathbf{R} = \mathbf{R}_{12}\mathbf{R}_{13}\mathbf{R}_{23}. \tag{33}$$

For this choice of rotations one can obtain the exact results for diagonalizing angles similar to the case discussed in Sec. II,

$$\frac{s_{12}}{c_{12}} = \frac{(\lambda_{33}-\lambda_1)\lambda_{12}-\lambda_{13}\lambda_{23}}{(\lambda_{33}-\lambda_1)(\lambda_{22}-\lambda_1)-\lambda_{23}^2},$$

$$\frac{s_{13}}{c_{13}} = \frac{\lambda_{13}c_{12}-\lambda_{23}s_{12}}{\lambda_{33}-\lambda_1},$$

$$\frac{s_{23}}{c_{23}} = \frac{\lambda_{13}s_{12}+\lambda_{23}c_{12}}{(\lambda_{33}-\lambda_2)c_{13}+(\lambda_{13}c_{12}+\lambda_{23}s_{12})s_{13}}. \tag{34}$$

For hierarchical Yukawa structures, conditions (**h1**)–(**h4**) give

$$\frac{s_{12}}{c_{12}} \approx \frac{\lambda_{12}-\lambda_{13}\frac{\lambda_{23}}{\lambda_{33}}}{\lambda_{22}-\frac{\lambda_{23}^2}{\lambda_{33}}},$$

$$\frac{s_{13}}{c_{13}} \approx \frac{\lambda_{13}-\lambda_{23}s_{12}}{\lambda_{33}},$$

$$\frac{s_{23}}{c_{23}} \approx \frac{\lambda_{23}}{\lambda_3}. \tag{35}$$

Comparing with the approximate angles (16), we see that the rotation angles (1,2) and (2,3) agree to leading order. Only the small (1,3) rotation changes.

V. CKM PARAMETRIZATIONS FOR THE HIERARCHICAL STRUCTURES

The most general CKM matrix can be written as a function of three angles and one phase. Various parametrizations of CKM exist today [11,28,29,30,13,31] in which these three angles and one phase appear in various places. It was noticed some time ago [32] that there are essentially twelve different parametrizations [12,33], which correspond to various ways of combining the three rotation angles in a particular parametrization. For each of the combinations there is a continuum of possibilities, depending on positioning of the one nontrivial CP violating phase in the parametrization.

Physics of the standard model clearly does not depend on which parametrization we use. However, if one goes beyond the standard model, it might turn out more practical to use a certain parametrization. In such a parametrization a particular prediction, such as a relation between CKM elements and quark masses, may be more transparent.

As was shown in [12], it is always possible to get any of the 12 possible parametrizations of the CKM matrix from any parametrizations of the unitary matrices that diagonalize up and down quark masses. However, such procedure may be quite complicated, and, in the process, possible relations between quark masses and CKM matrix elements may be lost. Only a clever choice of a particular parametrization may reveal clearly such predictions, and we discuss which one should be used in the case of hierarchical structures.

We defined hierarchical structures in the previous sections as the ones in which the diagonalizing angles s_{23} and s_{12} are much bigger than the third angle s_{13}. In order to discuss which CKM parametrization to use, we need to know exactly how much bigger they are since we need to estimate also the smallest elements V_{ub} and V_{td}, which will involve both s_{13} and products $s_{12}s_{23}$. We now discuss separately the relative sizes of these two elements

Case I: s_{13} rotations affecting V_{ub} or V_{td}

If s_{13} is of the order of $s_{12}s_{23}$ we cannot neglect this rotation when estimating V_{ub} and V_{td}. In this case, in the most general case when both up and down quark mass matrices need to be diagonalized with three rotations each, the analysis is quite complicated and one has to resort to the exact results [12]. However, in the case of simpler structures where one of the quark mass matrices is diagonalized with only one rotation, there are preferred CKM parametrizations.

