

## Long range forces from pseudoscalar exchange

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(Received 2 June 1998; published 29 September 1998)

Using dispersion theoretic techniques, we consider coherent long range forces arising from double pseudoscalar exchange among fermions. We find that Yukawa type coupling leads to  $1/r^3$  spin-independent attractive potentials whereas derivative coupling renders  $1/r^5$  spin-independent repulsive potentials.  
[S0556-2821(98)03721-7]

PACS number(s): 11.80.Fv, 11.55.Fv

### I. INTRODUCTION

Many extensions of the standard model predict the existence of light scalar particles. The axion may be the most debated one but there are also approximate Nambu-Goldstone fields associated with family symmetries, moduli fields, dilatons, or superpartners of the gravitino [1–4]. Exchange of such particles by ordinary matter will induce forces that extend over the Compton wavelength of the particle [2,5,6]. However, the effect will be felt by bulk matter only if the potential is spin independent so that forces can add up coherently over macroscopic distances. Now, a pseudoscalar particle, such as the axion, is coupled to fermions via a  $\gamma_5$  which, in the nonrelativistic limit, flips the spin. Therefore, single pseudoscalar exchange leads to spin-dependent forces that do not extend over macroscopic unpolarized bodies [3,5]. A double exchange of pseudoscalars on the other hand can coherently sum over a macroscopic sample of matter because it can leave the spin unflipped. The explicit form of these forces has been derived and their phenomenological consequences explored in previous work in the context of nonrelativistic “old fashioned perturbation theory” [7]. Here we reopen the question of pseudoscalar mediated forces in the light of the powerful dispersion theoretical techniques devised by Feinberg, Sucher and Au that make extensive use of full relativistic quantum field theory [8,9].

In Sec. II we give the necessary theoretical background which is nothing but a brief summary of the seminal work by Feinberg and Sucher. Section III is devoted to the Yukawa type interaction and Sec. IV deals with derivatively coupled scalars. We shall see that the two interactions produce quite different potentials [10], a fact that could not have been derived in a purely non-relativistic framework [11]. The paper ends with a brief summary and conclusions contained in Sec. V.

### II. DISPERSIVE FORCES

Following the general strategy devised by Feinberg, Sucher and Au [8,9], we define a potential in a given quantum field theory by equating the scattering amplitude for a two body process that follows from the usual Feynman rules, with the transition amplitude associated to a Schrödinger type equation solved in the manner of Lippmann and Schwinger. Let us be explicit and consider elastic scattering

of particles A and B with four-momenta  $p_a$  and  $p_b$  in the initial state and  $p'_a$  and  $p'_b$  in the final state. The Mandelstam variables are then

$$s = (p_a + p_b)^2, \quad t = Q^2, \quad u = (p_a - p'_b)^2 \quad (1)$$

with  $Q = p_a - p'_a = -p_b + p'_b$ .

In the center of momentum (c.m.) we write the momenta as

$$\begin{aligned} p_a &= (E_a, \mathbf{p}), & p_b &= (E_b, -\mathbf{p}) \\ p'_a &= (E_a, \mathbf{p}'), & p'_b &= (E_b, -\mathbf{p}'). \end{aligned} \quad (2)$$

Now  $s = W^2$  where  $W = E_a + E_b$  and  $t = -Q^2$  with  $Q = (0, \mathbf{Q})$ , whose physical region is

$$s \geq s_0 \quad \text{and} \quad -4\mathbf{p}^2 \leq t \leq 0 \quad (3)$$

where

$$\begin{aligned} s_0 &= (m_a + m_b)^2 \\ \mathbf{p}^2 &= [s - (m_a + m_b)^2][s - (m_a - m_b)^2]/4s. \end{aligned} \quad (4)$$

The transition from initial state  $i$  to final state  $f$  is described in quantum field theory by the transition matrix element

$$T_{fi} = N_f \mathcal{M}_{fi} N_i \quad (5)$$

where  $N_{f,i}$  are normalization factors of one particle states and  $\mathcal{M}_{fi}$  is the invariant Feynman amplitude.

The definition of our potential follows now from identifying this transition amplitude with

$$T_{fi} = \langle \mathbf{p}', -\mathbf{p}' | V + V(W - h_0 - V + i\epsilon)^{-1} V | \mathbf{p}, -\mathbf{p} \rangle \frac{m_a m_b}{E_a E_b} \quad (6)$$

where  $h_0$  is the sum of the free Dirac Hamiltonians for particles A and B. Here all quantities are referred to the c.m.

The Feynman amplitude  $\mathcal{M}$  is understood as a series expansion in (even) powers of the coupling constant associated to single, double, . . . particle exchange in the t-channel. We assume that the potential  $V$  also admits a series expansion

$$V = V^{(2)} + V^{(4)} + \dots \quad (7)$$

So we determine  $V$  order by order in perturbation theory through

$$\langle \mathbf{p}', -\mathbf{p}' | V^{(2)} | \mathbf{p}, -\mathbf{p} \rangle = \mathcal{M}_{fi}^{(2)} \quad (8)$$

$$\langle \mathbf{p}', -\mathbf{p}' | V^{(4)} | \mathbf{p}, -\mathbf{p} \rangle = \mathcal{M}_{fi}^{(4)} - \langle \mathbf{p}', -\mathbf{p}' | V^{(2)} (W - h_0 + i\epsilon)^{-1} V^{(2)} | \mathbf{p}, -\mathbf{p} \rangle. \quad (9)$$

Here we should point out a technicality. The potential we are after must have the form [12]

$$V = \Lambda_{++} U \Lambda_{++} \quad (10)$$

where  $\Lambda_{++} = \Lambda_{+;a} \Lambda_{+;b}$  is an operator that projects on the positive energy states of  $h_0$ . Because  $\Lambda_{++} | \mathbf{p}, -\mathbf{p} \rangle = | \mathbf{p}, -\mathbf{p} \rangle$  and  $\Lambda_{++}^2 = \Lambda_{++}$ , we can rewrite the previous equations that determine the potential as

$$\langle \mathbf{p}', -\mathbf{p}' | U^{(2)} | \mathbf{p}, -\mathbf{p} \rangle = \mathcal{M}_{fi}^{(2)} \quad (11)$$

$$\langle \mathbf{p}', -\mathbf{p}' | U^{(4)} | \mathbf{p}, -\mathbf{p} \rangle = \mathcal{M}_{fi}^{(4)} - \langle \mathbf{p}', -\mathbf{p}' | U^{(2)} (W - h_0 + i\epsilon)^{-1} \Lambda_{++} U^{(2)} | \mathbf{p}, -\mathbf{p} \rangle. \quad (12)$$

⋮

In principle the above equations permit an iterative determination of the potential to the desired order. However, we are not done yet because we would like to have our potential in position space and what we have is the operator  $U$  in the momentum representation. Therefore, we should Fourier transform our results back to configuration space; i.e., we wish to find  $U^{(n)}(r)$  such that

$$\langle \mathbf{p}', -\mathbf{p}' | U^{(n)} | \mathbf{p}, -\mathbf{p} \rangle = \int d\mathbf{r} e^{i\mathbf{Q}\cdot\mathbf{r}} U^{(n)}(r). \quad (13)$$

Inversion of the above equation, however, requires knowing the function  $\mathcal{M}$  for all values of the three-momentum. But we only know the scattering amplitude on shell, i.e. for  $\mathbf{p}^2 = \mathbf{p}'^2$ . We can use the fact that  $\mathcal{M}(s, t)$  is an analytic function of  $t$  and so analytically extend its domain beyond the physical region, i.e. for all values of  $\mathbf{Q}^2 = t$ .

