Critical region of the finite temperature chiral transition

J. B. Kogut
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801-3080 and Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, MIT, Cambridge, Massachusetts 02139

M. A. Stephanov
Institute for Theoretical Physics, SUNY, Stony Brook, New York 11794-3840

C. G. Strouthos
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801-3080
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We study a Yukawa theory with spontaneous chiral symmetry breaking and with a large number N of fermions near the finite temperature phase transition. Critical properties in such a system can be described by the mean field theory very close to the transition point. We show that the width of the region where non-trivial critical behavior sets in is suppressed by a certain power of $1/N$. Our Monte Carlo simulations confirm these analytical results. We discuss implications for the chiral phase transition in QCD. [S0556-2821(98)08819-5]

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I. INTRODUCTION

The transition in QCD separating the high temperature quark-gluon plasma phase from the low temperature hadronic phase has been studied intensively in the last decade. Understanding the properties of this transition is becoming increasingly important in view of recent experimental progress in the physics of ultrarelativistic heavy ion collisions.

Since the u and d quark masses are small, the dynamics of the finite temperature transition is affected by the phenomenon of chiral symmetry restoration which occurs in the limit when the quark masses are put to zero. In this limit QCD with two massless quarks has a global $SU(2)_L \times SU(2)_R$ symmetry which is spontaneously broken to $SU(2)_V$ at low temperatures. It can be argued that if the restoration of this spontaneously broken symmetry proceeds through a second-order phase transition, the critical properties of this transition are in the universality class of the classical O(4) spin model in three dimensions [1,2]. This means that the leading singular behavior of thermodynamic quantities can be predicted: i.e., it is given by universal O(4) critical exponents. However, universality does not answer more detailed questions such as how this criticality is approached and what is the width of the region of parameters in which this singular behavior sets in. These questions require more detailed knowledge of the dynamics of the theory.

In this paper we discuss a phenomenon which is related to the way the critical behavior sets in for a certain class of theories with a second-order chiral symmetry restoration transition at finite temperature, T_c. These are theories with a large number of fermion species, N. On general grounds, one could expect that the universal critical, or scaling, behavior near T_c sets in as soon as the correlation length, ξ, of the fluctuations of the chiral condensate exceeds $1/T_c$. However, as we show in this paper, if the number of fermions involved in the chiral symmetry breaking is large, the critical behavior which sets in when ξ exceeds $1/T_c$ is given by mean field theory, rather than by arguments based on dimensional reduction and universality as in [1]. The critical behavior given by these latter arguments sets in much later, closer to T_c, when the correlation length ξ exceeds N^x/T_c. Below we shall determine the value of the positive exponent x.

The phenomenon of non-trivial critical region suppression has been observed in the Yukawa model [3] and in the Gross-Neveu[1] model [4] using a large-N expansion and was confirmed by lattice Monte Carlo calculations in [4]. These large-N results predicting mean-field critical behavior were in an apparent contradiction with the general arguments of [1]. In this paper we show that both critical regimes are realized in the vicinity of T_c, but in separate scaling windows, one following the other.

II. LARGE N YUKAWA THEORY NEAR T_c

In this section we consider a general Yukawa theory in d dimensions, $2 < d \leqslant 4$, with a large number N of fermion species and at finite temperature T. As argued in [5], in the continuum limit both Yukawa model and Gross-Neveu, or Nambu–Jona-Lasinio, models with a four-fermion interaction define the same theory. In the absence of a bare fermion mass there is a (chiral) symmetry, which can be broken spontaneously at low temperature with a suitable choice of couplings. This symmetry is restored at some finite temperature T_c. We are interested in the nature of this phase transition.

In the absence of fermions (or when the Yukawa coupling is zero) the nature of the transition is rather well known. It depends on the symmetry of the model which is restored at

[1]In this paper we shall use the terms ''Gross-Neveu model'' and ''Nambu–Jona-Lasinio model'' interchangeably, especially when it concerns a theory with a four-fermion interaction in $2+1$ dimensions.

T_c. In this paper, for simplicity, we consider a model with Z_2 symmetry. Similar results will also apply to theories with other symmetry groups [e.g., SU(2)×SU(2), as in QCD with 2 massless quarks], as long as the temperature driven symmetry restoration transition is of second order.

One can argue that the critical behavior of a quantum theory of a scalar field near T_c is the same as in a classical scalar theory with the same symmetry. The argument is based on two expected properties of the model: dimensional reduction and universality. In the Euclidean formulation the quantum scalar field is defined in a d-dimensional box with the extent in the imaginary (Matsubara) time dimension equal to $1/T$. When the diverging correlation length, ξ, becomes much larger than $1/T_c$ long wavelength fluctuations of the field become effectively $(d-1)$-dimensional: i.e., on their scale the box looks like a $(d-1)$-dimensional ''pancake.'' Since such fluctuations determine the critical behavior in the model, one can expect that, by universality, the critical exponents are the same as in a $(d-1)$-dimensional theory with the same symmetry. This $(d-1)$-dimensional theory is obviously a classical field theory at finite temperature. One can also understand this realizing that classical thermal fluctuations whose energy is $O(k_B T)$ dominate over quantum fluctuations with energy $O(\hbar\omega)$ for soft modes of the field.

Another common way of describing this phenomenon in perturbation theory is to consider Fourier decomposition of the field into discrete Matsubara frequency components. A non-zero frequency acts as a mass term of order πT for the $(d-1)$-dimensional components of the field. Near T_c this mass is much larger than the mass of a component with zero Matsubara frequency. The dimensional reduction is then equivalent to the decoupling of the modes with non-zero frequencies.

We want to understand what happens in this theory near T_c when one turns on the Yukawa coupling. Dimensional reduction and universality arguments suggest that the critical behavior of the theory should not change. This is based on the observation that there is no zero Matsubara frequency for the fermion fields due to antiperiodic boundary condition in the Euclidean time. Therefore all fermion modes should decouple at T_c. In other words, the fermion fields do not have a classical limit and do not survive quantum-to-classical reduction at T_c [6].

