Tests of hypercubic fermion actions

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I have performed scaling tests using quenched spectroscopy of a family of fermion actions which have a hypercubic kinetic term, gauge connections built of smeared links, and an anomalous magnetic moment term. These actions show improved rotational invariance compared to the standard Wilson action and to the tadpole-improved clover action. Hyperfine splittings are improved compared to the standard Wilson action (at the level of a factor of three in the lattice spacing), and are about the same as for the tadpole-improved clover action. [S0556-2821(98)01221-1]

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I. INTRODUCTION

I report on tests of a family of fermion actions for lattice gauge theory simulations, which are designed to improve the scaling of hadron spectroscopy. While very expensive to simulate, they appear to more than repay their computational cost with better scaling behavior.

The actions which I tested were inspired by the fixed point (FP) action program [1,2] for fermions [3–10]. However, they are not FP actions. They are Wilson-fermion like, in the sense that they have four component spinors on all sites and there are no manifest symmetries which protect the bare quark mass from being additively renormalized.

The new features which I tested include

(1) a hypercubic kinetic energy term. Each fermion in the action communicates with $3^4 - 1 = 80$ nearest neighbors. This term improves the hadron dispersion relation compared to that from actions using a standard on-axis nearest neighbor coupling.

(2) Gauge connections built of very fat gauge links, links built by averaging the fundamental link variables over a local region and re-projecting them back onto the gauge group. This construction removes short distance fluctuations from the correlators during the simulation process, rather than attempting to divide them out at the end. This results in an observed, very small, additive mass renormalization of bare quantities, and I conjecture that all perturbative corrections to observables are considerably reduced. To include very fat links in a simulation with dynamical fermions using known technology might be quite expensive, but in quenched simulations the cost is minimal.

I also tested a complicated lattice anomalous magnetic momentum term in which the quark and antiquark are not fixed to the same site. Some kind of term is needed to correct the lattice free quark magnetic moment and to improve meson and baryon hyperfine splittings. However, my tests show that the standard "clover" term, suitably normalized, improves scaling as much as the complicated Pauli term needed to satisfy the FP equations.

The scaling tests in this paper are a bit non-standard (when compared to other studies of spectroscopy) and deserve some explanation. I am interested in scaling tests which are uncontaminated by extrapolations in volume or to the chiral limit. Thus I compare only simulations in fixed physical volume. Any volume would do, and so I choose a small one simply because these actions are expensive to simulate. To set the scale I use a gluonic observable because these are quenched simulations. As a choice of gluonic observables, one has T_c , the critical temperature for deconfinement, the string tension σ or the Sommer radius r_0 . Glueball or torelon measurements are just too costly. Of these observables, the ones associated with the potential (σ and r_0) require a fit to a function V(r); the choice of the fitting function can affect the results. At a very coarse lattice spacing this problem becomes more serious. Thus I use T_c to set the scale, and do simulations on lattices of fixed size $L=2/T_c$. With $\sqrt{\sigma} \approx 440$ MeV and $\sqrt{\sigma}/T_c \approx 1.60$, the scales are T_c =275 MeV, L=1.45 fm, and $2\pi/L=860$ MeV. Lattice spacings $aT_c = 1/2, 1/3, 1/4$ correspond to a = 0.36, 0.24, and 0.18 fm.

All the tests are performed at fixed physical quark mass (defined either by interpolating lattice data to a fixed value of m_{π}/m_{ρ} or to a fixed value of m_{π}/T_c). At very coarse lattice spacings, and with heavy (though still relativistic) quarks, scaling violations from conventional actions are very large. Thus only modest statistics are required to identify improvements—or lack thereof—compared to them.

Besides spectroscopy, I measure the meson or baryon dispersion relation. In lattices of fixed physical volume set by T_c , the physical momenta corresponding to the different allowed lattice modes are multiples of T_c , $a\vec{p} = 2\pi\vec{n}/L$ or $\vec{p} = \pi T_c \vec{n}$, if $L = 2/T_c$, and one can compare data with different lattice spacings at the same physical momentum. Wilson and clover fermions at $\beta < 6.0$ (using the Wilson gauge action) exhibit bad scaling or rotational invariance violations. The new actions are rotationally invariant even at a = 0.36 fm.