Suppose that the down quark mass matrix has only mixing between the first two families, for example of the Fritzsch-Weinberg-Wilczek-Zee type [19,34]

$$\mathbf{h}^d = \begin{pmatrix} 0 & F & 0 \\ F & E & 0 \\ 0 & 0 & D \end{pmatrix}, \tag{36}$$

so that the diagonalizing matrix is

$$\mathbf{V}^d = \mathbf{R}_{12}, \tag{37}$$

with

$$s_{12}^d \approx \sqrt{m_d/m_s}. \tag{38}$$

Then we should choose the three rotations that diagonalize the up quark matrix such that the first rotation is a s_{12}^u rotation, so that it is trivially combined with the s_{12}^d into a single

s_{12}. Of the other two rotations in the up diagonalizing matrix, one should be a s_{23} rotation (since we assume hierarchical matrices), but the choice for the last rotation and the order of rotations should be chosen only by the criteria of predictivity. Thus, for this case possible parametrizations are

$$\mathbf{V}=\mathbf{R}_{12}\mathbf{R}_{23}\mathbf{R}_{12},\quad \mathbf{R}_{13}\mathbf{R}_{23}\mathbf{R}_{12},\quad \mathbf{R}_{23}\mathbf{R}_{13}\mathbf{R}_{12}. \tag{39}$$

As an example let us assume that, in addition to the form (36) for the down quark mass matrix, the up type quark mass matrix is of the form given in example IV

$$\mathbf{h}^u=\begin{pmatrix} 0 & C' & C \\ C' & 0 & B \\ C & B & A \end{pmatrix}, \tag{40}$$

where $C'\simeq CB/A$. Here s_{13} is exactly of the order $s_{12}s_{23}$ and it needs to be included in the CKM matrix. Thus a choice for the up quark diagonalizing matrix is

$$\mathbf{R}^u=\mathbf{R}_{12}^u\mathbf{R}_{13}^u\mathbf{R}_{23}^u, \tag{41}$$

where, to leading order (see example IV),

$$s_{12}^u\simeq\sqrt{\frac{m_u}{m_c}},\quad s_{23}^u\approx\sqrt{\frac{m_c}{m_t}},\quad s_{13}^u\approx\sqrt{\frac{m_u}{m_t}}. \tag{42}$$

The CKM matrix that one obtains is

$$\mathbf{V}=R^{u\dagger}R^d=\mathbf{R}_{23}^{uT}\mathbf{R}_{13}^{uT}\mathbf{R}_{12}, \tag{43}$$

where $s_{12}=s_{12}^d-s_{12}^u$. This is the ''standard CKM parametrization'' of Chau, Keung and Maiani [30,28]. With the above predictions (38) and (42) the CKM elements are successfully reproduced.[6] A clear prediction here is

$$\left|\frac{V_{ub}}{V_{cb}}\right|\approx\frac{s_{13}^u}{s_{23}^u}\simeq\sqrt{\frac{m_u}{m_c}}. \tag{44}$$

Case II: s_{13} rotations too small to affect V_{ub} or V_{td} to leading order

In order for the angles s_{13}^u and s_{13}^d not to contribute to V_{ub} and V_{cb} to leading order, the following conditions must be satisfied [14]:

$$|s_{13}^d-s_{13}^u|\ll s_{12}^u|s_{23}^d-s_{23}^u|;|s_{13}^d-s_{13}^u|\ll s_{12}^d|s_{23}^d-s_{23}^u|. \tag{45}$$

The above relations are the exact conditions on λ_{13}^u and λ_{13}^d, but are a bit complicated to write down explicitly in terms of λ_{ij}. As a guideline, a somewhat less restricting, but more understandable condition $s_{13}\ll s_{12}s_{23}$ is obtained when

$$\lambda_{13}\ll\lambda_{12}s_{23}, \tag{46}$$

in both up and down sectors.

Thus in this case we need to consider only the s_{23} and s_{12} rotations. As was shown in Sec. IV, the values of these two angles do not depend on the order in which they are taken when diagonalizing a quark mass matrix. However, depending on the order two simple CKM parametrizations emerge, and we discuss them next in detail. If one diagonalizes up and down quark matrices with first (2,3) rotations and then (1,2) rotations, the CKM matrix is

$$\mathbf{V}=R^{u\dagger}R^d\approx R_{12}^{uT}R_{23}R_{12}^d, \tag{47}$$

where $\theta_{23}=\theta_{23}^d-\theta_{23}^u$. This is a parametrization of the CKM in terms of three rotation angles. What is nice is that these angles are directly related to angles of the original diagonalizing matrices. Since one can write an *exact* parametrization in terms of three rotation angles, the exact angles will differ from the above angles only by small corrections that are subleading to the (1,2) and (2,3) rotations that we used.

