

QCD sum rules at finite temperature

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We derive thermal QCD sum rules for the correlation function of two vector currents in the rho-meson channel. It takes into account the leading non-perturbative corrections from the additional operators, which appear due to the breakdown of Lorentz invariance at finite temperature. The mixing of the new operators has a drastic effect on their coefficients. The thermal average of all the operators can be related to that of the quark condensate and the energy density. The sum rules then yield the temperature dependence of the parameters of the ρ meson, namely its mass and coupling to the vector current. Our result is that these parameters are practically independent of temperature at least up to a temperature of 125 MeV. [S0556-2821(98)04921-2]

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I. INTRODUCTION

The QCD sum rules [1], proposed about two decades ago, prove remarkably successful in addressing the non-perturbative problems of hadron phenomenology. In this approach one considers the product of two local operators, such as the currents of the QCD theory. A sum rule is obtained by equating the dispersion relation for its vacuum matrix element at sufficiently large space-like momenta to the corresponding Wilson operator product expansion [2]. The higher dimension operators present in the expansion give rise to the non-perturbative, power corrections. The idea of extending these sum rules to finite temperature by replacing the vacuum matrix element with the thermal average naturally suggests itself.

The original work establishing the thermal QCD sum rules is that of Bochkarev and Shaposhnikov [3]. They recognized the importance of the (low energy) continuum states in the spectral function to account for the effects of the medium. On the basis of these sum rules they discussed the temperature dependence of the resonance parameters and the existence of phase transition. However, there arise additional operators in the Wilson expansion at finite temperature [4], which were not correctly incorporated in their sum rules: In effect, they calculated these new operator contributions perturbatively, which cannot be justified, particularly at low temperature.

The additional operators arise because of the breakdown of Lorentz invariance at finite temperature by the choice of the thermal rest frame, where matter is at rest at a definite temperature [5]. The residual $O(3)$ invariance naturally brings in additional operators. The expected behavior of the thermal averages of these Lorentz non-invariant (new) operators is somewhat opposite to those of the Lorentz invariant (old) ones: While the old operators start with non-zero values at zero temperature and decrease in magnitude with the rise of temperature, the new ones, on the other hand, are zero at zero temperature but grow rapidly with temperature. The importance of including these new operators in the thermal sum rules, particularly at not too low a temperature, is thus clear.

Although a number of works on thermal QCD sum rules exist by now, these are flawed with respect to the new operators: Either some of these are missed [6] or their mixing,

which changes their coefficients drastically, is not taken into account [7]. In a recent work [8] we applied a simple, configuration space method [9,10] to evaluate the Wilson coefficients of these new operators (up to dimension four) which appear at finite temperature in the short distance expansion of the product of two quark bilinear operators. Here we make use of this result to write and evaluate the thermal QCD sum rules, incorporating correctly the contributions from all the dimension four operators.

We consider the correlation function of the time ordered (T) product of two vector currents in the ρ -meson channel. The use of the T -product, rather than the retarded (or advanced) product, is a little complicated in writing down the spectral representation but has the advantage in perturbative calculations, for which we can apply the conventional formalism. Throughout this work we shall employ the real time formulation of the thermal field theory [11], which requires in general not only the physical fields but also the accompanying ‘‘ghost’’ fields. Since, however, we work to lowest order in perturbation expansion, ghost fields do not show up.

It is convenient to write difference sum rules by subtracting the vacuum sum rules from their finite temperature counterparts. For such sum rules the absorptive parts are expected to be saturated with the ρ -meson pole and the $\pi\pi$ -continuum [3], whose contributions we derive here for completeness. The thermal averages of the different operators reduce essentially to that of the quark condensate and the energy densities of quarks and gluons [12]. Chiral perturbation theory [13] has been used to calculate the temperature dependence of the quark condensate [14]. The difficulty with the energy densities is that while one of them is the total energy density, which can be obtained from a knowledge of the hadronic spectrum at low temperature, the other is an unphysical combination of the two densities. We need an additional input to relate this latter combination to the total energy density.

Once the power corrections are known, the difference sum rules give the temperature dependence of the ρ -meson parameters, namely its mass and its coupling with the vector current. With our simple saturation scheme, the sum rules can be used up to a temperature of about 125 MeV. The numerical evaluation shows that these parameters have negligible dependence on temperature.

In Sec. II we write the kinematic decomposition for the

thermal correlation function of two vector currents, derive the Landau representation [15] for the time ordered product and state the results of operator product expansion to derive finally the form of the thermal QCD sum rules. In Sec. III we obtain the contributions of ρ and $\pi\pi$ intermediate states to the spectral function. In Sec. IV we collect the information on thermal average of the operators present in the sum rules and evaluate the difference sum rules. In Sec. V we discuss how to extend the sum rules to higher temperature. In the Appendix we give the details of the evaluation of a limit stated earlier [3].

II. SUM RULES

We derive the QCD sum rules for the thermal average of the time ordered (T) product of two currents:

$$\mathcal{T}_{\mu\nu}^{ab}(q) = i \int d^4x e^{iq \cdot x} \langle T(V_\mu^a(x) V_\nu^b(0)) \rangle. \quad (2.1)$$

Here $V_\mu^a(x)$ is the vector current (in the ρ meson channel) in QCD theory:

$$V_\mu^a(x) = \bar{q}(x) \gamma_\mu(\tau^a/2) q(x), \quad (2.2)$$

$q(x)$ being the field of the u and d quark doublet and τ^a the Pauli matrices. The thermal average of an operator O is denoted by $\langle O \rangle$:

$$\langle O \rangle = \text{Tr} e^{-\beta H} O / \text{Tr} e^{-\beta H},$$

where H is the QCD Hamiltonian, β is the inverse of the temperature T (coinciding with the time ordering symbol) and Tr denotes the trace over any complete set of states.