Suppose  $\mathcal{M}(s, t)$  is analytic everywhere except for branch cuts on the real axis and, furthermore, that it vanishes for large  $|t|$ . Then, using Cauchy's theorem, we can write

$$\mathcal{M}^{(R)} = \frac{1}{\pi} \int_{t_0}^{\infty} dt' \frac{\rho^{(R)}(s, t')}{t' - t} \quad (14)$$

$$\mathcal{M}^{(L)} = \frac{1}{\pi} \int_{-\infty}^{\bar{t}_0} dt' \frac{\rho^{(L)}(s, t')}{t' - t} \quad (15)$$

where  $\rho(s, t) = [\mathcal{M}]_t / 2i$  is the spectral density and  $[\mathcal{M}]_t$  is the discontinuity of  $\mathcal{M}$  across the cut. Only the piece of the amplitude arising from the right hand cut will be of interest to us for only this piece leads to a long range potential [13].

Assuming that the basic relations (11) and (12) hold also in the extended domain, we can Fourier invert them as follows<sup>1</sup>:

$$U^{(n)}(r; s) = \frac{1}{(2\pi)^3} \int d\mathbf{Q} e^{-\mathbf{Q}\cdot\mathbf{r}} \mathcal{M}^{(n)}(s, -\mathbf{Q}^2). \quad (16)$$

We use now the spectral representation given before to obtain

$$U^{(n)}(r; s) = \frac{1}{4\pi^2 r} \int_{t_0}^{\infty} dt \rho^{(n)}(s, t) e^{-\sqrt{t}r} \quad (17)$$

where, to reach this final form, we conveniently changed the order in which integrals were done. In short, obtaining long range potentials amounts to calculating t-channel discontinuities in Feynman diagrams and performing a Laplace transform. We shall see how things work out in detail as we do our specific calculations in the next two sections.

### III. YUKAWA COUPLING

Our starting point is the Lagrangian density

$$\mathcal{L}_{int}^Y = -ig \bar{\Psi}(x) \gamma^5 \Psi(x) \Phi(x) \quad (18)$$

where  $\psi$  is a fermion field and  $\phi$  is the pseudoscalar field which we take to be massless.<sup>2</sup> The potential associated to single particle exchange is easily obtained from the discontinuity associated with the diagram in Fig. 1. The spectral density function  $\rho^{(2)}(s, t)$  is, in this case,

$$\rho^{(2)}(s, t) = \pi g^2 \bar{u}(p'_a) \gamma^5 u(p_a) \bar{u}(p'_b) \gamma^5 u(p_b) \delta(t). \quad (19)$$

After Laplace transforming we get the relativistic potential operator

<sup>1</sup>Notice that our generalized potential will depend on the parameter  $s$ .

<sup>2</sup>For scalars with mass, the long distance potentials are damped with Yukawa exponentials. In this case, our results are valid for distances on the order or smaller than the Compton wavelength of the exchanged particles.

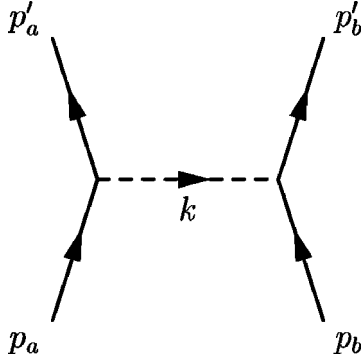


FIG. 1. Lowest order scattering amplitude. Single pseudo-scalar exchange.

$$U^{(2)} = \frac{g^2}{4\pi r} \gamma_a^0 \gamma_a^5 \gamma_b^0 \gamma_b^5 \quad (20)$$

where subindices make explicit that Dirac matrices act either on spinor A or spinor B.

The nonrelativistic limit of equation above leads to the well-known spin-dependent potential

$$V_{nr}^{(2)} = \frac{g^2}{4\pi r (2m_a)(2m_b)} \boldsymbol{\sigma}_a \cdot \nabla \otimes \boldsymbol{\sigma}_b \cdot \nabla \quad (21)$$

with  $m_a$  and  $m_b$  the masses of particles A and B respectively. What we are really interested in is  $U^{(4)}$ , i.e. the potential due to two-particle exchange. To this end we need the discontinuities of diagrams in Fig. 2 and the discontinuity of the subtraction term in Eq. (12) (iteration of the lowest order potential  $U^{(2)}$ ).

The Feynman amplitude associated to Fig. 2 can be written

$$\begin{aligned} \mathcal{M}^{(4)} &= \frac{i}{2!(2\pi)^4} \\ &\times \int d^4k d^4k' \delta^{(4)}(Q - k - k') \frac{1}{k^2 + i\epsilon} \frac{1}{k'^2 + i\epsilon} \\ &\times \mathcal{M}_a^C(-k, k'; P_a) \mathcal{M}_b^C(k, -k'; P_b) \end{aligned} \quad (22)$$

in terms of the Compton amplitude, depicted in Fig. 3:

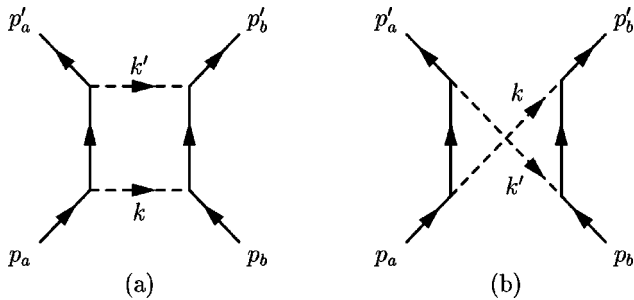


FIG. 2. Diagrams contributing to the  $\mathcal{O}(g^4)$  terms of the potential.

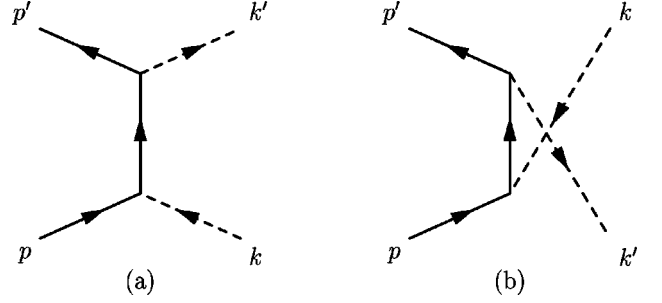


FIG. 3. Compton scattering amplitude diagrams.

$$\mathcal{M}^C(k, k'; P) = g^2 \bar{u}(p') \left[ \frac{\not{k}}{2p \cdot k} + \frac{\not{k}'}{2p \cdot k'} \right] u(p). \quad (23)$$

Making use of the Dirac equation and trading the propagators for their discontinuities, i.e.

$$\frac{1}{k^2 + i\epsilon} \rightarrow -2\pi i \delta(k^2) \Theta(k^0), \quad (24)$$

we arrive at

$$\begin{aligned} [\mathcal{M}^{(4)}]_i &= -\frac{ig^4}{8\pi^2} \int d\Phi \bar{u}'_a \left[ \frac{p_a \cdot (k' - k) \not{k}}{2p_a \cdot k' p_a \cdot k} \right] \\ &\times u_a \bar{u}'_b \left[ \frac{p_b \cdot (k' - k) \not{k}}{2p_b \cdot k' p_b \cdot k} \right] u_b \end{aligned} \quad (25)$$

with the two particle phase space explicitly given by

$$\begin{aligned} d\Phi &= \delta(Q - k - k') \delta(k^2) \delta(k'^2) \Theta(k^0) \\ &\times \Theta(k'^0) d^4k d^4k'. \end{aligned} \quad (26)$$