If one allows Yukawa matrices to be complex, one can show [12] that it amounts to putting one complex phase δ in the CKM (47) between the rotations. Also, the (2,3) rotation will now in general be only the absolute value of the sum of the (2,3) rotations with a relative complex phase

$$\theta_{23}=|\theta_{23}^d-e^{i\alpha}\theta_{23}^u| \tag{48}$$

so that the relation of that CKM angle with the original diagonalizing (2,3) angles gets blurred. However, to leading order, the (1,2) CKM angles *are* the (1,2) angles that diagonalize the up and down sector. For completeness, let us list a complex CKM generalization of Eq. (47):[7]

$$\begin{aligned}\mathbf{V}&=\mathrm{diag}(e^{-i\delta}11)\mathbf{R}_{12}^{Tu}\ \mathrm{diag}(e^{i\delta}11)\mathbf{R}_{23}\mathbf{R}_{12}^d\\ &=\begin{pmatrix} c_{12}^uc_{12}^d+s_{12}^ds_{12}^uc_{23}e^{-i\delta} & c_{12}^us_{12}^d-s_{12}^uc_{23}c_{12}^de^{-i\delta} & -s_{12}^us_{23}e^{-i\delta} \\ s_{12}^uc_{12}^de^{i\delta}-s_{12}^dc_{12}^uc_{23} & +s_{12}^us_{12}^de^{i\delta}+c_{12}^uc_{12}^dc_{23} & c_{12}^us_{23} \\ s_{12}^ds_{23} & -c_{12}^ds_{23} & c_{23}\end{pmatrix}.\end{aligned} \tag{49}$$

This parametrization appears in [13,14,15,16,17]. One can immediately write the following relations:

[6] I am interested here only in approximate relations. These relations are successful within a factor of 2 or 3, which can easily be accommodated in the original Yukawa matrices by numerical factors of order one.

[7] For any product of three angles there is a continuum of possibilities for placement of the phase δ.

$$\left|\frac{V_{ub}}{V_{cb}}\right|=\frac{s^u_{12}}{c^u_{12}};\quad \left|\frac{V_{td}}{V_{ts}}\right|=\frac{s^d_{12}}{c^d_{12}}. \tag{50}$$

Similarly, if one diagonalizes up and down quark matrices with first (1,2) rotations and then (2,3) rotations, the CKM matrix is

$$\mathbf{V}=R^{u\dagger}R^d\approx R^{uT}_{23}R_{12}R^d_{23}, \tag{51}$$

where $\theta_{12}=\theta^d_{12}-\theta^u_{12}$, so that the parametrization is again given in terms of three angles. The complex generalization can be written as

$$\mathbf{V}=\mathrm{diag}(11e^{-i\delta})\mathbf{R}^{uT}_{23}\ \mathrm{diag}(11e^{i\delta})\mathbf{R}_{12}\mathbf{R}^d_{23}$$
$$=\begin{pmatrix} c_{12} & s_{12}c^d_{23} & s_{12}s^d_{23} \\ -s_{12}c^u_{23} & c_{12}c^u_{23}c^d_{23}+s^u_{23}s^d_{23}e^{i\delta} & c_{12}c^u_{23}s^d_{23}-s^u_{23}c^d_{23}e^{i\delta} \\ -s_{12}s^u_{23}e^{-i\delta} & c_{12}c^d_{23}s^u_{23}e^{-i\delta}-s^d_{23}c^u_{23} & c_{12}s^d_{23}s^u_{23}e^{-i\delta}+c^u_{23}c^d_{23}\end{pmatrix}. \tag{52}$$

It is the parametrization originally proposed by Kobayashi and Maskawa [11]. It has the following predictions:

$$\left|\frac{V_{td}}{V_{cd}}\right|=\frac{s^u_{23}}{c^u_{23}};\quad \left|\frac{V_{ub}}{V_{us}}\right|=\frac{s^d_{23}}{c^d_{23}}. \tag{53}$$

A final note: the parametrization (52) was discarded in [33] on two grounds which I now want to argue to be unnecessary. Let us first assume there is no CP violating phase. The first requirement that there be only one $\mathbf{R}_{23}$ rotation in a particular parametrization because there should be just one angle when the first generation decouples is too strong, since in the parametrization (52) when the $\mathbf{R}_{12}$ approaches unity, the two $\mathbf{R}_{23}$ rotations trivially combine into a single one. Even with the CP violating phase between two $\mathbf{R}_{23}$ rotations, it is easy to write it as one $\mathbf{R}_{23}$ rotation between two phase transformations (see for example Ref. [12]). The second requirement that the CP violating phase should disappear when the first generation masses disappear is also too strong. What one should ask from a parametrization is that the phase disappears when the mixing between a certain generation with the other two generations disappears. From this standpoint both parametrizations (49) and (52) are acceptable: in parametrization (49) the phase disappears when the first generation does not mix with the rest $(\mathbf{R}_{12}=\mathbf{1})$, and similarly in parametrization (52) when the mixings with the third generation vanish $(\mathbf{R}_{23}=\mathbf{1})$.