A. Kinematics

At finite temperature Lorentz invariance is broken by the choice of a preferred frame of reference, viz., the matter rest frame where temperature is defined. But the bookkeeping with indices becomes simpler if we restore it formally by introducing the four-velocity u_μ of the matter [5]. [In the matter rest frame $u_\mu = (1, 0, 0, 0)$.] The time and space components of q_μ are then raised to the Lorentz scalars, $\omega = u \cdot q$ and $\bar{q} = \sqrt{\omega^2 - q^2}$. We shall, however, return to the matter rest frame while doing all actual computations.

In such a Lorentz invariant framework, there are two independent conserved kinematic covariants [16], which we choose as

$$P_{\mu\nu} = -\eta_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} - \frac{q^2}{q^2} \tilde{u}_\mu \tilde{u}_\nu, \quad Q_{\mu\nu} = \frac{q^4}{q^2} \tilde{u}_\mu \tilde{u}_\nu,$$

where $\tilde{u}_\mu = u_\mu - \omega q_\mu / q^2$. The invariant decomposition of $\mathcal{T}_{\mu\nu}^{ab}$ is then given by

$$\mathcal{T}_{\mu\nu}^{ab}(q) = \delta^{ab} (Q_{\mu\nu} T_l + P_{\mu\nu} T_t), \quad (2.3)$$

where the invariant amplitudes $T_{l,t}$ are functions of q^2 and ω .

Notice that the kinematic covariants $P_{\mu\nu}$ and $Q_{\mu\nu}$ are free from singularities at $q^2=0$ (and also at $\bar{q}^2=0$). This is convenient at finite temperature as there are dynamical singularities extending up to $q^2=0$. With this choice of kinematic covariants, the dynamical singularities reside only in the invariant amplitudes.

To extract the invariant amplitudes from $\mathcal{T}_{\mu\nu}^{ab}$, it is convenient to form the scalars $\Pi_{1,2}$:

$$\delta^{ab} \Pi_1(q) = \mathcal{T}_\mu^{\mu ab}(q), \quad \delta^{ab} \Pi_2(q) = u^\mu \mathcal{T}_{\mu\nu}^{ab}(q) u^\nu.$$

Then the invariant amplitudes are given by

$$T_l = \frac{1}{q^2} \Pi_2, \quad T_t = -\frac{1}{2} \left(\Pi_1 + \frac{q^2}{q^2} \Pi_2 \right). \quad (2.4)$$

In the limit $\bar{q}=0$, the amplitudes T_l and T_t are related. To see this we write the spatial components of $\mathcal{T}_{\mu\nu}$ as

$$\mathcal{T}_{ij}^{ab}(q) = \delta^{ab} [\delta_{ij} T_t - \hat{q}_i \hat{q}_j (T_t - q_0^2 T_l)],$$

where \hat{q}_i is the i th component of the unit vector along \vec{q} . As $|\vec{q}| \rightarrow 0$, there cannot be any dependence on $|\vec{q}|$, getting

$$T_t(q_0, |\vec{q}|=0) = q_0^2 T_l(q_0, |\vec{q}|=0). \quad (2.5)$$

B. Spectral representation

Let us obtain the spectral representation for the correlation function in q_0 at fixed $|\vec{q}|$. First, evaluate the trace over a complete set of eigenstates of four-momentum, when it becomes a sum over forward amplitudes weighted by the corresponding Boltzmann factors. Then insert the same set of complete states between the currents to extract its x -dependence which is then integrated out. Introducing a δ -function in q_0 , the result can be written as

$$\mathcal{T}_{\mu\nu}^{ab}(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dq'_0 \left(\frac{M_{\mu\nu}^{ab}(q'_0, |\vec{q}|)}{q'_0 - q_0 - i\epsilon} - \frac{M_{\nu\mu}^{ba}(q'_0, |\vec{q}|)}{q'_0 - q_0 + i\epsilon} \right), \quad (2.6)$$

where the expression for $M_{\mu\nu}^{ab}$ with the double sum over states may be converted back to the form

$$M_{\mu\nu}^{ab}(q) = \int d^4x e^{iqx} \langle V_\mu^a(x) V_\nu^b(0) \rangle. \quad (2.7)$$

Using the double sum representation, it is easy to show that

$$M_{\nu\mu}^{ba}(q_0, |\vec{q}|) = e^{-\beta q_0} M_{\mu\nu}^{ab}(q_0, |\vec{q}|). \quad (2.8)$$

The opposite sign of $i\epsilon$ in the two terms in Eq. (2.6) is typical of T -products. As a result the imaginary part of $\mathcal{T}_{\mu\nu}^{ab}$ is given by the sum, $\frac{1}{2}(M_{\mu\nu}^{ab} + M_{\nu\mu}^{ba})$, while the principal value integral contains the difference, $\frac{1}{2}(M_{\mu\nu}^{ab} - M_{\nu\mu}^{ba})$. But the two are related by (2.8):

$$M_{\mu\nu}^{ab} - M_{\nu\mu}^{ba} = (M_{\mu\nu}^{ab} + M_{\nu\mu}^{ba}) \tanh(\beta q_0/2). \quad (2.9)$$