The outline of the paper is as follows: In Sec. II I present a general discussion of the construction of a FP fermion action. In Sec. III I describe the new features of these actions. Section IV is devoted to scaling tests, and I make some tentative conclusions in Sec. V. I describe the (new) FP gauge action used in these simulations in the Appendix.

II. FINDING FIXED-POINT FERMION ACTIONS

The main problem in doing simulations with actions containing many parameters is that one has to have some organizational scheme which keeps track of them, and it may be useful to understand the theoretical background of the actions (if any) before proceeding to tests.

To find a FP action for QCD, I begin with a set of fermionic $(\psi_n, \bar{\psi}_n)$ and gauge field $(U_{\mu}(n))$ variables defined on a fine lattice (or perhaps in the continuum). I assume a fine action

$$S = \beta S_g(U) + \overline{\psi}_i \Delta(U)_{ij} \psi_j. \qquad (2.1)$$

 S_g is the gauge action, i,j label sites, and $\Delta(U)$ is the fermion action. I then integrate out these degrees of freedom to construct an action involving coarse-grained variables Ψ_{n_b} , $\overline{\Psi}_{n_b}$ and $V_{\mu}(n_b)$. This is done using a renormalization group kernel

$$T = \beta T_g(U, V) + \kappa \sum_{n_b} (\bar{\Psi}_{n_b} - \bar{\psi}_n \Omega_{n, n_b}^{\dagger}) (\Psi_{n_b} - \Omega_{n_b, n'} \psi_n').$$
(2.2)

 T_g is the blocking kernel for the gauge fields and Ω is some local averaging function, and κ is a parameter of the renormalization group theory (RGT). The renormalization group equation

$$e^{-S'} = \int d\psi d\bar{\psi} dU e^{-(T+S)}$$
(2.3)

has a pure gauge FP at $g^2=0$ ($\beta \rightarrow \infty$). In that limit the gauge action dominates the integral; its RG equation is given by the same steepest-descent equation as for a pure gauge model

$$S^{FP}(V) = \min_{\{U\}} (S^{FP}(U) + T(U,V)), \qquad (2.4)$$

while the fermions sit in the gauge-field background. Their action remains quadratic in the field variables, and the transformation of the fermion action is given most easily in terms of the propagator

$$(\Delta'(V))_{n_b,n_b'}^{-1} = \frac{1}{\kappa} \delta_{n_b,n_b'} + \Omega(U)_{n_b,n} (\Delta(U))_{n,n'}^{-1} \Omega(U)_{n',n_b'}^T$$
(2.5)

where $\{U\}$ is the field configuration which minimizes Eq. (2.4) for a given $\{V\}$.

In all my work I have focused on a scale factor 2 RG transformation, for gauge fields the so-called "Type-1" transformation of Ref. [2].

One can determine the FP vertex by expanding Eq. (2.4) in a power series in the gauge coupling g and keeping only the lowest order term

$$S_{F} = \sum_{x,y} \bar{\psi}(x) \bigg[(\Delta^{(0)}(x-y)) + ig \sum_{z} \sum_{\mu} \Delta^{(1)}_{\mu}(x-y,z-y) A_{\mu}(z) \bigg] \psi(y). \quad (2.6)$$

This has been presented in Refs. [6, 7] for a particular RGT. Having a parameterization of the FP vertex which appears to be valid for smooth gauge configurations, one can then attempt to find a FP action appropriate to rough configurations. However, the FP vertices are quite complicated, and at the couplings appropriate to real simulations, a naive exponentiation of the FP vertex does not result in a good solution to a FP action. Instead, I short-cut the procedure by going directly to a nonperturbative construction.