However, as was demonstrated in [4], the fermions do affect the behavior near T_c in a certain way. We shall show that this happens when there are ''too many'' of them. The theory can be solved in the limit when the number of fermions, N, is large. In this limit the theory has mean-field critical behavior near T_c [4,3]. This is different from the critical behavior of the corresponding classical scalar field theory. The mean-field behavior was also observed in numerical Monte Carlo calculations near T_c [4].

Here we show that such mean-field behavior can be reconciled with the standard arguments of dimensional reduction and universality. The phenomenon which leads to an apparent contradiction is the suppression of the width of the non-mean-field critical region by a power of $1/N$.

We consider the following model with one-component scalar field and Z_2 symmetry in d dimensional Euclidean space:

$$\mathcal{L}=\frac{1}{2}(\partial\phi)^2+\frac{1}{2}\mu^2\phi^2+\lambda\phi^4+\sum_{f=1}^{N}\bar{\psi}_f(\not{\partial}+g\phi)\psi_f. \quad (1)$$

We regularize the model by some momentum cutoff, Λ. The cutoff can be removed if $d<4$ but we shall keep it finite to compare with a corresponding lattice theory. There are two other important scales in the theory: the temperature, T, and the physical mass, m, which we identify with the mass of thermal excitations of the scalar field. Near T_c this mass, m, is significantly different from the zero temperature mass m_0, since m vanishes at the critical temperature. It is the mass m, or the correlation length $1/m$, which is important for the critical behavior. The mass m is a natural measure (more natural than, say, $T-T_c$) of the distance from the criticality.[2] Therefore, near the finite temperature phase transition we have the following hierarchy of scales: $\Lambda \gg T \gg m$.

Let us consider the renormalization group (RG) evolution of the couplings from the scale of Λ down to the scale of T and then from T down to m. We focus on the quartic self-coupling of the scalar field, λ. The evolution from the scale Λ to the scale T is governed by the RG equations of the d-dimensional quantum Yukawa model. After that, at the scale of T, we pass through a crossover region due to the fact that the fermions and non-zero Matsubara frequencies of the scalar fields do not contribute to the evolution below T (the decoupling). The evolution below T is governed by the RG equations of the scalar ϕ^4 theory in $d-1$ dimensions.

If the window of scales between Λ and T is wide enough (as it is in the continuum limit $\Lambda\to\infty$), the value of the renormalized coupling λ at the scale T, $\lambda(T)$, is close to the infrared fixed point of the d-dimensional Yukawa theory. In the large-N limit one can calculate this value [7]:

$$\lambda(T)\sim\frac{(4-d)T^{4-d}}{N} \quad \text{for} \quad 2<d<4. \quad (2)$$

The case $d=4$ is special. The infrared fixed point is trivial and is approached logarithmically as $\Lambda/T\to\infty$:

$$\lambda(T)\sim\frac{1}{N\,\ln(\Lambda/T)} \quad \text{for} \quad d=4. \quad (3)$$

This value provides the starting point, $\lambda_{d-1}=T\lambda(T)$, for the evolution of this coupling below T in the ϕ^4 theory in $d-1$ dimensions. For large N this coupling is small. As we shall see shortly, this is the reason why the critical region where one observes non-trivial critical behavior is reached only very close to the phase transition.

The phenomenon of the suppression of the width of a non-trivial critical region is common in condensed matter

[2]The whole theory of critical scaling is based on this observation. We shall also use this fact more explicitly when we consider a lattice theory.

physics [8]. BCS superconductors provide the most well-known example of a system where criticality near T_c is described by the Landau-Ginzburg mean-field theory. The width, ΔT, of the region around T_c where the mean-field description breaks down is tiny. There are systems where the width of the non-trivial critical region is small but measurable. In such systems one can observe the crossover between a mean-field and a non-trivial scaling.

The quantitative relation between the size of the non-trivial, non-mean-field critical region, the Ginzburg region, and certain parameters of a given system is known as the Ginzburg criterion. In superconductors such a parameter is a small ratio T/E_F: i.e., the width of the Ginzburg region is suppressed by a power of this parameter. In this paper, we show that in a field theory with large number of fermions, such as Eq. (1), the width of the Ginzburg region is suppressed by a power of $1/N$.

The Ginzburg criterion can be obtained by estimating the effects of fluctuations within the mean-field approximation. When the fluctuations become large the mean-field approximation breaks down because of self-inconsistency. A measure of the importance of the fluctuations, or the size the corrections to the mean-field, is the value of the effective selfcoupling of the scalar field, λ. If the coupling λ is small, the effect of the fluctuations is also small and the theory can be described by the mean-field approximation very well. However, for the ϕ^4 theories in less than four dimensions[3] fluctuations always become important close enough to the phase transition. This happens roughly when the coupling λ_{d-1} on the scale of m is not small anymore. Since λ_{d-1} has nonzero dimension equal to $5-d$, it should be compared to m^{5-d}. In this way, and with the help of Eq. (2), one arrives at the following Ginzburg criterion for the applicability of the mean-field scaling:

$$m \gg \frac{T}{N^x}, \quad x = \frac{1}{5-d}. \tag{4}$$

In the special case of $d=4$, using Eq. (3) one finds

$$m \gg \frac{T}{N \ln(\Lambda/T)} \quad \text{for} \quad d=4. \tag{5}$$

Alternatively, one can compare the size of the one-loop correction in the effective ϕ^4 theory in $d-1$ dimensions to the bare λ_{d-1}. The loop correction becomes important when Eq. (4) is violated.[4] Indeed, consider the contribution, $\Delta\lambda$, of a graph such as in Fig. 1 to the effective quartic coupling λ. This contribution diverges when $m \to 0$:

$$\Delta\lambda \sim \frac{T[\lambda(T)]^2}{m^{5-d}}, \tag{6}$$

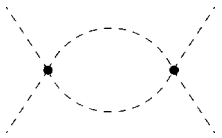

FIG. 1. An example of a graph whose contribution breaks the mean field approximation and the large-N expansion near T_c.

where we integrated over fluctuations in the window between T and m with $m \ll T$. Thus, this one-loop contribution, $\Delta\lambda$, is negligible compared to $\lambda(T)$ if $T\lambda(T) \ll m^{5-d}$, which together with Eq. (2) leads to Eq. (4). Note also that, since $\lambda(T) \sim 1/N$, the contribution of the graph in Fig. 1 should be subleading in the large-N expansion. Therefore, the large-N expansion breaks down when Eq. (4) is violated.