It is convenient to do the integrals in the c.m. of the pseudo-scalars, i.e. to go to the t-channel, and then use crossing symmetry to recover the original amplitude. We follow here the notation in Ref. [14] where they deal with a related problem. First define momenta as

$$\begin{aligned} p_a &= \left( \frac{\sqrt{t}}{2}, \mathbf{p} \right), & p_{\bar{a}} &= -p'_a = \left( \frac{\sqrt{t}}{2}, -\mathbf{p} \right) \\ p_{\bar{b}} &= -p_b = \left( \frac{\sqrt{t}}{2}, -\mathbf{p}' \right), & p'_b &= \left( \frac{\sqrt{t}}{2}, \mathbf{p}' \right) \\ k &= \left( \frac{\sqrt{t}}{2}, \mathbf{k} \right), & k' &= \left( \frac{\sqrt{t}}{2}, -\mathbf{k} \right) \end{aligned} \quad (27)$$

and introduce next the unit imaginary vectors

$$\mathbf{p} = i \xi_a m_a \hat{\mathbf{p}} \quad (28)$$

$$\mathbf{p}' = i \xi_b m_b \hat{\mathbf{p}}' \quad (29)$$

with  $\xi_{a,b} \equiv \sqrt{1 - t/4m_{a,b}^2}$  and  $\hat{\mathbf{p}}, \hat{\mathbf{p}}'$  are unitary complex vectors verifying  $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}' = -1$  so that all particles are on-shell.

Now the discontinuity can be put in the form

$$[\mathcal{M}^{(4)}]_t = \frac{ig^4}{4\pi bt} \int \frac{d\Omega}{4\pi} \frac{x_a x_b}{d_a d_b} \bar{u}'_a \mathbf{k} u_a \bar{u}'_b \mathbf{k} u_b \quad (30)$$

where, to simplify expressions, we use

$$\begin{aligned} b &\equiv m_a \xi_a m_b \xi_b \\ x_a &\equiv \hat{\mathbf{p}} \cdot \hat{\mathbf{k}} \quad x_b \equiv \hat{\mathbf{p}}' \cdot \hat{\mathbf{k}} \\ d_{a,b} &\equiv \tau_{a,b}^2 + x_{a,b}^2 \\ \tau_{a,b} &\equiv \frac{\sqrt{t}}{2\xi_{a,b} m_{a,b}}. \end{aligned} \quad (31)$$

The integration to be carried out is an angular average. We use the shorthand  $\int (d\Omega/4\pi) f \equiv \langle f \rangle$ . Hence, the discontinuity takes the form

$$[\mathcal{M}^{(4)}]_t = \frac{ig^4}{4\pi bt} \bar{u}'_a \gamma_\mu u_a \bar{u}'_b \gamma_\nu u_b T^{\mu\nu} \quad (32)$$

with  $T^{\mu\nu} \equiv \langle (x_a x_b / d_a d_b) k^\mu k^\nu \rangle$ .

Lorentz covariance dictates the following decomposition:

$$\begin{aligned} T^{\mu\nu} &= a_1 P_a^\mu P_a^\nu + a_2 P_b^\mu P_b^\nu + a_3 (P_a^\mu P_b^\nu + P_b^\mu P_a^\nu) + a_4 g^{\mu\nu} \\ &+ a_5 Q^\mu Q^\nu + a_6 (Q^\mu P_a^\nu + P_a^\mu Q^\nu) \\ &+ a_7 (Q^\mu P_b^\nu + P_b^\mu Q^\nu) \end{aligned} \quad (33)$$

in terms of the three independent momenta

$$P_a \equiv p_a + p'_a, \quad P_b \equiv p_b + p'_b, \quad Q \equiv k + k'. \quad (34)$$

The coefficients  $a_i$  can be found to be combinations of scalar integrals as shown in the Appendix. Now, in the c.m. of the incident particles, the relations

$$\begin{aligned} \bar{u}'_a \mathbf{P}_a u_a \bar{u}'_b \mathbf{P}_a u_b &= 4m_a \bar{u}'_a u_a \bar{u}'_b (W\gamma_0 - m_b) u_b \\ \bar{u}'_a \mathbf{P}_b u_a \bar{u}'_b \mathbf{P}_b u_b &= 4m_b \bar{u}'_a (W\gamma_0 - m_a) u_a \bar{u}'_b u_b \\ \bar{u}'_a \mathbf{P}_a u_a \bar{u}'_b \mathbf{P}_b u_b &= 4m_a m_b \bar{u}'_a u_a \bar{u}'_b u_b \\ \bar{u}'_a \mathbf{P}_b u_a \bar{u}'_b \mathbf{P}_a u_b &= 4\bar{u}'_a (W\gamma_0 - m_a) u_a \bar{u}'_b (W\gamma_0 - m_b) u_b \\ \bar{u}'_a \mathbf{Q} u_a &= 0 \\ \bar{u}'_b \mathbf{Q} u_b &= 0 \end{aligned} \quad (35)$$

are easily established with the help of the Dirac equation. This leads directly to

$$\begin{aligned} [\mathcal{M}^{(4)}]_t &= \frac{ig^4}{4\pi bt} \bar{u}'_a \bar{u}'_b [4m_a m_b (2a_3 - a_1 - a_2) \\ &+ 4m_b W \gamma_a^0 (a_2 - a_3) + 4m_a W \gamma_b^0 (a_1 - a_3) \\ &+ \gamma_a^0 \gamma_b^0 (4W^2 a_3 + a_4) - \boldsymbol{\gamma}_a \boldsymbol{\gamma}_b a_4] u_a u_b. \end{aligned} \quad (36)$$

This discontinuity is a complex function since the  $a_i$  are complex and hence adds an imaginary component to the spectral density which would finally contribute an imaginary piece to the potential. Inspection of Eq. (36) immediately tells us that the offending piece comes from the imaginary parts of the  $a_i$ . But we should recall that we still have to subtract the contribution from the iterated lowest order potential. It turns out that its imaginary part exactly cancels the unwanted contribution coming from Eq. (36). Indeed we have explicitly checked this to be the case. However, in order to make this paper not too lengthy, we do not include the intermediate steps of the calculation. We only report on the result: i.e.,

$$\begin{aligned} \Re[\mathcal{M}^{(4)}]_t &= \frac{g^4}{8pW} \bar{u}'_a \bar{u}'_b \left[ \frac{t - 4p^2}{(4p^2 + t)^2} (E_a \gamma_a^0 - m_a) \right. \\ &\left. \times (E_b \gamma_b^0 - m_b) - \frac{p^2}{4p^2 + t} \boldsymbol{\gamma}_a \cdot \boldsymbol{\gamma}_b \right] u_a u_b. \end{aligned} \quad (37)$$

The relevant contribution to the long range potential comes from the real parts of the  $a_i$ , that is, the imaginary part of Eq. (36), once the contribution of the iterated potential has been subtracted. Let us elaborate on the iteration amplitude:

$$\mathcal{M}_I = \langle \mathbf{p}', -\mathbf{p}' | U^{(2)} (W - h_0 + i\epsilon)^{-1} \Lambda_{++} U^{(2)} | \mathbf{p}, -\mathbf{p} \rangle. \quad (38)$$

This formal expression can be recast in the explicit form<sup>3</sup>

$$\begin{aligned} \mathcal{M}_I &= \frac{g^4}{8\pi^2} \int l^2 dl \int \frac{d\Omega}{4\pi} u_a^\dagger u_b^\dagger \gamma_a^0 (E_a \gamma_a^0 - \boldsymbol{\gamma}_a \cdot \mathbf{l} - m_a) \\ &\times \gamma_b^0 (E_b \gamma_b^0 + \boldsymbol{\gamma}_b \cdot \mathbf{l} - m_b) u_a u_b \mathcal{C}(p, l) \frac{1}{\mathbf{q}^{l^2}} \frac{1}{\mathbf{q}^2} \end{aligned} \quad (39)$$

with  $\mathbf{q} \equiv \mathbf{p} - \mathbf{l}$  and  $\mathbf{q}' \equiv \mathbf{p}' - \mathbf{l}$  and  $\mathcal{C}(p, l) \equiv 1/E'_a E'_b (W - W' + i\epsilon)$ .