I test actions by generating coarse configurations $\{V\}$ and solving Eq. (2.4) to form a set of fine links $\{U\}$. I assume some functional form for the fine fermion action $\Delta_f(U)$, then find the propagator on the coarse configuration from a test source *S* by solving the RG equation Eq. (2.5):

$$\psi_{n_b} = (\Delta'(V))_{n_b, n_b'}^{-1} S_{n_b'}.$$
(2.7)

I use a source of Gaussian random numbers distributed over the lattice.

I then parametrize the coarse action $\Delta_c(V)$ with a set of parameters multiplying operators and minimize the "goodness of fit"

$$\chi^2 = \sum |\Delta_c \psi - S|^2 \tag{2.8}$$

or

$$r^{\dagger}r = \chi^2 / S^{\dagger}S \tag{2.9}$$

[the latter is the familiar normalized (squared) residue minimized by all sparse matrix inverters] by varying parameters in Δ_c . Some fits let the free field kinetic parameters be free parameters. Typically, however, I have had more success by fixing the overall normalization of the parameters in the action to their free field values and tuning parameters in the trial gauge connections.

As the reader can see, it is necessary to parametrize both Δ_f and Δ_c . At the FP, $\Delta_f = \Delta_c$. Because the fine link variables are smooth at short distances, Δ_f naively resembles some gauge invariant extension of the free action and the FP vertex, and this fact can motivate an initial choice for the fine action. Because blocking carries the action towards its FP, $(\Delta'(V))_{n_b,n_b'}^{-1}$ is a better approximation to the FP propagator than Δ_f^{-1} is. One can pick an initial Δ_f , test many parametrizations of Δ_c , choose the best, and (in principle) iterate towards a solution.

As an example of a test, compare Fig. 1. On the fine lattice the bare mass is $m_0 = 0.04$. The input bare mass of the coarse action is varied along each curve. (An exact parametrization of a FP action would put the minimum of χ^2 at a coarse mass whose value is twice the fine mass.) The coarse



FIG. 1. A test of the fixed point structure of a candidate FP action. The curves correspond to a fine action with a bare mass $m_0 = 0.04$. The bare mass of the coarse action is tuned. The families of curves correspond to decreasing β 's as they ascend. If the bare mass were doubling under an RGT, the optimal coarse action would have bare mass 0.08.

and fine actions are identical (they are action C of Sec. IV). The curves represent different coarse couplings: in ascending order, they are $\beta = 20$, 10, 5, 3.70 ($\beta_c(N_t=4)$), 3.50 ($\beta_c(N_t=3)$), and 3.092 ($\beta_c(N_t=2)$).

All of my studies testing actions are done on small lattices, 3^4 or 4^4 (coarse) volumes. I have not included instanton configurations in any fitting. The fit to a FP action gradually worsens as one moves towards stronger couplings, and the fits become less sensitive to the particular choice of parametrization. Presumably all the parametrizations I have tested are leaving out some common feature which is important for rough configurations. It is also the case that the gauge action is a bad parametrization of a FP action for β $<\beta_c(N_t \approx 3)$.

III. INGREDIENTS OF THE ACTIONS

A. Hypercubic kinetic term

The fermionic free field action has the generic form

$$\Delta_0(x) = \lambda(x) + i \sum_{\mu} \gamma_{\mu} \rho_{\mu}(x). \qquad (3.1)$$

It is constructed by finding some free FP action [3–6]. I begin with a continuum action for fermions which has no doublers and is chirally symmetric. I construct an action on a coarser distance scale Δ' by iterating the free field limit of the FP equation (2.5). I select a blocking kernel, iterate the RGT to find a fixed point action, and then tune parameters in Ω to make the action maximally local. I have used a factorof-two rescaling in which Ω is restricted to a hypercube: Ω_{ij} is nonzero only if $j=i\pm\mu, i\pm\mu\pm\nu, \ldots i\pm\mu\pm\nu\pm\lambda\pm\sigma$, and $\Omega_{ij}=c((i-j)_1,(i-j)_2,(i-j)_3,(i-j)_4)$. Each site communicates to $3^4-1=80$ neighbors.