The Ginzburg criterion tells us that for masses m inside the window $T \gg m \gg T/N^x$ the mean-field scaling holds, while for smaller masses $m \ll T/N^x$ (i.e., closer to the transition) the non-trivial $d-1$ Ising scaling sets in. We see that the size of this latter, non-trivial critical region is suppressed at large N.[5]

III. LATTICE THEORY

In this section we analyze how the effect of the suppression of the non-trivial critical region manifests itself in the lattice formulation of the theory. For simplicity, we shall discuss the case $d=3$. The generalization to arbitrary d, $2<d\leq 4$, can be done as in the previous section. We consider the following lattice discretization of the theory (1) in $d=3$:

$$S = \sum_{\tilde{x}} \left(-\kappa \sum_{\mu} \phi_{\tilde{x}+\hat{\mu}} \phi_{\tilde{x}} + \lambda_{\rm lat} \phi_{\tilde{x}}^4 + \frac{\beta N}{4} \phi_{\tilde{x}}^2 \right) + \sum_{i=1}^{N/2} \left(\sum_{x,y} \bar{\chi}_x^i M_{x,y} \chi_y^i + \frac{1}{8} \sum_x \bar{\chi}_x^i \chi_x^i \sum_{\langle \tilde{x},x \rangle} \phi_{\tilde{x}} \right), \tag{7}$$

where χ^i and $\bar{\chi}^i$ are Grassmann-valued staggered fermion fields defined on the lattice sites; the scalar field ϕ is defined on the dual lattice sites, and the symbol $\langle \tilde{x},x \rangle$ denotes the set of 8 (i.e., 2^d) dual lattice sites $\tilde{x}$ surrounding the direct lattice site x. The fermion kinetic (hopping) matrix M is given by

$$M_{x,y} = \frac{1}{2} \sum_{\mu} \eta_\mu(x) [\delta_{y,x+\hat{\mu}} - \delta_{y,x-\hat{\mu}}], \tag{8}$$

where $\eta_\mu(x)$ are the Kawamoto-Smit phases $(-1)^{x_1 + \cdots + x_{\mu-1}}$. The cubic lattice has L_s lattice spacings a in spatial directions and L_t lattice spacings in the temporal direction. The cutoff scale can be defined as $\Lambda = 1/a$, the

[3]Here again the case of four dimensions is special. However, we are now discussing theories in $d-1$ dimensions and $1<d-1\leq 3$.

[4]In fact, the criterion (5) is well-known in the form $\lambda T_c/m \ll 1$ [9] as the criterion for the applicability of perturbation theory near a finite temperature phase transition.

[5]Note that there is no proportionality constant in the Ginzburg criterion (4). This constant would depend on the definition of the boundary of the mean-field region which is naturally ambiguous. The Ginzburg criterion tells us how this boundary moves as a function of N.

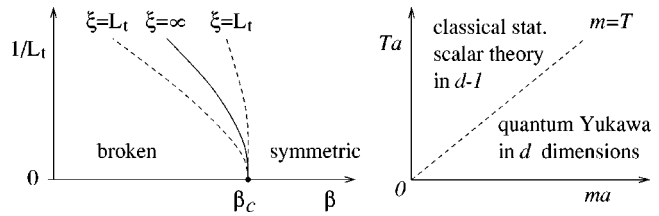

FIG. 2. A schematic phase diagram (a) of a lattice Yukawa theory in the plane of $Ta \equiv 1/L_t$ and the lattice action parameter β. The solid line is the phase boundary. The dashed lines show the location of points where the correlation length, ξ, reaches L_t. The same phase diagram (b) but the distance from the critical line is expressed in terms of a natural variable $ma = 1/\xi$ and only one side (either symmetric or broken) is shown. Various continuum limits correspond to approaching the origin in (b). The slope determines the ratio T/m in the resulting continuum theory. The values of L_t and thus of Ta are discrete, but this is of no importance to our discussion.

temperature is given by $T = 1/(L_t a)$ and the mass is $m = 1/(\xi a)$, where ξ is the correlation length of the scalar field. To reach a continuum limit one has to satisfy the following two conditions: $\Lambda \gg T$ and $\Lambda \gg m$. The condition $\Lambda \gg T$ requires a lattice with sufficiently large $L_t \gg 1$. The parameters of the action should then be tuned towards their critical values where the correlation length $\xi \gg 1$. This satisfies the condition $\Lambda \gg m$.

The condition $\xi \gg 1$, or $ma = 1/\xi \to 0$, specifies a 2-dimensional critical surface in the space of 3 parameters κ, $\lambda_{\rm lat}$ and β. One expects that a continuum limit taken at any generic point of this surface defines the same theory [5,10]. (This is the meaning of the equivalence between Yukawa and four-fermion theories.) Therefore we can fix two out of three bare parameters, $\kappa = \lambda_{\rm lat} = 0$, and tune a single parameter β to criticality: $1/\xi \to 0$.

The phase diagram as a function of β and $Ta = 1/L_t$ looks like in Fig. 2(a). A natural measure of the distance from the criticality is $ma = 1/\xi$. Trading the lattice parameter β for ma we obtain the phase diagram of Fig. 2(b).

The line $T = m$ separates the regions of quantum and classical behavior or, equivalently, the regions of d-dimensional and $(d-1)$-dimensional behavior. Below this line, when $T \ll m$, the correlation length ξ is smaller than the extent in the time direction L_t and the system behaves as a quantum Yukawa model in $2+1$ dimensions. Above the line, when $T \gg m$, the correlation length ξ is much larger than L_t, the system looks like a ''pancake'' and behaves as a 2-dimensional classical statistical theory of a scalar field.