The integration over momentum  $\mathbf{l}$  reflects the fact that we have inserted a complete set of plane wave intermediate states in Eq. (38). We have also used

$$\Lambda_{+,a}(\mathbf{l}) = \frac{E'_a + \boldsymbol{\gamma}_a \boldsymbol{\gamma}_a \cdot \mathbf{l} + \gamma_a^0 m_a}{2E'_a} \quad (40)$$

where  $E'_a = \sqrt{m_a^2 + l^2}$ .

The iteration amplitude can be conveniently put as follows:

<sup>3</sup>This integral as it stands is infrared divergent. A fictitious mass regulator is understood to be introduced in the scalar propagators which is set to zero after the integrations are performed.

$$\mathcal{M}_I = \frac{g^4}{8\pi^2} \int l^2 dl \bar{u}'_a \bar{u}'_b \{ (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) \} \quad \mathcal{V} \equiv \int \frac{d\Omega}{4\pi} \mathbf{1} \frac{1}{\mathbf{q}^2} \frac{1}{\mathbf{q}^2} \quad (43)$$

$$\times \mathcal{L} - \boldsymbol{\gamma}_a \cdot \mathcal{V} (E'_b \gamma_b^0 - m_b) \quad \mathcal{Y}_{ij} \equiv \int \frac{d\Omega}{4\pi} l_i l_j \frac{1}{\mathbf{q}^2} \frac{1}{\mathbf{q}^2} \quad (44)$$

$$+ (E'_a \gamma_a^0 - m_a) \boldsymbol{\gamma}_b \cdot \mathcal{V} - \gamma'_a \gamma'_b Y_{ij} u_a u_b \mathcal{C}(p, l) \quad (41)$$

where

$$\mathcal{L} \equiv \int \frac{d\Omega}{4\pi} \frac{1}{\mathbf{q}^2} \frac{1}{\mathbf{q}^2} \quad (42)$$

and they can be found in the Appendix.

A little bit of Dirac algebra and the results in the Appendix allow us to write the discontinuity of  $\mathcal{M}_I$  as

$$[\mathcal{M}_I]_t = \frac{g^4}{8\pi^2} \int l^2 dl \bar{u}'_a \bar{u}'_b \left\{ (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) - 2 \frac{p^2 + l^2}{4p^2 + t} [(E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) + (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b)] \right. \\ \left. + \frac{4}{4p^2 + t} \left( 2 \frac{(p^2 + l^2)^2}{4p^2 + t} - l^2 \right) (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) + \boldsymbol{\gamma}_a \cdot \boldsymbol{\gamma}_b \left( \frac{(p^2 + l^2)^2}{4p^2 + t} - l^2 \right) \right\} [\mathcal{L}]_t u_a u_b \mathcal{C}(p, l). \quad (45)$$

The explicit form for  $[\mathcal{L}]_t$  is given in the Appendix. Note that in Eq. (45) we have

$$\frac{1}{W - W' + i\epsilon} = \wp \left( \frac{1}{W - W'} \right) - i\pi \delta(W - W'). \quad (46)$$

The Dirac delta piece gives a contribution that, as already advertised, will exactly cancel the real part of the fourth order discontinuity function (37). The principal part integral can be cast in the form

$$i\Im[\mathcal{M}_I]_t = \frac{ig^4}{16\pi\sqrt{t}(4p^2 + t)^2} \wp \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \mathcal{C}(p, l) \bar{u}'_a \bar{u}'_b \mathcal{N} u_a u_b \quad (47)$$

where we changed the integration variable via the relation

$$l^2 = \frac{a' + b'x}{2} \equiv \frac{1}{2} [(2p^2 + t) + \sqrt{t(4p^2 + t)}x] \quad (48)$$

and we used the shorthand

$$\mathcal{N} \equiv (4p^2 + t)^2 (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) - 2(4p^2 + t)(p^2 + l^2) \\ \times [(E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) + (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b)] + 4[2(p^2 + l^2)^2 - l^2(4p^2 + t)] \\ \times (E'_a \gamma_a^0 - m_a)(E'_b \gamma_b^0 - m_b) - \boldsymbol{\gamma}_a \cdot \boldsymbol{\gamma}_b (4p^2 + t)[(4p^2 + t)l^2 - (p^2 + l^2)^2]. \quad (49)$$

It is convenient now to split the function  $\mathcal{C}(p, l)$  as

$$\mathcal{C}(p, l) \equiv \mathcal{C}_1(p, l) + \mathcal{C}_2(p, l) \equiv \frac{2}{W(p^2 - l^2)} + \frac{1}{E'_a E'_b W} \left( \frac{p^2 + l^2 + m_a^2 + m_b^2}{E_a E_b + E'_a E'_b} + \frac{W'}{W + W'} \right). \quad (50)$$

The integral above cannot be done exactly and we will expand the integral in a power series in  $t$  and  $p^2$ . This is a licit procedure because we will perform a Laplace transform that heavily weighs the small  $t$  region of the spectral function when determining the long range (large  $r$ ) potential and we will eventually take the non-relativistic limit of the potential, i.e. for  $p^2 \sim 0$ . Furthermore, each extra power of  $t$  or  $p^2$  implies a correction to the potential with an extra power of  $r^{-1}$ . We see from Eq. (50) that the calculation will involve doing integrations of the type

$$I_{c1}(n) \equiv \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \frac{l^n}{p^2 - l^2} \quad (51)$$

$$I_{c2}(n) \equiv \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} l^n. \quad (52)$$

The explicit results of the integrals needed in our calculation are also given in the appendix. Armed with all this artillery we find, for the  $C_1$  piece of the discontinuity  $[\mathcal{M}_I]_t$ ,

$$\begin{aligned} i\Im[\mathcal{M}_I^{C_1}]_t = & \frac{ig^4}{8\sqrt{t}(4p^2+t)^2(m_a+m_b)} \bar{u}'_a \bar{u}'_b \left\{ m_a m_b \left[ 4t - \frac{2tp^2}{m_a m_b} + \frac{p^4 t}{2m_a m_b} \left( \frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_a m_b} \right) \right] \right. \\ & + \gamma_a^0 m_a m_b \left[ -4t + \frac{2(m_a - m_b)}{m_a^2 m_b} t p^2 + \frac{t^3}{32m_a^4} + \frac{t^2 p^2}{4m_a^4} + \frac{-m_a^3 - m_a^2 m_b + m_a m_b^2 + 2m_b^3}{2m_a^4 m_b^3} t p^4 \right] \\ & + \gamma_b^0 m_a m_b \left[ -4t + \frac{2(m_b - m_a)}{m_a m_b^2} t p^2 + \frac{t^3}{32m_b^4} + \frac{t^2 p^2}{4m_b^4} + \frac{2m_a^3 + m_a^2 m_b - m_a m_b^2 - m_b^3}{2m_a^3 m_b^4} t p^4 \right] \\ & + \gamma_a^0 \gamma_b^0 m_a m_b \left[ 4t + \left( \frac{2}{m_a^2} + \frac{2}{m_b^2} - \frac{2}{m_a m_b} \right) t p^2 - t^3 \left( \frac{1}{32m_a^4} + \frac{1}{32m_b^4} + \frac{1}{8m_a^2 m_b^2} \right) - t^2 p^2 \left( \frac{1}{4m_a^4} + \frac{1}{4m_b^4} + \frac{1}{m_a^2 m_b^2} \right) \right. \\ & \left. - t p^4 \frac{2m_a^4 + m_a^3 m_b + m_a^2 m_b^2 + m_a m_b^3 + 2m_b^4}{2m_a^4 m_b^4} \right] + \gamma_a \cdot \gamma_b \left[ 2tp^2 + \frac{t^2}{2} - \frac{t^2 p^2}{4m_a m_b} - \frac{tp^4}{m_a m_b} \right] \left. \right\} u_a u_b \quad (53) \end{aligned}$$