FIG. 2. Dispersion relation E(p) vs |p| for $m_0=0$ for the hypercubic action.

There are many good parametrizations, resulting in fairly local FP actions. However, one ultimately wants to use these actions in simulations, and the action must be somehow truncated. There are a number of (subjective) criteria to select a good RGT, based on the properties of the truncated action (which the RGT does not know about): a good dispersion relation, $E(p) = |\vec{p}|$ out to large $|\vec{p}|$ with no complex roots, good free-field thermodynamics, $P = 1/3\sigma T^4$ even at large discretization, etc. I tuned the RGT to optimize these criteria, and my choice is given by c(1,0,0,0) = 0.03, c(1,1,0,0) = 0.01, c(1,1,1,0) = 0.005, c(1,1,1,1) = 0.0025, and $\kappa = 44.0$.

The couplings of the FP action for my RGT (for massless fermions) fall off exponentially with $r = \sqrt{\Sigma_{\mu} x_{\mu}^2}$. The largest entries at distance r=2 are at location $x = (\pm 1, \pm 1, \pm 1, \pm 1)$. The smallest truncation which accurately reproduces the main features of this FP action is to an action which sits on a hypercube—that is, the free field action is to be param-



FIG. 3. Dispersion relation E(p) vs |p| for massless Wilson fermions.

etrized with five nonzero λ 's and four nonzero ρ 's, corresponding to each of the nonzero offsets. An example of a dispersion relation for this hypercubic action is compared to the Wilson action in Figs. 2 and 3. I show both branches of the hypercubic action's dispersion relation; all roots are real. The non-truncated FP action has a perfect dispersion relation $E = |\vec{p}|$ for all \vec{p} .

For massive fermions, I need an action which is on a renormalized trajectory (RT) for some RGT (with the mass scaling at each step with the ratio of lattice spacings). To reach the RT, one can begin with an action which has a very small mass but is otherwise close to a FP action, perform a series of blockings, and follow it out.

One complication with this procedure is that an action which is local for a small mass can block into an action for large mass which is not short range. To avoid this, I take an RG transformation whose parameters are functions of the mass and tune the parameter(s) to insure a local action at each blocking step. The resulting λ 's and ρ 's are smooth functions of the mass. Again, the dispersion relation for hypercubic approximations to RT actions are well behaved out to large $|\vec{p}|$.

Plots of the variation of the parameters—or the tables of numbers corresponding to them—are by themselves not very useful for calculation. Rather than give them, I will immediately present a simple parametrization of the couplings.

My choice of parametrization is to regard all the ρ 's and λ 's as linear functions of the bare mass and to determine $\lambda(0,0,0,0)$ for a positive bare mass by solving the dispersion relation for it: defining

$$R = 2\rho_0(1,0,0,0) + 12\rho_0(1,1,0,0) + 24\rho_0(1,1,1,0) + 16\rho_0(1,1,1,1)$$
(3.2)

$$A_s = 2\lambda(1,0,0,0) + 12\lambda(1,1,0,0) + 24\lambda(1,1,1,0) + 16\lambda(1,1,1,1)$$
(3.3)

$$D = 8\lambda(1,0,0,0) + 24\lambda(1,1,0,0) + 32\lambda(1,1,1,0) + 16\lambda(1,1,1,1)$$
(3.4)

the pole of the propagator $[\lambda^2(p) + \sum_{\mu} \rho_{\mu}^2(p) = 0$ at $p_{\mu} = (im_0, 0, 0, 0)]$ is at

$$\lambda(0,0,0,0) = -A_s(\cosh(m_0) - 1) - R \sinh(m_0) - D \qquad (3.5)$$

(with the convention that all $\rho'_j s$ are negative). My linear parametrization runs over $0 < m_0 < 0.32$, which is the useful range of mass for light hadron spectroscopy for lattice spacings $aT_c = 1/2$ to 1/4.