We thus get the Landau representation for the T product at finite temperature [15],

$$\begin{aligned} \mathcal{T}_{\mu\nu}^{ab}(q_0, |\vec{q}|) &= i \operatorname{Im} \mathcal{T}_{\mu\nu}^{ab}(q_0, |\vec{q}|) \\ &+ P \int_{-\infty}^{+\infty} dq'_0 \frac{N_{\mu\nu}(q'_0, |\vec{q}|)}{q'_0 - q_0}, \end{aligned} \quad (2.10)$$

where, for brevity, we write

$$N_{\mu\nu}^{ab}(q) \equiv \delta^{ab} N_{\mu\nu}(q) = \pi^{-1} \operatorname{Im} \mathcal{T}_{\mu\nu}^{ab}(q) \tanh(\beta q_0/2). \quad (2.11)$$

Expressing $q'_\mu = q_\mu + (q'_0 - q_0)u_\mu$, we recover the representation for the invariant amplitudes $T_{l,t}$. Further, using the symmetry properties, $\operatorname{Im} T_{l,t}(-q_\mu) = \operatorname{Im} T_{l,t}(q_\mu)$ and going over to imaginary values of q_0 ($q_0^2 = -Q_0^2$, $Q_0^2 > 0$), for which $N_{\mu\nu}$ vanishes, these become

$$T_{l,t}(q_0^2, |\vec{q}|) = \int_0^\infty dq'_0 \frac{N_{l,t}(q'_0, |\vec{q}|)}{q'_0{}^2 + Q_0^2}. \quad (2.12)$$

It may actually require subtractions, but it does not affect the Borel transformed sum rules we shall write below.

C. Operator product expansion

The contributions of operators of dimension four to $T_{l,t}$ are obtained from the short distance expansion and improved upon by the renormalization group equation in a previous work [8]. Including the unit operator (the perturbative contribution), T_l is given for large Euclidean momenta by ($Q^2 = -q^2$)

$$\begin{aligned} T_l &= -\frac{1}{8\pi^2} \ln\left(\frac{Q^2}{\mu^2}\right) \\ &+ \frac{1}{Q^4} \left(\hat{m} \langle \bar{u}u \rangle + \frac{\langle G^2 \rangle}{24} + \frac{4}{16+3n_f} \right. \\ &\left. \times \{ \langle \Theta \rangle + a(Q^2) \langle 16\Theta^f/3 - \Theta^g \rangle \} \right). \end{aligned} \quad (2.13)$$

T_t is also given by the same expression, except for an overall factor of $-Q^2$ and a factor of $(1 - 2\vec{q}^2/Q^2)$ multiplying the term with Θ 's. Here \hat{m} is the degenerate mass of the u and the d quarks and $\frac{1}{2} \langle \bar{q}q \rangle = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$. The operator G^2 is quadratic in the gauge field strength $G_{\mu\nu}^a$, ($a=1, \dots, 8$), $G^2 = (\alpha_s/\pi) G_{\mu\nu}^a G^{a\mu\nu}$, with $\alpha_s = g^2/4\pi$, where g is the gauge coupling constant. Along with these two operators already contributing to zero temperature, we now have the linear combinations of two new ones, $\Theta^{f,g} \equiv u^\mu \Theta_{\mu\nu}^{f,g} u^\nu$, where $\Theta_{\mu\nu}^{f,g}$ are the energy-momentum tensors for quarks and gluons. The corresponding components for the total tensor is $\Theta = n_f \Theta^f + \Theta^g$, where n_f is the number of effective quark flavors. The logarithmic Q^2 dependence of $a(Q^2)$ arise due to the anomalous dimension d of the operator $(16\Theta^f/3 - \Theta^g)$,

$$a(-q^2) = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(-q^2)} \right)^{-d/2b},$$

$$d = \frac{4}{3} \left(\frac{16}{3} + n_f \right), \quad b = 11 - \frac{2}{3} n_f,$$

where μ ($\simeq 1$ GeV) is the scale at which all renormalizations are carried out.

Note the mixing of the operators Θ^f and Θ^g in Eq. (2.13) under the renormalization group, which is well-known in the context of deep inelastic scattering [17]. In the operator product expansion of two quark currents, the Wilson coefficients of Θ^f and Θ^g are, to leading order, of zeroth and first order in α_s , arising from Born and one-loop graphs respectively. However, due to operator mixing, the coefficients are drastically changed in that both Θ and $(16\Theta^f/3 - \Theta^g)$ have coefficients with leading terms of zeroth order in α_s . In Eq. (2.13) we retain only these leading terms.

D. Sum rules

We now equate the spectral representation and the result of operator product expansion for the amplitudes T_l and T_t at sufficiently high Q_0^2 . Taking Borel transform we arrive at the thermal QCD sum rules [3]. For T_l we get

$$\frac{1}{M^2} \int_0^\infty dq_0^2 e^{-q_0^2/M^2} N_l(q_0, |\vec{q}|) = e^{-|\vec{q}|^2/M^2} \left(\frac{1}{8\pi^2} + \frac{\langle O \rangle}{M^4} \right), \quad (2.14)$$

where $\langle O \rangle$ is the non-perturbative contribution of higher dimension operators:

$$\begin{aligned} \langle O \rangle &= \hat{m} \langle \bar{u}u \rangle + \frac{\langle G^2 \rangle}{24} \\ &+ \frac{4}{16+3n_f} \{ \langle \Theta \rangle + a(M^2) \langle 16\Theta^f/3 - \Theta^g \rangle \}, \end{aligned} \quad (2.15)$$

and a similar one for T_t . Each is a two parameter sum rule, dependent not only on T but also on $|\vec{q}|$.