and, for the  $C_2$  piece,

$$\begin{aligned} i\Im[\mathcal{M}_I^{C_2}]_t = & \frac{ig^4}{16\sqrt{t}(4p^2+t)^2(m_a+m_b)m_a m_b} \bar{u}'_a \bar{u}'_b \left\{ \gamma_a^0 m_a m_b \left[ \frac{m_a^2 + m_a m_b + m_b^2}{16m_a^3 m_b} t(4p^2+t)^2 \right] \right. \\ & + \gamma_b^0 m_a m_b \left[ \frac{m_a^2 + m_a m_b + m_b^2}{16m_a m_b^3} t(4p^2+t)^2 \right] - \gamma_a^0 \gamma_b^0 \left[ \frac{m_a^4 + m_a^3 m_b + 2m_a^2 m_b^2 + m_a m_b^3 + m_b^4}{16m_a^2 m_b^2} t(4p^2+t)^2 \right] \\ & \left. - \gamma_a \cdot \gamma_b \left[ \frac{m_a^2 + m_a m_b + m_b^2}{16m_a m_b} t(4p^2+t)^2 \right] \right\} u_a u_b. \quad (54) \end{aligned}$$

In both equations above we kept terms only up to the powers of  $t$  and  $p^2$  that will be needed, either in this section or in the next section, to obtain the leading two-particle exchange potential. In this respect Eq. (54) does not contribute to the leading potential just under scrutiny. To obtain the spectral density  $\rho^{(4)}$ , we must finally perform the subtraction

$$\rho^{(4)}(s,t) = \frac{\Im[\mathcal{M}^{(4)}]_t - \Im[\mathcal{M}_I]_t}{2}. \quad (55)$$

Recall that what enters  $\rho^{(4)}$  is the real part of the  $a_i$  in Eq. (36). Although the integrals that go in the  $a_i$  are exactly given in the Appendix, the required subtraction (55) and final Laplace transformation (17) demand that we here also expand the integrands in a power series in  $t$  and  $p^2$ . After some lengthy algebra, we arrive at the final form for the imaginary part of  $[\mathcal{M}]_t$ :

$$\Im[\mathcal{M}]_t = \Im[\mathcal{M}]_t^{odd} + \Im[\mathcal{M}]_t^{even} \quad (56)$$

where

$$\begin{aligned} \Im[\mathcal{M}^{(4)}]_t^{odd} = & \frac{g^4 m_a m_b}{4(m_a+m_b)\sqrt{t}(4p^2+t)^2} \bar{u}'_a \bar{u}'_b \left\{ \left[ 2t - \frac{tp^2}{m_a m_b} + \frac{-3m_a^4 + 2m_a^2 m_b^2 - 3m_b^4}{4m_a^4 m_b^4} t p^4 \right. \right. \\ & + \frac{-3m_a^4 - m_a^3 m_b + m_a^2 m_b^2 - m_a m_b^3 - 3m_b^4}{8m_a^4 m_b^4} t^2 p^2 + \frac{-3m_a^4 - m_a^3 m_b + m_a^2 m_b^2 - m_a m_b^3 - 3m_b^4}{64m_a^4 m_b^4} t^3 \left. \right] \\ & + \gamma_a^0 \left[ -2t + \frac{m_a - m_b}{m_a^2 m_b} t p^2 + \frac{3m_a + 4m_b}{4m_a m_b^4} t p^4 + \frac{3m_a^4 + 5m_a^3 m_b + m_a^2 m_b^2 - m_a m_b^3 - m_b^4}{8m_a^4 m_b^4} t^2 p^2 \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{3m_a^4 + 5m_a^3m_b + m_a^2m_b^2 - m_am_b^3 - m_b^4}{64m_a^4m_b^4} t^3 \Big] \\
& + \gamma_b^0 \left[ -2t + \frac{m_b - m_a}{m_am_b^2} tp^2 + \frac{4m_a + 3m_b}{4m_a^4m_b} tp^4 + \frac{-m_a^4 - m_a^3m_b + m_a^2m_b^2 + 5m_am_b^3 + 3m_b^4}{8m_a^4m_b^4} t^2 p^2 \right. \\
& \left. + \frac{-m_a^4 - m_a^3m_b + m_a^2m_b^2 + 5m_am_b^3 + 3m_b^4}{64m_a^4m_b^4} t^3 \right] \\
& + \gamma_a^0 \gamma_b^0 \left[ 2t + \frac{m_a^2 - m_am_b + m_b^2}{m_a^2m_b^2} tp^2 + \frac{tp^4}{m_a^2m_b^2} + \frac{m_a^4 + m_a^3m_b + m_a^2m_b^2 + m_am_b^3 + m_b^4}{8m_a^4m_b^4} t^2 p^2 \right. \\
& \left. + \frac{m_a^4 + m_a^3m_b + m_a^2m_b^2 + m_am_b^3 + m_b^4}{64m_a^4m_b^4} t^3 \right] \\
& + \gamma_a \cdot \gamma_b \left[ \frac{p^2 t}{m_am_b} + \frac{t^2}{4m_am_b} + \frac{m_a^2 + 4m_am_b + m_b^2}{2m_a^3m_b^3} tp^4 + \frac{3m_a^2 + 5m_am_b + 3m_b^2}{8m_a^3m_b^3} t^2 p^2 + \frac{2m_a^2 + m_am_b + 2m_b^2}{32m_a^3m_b^3} t^3 \right] \Big\} u_a u_b
\end{aligned} \tag{57}$$

contains the odd powers of  $\sqrt{t}$  and

$$\begin{aligned}
\Im[\mathcal{M}^{(4)}]_t^{even} = \frac{g^4}{4\pi} \bar{u}'_a \bar{u}'_b \Big\{ & - \frac{(m_a + m_b) \gamma_b^0 - m_b}{6m_a^2m_b} - \frac{(m_a + m_b) \gamma_a^0 - m_a}{6m_am_b^2} + \frac{\gamma_a^\mu \gamma_\mu^b}{4m_am_b} \\
& - \frac{m_am_b + [(m_a + m_b) \gamma_a^0 - m_a][(m_a + m_b) \gamma_b^0 - m_b]}{12m_a^2m_b^2} \Big\} u_a u_b
\end{aligned} \tag{58}$$

contains the even powers of  $\sqrt{t}$ . We did this separation to emphasize that, after the subtraction in Eq. (55), only the term (58) survives to leading non-vanishing order. Indeed, Eq. (57) coincides exactly with Eq. (53) if we neglect terms beyond  $t^2$ ,  $p^4$ , or  $tp^2$ .