The big problem in parametrizing an approximate action is that the additive renormalization of the quark mass for $g^2 \neq 0$ drives the critical quark mass m_q^c negative, outside the region where the solution of the FP equation is meaningful. With Wilson fermions, and one free parameter (κ) this is not a problem (one just tunes κ above κ_c without otherwise altering the action) but with these complicated actions there is not a clear cut way to proceed. In principle, all the parameters in the action should depend on properties of the gauge field (for example, on the local value of the plaquette). To circumvent this, I make an arbitrary choice: I assume that the parameters continue to vary linearly with the bare mass, and I determine $\lambda(0,0,0,0)$ by continuing the low-mass limit of Eq. (3.5) to a negative mass.

$$\lambda(0,0,0,0) = -Rm_0 - D. \tag{3.6}$$

Since $m_q^c \approx -0.4$ at $\beta_c(N_t=4)$ and moves toward zero at bigger β 's, $\lambda(0,0,0,0)$ is basically a linear function of m_0 .

This parametrization is not appropriate for studying charm with this action (where $am_0=2$ to 4 depending on the lattice spacing). The parameters of the action on the RT are

smooth functions of m_0 , so one could probably construct a more complicated (polynomial or exponential) fit to them.

Table I gives the parameters of the linear fit.

B. Very fat links

Measurements of pure gauge observables (the potential or glueball masses) suffer from noise arising from the short distance fluctuations of the gauge fields. A good cure for this problem has been known for many years: define new link variables which do not couple to the UV sector of the lattice variables, and which have the same IR properties as the original variables. An example of such a variable is an APEblocked link [11]

$$V^{n+1}_{\mu}(x) = (1-\alpha) V^{n}_{\mu}(x) + \alpha/6 \sum_{\nu \neq \mu} (V^{n}_{\nu}(x) V^{n}_{\mu}(x+\hat{\nu}) V^{n}_{\nu}(x+\hat{\mu})^{\dagger} + V^{n}_{\nu}(x-\hat{\nu})^{\dagger} V^{n}_{\mu}(x-\hat{\nu}) V^{n}_{\mu}(x-\hat{\nu}+\hat{\mu}))$$
(3.7)

[with $V^0_{\mu}(x) = U_{\mu}(x)$ and $V^{n+1}_{\mu}(x)$ projected back onto SU(3)]. It is also known that for best results, both α and the maximum number of blocking steps N should increase as the lattice spacing decreases.

Fermions also suffer from bad UV behavior, and their symptoms include the breaking of flavor symmetry (for staggered fermions), large additive renormalization of the bare mass (for Wilson fermions), and large renormalizations of currents (for any kind of fermion). The tadpole improvement program [12] was originally designed to estimate or compute these large UV effects and subtract (or divide) them out during the conversion from the lattice calculation to continuum number.

TABLE I. Linear parametrization of the couplings of the hypercubic action: $\lambda(x_0, x_1, x_2, x_3) = \lambda_a + m_0 \lambda_b$, $\rho_0(x_0, x_1, x_2, x_3) = \rho_a + m_0 \rho_b$.

offset	λ_a	λ_b	$ ho_a$	$ ho_b$
0000	2.256756	-0.9863		
$1 \ 0 \ 0 \ 0$	-0.1122	0.0741	-0.1464	0.1300
$1 \ 1 \ 0 \ 0$	-0.0323	0.0271	-0.0329	0.0303
$1 \ 1 \ 1 \ 0$	-0.0144	0.0141	-0.0101	0.0096
1111	-0.0072	0.0076	-0.0035	0.0033

However, recent evidence suggests that it may in some cases be better to remove the UV fluctuations directly from the simulations. This evidence is the partial restoration of flavor symmetry breaking for staggered fermions by replacing the link by an N=1 APE-blocked link, as shown by Ref. [13] and (with a slightly different averaging) by Ref. [14]. These authors restrict themselves to N=1, presumably because they wish to use their actions for simulations with dynamical fermions. However, if one is interested in quenched simulations, one can APE-block to any desired level, with a tiny overhead, simply by pre-computing and storing the APE-blocked links. Then if N>1 improves UV behavior, one is free to use it.