Now consider changing the parameter β on a given lattice, i.e., at fixed $Ta = 1/L_t$, so that we move along a trajectory such as ABC in Fig. 3. The effective (long distance) coupling $\lambda(\xi)$ follows the evolution governed by the RG equations of the Yukawa model from A to B. Near the point B it reaches some value, $\lambda(L_t)$, which, if L_t is large enough, is given by the infrared fixed point and is $O(1/N)$. As we continue to increase ξ from B to C the coupling $\lambda(\xi)$ evolves according to the RG equations of the 2-dimensional ϕ^4 theory. Since $\lambda(L_t) \sim 1/N$, in order to reach the non-trivial critical region one needs to go to the correlation length ξ

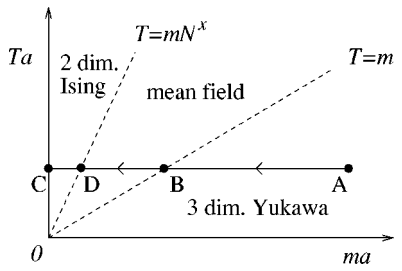

FIG. 3. The same phase diagram as in Fig. 2(b) for a Yukawa theory with large N. The trajectory ABC corresponds to changing some lattice parameter to approach criticality on a lattice with a given $L_t \equiv 1/Ta$. The point A corresponds to the correlation length $\xi \equiv 1/ma \sim 1$, $\xi(B) = L_t$, $\xi(D) = L_t N^x$, and $\xi(C) = \infty$. In $d = 3$, $x = 1/2$.

$\sim L_t N^x$, with $x = 1/2$ in $d = 3$, according to the Ginzburg criterion (4).

Thus the line of a crossover with a slope $Ta/ma = N^x$ divides the region, $T > m$, of a 2-dimensional, or classical, behavior into two subregions, or windows: one window, where the mean-field approximation works ($L_t \ll \xi \ll L_t N^x$), and another window, where the non-trivial critical behavior sets in ($\xi \gg L_t N^x$).

IV. MONTE CARLO SIMULATIONS

We performed Monte Carlo simulations of the Gross-Neveu model in $d = 2+1$ dimensions at finite temperature to test the results of the analysis of the previous section. We chose the Z_2 four-fermion model because it is relatively easy to simulate. We used a hybrid Monte Carlo method described in [11], which proved to be very efficient for our purposes. Since the chiral symmetry is discrete, we were able to simulate the model directly in the chiral limit, i.e., setting the bare fermion mass to zero. This allowed a particularly accurate determination of the critical properties. The action is that of the Yukawa lattice theory (7) with $\kappa = \lambda = 0$, and we tuned β to reach criticality. The long-wavelength (continuum limit) behavior of such a theory is determined by the infrared fixed point, which is the same [5,10] in the more general Yukawa model (7) and in the Gross-Neveu model.

We used the following two methods to optimize the performance of the hybrid Monte Carlo procedure. The first method consisted of tuning the effective number of fermion flavors N', which is used during the integration of the equations of motion along a microcanonical trajectory, so as to maximize the acceptance rate of the Monte Carlo procedure for a fixed microcanonical time-step $d\tau$. As the lattice size was increased, the time step $d\tau$ had to be taken smaller and the optimal N' approached N. For example, for an $N = 4$ theory on a 6×36^2 lattice the choices $d\tau = 0.15$ and $N' = 4.036$ gave acceptance rates greater than 95% for all couplings of interest. To maintain this acceptance rate on a 6×60^2 lattice we used $d\tau = 0.11$ and $N' = 4.016$, while on a 6×80^2 lattice we used $d\tau = 0.10$ and $N' = 4.012$. In the second method the Monte Carlo procedure was optimized by choosing the trajectory length τ at random from a Poisson distribution with the mean equal to $\bar{\tau}$. This method of opti-

mization, which guarantees ergodicity, was found to decrease autocorrelation times dramatically [12]. For most of our runs we used the average trajectory length $\overline{\tau} \simeq 2.0$. As usual, the errors were calculated by the jackknife blocking, which accounts for correlations in a raw data set.

As will be seen below, we used values of the lattice coupling β sufficiently close to the critical value, β_c, so that we are close to the continuum limit $\Lambda \gg m$, where m is the thermal mass, and the scaling behavior is not affected by lattice artifacts. In addition, we verified that also another important physical parameter, the zero-temperature mass, m_0, is sufficiently smaller than the cutoff Λ. We ran on lattices with large L_t, such as 20^3 with $N=4$ for $\beta=0.600-0.750$, and 16^3 with $N=24$ for $\beta=0.725-0.875$, and determined the value of the scaling exponent β_m. For $N=12$ we can use the results from [11]. We found values in agreement with the analytical prediction $\beta_m = 1 + \mathcal{O}(1/N^2)$ for the $T=0$ scaling [11]. This confirms that for our values of the coupling β the lattice theory remains in the scaling window for all range of temperatures down to $T=0$ and effects of the lattice are negligible.

A. Exponents from finite size scaling

The finite size scaling (FSS) analysis is a well-established tool for studying critical properties of phase transitions [13]. The critical, singular behavior in a statistical system is caused by the divergence of the correlation length ξ. On a finite lattice the correlation length is limited by the size of the system and, consequently, no true criticality can be observed. However, if the size, L_s, of the lattice is large, a qualitative change in the behavior of the system occurs when $\xi \sim L_s$. For $1 \ll \xi \ll L_s$ the behavior of the system is almost the same as in the bulk ($L_s = \infty$). However, when $\xi \sim L_s$ the behavior of the system reflects the size and the shape of the box to which it is confined. The dependence of a given thermodynamic observable, P, on the size of the box, L_s, is singular and, according to the FSS hypothesis, is given by

$$P(t, L_s) = L_s^{\rho_P/\nu} Q_P(t L_s^{1/\nu}), \tag{9}$$

where t is the distance from the critical point, $t=(\beta_c - \beta)/\beta_c$; ν is the standard exponent of the correlation length, $\xi \sim t^{-\nu}$; and Q_P is a scaling function, which is not singular at zero argument. The exponent ρ_P is the standard critical exponent for the quantity P: $P \sim t^{-\rho_P}$. Studying the dependence on the size of the box, L_s, and using Eq. (9) one can determine such exponents.