The final step involves the Laplace transformation indicated by Eq. (17). Using the general formula

$$\int_0^\infty t^n e^{-\sqrt{t}r} dt = \frac{2(2n+1)!}{r^{2n+2}} \tag{59}$$

we get

$$\begin{aligned}
U^{(4)}(r; s) = \frac{g^4}{16\pi r^3} \bar{u}'_a \bar{u}'_b \Big\{ & - \frac{(m_a + m_b) \gamma_b^0 - m_b}{6m_a^2m_b} - \frac{(m_a + m_b) \gamma_a^0 - m_a}{6m_am_b^2} + \frac{\gamma_a^\mu \gamma_\mu^b}{4m_am_b} \\
& - \frac{m_am_b + [(m_a + m_b) \gamma_a^0 - m_a][(m_a + m_b) \gamma_b^0 - m_b]}{12m_a^2m_b^2} \Big\} u_a u_b
\end{aligned} \tag{60}$$

which leads, in the non-relativistic limit and concentrating only on the spin-independent terms of Eq. (60), to the long-range attractive potential

$$V_{nr}^{(4)} = - \frac{g^4}{64\pi^3 r^3 m_a m_b} \mathbf{1}_2^a \otimes \mathbf{1}_2^b \tag{61}$$

where this operator is supposed to act between two-component Pauli spinors.

#### IV. DERIVATIVE COUPLING

In this section we consider the interaction Lagrangian

$$\mathcal{L}_{int}^{der} = \frac{g}{2m} \bar{\Psi}(x) \gamma_\mu \gamma^5 \Psi(x) \partial^\mu \Phi(x), \tag{62}$$

which is how Goldstone bosons couple to fermions.

This derivative coupling leads to the same one particle exchange Feynman amplitude as before and therefore to the same lowest order potential (20). Hence the iteration amplitude will be also identical. However, the two particle exchange amplitude (see Fig. 2) is different because the Compton amplitude that goes into Eq. (22) is different. Indeed, the Compton amplitude, corresponding to Fig. 3, is now

$$\mathcal{M}^C(k, k'; P) = g^2 \bar{u}(p') \left[ \frac{\not{k}}{2p \cdot k} + \frac{\not{k}'}{2p \cdot k'} - \frac{1}{m} \right] u(p). \quad (63)$$

This amplitude differs from Eq. (23) by an extra term proportional to  $m^{-1}$ .

We introduce this amplitude in Eq. (22) and replace the massless propagators by Dirac deltas to obtain the discontinuity function

$$\begin{aligned} [\mathcal{M}^{(4)}]_t = & -\frac{ig^4}{8\pi^2} \int d\Phi \bar{u}'_a \left[ \frac{p_a \cdot (k' - k) \not{k}}{2p_a \cdot k' p_a \cdot k} \right] u_a \bar{u}'_b \left[ \frac{p_b \cdot (k' - k) \not{k}}{2p_b \cdot k' p_b \cdot k} \right] u_b \\ & - \frac{ig^4}{8\pi^2} \int d\Phi \left\{ \bar{u}'_a u_a \bar{u}'_b u_b \frac{1}{m_a m_b} - \frac{1}{m_a} \bar{u}'_a u_a \bar{u}'_b \left[ \frac{p_b \cdot (k' - k)}{2p_b \cdot k p_b \cdot k'} \not{k} \right] u_b - \frac{1}{m_b} \bar{u}'_a \left[ \frac{p_a \cdot (k' - k)}{2p_a \cdot k p_a \cdot k'} \not{k} \right] u_a \bar{u}'_b u_b \right\}. \end{aligned} \quad (64)$$

The first piece is exactly what we had in the last section. We call  $[\Delta \mathcal{M}^{(4)}]_t$  the extra added piece that involves the integrations

$$-\frac{ig^4}{8\pi^2} \int d\Phi \bar{u}'_a u_a \bar{u}'_b u_b \frac{1}{m_a m_b} = -\frac{ig^4}{16\pi m_a m_b} \bar{u}'_a u_a \bar{u}'_b u_b \quad (65)$$

and

$$\begin{aligned} \frac{ig^4}{8\pi^2 m_a} \int d\Phi \bar{u}'_a u_a \bar{u}'_b \left[ \frac{p_b \cdot (k' - k)}{2p_b \cdot k p_b \cdot k'} \not{k} \right] u_b + (a \leftrightarrow b) &= \frac{ig^4}{8\pi^2 m_a} \int d\Phi \bar{u}'_a u_a \bar{u}'_b \not{k} u_b \frac{i\sqrt{t} \xi_b m_b x_b}{\frac{t}{2} \left( \frac{t}{4} + (\xi_b m_b x_b)^2 \right)} + (a \leftrightarrow b) \\ &= -\frac{g^4}{8\pi m_a \sqrt{t} \xi_b m_b} \bar{u}'_a u_a \bar{u}'_b \gamma_\mu u_b \left\langle \frac{x_b}{d_b} k^\mu \right\rangle + (a \leftrightarrow b) \\ &= \frac{ig^4}{16\pi m_a m_b \xi_b^2} \left[ 1 - \tau_b \arctan \left( \frac{1}{\tau_b} \right) \right] \bar{u}'_a u_a \bar{u}'_b u_b + (a \leftrightarrow b). \end{aligned} \quad (66)$$

The last line in Eq. (66) is reached by demanding Lorentz covariance to write

$$\left\langle \frac{x_b}{d_b} k^\mu \right\rangle = a_b P_a^\mu + b_b P_b^\mu + c_b Q^\mu, \quad (67)$$

by solving for the coefficients as explained in the Appendix, and by using the Dirac equation.

Putting things together,

$$[\Delta \mathcal{M}^{(4)}]_t = \frac{ig^4}{16\pi m_a m_b} \bar{u}'_a u_a \bar{u}'_b \left\{ \frac{1}{\xi_a^2} \left[ 1 - \tau_a \arctan \left( \frac{1}{\tau_a} \right) \right] + \frac{1}{\xi_b^2} \left[ 1 - \tau_b \arctan \left( \frac{1}{\tau_b} \right) \right] - 1 \right\} u_a u_b. \quad (68)$$

To leading order in  $t$  and  $p^2$  we have

$$[\Delta \mathcal{M}^{(4)}]_t = \frac{ig^4}{16\pi m_a m_b} \bar{u}'_a u_a \bar{u}'_b u_b. \quad (69)$$

In the non-relativistic limit this contributes the quantity

$$\Delta V_{nr}^{(4)} = \frac{1}{4\pi^2 r} \int_0^\infty \Delta \rho e^{-\sqrt{r} t} dt = \frac{g^4}{64\pi^3 r^3 m_a m_b} \quad (70)$$

which exactly cancels contribution (61), i.e.,

$$V_{der;nr}^{(4)} = V_{Y;nr}^{(4)} + \Delta V_{nr}^{(4)} = 0 + \mathcal{O}(r^{-4}). \quad (71)$$



Hence, to find the form for the potential in the case under scrutiny, we must go to the next order in our series expansions. What we need now is to consistently take into account the previously neglected terms in the spectral density

$$\rho_{der}^{(4)} \equiv \frac{[\mathcal{M}_Y^{(4)} + \Delta\mathcal{M}^{(4)} - \mathcal{M}_I]_t}{2i}. \quad (72)$$