It is easy to understand why the fat links suppress UV fluctuations [15]. Each term in the action in coordinate space

$$\mathcal{L}_{I} = \frac{1}{2a} \sum_{x,y,z} \overline{\psi}(x) \Gamma \dots U_{\mu}(x+y) \dots \psi(x+z) \quad (3.8)$$

can be expanded as a power series in g

$$\mathcal{L}_{I} = \frac{1}{2a} \sum_{x,y,z} \bar{\psi}(x) \Gamma \dots \left[1 + igaA_{\mu}(x+y+a/2\hat{\mu}) - \frac{1}{2}(ag)^{2}A_{\mu}(x+y+a/2\hat{\mu})^{2} \dots \right] \dots \psi(x+z)$$
(3.9)

which in momentum space becomes $\mathcal{L}_I = \mathcal{L}_I^1 + \mathcal{L}_I^2$ with

$$\mathcal{L}_{I}^{1} = i \frac{g}{2} \int_{p,q} \overline{\psi}(p) \Gamma \int_{k} \delta^{4}(k+q-p) A_{\mu}(k)$$
$$\times e^{i(y+a/2\hat{\mu})k} \psi(q) e^{iqz}$$
(3.10)

$$\mathcal{L}_{I}^{2} = \frac{g^{2}a}{2} \int_{p,q} \overline{\psi}(p) \Gamma \int_{k_{1},k_{2}} \delta^{4}(k_{1}+k_{2}+q-p) A_{\mu}(k_{1})$$
$$\times A_{\mu}(k_{2}) e^{i(y+a/2\hat{\mu})(k_{1}+k_{2})} \psi(q) e^{iqz}.$$
(3.11)

Smearing the link over a distance r_0 makes the replacement

$$A_{\mu}\left(r+\frac{a}{2}\ \hat{\mu}\right) \rightarrow \sum_{\mu,\nu} \sum_{w} h_{\mu\nu}(w) A_{\nu}\left(r+w+\frac{a}{2}\ \hat{\nu}\right) \quad (3.12)$$

$$A_{\mu}(k) \to \sum_{\mu,\nu} H_{\mu\nu}(k) A_{\nu}(k) e^{i(y+a/2(\hat{\nu}-\hat{\mu})k)} \qquad (3.13)$$

(and a similar formula for \mathcal{L}_I^2) where the form factor is $H_{\mu\nu}(k) = \sum_r h_{\mu\nu}(r)e^{ikr}$. Essentially any smearing function suppresses the vertex at $k > \pi/r_0$. In the language of Ref. [12], tadpoles contribute beyond their naive strength because the UV divergence of the gluon loop compensates for the *a*-dependence of the vertex; smearing suppresses the coupling of the fermion to high momentum gluons.

The reader might be (justifiably) concerned about the existence of a transfer matrix for these actions. As far as I know, no FP action has a transfer matrix connecting neighboring time slices t and t+1. This can be seen even for free scalar field theory [1]: the couplings of spins on different sites, the analogues of the ρ 's and λ 's of Eq. (3.1), die away exponentially with distance. Any finite truncation of a FP action (for r < L) presumably has a transfer matrix connecting t and t+L. For the free field hypercubic action, L=1, just like the usual Wilson or staggered actions. Fattening the links makes the range of the action much greater. With N levels of APE blocking, the action extends over a range $\pm N$ sites before it cuts off. Because this blocking is basically a random walk, the effective range of the action is $O(\alpha \sqrt{N})$ $\simeq 1.2$ sites for the particular parameters I used. Except at the largest lattice spacing, a standard fitting procedure for spectroscopy sees an apparent plateau in effective masses at T \geq 3 lattice spacings' separation of source and sink. It may be that the right thing to do is to be more careful, but I saw nothing in any of my spectroscopy computations which indicated any problems with a naive approach.