In the thermal rest frame the thermal average of the new operators are energy densities, which increase rapidly with temperature. Earlier works on thermal QCD sum rules were flawed, as these contributions were not properly included.

III. ABSORPTIVE PARTS

We work below the critical temperature, where hadrons constitute the physical spectrum. As with the vacuum sum rules, the dominant contribution to the spectral function is given by the ρ -meson. We also calculate the contribution of the non-resonant $\pi\pi$ -continuum. The question of other significant contributions will be discussed in Sec. IV.

A. ρ -pole

The coupling of the vector current to the ρ -meson is given by

$$\langle 0 | V_\mu^a | \rho^b \rangle = \delta^{ab} F_\rho m_\rho \epsilon_\mu, \quad (3.1)$$

where ϵ_μ is the polarization vector of ρ of mass m_ρ . The experimental value of F_ρ as measured by the electronic decay rate of ρ^0 [18] is $F_\rho = 153.5$ MeV.

A simple way to calculate the absorptive part of the ρ -pole diagram is to note the field-current identity of $V_\mu^a(x)$ with the rho-meson field ρ_μ^a ,

$$V_\mu^a(x) = m_\rho F_\rho \rho_\mu^a(x), \quad (3.2)$$

which reproduces Eq. (3.1). Then the ρ -meson contribution is given essentially by its thermal propagator

$$\begin{aligned} T_{\mu\nu}^{ab}(q) &= i m_\rho^2 F_\rho^2 \int d^4x e^{iqx} \langle \rho_\mu^a(x) \rho_\nu^b(0) \rangle \\ &= i \delta^{ab} m_\rho^2 F_\rho^2 \left(-\eta_{\mu\nu} + \frac{q_\mu q_\nu}{m_\rho^2} \right) \Delta_{11}^\rho(q), \end{aligned} \quad (3.3)$$

where $\Delta_{11}^\rho(q)$ is the 11-component of a scalar field propagator with mass m_ρ :

$$\Delta_{11}^\rho(q) = \frac{i}{q^2 - m_\rho^2 + i\epsilon} + 2\pi n(\omega_q) \delta(q^2 - m_\rho^2), \quad (3.4)$$

and $n(\omega_q)$ is the Bose distribution function, $n(\omega_q) = (e^{\beta\omega_q} - 1)^{-1}$, $\omega_q = \sqrt{q^2 + m_\rho^2}$. We then get

$$\begin{pmatrix} N_l \\ N_t \end{pmatrix} = \begin{pmatrix} 1 \\ m_\rho^2 \end{pmatrix} F_\rho^2 \delta\{q_0^2 - (|\vec{q}|^2 + m_\rho^2)\}. \quad (3.5)$$

Loop corrections at finite temperature will make m_ρ and F_ρ temperature dependent, modifying them, in general, differently in the longitudinal and transverse amplitudes. These modifications may be obtained by calculating the appropriate loop graphs. Here we shall find them by evaluating our sum rules.

B. $\pi\pi$ -continuum

The $\pi\pi$ -contribution to the amplitudes describes the interaction of the current with the particles in the medium, which are predominantly pions. Chiral perturbation theory [13] gives the contribution of the pion field $\phi^a(x)$ to the vector current, which, to lowest order, is

$$V_\mu^a(x) = \epsilon^{abc} \phi^b(x) \partial_\mu \phi^c(x).$$

Then the pions contribute to the correlation function as

$$\begin{aligned} T_{\mu\nu}^{ab}(q) &= i \delta^{ab} \int \frac{d^4k}{(2\pi)^4} (2k-q)_\mu (2k-q)_\nu \\ &\quad \times \Delta_{11}^\pi(k) \Delta_{11}^\pi(k-q), \end{aligned} \quad (3.6)$$

where Δ_{11}^π is again the 11-component of the scalar propagator (3.4) but with mass m_π . Its imaginary part can be obtained by the cutting rules at finite temperature [19]. Here we obtain it directly for this simple amplitude. We express Δ_{11}^π as

$$\Delta_{11}^\pi(k) = \{1 + n(\omega_k)\} D(k) + n(\omega_k) D^*(k), \quad (3.7)$$

where $D(k)$ is the zero temperature propagator, $D(k) = i/(k^2 - m_\pi^2 + i\epsilon)$, and carry out the k_0 -integration [20]. The imaginary part may then be read off as

$$\text{Im } T_{\mu\nu}^{ab}(q) = \delta^{ab} \{L_{\mu\nu}(q) + L_{\mu\nu}(-q)\}, \quad (3.8)$$

where

$$\begin{aligned} L_{\mu\nu}(q) &= \pi \int \frac{d^3k}{(2\pi)^3} \frac{(2k-q)_\mu (2k-q)_\nu}{4\omega_1\omega_2} [\{(1+n_1)(1+n_2) \\ &\quad + n_1 n_2\} \delta(q_0 - \omega_1 - \omega_2) + \{(1+n_1)n_2 \\ &\quad + (1+n_2)n_1\} \delta(q_0 - \omega_1 + \omega_2)]. \end{aligned} \quad (3.9)$$

Here $n_1 \equiv n(\omega_1)$, $n_2 \equiv n(\omega_2)$ with $\omega_1 = \sqrt{k^2 + m_\pi^2}$, $\omega_2 = \sqrt{(\vec{k} - \vec{q})^2 + m_\pi^2}$. The time component of k_μ in the tensor structure is understood to be given by $k_0 = \omega_1$.