So we collect the relevant pieces in Eqs. (57), (53), and (54) in addition to

$$[\Delta\mathcal{M}]_t^{(4)} \sim \dots - \frac{ig^4\sqrt{t}}{16m_a m_b} \bar{u}'_a \bar{u}'_b \left( \frac{1}{4m_a} + \frac{1}{4m_b} \right) u_a u_b \quad (73)$$

which is the next to leading term in the expansion of Eq. (68). The result is

$$\begin{aligned} \rho_{der}^{(4)} \equiv \frac{[\mathcal{M}_Y^{(4)} + \Delta\mathcal{M} - \mathcal{M}_I]_t}{2i} &= \frac{g^4 m_a m_b \sqrt{t}}{8(m_a + m_b)} \bar{u}'_a \bar{u}'_b \left\{ - \frac{3m_a^4 + 5m_a^3 m_b + 7m_a^2 m_b^2 + 5m_a m_b^3 + 3m_b^4}{64m_a^4 m_b^4} \right. \\ &+ \gamma_a^0 \frac{3m_a^4 + 5m_a^3 m_b - 2m_a m_b^3 - 3m_b^4}{64m_a^4 m_b^4} + \gamma_b^0 \frac{-3m_a^4 - 2m_a^3 m_b + 5m_a m_b^3 + 3m_b^4}{64m_a^4 m_b^4} \\ &\left. + \gamma_a^0 \gamma_b^0 \frac{3m_a^4 + 2m_a^3 m_b + 7m_a^2 m_b^2 + 2m_a m_b^3 + 3m_b^4}{64m_a^4 m_b^4} + \text{spin-dependent terms of the kind } \boldsymbol{\gamma}_a \cdot \boldsymbol{\gamma}_b \right\} u_a u_b. \quad (74) \end{aligned}$$

This spectral density nonetheless gives a vanishing spin-independent potential in the static approximation, i.e.

$$V_{der:nr}^{(4)} = 0 + \mathcal{O}(r^{-5}). \quad (75)$$

The first non-vanishing contribution to the spin-independent potential arises from the part in the spectral density which is linear in  $t$ . Indeed, the explicit form of the spectral density reads

$$\begin{aligned} \rho^{(4)} \equiv \frac{[\mathcal{M}_Y^{(4)} + \Delta\mathcal{M}^{(4)}]_t}{2i} &= \frac{g^4}{8\pi} \frac{1}{240m_a^4 m_b^4} \bar{u}'_a \bar{u}'_b \{ 4[(-3m_a^4 - 2m_a^3 m_b + 2m_a^2 m_b^2 - 2m_a m_b^3 - 3m_b^4)p^2 \\ &+ (6m_a^4 + 3m_a^3 m_b + 4m_a^2 m_b^2 + 3m_a m_b^3 + 6m_b^4)t] \\ &+ \gamma_a^0 2(m_a + m_b)[(6m_a^3 - 2m_a^2 m_b + 3m_a m_b^2 - 4m_b^3)p^2 + (-12m_a^3 + 6m_a^2 m_b - 4m_a m_b^2 + 3m_b^3)t] \\ &+ \gamma_b^0 2(m_a + m_b)[(-4m_a^3 + 3m_a^2 m_b - 2m_a m_b^2 + 6m_b^3)p^2 + (3m_a^3 - 4m_a^2 m_b + 6m_a m_b^2 - 12m_b^3)t] \\ &+ \gamma_a^0 \gamma_b^0 [(8m_a^4 + 2m_a^3 m_b - 12m_a^2 m_b^2 + 2m_a m_b^3 + 8m_b^4)p^2 + (-6m_a^4 + 2m_a^3 m_b - 9m_a^2 m_b^2 + 2m_a m_b^3 - 6m_b^4)t] \} u_a u_b \quad (76) \end{aligned}$$

where no iterated second order amplitude contributes to this order, and where we picked the term proportional to  $t$  in the expansion of Eqs. (65) and (66).

If we use now

$$\bar{u}'_a \gamma_a^0 u_a \approx \bar{u}'_a [1 + \mathcal{O}(p^2)] u_a \quad (77)$$

and pass to the static limit, we find

$$\rho_{nr}^{(4)} = \frac{t}{32m_a^2 m_b^2} \mathbf{1}_a \otimes \mathbf{1}_b \quad (78)$$

which, upon Laplace transformation, leads to

$$V_{der:nr}^{(4)} = \frac{3g^4}{128\pi^3 m_a^2 m_b^2} \frac{1}{r^5} \quad (79)$$

for the desired spin-independent long range potential. Note that, as opposed to the Yukawa type coupling potential (61), the derivative interaction leads to a repulsive potential.

## V. CONCLUSIONS

Very light particles can mediate forces extending over distances on the order of their Compton wavelength. If this range is macroscopic, unpolarized bulk matter will only experience the effect of spin-independent interactions. It is a well-known fact that the Yukawa potential due to pseudoscalar exchange depends on spin and as a consequence no coherent effects do arise on a macroscopic scale, unless of course our sample is polarized [5]. However, residual (van der Waals type) forces may arise between macroscopic bodies in the case of pseudoscalar mediated interactions, due to the exchange of two quanta ‘‘at the same time.’’ The double helicity flip involved eventually makes the resulting effective

potential spin-independent [7].

In the preceding sections we have established, with the help of the formalism developed by Feinberg and Sucher, the large distance behavior of such residual forces, i.e. those associated to double pseudoscalar exchange. We have considered two different basic couplings of those scalars to matter fermions. On the one hand we took the ordinary Yukawa coupling (e.g., this is the way the Higgs particle couples to fermions) and on the other we considered the derivative coupling (e.g., the axion-fermion interaction). Both couplings reduce to the same spin-flip interaction in the static non-relativistic limit. And both interactions also produce identical one-particle exchange (spin-dependent) potentials. In spite of this fact, we have explicitly shown that the spin-independent two-particle exchange potential is substantially different in both cases. Indeed, for Yukawa coupling we derive a  $1/r^3$  attractive long distance behavior whereas for the derivative coupling the potential, which is now repulsive, falls off as  $1/r^5$ . Since, as emphasized, double exchange will lead to residual macroscopic effects, these effects will be quite different in both cases. So we have found still another instance where the interaction of pseudoscalars to fermions can be discriminated. Other places are soft pion emission in proton-proton scattering or axion bremsstrahlung in a supernova core [4,15].

Of course, the effects just reported are extremely small for the light scalars presently contemplated in particle physics, such as the axion, and thus their experimental detection is beyond the reach of present technology. However, there is much activity and interest on the experimental front and experiments have been designed and performed that explore the sub-centimeter and sub-millimeter regime with an ever-increasing sensitivity [16,17]. And on the theoretical side, the completion of the particle physics paradigm may still require new superlight scalar particles to exist.

#### ACKNOWLEDGMENTS

Work partially supported by the CICYT Research Project AEN95-0882. F.F. acknowledges the CIRIT for financial support.

#### APPENDIX

##### Coefficients $a_i$

The coefficients in the tensor decomposition (33) read

$$\begin{aligned}
 a_1 &= \frac{t(I_1 - 2I_5 - I_6 + 4yI_4 - y^2I_1 - y^2I_6)}{16m_a^2\xi_a^2(1-y^2)^2} \\
 a_2 &= \frac{t(I_1 - I_5 - 2I_6 + 4yI_4 - y^2I_1 - y^2I_5)}{16m_b^2\xi_b^2(1-y^2)^2} \\
 a_3 &= \frac{t(-I_4 - yI_1 + 2yI_5 + 2yI_6 - 3y^2I_4 + y^3I_1)}{16b(1-y^2)^2} \\
 a_4 &= -\frac{t(I_1 - I_5 - I_6 + 2yI_4 - y^2I_1)}{4(1-y^2)} \quad (A1)
 \end{aligned}$$

with  $y \equiv \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$ , in terms of various angular integrals in the set

$$\begin{aligned}
 I_0 &\equiv \left\langle \frac{1}{d_a d_b} \right\rangle, & I_1 &\equiv \left\langle \frac{x_a x_b}{d_a d_b} \right\rangle, & I_2 &\equiv \left\langle \frac{x_a^2}{d_a d_b} \right\rangle \\
 I_3 &\equiv \left\langle \frac{x_b^2}{d_a d_b} \right\rangle, & I_4 &\equiv \left\langle \frac{x_a^2 x_b^2}{d_a d_b} \right\rangle, & I_5 &\equiv \left\langle \frac{x_a^3 x_b}{d_a d_b} \right\rangle \\
 I_6 &\equiv \left\langle \frac{x_a x_b^3}{d_a d_b} \right\rangle. \quad (A2)
 \end{aligned}$$

These results are obtained after repeated contraction of Eq. (33) with the independent momenta in our problem and after solving the resulting algebraic system of equations.