C. Nonlocal Pauli term

Lattice fermions have a magnetic moment which is anomalously small due to lattice artifacts. One can parametrize the vertex through the interaction of a fermion with an infinitesimal magnetic field *B*: the pole in the propagator will be at $E = m_0 + B/2m_B$, where m_B is the so-called magnetic mass. I write the momentum-space interaction term as

$$\bar{\psi}(p)i\Delta^{1}_{\mu}(p,-p')A_{\mu}(p-p')\psi(p')$$
(3.14)

and expand the vertex in Dirac space as

$$\Delta^{1}_{\mu}(k,-p) = f_{\mu,0}(k,-p) + f_{\mu,\nu}(k,-p) \gamma_{\nu} + \sum_{\rho < \nu} f_{\mu,\rho\nu}(k,-p) \gamma_{\rho} \gamma_{\nu} + f_{\mu,5}(k,-p) \gamma_{5} + f_{\mu,\nu5}(k,-p) \gamma_{\nu} \gamma_{5}$$
(3.15)

(with an identical labeling for the decomposition in coordinate space). A lot of algebra [5] gives

$$m_B = -\frac{\lambda(im_0)[\lambda'(im_0) - \rho'_0(im_0)]}{\rho'_2(im_0)f_{1,1}(im_0) - ic^1_{12}(im_0)\lambda(im_0)} \quad (3.16)$$

where

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or

TABLE II. Nonlocality of the Pauli term $ic_{12}^{1}(x,im)$ for actions along the RT.

x	$m_0 = 0.08$	$m_0 = 0.16$	$m_0 = 0.32$	$m_0 = 0.64$
0 0 0 0	-0.0913	-0.0840	-0.0709	-0.0527
$1 \ 0 \ 0 \ 0$	-0.2490	-0.2269	-0.1883	-0.1278
1 1 0 0	-0.2980	-0.2710	-0.2253	-0.1479
1 1 1 0	-0.1641	-0.1493	-0.1248	-0.0816
1111	-0.0318	-0.0291	-0.0246	-0.0167
Sum in hypercube: Total:	-0.8343 -0.8974	-0.7604 - 0.8182	-0.6340 -0.6837	-0.4269 -0.4807

$$\lambda(im) = \sum_{n} e^{mn_0} \lambda(n)$$
(3.17)

$$\lambda'(im) = \sum_{n} n_0 e^{mn_0} \lambda(n)$$
(3.18)

$$\rho_0'(im) = -i\sum_n n_0 e^{mn_0} \rho_0(n)$$
(3.19)

$$\rho_2'(im) = -i\sum_n n_2 e^{mn_0} \rho_2(n)$$
(3.20)

$$f_{1,1}(im) = \sum_{xy} e^{mx} f_{1,1}(x,y)$$
(3.21)

$$ic_{12}^{1}(im) = \sum_{xy} (x - 2y)_2 e^{mx} f_{1,12}(x,y)$$
 (3.22)

are all real.

All approximate FP vertices I have seen have a complicated Pauli term with sizable contributions when the quark



FIG. 4. $m_{\rho}/T_c \operatorname{vs} m_{\pi}/T_c$ for the Wilson action. Data are labeled with octagons for $aT_c = 1/2$, diamonds for $aT_c = 1/3$, crosses for $aT_c = 1/4$, squares for $aT_c = 1/8$, and fancy crosses for $aT_c = 1/12$.



FIG. 5. m_N/T_c vs. m_{π}/T_c for the Wilson action. Data are labeled as in Fig. 4.

and antiquark do not sit on the same lattice site. As an example, Table II shows the contribution of various fermion offsets to $ic_{12}^1(x,im)$ for a vertex based on the RGT I have been using in this paper. The normalization appears to be reasonably well saturated by fermion offsets over a cube. The gauge connections are very complicated.

However, it is not clear how important the FP version of the Pauli term will be in spectroscopy. I therefore studied three possibilities:

(1) No Pauli term at all. This turns out to give hyperfine interactions which are too small.