With the help of the δ -functions we can rewrite Eq. (3.9) to get $N_{\mu\nu}$ defined by Eq. (2.11) for $q_0 > 0$,

$$\begin{aligned} N_{\mu\nu}(q) &= \int \frac{d^3k}{(2\pi)^3} \frac{(2k-q)_\mu (2k-q)_\nu}{4\omega_1\omega_2} \\ &\quad \times [(1+n_1+n_2) \delta(q_0 - \omega_1 - \omega_2) \\ &\quad + (n_2 - n_1) \delta(q_0 - \omega_1 + \omega_2)]. \end{aligned} \quad (3.10)$$

In this form the factors involving the density distributions can be interpreted in terms of pion absorption from and emission into the medium [21]. Since the first and the second δ -functions in Eq. (3.10) contribute to time-like ($q^2 \geq 4m_\pi^2$) and space-like ($q^2 < 0$) regions respectively, we write it as

$$\begin{aligned} N_{\mu\nu}(q) &= \int \frac{d^3k}{(2\pi)^3 2\omega_1} (2k-q)_\mu (2k-q)_\nu \\ &\quad \times [(1+n_1+n_2) \theta(q^2 - 4m_\pi^2) \\ &\quad + (n_2 - n_1) \theta(-q^2)] \delta((q_0 - \omega_1)^2 - \omega_2^2). \end{aligned} \quad (3.11)$$

Thus in addition to the usual cut, $4m_\pi^2 + |\vec{q}|^2 \leq q_0^2 \leq \infty$, the amplitude at finite temperature acquires a new short cut, $0 \leq q_0^2 \leq |\vec{q}|^2$ [21,3].

The angular integration is carried out using the δ -function, when the constraint $|\cos \theta_{q,k}| \leq 1$ results in a θ -function, $\theta[-q^2(\omega_1 - \omega_+)(\omega_1 - \omega_-)]$, where

$$\omega_\pm = \frac{1}{2}(q_0 \pm |\vec{q}|v), \quad v(q^2) = \sqrt{1 - 4m_\pi^2/q^2}. \quad (3.12)$$

We thus get

$$N_{\mu\nu}(q) = \frac{1}{2|\vec{q}|} \int_{\omega_-}^{\omega_+} \frac{d\omega_1}{(2\pi)^2} (2k-q)_\mu (2k-q)_\nu (1+n_1+n_2) \theta(q^2-4m_\pi^2) + \frac{1}{2|\vec{q}|} \times \int_{\omega_+}^{\infty} \frac{d\omega_1}{(2\pi)^2} (2k-q)_\mu (2k-q)_\nu (n_2-n_1) \theta(-q^2). \quad (3.13)$$

It is now simple to extract the absorptive parts of the invariant amplitudes by using (2.4). Changing the variable ω_1 to x given by $\omega_1 = \frac{1}{2}(q_0 + |\vec{q}|x)$, we get

$$\begin{pmatrix} N_l^+ \\ N_l^- \end{pmatrix} = \frac{v^3}{48\pi^2} \begin{pmatrix} 1 \\ q^2 \end{pmatrix} + \begin{pmatrix} \bar{N}_l^+ \\ \bar{N}_l^- \end{pmatrix}, \quad \text{for } q^2 > 4m_\pi^2 \quad (3.14)$$

with

$$\begin{pmatrix} \bar{N}_l^+ \\ \bar{N}_l^- \end{pmatrix} = \frac{1}{32\pi^2} \int_{-v}^v dx \begin{pmatrix} 2x^2 \\ q^2(v^2-x^2) \end{pmatrix} n((|\vec{q}|x+q_0)/2) \quad (3.15)$$

and

$$\begin{pmatrix} N_l^- \\ N_l^+ \end{pmatrix} = \frac{1}{64\pi^2} \int_v^\infty dx \begin{pmatrix} 2x^2 \\ q^2(v^2-x^2) \end{pmatrix} \{n((|\vec{q}|x-q_0)/2) - n((|\vec{q}|x+q_0)/2)\}, \quad \text{for } q^2 \leq 0. \quad (3.16)$$

The superscripts (\pm) on N denote time-like and space-like q_μ respectively, where they are non-vanishing. The first term on the right of Eq. (3.14) arising from the unity in the factor $(1+n_1+n_2)$ in Eq. (3.13) is the zero temperature contribution of the $\pi\pi$ state. Evaluated here in a non-covariant way it, of course, agrees with the covariant evaluation of the Feynman amplitude (3.6) with $\Delta_{11}^\pi(k)$ replaced by the vacuum propagator $D(k)$ [13].

C. Explicit sum rules

Let us now write explicitly the sum rule (2.14) for T_l . Saturating N_l with the above contributions it becomes

$$F_\rho^2(T) e^{-m_\rho^2(T)/M^2} + \frac{1}{48\pi^2} \int_{4m_\pi^2}^\infty dq_0^2 e^{-q_0^2/M^2} v^3 (q_0^2 + |\vec{q}|^2) + e^{|\vec{q}|^2/M^2} \left(\int_{4m_\pi^2 + |\vec{q}|^2}^\infty dq_0^2 e^{-q_0^2/M^2} \bar{N}_l^+(q_0, |\vec{q}|) + \int_0^{|\vec{q}|^2} dq_0^2 e^{-q_0^2/M^2} N_l^-(q_0, |\vec{q}|) \right) = \frac{M^2}{8\pi^2} + \frac{\langle O \rangle}{M^2}. \quad (3.17)$$

As the temperature goes to zero, the two terms in brackets go to zero and the thermal average of the operators on the right become the vacuum expectation values, recovering the familiar vacuum sum rule. The integral on the left is the non-resonant 2π contribution and is small compared to the resonance contribution.