The angular integrals  $I_i$  are given next. The explicit calculation of  $I_i$  for  $i < 5$  is given in [13],  $I_5$  can be found in [14] and  $I_6$  follows trivially from  $I_5$  by interchanging  $a$  and  $b$ . For the first two integrals from the set (A2) one gets

$$I_0 = \frac{F_+ + \pi N_+^{-1}}{2\tau_a \tau_b}, \quad I_1 = \frac{F_- + \pi N_+^{-1}}{2} \quad (A3)$$

where we have defined

$$\begin{aligned}
 F_{\pm} &= \pm N_-^{-1} \arctan\left(\frac{N_-}{D_+}\right) - N_+^{-1} \arctan\left(\frac{N_+}{D_-}\right) \\
 N_+ &= -\frac{ips^{1/2}}{b} \\
 N_- &= -\frac{i\sqrt{p^2 s + byt}}{b} \\
 D_{\pm} &= y \pm \tau_a \tau_b \quad (A4)
 \end{aligned}$$

and the rest is given in terms of  $I_0$  and  $I_1$  by

$$\begin{aligned}
 I_2 &= \frac{1}{\tau_b} \arctan\left(\frac{1}{\tau_b}\right) - \tau_a^2 I_0 \\
 I_3 &= \frac{1}{\tau_a} \arctan\left(\frac{1}{\tau_a}\right) - \tau_b^2 I_0 \\
 I_4 &= 1 - \tau_a \arctan\left(\frac{1}{\tau_a}\right) - \tau_b \arctan\left(\frac{1}{\tau_b}\right) \\
 &\quad + \tau_a^2 \tau_b^2 I_0 \\
 I_5 &= y \left[ 1 - \tau_b \arctan\left(\frac{1}{\tau_b}\right) \right] - \tau_a^2 I_1 \\
 I_6 &= y \left[ 1 - \tau_a \arctan\left(\frac{1}{\tau_a}\right) \right] - \tau_b^2 I_1. \quad (A5)
 \end{aligned}$$

**Angular integrals entering the iterated amplitude**

We start with Eq. (42). Its discontinuity  $[L]_t$  has been obtained in [13]:

$$[\mathcal{L}]_t = i \left[ \frac{\pi}{l\sqrt{t}} \frac{1}{\sqrt{(l_+^2 - l^2)(l^2 - l_-^2)}} \right] \Theta(l_+^2 - l^2) \Theta(l^2 - l_-^2) \quad (\text{A6})$$

with

$$l_{\pm}^2 = \frac{a' \pm b'}{2}, \quad a' = 2p^2 + t, \quad b' = \sqrt{t(4p^2 + t)}. \quad (\text{A7})$$

For  $p^2 = l^2$ , simplifies to

$$[\mathcal{L}]_t|_{p^2=l^2} = \frac{i\pi}{p^2 t}. \quad (\text{A8})$$

We turn now to Eq. (43). It has been calculated in [14]. It is symmetric in  $\mathbf{p}$  and  $\mathbf{p}'$ . Hence it can be cast in the form

$$\mathcal{V} = v\mathbf{P} \quad \text{with} \quad \mathbf{P} \equiv \mathbf{p} + \mathbf{p}' \quad (\text{A9})$$

where  $v$  is given by

$$v = \frac{1}{p^2} \left( (p^2 + l^2)\mathcal{L} - \frac{1}{4pl} \ln \left[ \frac{(p+l)^2}{(p-l)^2} \right] \right). \quad (\text{A10})$$

The associated discontinuity is then

$$[\mathcal{V}]_t = [v]_t \mathbf{P} = \frac{p^2 + l^2}{4p^2 + t} [\mathcal{L}]_t \mathbf{P}. \quad (\text{A11})$$

Finally let us discuss Eq. (44). This integral admits the general decomposition

$$Y_{ij} = aP_i P_j + bQ_i Q_j + c\delta_{ij} \quad (\text{A12})$$

in terms of the vectors  $\mathbf{P}$ , defined above, and  $\mathbf{Q} \equiv \mathbf{p} - \mathbf{p}'$ . Clearly, we can write

$$\begin{aligned} P_i P_j Y^{ij} &= a(P^2)^2 + cP^2 \\ Q_i Q_j Y^{ij} &= b(Q^2)^2 + cQ^2 \\ \delta_{ij} Y^{ij} &= aP^2 + bQ^2 + 3c \end{aligned} \quad (\text{A13})$$

(we used  $\mathbf{P} \cdot \mathbf{Q} = 0$ ). The scalar integrals on the left are, respectively,

$$\begin{aligned} P_i P_j Y^{ij} &= \int \frac{d\Omega}{4\pi} \frac{(\mathbf{P} \cdot \mathbf{l})^2}{(\mathbf{q}'^2)(\mathbf{q}^2)} \\ &= (p^2 + l^2)\mathcal{L} - \frac{p^2 + l^2}{2pl} \ln \left[ \frac{(p+l)^2}{(p-l)^2} \right] \\ &\quad + \frac{4p^2 + t}{4p^2} + \frac{p^2 + l^2}{16p^3 l} t \ln \left[ \frac{(p+l)^2}{(p-l)^2} \right] \\ Q_i Q_j Y^{ij} &= \frac{1}{4} \left( -2 + \frac{2p^2 + t}{p^2} \right. \\ &\quad \left. + \frac{p^2 + l^2}{4p^3 l} t \ln \left[ \frac{(p+l)^2}{(p-l)^2} \right] \right) \\ \delta_{ij} Y^{ij} &= l^2 \mathcal{L}. \end{aligned} \quad (\text{A14})$$

Now the three equations can be solved for  $a$ ,  $b$ , and  $c$ . Recall that we only need the discontinuity of Eq. (44), i.e. the discontinuities of  $a$ ,  $b$ , and  $c$ . They are

$$\begin{aligned} [a]_t &= \frac{1}{4p^2 + t} \left( 2 \frac{(p^2 + l^2)^2}{4p^2 + t} - l^2 \right) [\mathcal{L}]_t \\ [b]_t &= \frac{[c]_t}{t} [\mathcal{L}]_t \\ [b]_t &= \left( l^2 - \frac{(p^2 + l^2)^2}{4p^2 + t} \right) [\mathcal{L}]_t. \end{aligned} \quad (\text{A15})$$

**Integrals  $I_{c1}$  and  $I_{c2}$** 

Here we display the explicit solutions of Eqs. (51) and (52) for  $n=0,2,4,6$ :

$$\begin{aligned} I_{c1}(0) &= 0 \\ I_{c1}(2) &= -\pi \\ I_{c1}(4) &= -\frac{\pi}{2}(4p^2 + t) \\ I_{c1}(6) &= -\frac{\pi}{8}[(4p^2 + t)^2 + 2(2p^2 + t)^2] \end{aligned} \quad (\text{A16})$$

$$\begin{aligned} I_{c2}(0) &= \pi \\ I_{c2}(2) &= \frac{\pi}{2}(2p^2 + t) \\ I_{c2}(4) &= \frac{\pi}{8}(8p^4 + 12p^2 t + 3t^2) \\ I_{c2}(6) &= \frac{\pi}{16}(16p^6 + 48p^4 t + 30p^2 t^2 + 5t^3). \end{aligned} \quad (\text{A17})$$

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