We simulated the model with $N=12$ fermion flavors at β close to the critical coupling β_c. The lattice sizes ranged from $L_s=12$ to 40 for $L_t=4$, and $L_s=18$ to 50 for $L_t=6$. Periodic boundary conditions in the spatial directions were used. Details of the $L_t=6$ runs are listed in Table I. To perform our study most effectively we used the histogram reweighting method [14] which enables us to calculate the observables in a region of β around the simulation coupling $\beta_{\rm sim}$.

TABLE I. Simulations for the FSS analysis with $L_t=6$ and $N=12$.

L_s	$\beta_{\rm sim}$	Trajectories
18	0.7744	130,000
18	0.7763	100,000
24	0.7744	100,000
24	0.7764	80,000
30	0.7744	120,000
30	0.7764	130,000
40	0.7754	140,000
50	0.7754	152,000

1. Exponent ν

We used the following thermodynamic observables [15] to determine the values of ν and the critical coupling β_c:

$$V_1 \equiv 4[\phi^3] - 3[\phi^4], \quad V_2 \equiv 2[\phi^2] - [\phi^4],$$
$$V_3 \equiv 3[\phi^2] - 2[\phi^3], \tag{10}$$

where

$$[\phi^n] \equiv \ln \frac{\partial \langle \phi^n \rangle}{\partial \beta}. \tag{11}$$

One can easily find that

$$V_j \approx (1/\nu) \ln L_s + \mathcal{V}_j(t L_s^{1/\nu}), \tag{12}$$

for $j=1,2,3$. At β_c, i.e., $t=0$, the last term on the right-hand side (RHS), $\mathcal{V}_j(0)$, is a constant independent of L_s. Scanning over a range of β's and looking for the value of β at which the slope of V_j versus $\ln L_s$ is j-independent, as it is in Fig. 4, we found $\nu = 1.00(3)$ and $\beta_c = 0.7762(15)$ for $L_t=6$, and $\nu = 1.00(2)$ and $\beta_c = 0.682(2)$ for $L_t=4$.

These values of ν are in a very good agreement with the two-dimensional Ising value $\nu=1$. This confirms that the behavior of the system sufficiently close to criticality is non-

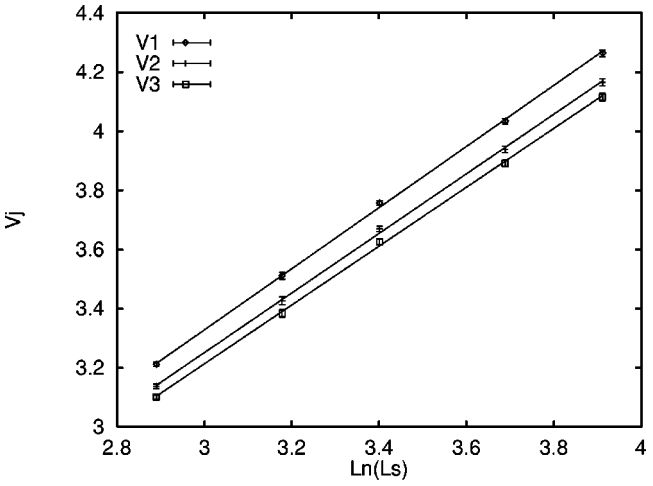

FIG. 4. Finite size dependence of V_j at $\beta=0.7762$ for $L_t=6$. All three lines have almost equal slopes.

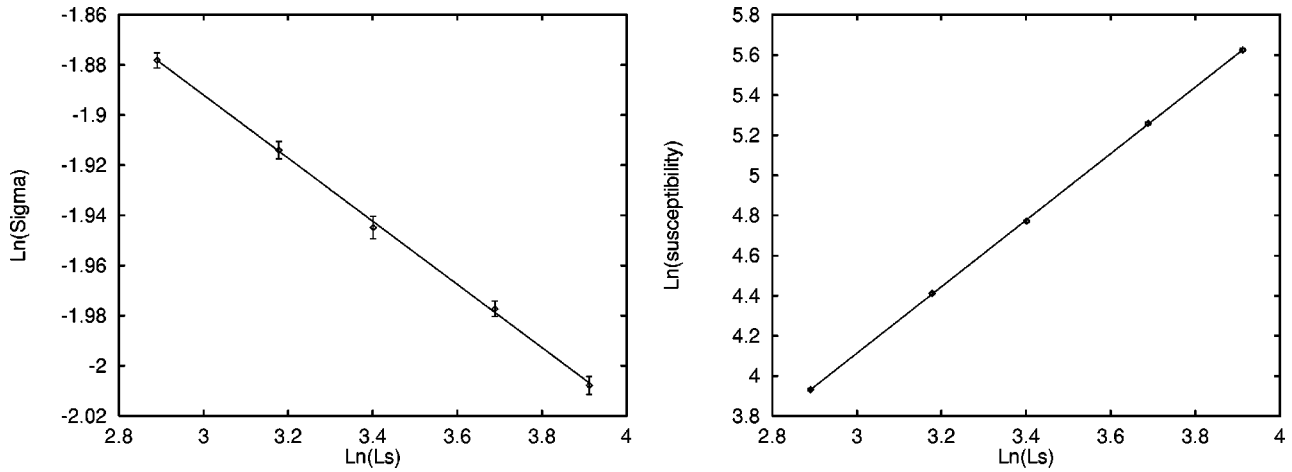

FIG. 5. Left: the best linear fit for $\ln\Sigma$ vs $\ln L_s$ in the minimum of χ^2 ($\beta=0.7747$). Right: the same for $\ln\chi$ vs $\ln L_s$ ($\beta=0.7753$).

trivial (in the mean field theory, $\nu=1/2$) and is given by the arguments of dimensional reduction and universality.