As $|\vec{q}| \rightarrow 0$, the sum rule (3.17) simplifies considerably. The limit for the second integral in bracket is given in Ref. [3]. Here we derive it in the Appendix. The sum rules for T_l and T_l then become

$$F_\rho^2(T) e^{-m_\rho^2(T)/M^2} + I_0(M^2) + I_T(M^2) = \frac{M^2}{8\pi^2} + \frac{\langle O \rangle}{M^2}, \quad (3.18)$$

and

$$m_\rho^2(T) F_\rho^2(T) e^{-m_\rho^2(T)/M^2} + J_0(M^2) + J_T(M^2) = \frac{M^4}{8\pi^2} - \langle O \rangle, \quad (3.19)$$

where

$$I_0(M^2) = \frac{1}{48\pi^2} \int_{4m_\pi^2}^\infty ds e^{-s/M^2} v^3,$$

$$J_0(M^2) = \frac{1}{48\pi^2} \int_{4m_\pi^2}^\infty ds s e^{-s/M^2} v^3,$$

$$I_T(M^2) = \frac{1}{24\pi^2} \int_{4m_\pi^2}^\infty ds \{e^{-s/M^2} v^3 + v(3-v^2)/2\} n(\sqrt{s}/2),$$

$$J_T(M^2) = \frac{1}{24\pi^2} \int_{4m_\pi^2}^{\infty} ds s e^{-s/M^2} v^3 n(\sqrt{s}/2), \quad (3.20)$$

with $v \equiv v(s) = \sqrt{1 - 4m_\pi^2/s}$. Observe that the sum rules (3.18), (3.19) are not independent, in agreement with the relation (2.5): the second one is obtained by differentiating the first with respect to $1/M^2$.

IV. EVALUATION OF SUM RULES

In the above sum rules we have approximated the absorptive parts of the amplitudes by the ρ -meson pole and the 2π continuum, while we retain only the contributions of the unit operator (the perturbative result) and of all the operators of dimension four in the operator product expansion. To check this saturation scheme, let us compare the zero temperature limit of our sum rules with the corresponding vacuum sum rules [1]. The latter include, in addition, the rather large contribution from the high energy continuum beyond 1.5 GeV, as indicated by the experimental data, as well as the contribution of a quark operator of dimension six, which is also large because of its origin in Born (rather than loop) diagrams. Since we do not include any of these contributions, we cannot expect the sum rules as written above to be well saturated.

Rather than incorporate these contributions, we isolate the thermal effects by considering the difference sum rules, obtained by subtracting out the vacuum sum rules from the corresponding finite temperature ones. Then the contribution to the absorptive parts beyond 1.5 GeV, being temperature independent, cancels out in the difference. Thus it is the temperature dependent contributions of the ρ -meson and of the 2π continuum, which should dominate the absorptive parts of these sum rules. Also the dimension six quark operator, O_6 say, contributes an amount $\sim \langle O_6 \rangle - \langle O_6 | 0 \rangle$, which is insignificant in the immediate neighborhood of $T=0$ and as the temperature rises, this contribution is overwhelmed by that of the two-gluon and other Lorentz non-invariant operators, as our estimates below for these operators show.

The difference sum rules for the two invariant amplitudes allow us to calculate the temperature dependence of the ρ -meson parameters,

$$\begin{aligned} \Delta m_\rho(T) &\equiv m_\rho(T) - m_\rho \\ &= \frac{m_\rho e^{m_\rho^2/M^2}}{2F_\rho} \left\{ I_T - \frac{J_T}{m_\rho^2} - \left(\frac{1}{m_\rho^2} + \frac{1}{M^2} \right) \overline{\langle O \rangle} \right\}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \Delta F_\rho(T) &\equiv F_\rho(T) - F_\rho \\ &= -\frac{e^{m_\rho^2/M^2}}{2F_\rho} \left\{ \frac{J_T}{M^2} + \left(1 - \frac{m_\rho^2}{M^2} \right) I_T + \frac{m_\rho^2}{M^4} \overline{\langle O \rangle} \right\}, \end{aligned} \quad (4.2)$$

with

$$\begin{aligned} \overline{\langle O \rangle} &= \hat{m} \overline{\langle \bar{u}u \rangle} + \frac{\overline{\langle G^2 \rangle}}{24} \\ &+ \frac{4}{16+3n_f} \{ \langle \Theta \rangle + a(M^2) \langle 16\Theta^f/3 - \Theta^g \rangle \}, \end{aligned} \quad (4.3)$$

where the bar on the operators indicates subtraction of their vacuum expectation values. Here we insert the experimental values for m_ρ and F_ρ , as these are well reproduced by the vacuum sum rules.