VI. DISCUSSION AND CONCLUSIONS

In conclusion, I have defined general hierarchical structures of the Yukawa matrices with the four conditions (**h1**)–(**h4**). I defined them as structures with hierarchy in Yukawa elements and their eigenvalues, and by demanding that the (1,3) rotation be much smaller than the (1,2) and (2,3) rotations. Then such structures can be diagonalized with the same (1,2) and (2,3) rotations in any order.

In addition to the more familiar Fritzsch structures, examples of such structures include matrices which have non-negligible (1,3) elements. They can be used to describe quark masses and mixings and examples of such matrices were given in Sec. III (examples III and IV.). It is interesting to note that it is possible to build such matrices in a U(2) flavor theory similar to [16], with an additional doublet flavon [35].

We studied the diagonalization of hierarchical quark mass matrices and the CKM parametrizations that naturally follow from such matrices. When the s_{13} contributions to V_{ub} and V_{td} cannot be neglected, any CKM parametrization with at least one (2,3) and one (1,2) rotation may appear useful, that is it will depend on the underlying flavor theory which of the CKM parametrizations will be most transparent to predictions of the theory.

If, further, the (1,3) rotations can be neglected in the CKM, two possible CKM parametrizations that simply relate the CKM elements to the diagonalizing angles appear, namely $\mathbf{R}_{12}\mathbf{R}_{23}\mathbf{R}_{12}$ and $\mathbf{R}_{23}\mathbf{R}_{12}\mathbf{R}_{23}$. Which one of the two parametrizations should one use? This will depend on the underlying flavor theory. If the theory has nice predictions for the (1,2) rotations in terms of quark masses one should use the parametrization (47). Examples of this type of theories include generalized Fritzsch structures (examples I and II in Sec. III) where $s^d_{12}\approx\sqrt{m_d/m_s}$ and $s^u_{12}\approx\sqrt{m_u/m_c}$. In this case, we get the clear predictions [36,37,13,14]

$$\left|\frac{V_{ub}}{V_{cb}}\right|\approx\sqrt{\frac{m_u}{m_c}};\quad \left|\frac{V_{td}}{V_{ts}}\right|\approx\sqrt{\frac{m_d}{m_s}}. \tag{54}$$

However, one can also have flavor theories of the second and third generation masses where, with first generation masses being small and their estimation not so reliable. Then the second case might be more applicable if there are clear predictions of the (2,3) rotations in terms of quark masses. For example if the up quark matrix is of the type shown in example I, and down quark matrix of type II we get relations [16,38] $s_{23}^d \simeq m_s/m_b$ and $s_{23}^u \approx \sqrt{m_c/m_t}$. The predictions are then visible in case II where

$$\left|\frac{V_{td}}{V_{cd}}\right| \approx \sqrt{\frac{m_c}{m_t}}; \quad \left|\frac{V_{ub}}{V_{us}}\right| \simeq \frac{m_s}{m_b}. \tag{55}$$

Of course, physics of the standard model does not depend on which CKM parametrization one uses. However, if there is a flavor theory, as we strongly believe, it will hopefully reduce the number of parameters and produce some predictions. Then, it is important to have clear and simple formulas for the predictions, and this depends on which parametrizations one uses. Which one of the two parametrizations one should use depends on the underlying flavor theory, i.e. on which diagonalizing angles one can relate to the quark masses.

ACKNOWLEDGMENTS

I would like to thank K. S. Babu and Atsushi Yamada for valuable comments, as well as Lawrence Hall and the LBNL theory group for hospitality during my stay in Berkeley. Part of this work was done during the CPMASS97 School and Workshop in Lisbon and the ICTP Extended Workshop on Highlights in Astroparticle Physics. This work is partially supported by EEC grant under the TMR contract ERBFMRX-CT960090.

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