2. Exponents β_m and γ

In this subsection we consider the following two exponents: the exponent β_m of the order parameter, $\Sigma\equiv\langle\phi\rangle\sim t^{\beta_m}$, and the exponent γ of the susceptibility, $\chi\sim t^{-\gamma}$. According to Eq. (9), at the critical point, $t=0$, the order parameter, Σ, and its susceptibility, χ, scale with L_s as $\Sigma\sim L_s^{-\beta_m/\nu}$ and $\chi\sim L_s^{\gamma/\nu}$.

At each value of β we made a linear χ^2-fit for $\ln\Sigma$ and $\ln\chi$ versus $\ln L_s$. The locations of the minima of χ^2/DOF as a function of β provide estimates of β_c. We estimated the error in β_c and the critical exponents by looking at the values of β that increase χ^2/DOF by 1. An estimate of the error in β_c is $\min(\chi^2/\mathrm{DOF})+1$, which also gives the error on the critical exponents. Fits at $L_t=6$ for the order parameter and susceptibility gave $\beta_m/\nu=0.12(6)$, $\beta_c=0.7747(15)$ and $\gamma/\nu=1.66(9)$, $\beta_c=0.7750(15)$ respectively. Similar results at $L_t=4$ are $\beta_m/\nu=0.16(7)$, $\beta_c=0.6805(15)$ and $\gamma/\nu=1.57(9)$, $\beta_c=0.6817(15)$.

The critical exponents which we found are in good agreement with the two-dimensional Ising exponents: $\beta_m/\nu=0.125$ and $\gamma/\nu=1.75$ (in the mean field theory: $\beta_m/\nu=1$ and $\gamma/\nu=2$). The values of β_c extracted from this analysis are also consistent with the estimate of $\beta_c=0.7765(15)$ evaluated from the analysis of V_j. Figure 5 shows the best linear fits of finite size dependence of $\ln\Sigma$ and $\ln\chi$ at $\beta=0.7747$ and $\beta=0.7753$ respectively.

B. Scaling in the broken phase

Our task in this section is to check the analytical result of Eq. (4), which is equivalent to

$$\xi\ll L_t N^x, \quad x=1/2. \tag{13}$$

In other words, we want to determine the position of the crossover from the mean-field (MF) critical behavior to the non-trivial 2D Ising critical behavior as a function of N.

A straightforward way to do this is to study the dependence of the order parameter, Σ, on β. Since Σ vanishes at the critical point, it can be thought of as a measure of the distance from the criticality. We expect that for sufficiently small Σ, i.e., close to β_c, this dependence should be given by a power-law scaling with the exponent $\beta_m=1/8$ of the 2D Ising model: $\Sigma\sim(\mathrm{const}-\beta)^{1/8}$. This corresponds to the region CD in the diagram in Fig. 3, i.e., $\xi\gg L_t N^x$. For larger Σ, farther away from criticality, the MF scaling holds: $\Sigma\sim(\mathrm{const}-\beta)^{1/2}$ (with some other value of const). This is the region DB in Fig. 3, i.e., $L_t\ll\xi\ll L_t N^x$. For even larger Σ we should see scaling corresponding to the fixed point of the 3D Gross-Neveu model [11]: $\Sigma\sim(\mathrm{const}-\beta)^1$. In this region $1\ll\xi\ll L_t$.

Since we are interested in the boundary of the mean-field scaling region, we can find ξ from Σ using a well-known relation between them, $\xi=1/(2\Sigma)$ [16], which holds inside the mean-field region. Thus we avoid direct measurements of

TABLE II. The data for the scaling of Σ in the broken phase. $N=4$.

β	Σ	L_s	Trajectories
0.595	0.5008(4)	36	30,000
0.600	0.4871(4)	36	40,000
0.605	0.4724(4)	36	40,000
0.610	0.4578(4)	36	40,000
0.615	0.4426(4)	36	50,000
0.620	0.4268(4)	36	54,000
0.625	0.4106(3)	36	70,000
0.630	0.3928(4)	36	64,000
0.635	0.3736(7)	36	70,000
0.640	0.3524(5)	36	110,000
0.645	0.3297(8)	36	130,000
0.6475	0.3162(9)	36	150,000
0.650	0.3016(10)	48	54,000
0.652	0.2858(15)	60	41,000
0.654	0.2647(26)	80	25,400
0.656	0.2398(30)	80	30,000

TABLE III. The data for the scaling of Σ in the broken phase. $N=12$.

β	Σ	L_s
0.640	0.5112(2)	30
0.650	0.4911(3)	30
0.660	0.4701(2)	30
0.670	0.4490(3)	30
0.680	0.4276(2)	30
0.690	0.4061(3)	30
0.700	0.3836(2)	30
0.710	0.3608(4)	30
0.720	0.3368(4)	30
0.730	0.3119(4)	30
0.740	0.2847(4)	30
0.750	0.2557(8)	30
0.755	0.2393(5)	30
0.760	0.2199(5)	30
0.764	0.2048(2)	60
0.766	0.1951(3)	60
0.768	0.1844(3)	60
0.770	0.1725(3)	60
0.772	0.1575(5)	60
0.774	0.1388(7)	60

ξ, which are much harder than measurements of Σ.

We studied the dependence of Σ on β at three different values of $N=4,12,24$ on lattices with fixed $L_t=6$. Ideally, in order to resolve all three critical scaling regions, or windows, we need to provide $1 \ll L_t \ll L_t N^x \ll L_s$. For finite L_t, L_s, and N_x these regions are squeezed, but for the values we used one can clearly resolve the MF region with the crossover towards the 2D Ising region.

TABLE IV. The data for the scaling of Σ in the broken phase. $N=24$.