We now collect information on the operator contributions. The vacuum expectation value of the chiral condensate $\langle 0 | \bar{u}u | 0 \rangle$ is known from the partially conserved axial-vector current (PCAC) relation of Gell-Mann, Oakes and Renner:

$$2\hat{m} \langle 0 | \bar{u}u | 0 \rangle = -F_\pi^2 m_\pi^2,$$

where $\hat{m} = 1/2(m_u + m_d)$ and the pion decay constant F_π , defined by

$$\langle 0 | A_\mu^a | \pi^b(q) \rangle = i q_\mu \delta^{ab} F_\pi,$$

has the value $F_\pi = 93.3$ MeV. Taking $\hat{m} = 7$ MeV [22], we get $\langle 0 | \bar{u}u | 0 \rangle = -(225 \text{ MeV})^3$. The vacuum expectation value of the two-gluon operator, as determined from the QCD sum rules [1], is $\langle 0 | G^2 | 0 \rangle = (330 \text{ MeV})^4$.

The operator G^2 is related to the trace of the energy momentum tensor $\Theta_{\mu\nu}$ by the trace anomaly. Normalizing it to zero vacuum expectation value and taking the thermal average, it gives

$$\begin{aligned} \overline{\langle G^2 \rangle} &\equiv \langle G^2 \rangle - \langle 0 | G^2 | 0 \rangle \\ &= -\frac{8}{9} \left(\langle \Theta_\mu^\mu \rangle - \sum_q m_q \overline{\langle \bar{u}u \rangle} \right). \end{aligned} \quad (4.4)$$

The trace at finite temperature is given by $\langle \Theta_\mu^\mu \rangle = \langle \Theta \rangle - 3P$ where $\langle \Theta \rangle$ is the energy density and P the pressure.

The temperature dependence of both $\overline{\langle \bar{u}u \rangle}$ and $\langle \Theta \rangle$ have been calculated in chiral perturbation theory [14]. Corrections due to nonzero quark masses as well as the contributions of the massive states (K, η, ρ, \dots) have also been incorporated. However, as the authors point out, the validity of the perturbation theory and the use of dilute gas approximation to calculate the contribution of the massive states restrict these results to within a temperature of about 150 MeV. Thus the critical temperature T_c is, strictly speaking, beyond the range of validity of their calculation. Since, however, the order parameter $\overline{\langle \bar{u}u \rangle}$ falls rapidly at the upper end of this range, one has only to extrapolate it a little further to get $T_c = 190$ MeV.

Besides the total energy density $\langle \Theta \rangle$, there also occurs the thermal average, $\langle 16\Theta^f/3 - \Theta^g \rangle$. The last one cannot be calculated without further input, at least in the hadronic phase. Now both naive counting of the degrees of freedom and empirical study of the pion structure function [4] suggest the quark fraction of the energy density to be about half of the

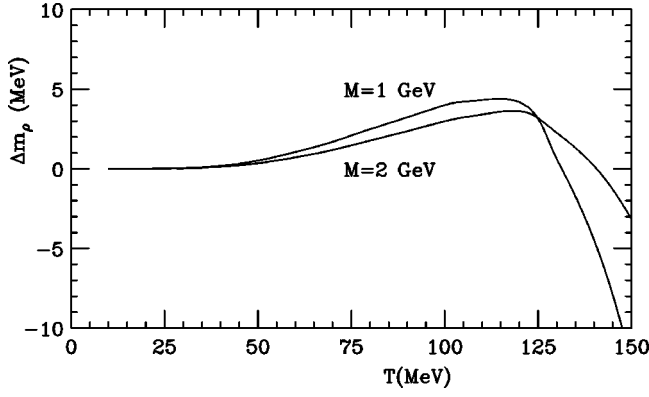


FIG. 1. Shift in the rho-meson mass as a function of temperature for $M^2=1 \text{ GeV}^2$ and $M^2=4 \text{ GeV}^2$.

total. So we assume $n_f \langle \Theta^f \rangle = \chi_f \langle \Theta \rangle$, with $\chi_f = .5$, where $\langle 16\Theta^f/3 - \Theta^g \rangle = \{(16/3n_f + 1)\chi_f - 1\} \langle \Theta \rangle$.

As with the zero temperature sum rules, the results are expected to be stable in a region in M^2 , which is neither too high to make the continuum contribution large relative to the resonance contribution nor too low to emphasize the neglected power corrections of higher order. Since the high energy continuum contribution gets cancelled in the difference sum rules, the region of M^2 may be extended somewhat at the upper end. Figs. 1 and 2 show the evaluation for M^2 equal to 1 GeV^2 and 4 GeV^2 . The results for Δm_ρ and ΔF_ρ are rather stable for temperatures up to about 125 MeV. At higher temperatures the results, particularly for Δm_ρ appear unstable. Closer observation reveals here a large cancellation between the 2π contribution and the leading power correction we have retained. Thus the non-leading power correction become important here, whose inclusion may restore the stability in M^2 to higher temperatures.

V. DISCUSSIONS

In this work we have written the thermal QCD sum rules for the two point correlation function of vector currents in the ρ -channel, including the leading power correction due to all the operators of dimension four. Because of the loss of Lorentz invariance at finite temperature, two new (Lorentz noninvariant) operators creep up in the Wilson expansion, in addition to the two old (Lorentz invariant) ones, already existing in the vacuum sum rules. Thus compared to the two numbers, $\langle 0|\bar{u}u|0\rangle$ and $\langle 0|G^2|0\rangle$ in the vacuum sum rules, we have now four temperature dependent quantities, $\langle \bar{u}u \rangle$, $\langle G^2 \rangle$, $\langle \Theta \rangle$ and $\langle 16\Theta^f/3 - \Theta^g \rangle$ in the thermal sum rules. All these thermal averages can be evaluated from chiral perturbation theory, supplemented by contributions from the massive states. An extra input is needed only for the last unphysical operator. The sum rules can then be used to determine the temperature dependence of the ρ -meson parameters. Our evaluation shows that the mass of the ρ -meson and its coupling with the vector current remain practically unaffected by the rise of temperature up to at least 125 MeV. The absence of the shift in the mass appears to agree with one set of results obtained in Ref. [23].