β	Σ	L_s	Trajectories
0.6700	0.4756(4)	24	3,000
0.6800	0.4560(4)	24	3,000
0.6900	0.4359(4)	24	3,000
0.7000	0.4162(4)	24	3,000
0.7100	0.3963(4)	24	3,000
0.7200	0.3760(5)	24	3,000
0.7300	0.3551(5)	24	3,000
0.7400	0.3325(5)	24	3,000
0.7500	0.3101(5)	24	3,000
0.7600	0.2866(3)	24	8,000
0.7700	0.2617(4)	24	8,000
0.7800	0.2345(5)	24	8,000
0.7900	0.2035(5)	24	10,000
0.7950	0.1874(3)	36	15,000
0.8000	0.1676(3)	36	20,000
0.8025	0.1568(4)	36	20,000
0.8050	0.1437(4)	48	24,000
0.8075	0.1267(10)	60	12,000
0.8100	0.1074(15)	60	22,000
0.8112	0.0914(28)	72	10,000

TABLE V. Goodness of linear fits of Σ^2 vs β for various ranges of β. $N=4$.

No. of points	β range	Goodness
4	0.595–0.610	0.92
5	0.595–0.615	0.95
6	0.595–0.620	0.98
7	0.595–0.625	0.99
8	0.595–0.630	0.96
9	0.595–0.635	0.21
10	0.595–0.640	10^{-5}
11	0.595–0.645	10^{-11}

In contrast to the FSS analysis where $\xi \sim L_s$, now we need to keep $\xi \ll L_s$ since we are studying bulk critical behavior. We monitored each simulation run for vacuum tunneling events, which signal that L_s is not big enough. Away from the critical point such events were so rare that good measurements of Σ and its susceptibility χ were possible. At couplings near the phase transition we increased L_s to suppress tunneling. A large number N, such as $N=24$, suppresses fluctuations and allows an accurate study within reasonable amount of computer resources. On the other hand, for smaller N, such as $N=4$, the simulations were also very efficient because the crossover to the 2D Ising behavior starts at a smaller correlation length. The data from these simulations is shown in Tables II, III, and IV.

Since $\beta_m=1/2$ in the MF region, we fitted Σ^2 with a linear function of β. The linearity made the fitting procedure very efficient. We evaluated the goodness of fit for data in various ranges of β in order to find the boundaries of the MF region (see Tables V, VI, and VII). The drop in the goodness of fit when new data are added implies that the new points deviate from the MF behavior and they belong either to the Gross-Neveu–MF crossover region or to the MF–2D-Ising crossover region. Figure 6 shows the data for Σ^2 versus β, where the straight lines represent the best linear fits in the MF region. All three graphs show the MF–2D-Ising crossovers. The graphs for $N=12,24$ also clearly show the Gross-Neveu–MF crossover.

TABLE VI. The same as Table V but for $N=12$.

No. of points	β range	Goodness
5	0.70–0.74	4×10^{-3}
6	0.70–0.75	5×10^{-3}
5	0.71–0.75	0.20
4	0.72–0.75	0.67
5	0.72–0.755	0.77
6	0.72–0.760	0.60
7	0.72–0.762	0.40
8	0.72–0.764	5×10^{-4}
9	0.72–0.766	2×10^{-15}

TABLE VII. The same as Table V but for $N=24$.

No. of points	β range	Goodness
7	0.74–0.795	6×10^{-4}
8	0.74–0.800	1.2×10^{-3}
6	0.75–0.795	0.28
7	0.75–0.800	0.24
3	0.76–0.780	0.68
4	0.76–0.790	0.91
5	0.76–0.795	0.70
6	0.76–0.800	0.38
7	0.76–0.8025	0.17
8	0.76–0.8050	1.2×10^{-8}
9	0.76–0.8075	5×10^{-16}

We used the histogram reweighting method in order to extract the value of Σ at the ''border'' of the MF region with the 2D Ising region. The quality of the fit drops very sharply. This allowed us to use a very conservative estimate of the error, such as the width of the region of Σ in which the quality of the fit drops down to 1% (from, e.g., 90% in the case of $N=4$). The center of this region of the drop is taken as the boundary of the MF scaling window. We find $\Sigma_{\rm MF}=0.377(11)$ for $N=4$, $\Sigma_{\rm MF}=0.213(4)$ for $N=12$, and $\Sigma_{\rm MF}=0.168(20)$ for $N=24$. It is clear that the non-trivial 2D Ising region is squeezed as N increases. In order to check the analytical prediction of Eq. (4), or (13), we fitted our results to the form $\Sigma_{\rm MF}=\text{const}\times N^{-x}$ (see Fig. 7). We found $x=0.51(3)$ which is in agreement with the analytical prediction $x=0.5$.

V. DISCUSSION AND CONCLUSIONS

In this paper we demonstrated, using analytical arguments and Monte Carlo simulations, that the width of the region near the finite temperature chiral phase transition where a non-trivial critical behavior sets in is suppressed in theories

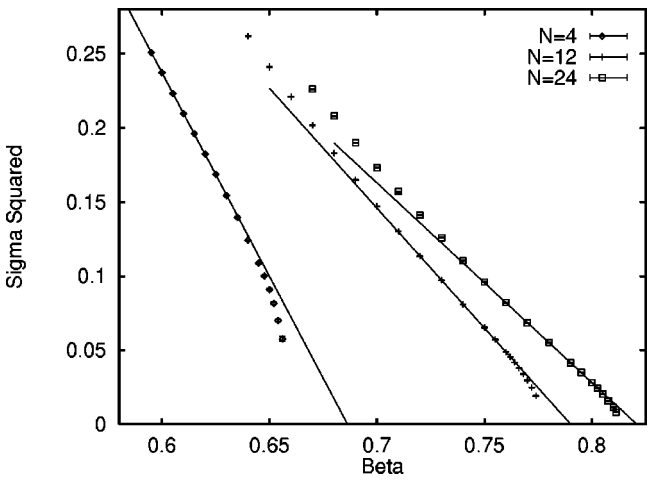

FIG. 6. Order parameter squared vs β for lattice theories with $N=4,12,24$. The straight lines are the fits to the data in the mean-field regions.

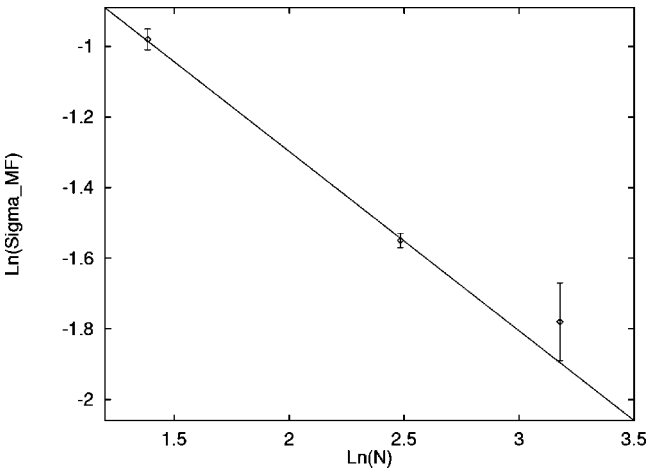

FIG. 7. The best power-law fit (goodness 0.26) for the dependence of the boundary of the mean-field region, $\Sigma_{\rm MF}$, on the number of fermions, N.

with a large number of fermions, N. This phenomenon explains an apparent contradiction between the large-N expansion [3,4], predicting mean-field scaling, and more general arguments based on dimensional reduction and universality [1], predicting the non-trivial scaling of a scalar theory in $d-1$ dimensions.

A tentative explanation of [3] was based, somewhat implicitly, on the expectation that the large-N expansion breaks down and cannot predict correct non-trivial exponents. Our key point is that we can say *when*, and as a function of what parameter (i.e., as a function of the distance from the criticality, m/T), the large-N expansion breaks down. What is, perhaps, even more important, is that we show that there does exist a region where the large-N expansion *is* valid. This region (in the space of m/T) of the mean-field behavior squeezes out the region of the true non-trivial critical behavior.[6]

One can also look at the whole problem as a question of the order of limits. If the limit $N\to\infty$ is taken before $m/T\to0$, the non-trivial region disappears, and all critical behavior is given by mean-field theory. If, on the other hand, the limit $m/T\to0$ is taken at fixed N, the true non-trivial scaling will hold. These effects can be clearly seen in our Monte Carlo studies.

A helpful analogy can be drawn with the question of the long distance behavior of the scalar field correlator in the original 2-dimensional Gross-Neveu model with a U(1) symmetry at large N [18]. The large-N expansion predicts spontaneous symmetry breaking and long-range order, which is in clear contradiction with the Mermin-Wagner theorem. As was shown by Witten [19], there is an ''almost long-range order'' in the system: i.e., the correlator falls off like $r^{-1/N}$. Looking at this expression, one can easily see the interplay between the limits $r\to\infty$ and $N\to\infty$.

We applied our arguments to a specific example of a

[6]It was conjectured in [17] that the Ginzburg region of non-trivial scaling could turn out to be small.

Yukawa theory, but the mechanism responsible for this phenomenon is clearly more universal and may apply, in particular, to QCD, provided that the number of colors, N_c, is large. The Yukawa or the Nambu–Jona-Lasinio theory is well-known to provide a very good description for the phenomenon of chiral symmetry breaking and restoration. One can then think of the theory we considered as an effective description of the degrees of freedom participating in chiral symmetry breaking.

As we have seen, the role of the fermions is to screen the effective self-coupling of the scalar field, λ. The strength of this effect depends on two factors: (i) a large N and (ii) a large window of scales between the cutoff of the effective theory, Λ, and the temperature, T. Let us see if these conditions are satisfied in QCD. The scale of the spontaneous symmetry breaking is of order $\Lambda \sim 1\,\mathrm{GeV}$, while $T_c \approx 160$ MeV. Thus, there is almost an order of magnitude window between Λ and T, which is presumably sufficient to drive the effective self-coupling of the scalar field to its infrared fixed point at the scale of T_c. How small this value is now depends on the number of the fermions [condition (i)]. In QCD the value $N_c=3$, though not very large, can be considered large in some cases. It would be interesting to see if this phenomenon could be rigorously shown to occur in the limit $N_c \to \infty$, which is very plausible.

The effect of the suppression of the width of the non-trivial critical region in QCD may lead to the following prediction: only mean-field [but no non-trivial $O(4)$] scaling behavior could be seen because of non-zero quark masses, m_q. On the one hand, the mean-field scaling would hold until a relatively large thermal correlation length. On the other hand, this correlation length in the real world is limited by the quark masses, m_q.

The following crude estimates can serve to illustrate this point.[7] A true criticality is never reached near T_c, because m_q plays the same role as the external ordering magnetic field in a ferromagnet. The largest thermal correlation length in units of $1/T$, T/m, can be estimated using the analogy to a ferromagnet and the mean-field value of $\nu/(\beta\delta)=1/3$, as:[8]

$$\frac{T}{m} \sim \left(\frac{T_c}{m_q}\right)^{\nu/\beta\delta} \approx \left(\frac{160\ \mathrm{MeV}}{5\ \mathrm{MeV}}\right)^{1/3} \sim 3. \tag{14}$$

The fact that this number is not large could be guessed by observing that the zero temperature pion masses, which are driven by m_q, are as large as T_c. This largest correlation length may turn out to be smaller than the one required for the crossover to non-trivial scaling region, $T/m \sim N_c \ln(\Lambda/T_c) \sim 6$, according to Eq. (5). In this case, the (near-)critical behavior observed in the window allowed by non-zero quark masses [according to Eq. (14), roughly, $1 < T/m < 3$] will be given entirely by the mean-field scaling.

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[7] To make these estimates quantitative one needs to find numerical coefficients, which are determined by non-universal dynamical properties of QCD and by the definition of the boundary of the mean-field region.

[8] We use the definition of the exponents, $m \sim t^{\nu}$, $\Sigma \sim t^{\beta}$, and $\Sigma \sim m_q^{1/\delta}$, together with the central postulate of the scaling theory: critical properties are determined by the correlation length, $1/m$.

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