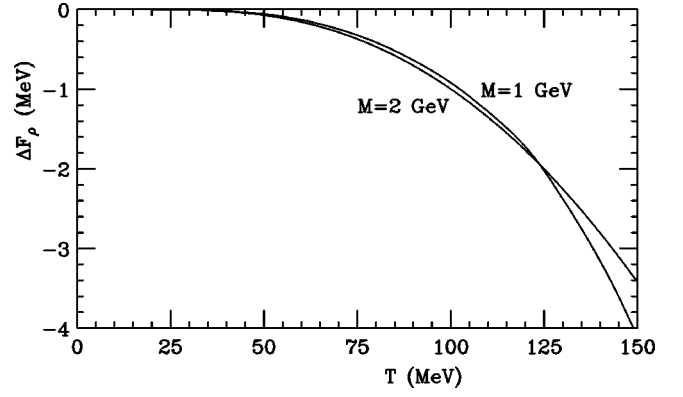


FIG. 2. Shift in the coupling of rho-meson with the vector current as a function of temperature for $M^2=1 \text{ GeV}^2$ and $M^2=4 \text{ GeV}^2$.

Unfortunately the sum rules, as they stand, cannot be extended up to the critical temperature. One reason contributing to this restriction has to do with the evaluation of the thermal average of the operators, as we already discussed in the last section. What restricts it further is their instability under a change of M^2 for temperatures above 125 MeV. Even in this restricted temperature range, the numerical evaluation shows that the new operators are significant. In fact, for temperatures above say 70 MeV, the new contributions overwhelm the old ones in the difference sum rules. This situation necessitates a reanalysis of all earlier results based on thermal QCD sum rules, where the new operators are not properly taken into account.

We now discuss a possible way to extend the sum rules to higher temperatures. In the vacuum sum rules the operators of dimension four (and higher) provide corrections to the leading (perturbative) result. But in the difference sum rules it is these corrections which become the leading contribution. One thus expects that by including nonleading contributions from higher dimension operators along with those of dimension four already included, the sum rules would be stable against variations in M^2 over a wider range of temperature. This inclusion is all the more necessary for sum rules like the one for Δm_ρ , where the leading operator contribution cancels largely with that of $\pi\pi$ -continuum.

The higher dimension operators will, of course, complicate the evaluation of the sum rules in that we have to know the temperature dependence of their thermal averages. Also there is more proliferation of operators than what is generally thought. The procedure in the literature [6] of including dimension six quark operators and excluding the Lorentz noninvariant gluon operators [24], because of the smallness of their coefficients by a factor of $\alpha_s(M^2)$, is not justified. For, as we have seen, the quark and the gluon operators mix under a change of scale, so that after a renormalization group improvement, both the coefficients are of the same order in $\alpha_s(M^2)$.

A way to proceed here is to write the entire set of sum rules by considering two-point functions of not only the vector quark bilinear but also the others, like the scalar, tensor, etc. All of these sum rules receive contributions from a few resonances and the operators from the same set. The simul-

taneous evaluation of all these sum rules is expected to provide a self-consistent check on the thermal average of the operators. Further, using quark bilinears of appropriate chiralities, one can get sum rules without any of the gluon operators [25]. These sum rules should prove easier to evaluate and would also check the saturation scheme in a simpler context.

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APPENDIX

Here we calculate the limit of the second integral in brackets in (3.17):

$$A = \frac{1}{32\pi^2} \int_0^{|\vec{q}|^2} dq_0^2 e^{-(q_0^2 - |\vec{q}|^2)/M^2} \int_v^\infty dx \times x^2 \{n((|\vec{q}|x - q_0)/2) - n((|\vec{q}|x + q_0)/2)\} \quad (\text{A1})$$

as $|\vec{q}| \rightarrow 0$. It is convenient to change the integration variables q_0 and x to λ and u respectively by $q_0 = \lambda|\vec{q}|$ and $|\vec{q}|x = u$, getting

$$A = \frac{1}{32\pi^2} \int_0^1 d\lambda^2 e^{-(\lambda^2 - 1)|\vec{q}|^2/M^2} \int_l^\infty du \times u^2 \frac{1}{|\vec{q}|} \{n((|\vec{q}|x - q_0)/2) - n((|\vec{q}|x + q_0)/2)\},$$

$$l = \sqrt{|\vec{q}|^2 + \frac{4m_\pi^2}{1 - \lambda^2}}.$$

As $|\vec{q}| \rightarrow 0$, it becomes

$$A \rightarrow -\frac{1}{16\pi^2} \int_0^1 d\lambda^2 \lambda \int_{2m_\pi/\sqrt{1-\lambda^2}}^\infty du u^2 \frac{dn(u/2)}{du} = \frac{1}{48\pi^2} \int_{4m_\pi^2}^\infty ds v (3 - v^2) n(\sqrt{s}/2). \quad (\text{A2})$$

To get the second line we integrate by parts over u and interchange the order of integration in the remaining double integral.

The limit of the corresponding integral for T_i is zero, because of the presence of q^2 in the integrand.

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