(2) Keep only the on-site part of the Pauli term (the standard clover term) but choose its normalization so that $m_B = m_0$. This is not a FP action. The gauge links will be fattened like the rest of the links in the action.

(3) Restrict the Pauli term to offsets which span a cube. For each offset, sum over all the minimum-length paths (with their sign factors) which contribute to the Pauli term.



FIG. 6. Bare squared pion mass (upper set of curves) and quark mass from Eq. (4.5) (lower set of curves) vs bare quark mass for action A, at β_c for $N_t = 2$ (squares), 3 (diamonds), 4 (crosses).



FIG. 7. Bare squared pion mass (upper set of curves) and quark mass from Eq. (4.5) (lower set of curves) vs bare quark mass for action C, at β_c for $N_t=2$ (squares), 3 (diamonds), 4 (crosses).

Choose the relative normalization of the terms to match the FP vertex and the overall normalization to fix $m_B = m_0$. Fatten the links if necessary. This choice (hereafter called a "full Pauli action") is a much better approximation to a FP action than the second choice ("clover action"), but if the clover action performs as well in a test, it is the action of choice. It is still only loosely related to a true FP action.

For $m_0 > 0$ the constraint $m_B = m_0$ fixes the normalization of the Pauli term. I find that the normalization varies roughly linearly with the bare quark mass. I choose (arbitrarily) to keep the same linear dependence with m_0 even for a negative bare mass. This would be equivalent in the standard clover action to making the size of the clover term a function of the hopping parameter κ , rather than a function of β . From a practical point of view the difference is slight: as one varies the gauge coupling in a simulation, the value of the bare quark corresponding to a particular physical hadron mass shifts, becoming (typically) more negative as β decreases. The input coefficient of the clover term becomes larger as β







FIG. 9. m_N/T_c vs m_π/T_c for action A.

decreases, so the net result is that one can as well say that the clover term tracks the bare quark mass, as to say that the it tracks the bare coupling. At zero gauge coupling FP actions do not have a mass-independent Pauli normalization and so the standard practice of making the normalization mass independent is unnatural. Presumably one could do simulations with a standard action, such as the Wilson-plus-clover action, tuning m_B to equal m_0 , although it is hard to see the point of doing this as long as the dispersion relation is imperfect.

The linear parametrization reproduces $m_B = m_0$ to within five per cent for $m_0 < 0.4$. For the clover hypercubic action I could have simply set $m_B = m_0$ for $m_0 > 0$ by inverting Eq. (3.16), although I did not do that.

IV. SCALING TESTS

A. Survey of actions tested

Most of the quenched spectroscopy has been done using a new parametrization of a FP gauge action for SU(3). In the Appendix I tabulate the critical temperature, string tension,



FIG. 10. m_{ρ}/T_c vs m_{π}/T_c for action C.



FIG. 11. m_N/T_c vs m_π/T_c for action C.

and Sommer parameters for the gluonic action, so the reader can convert to his favorite scaling variable. One test has been done using the original parametrization of the action presented in Ref. [16], and some tests use the SU(3) gauge action of our recent work on instantons [17]. I have made rough measurements of its $\beta_c(N_t)$ for deconfinement to set the scale. All the fermionic actions are made gauge-invariant by replacing the offsets by an average over the shortest distance gauge paths. For example,

$$\overline{\psi}(x)\psi(x+\hat{\mu}+\hat{\nu}) \rightarrow \frac{1}{2} \overline{\psi}(x)[V_{\mu}(x)V_{\nu}(x+\hat{\mu})$$
$$+V_{\nu}(x)V_{\mu}(x+\hat{\nu})]\psi(x+\hat{\mu}+\hat{\nu}) \quad (4.1)$$

where $V_{\mu}(x)$ is either one of the original links or an APEblocked link.

The cost of a hypercubic action per iteration step during matrix inversion is about 20 times as expensive as the usual Wilson action, since there are more neighbors and the Dirac connections are not projectors. Actions with the complicated Pauli term are about 56 times as expensive as the usual Wilson action. All the gauge connections are pre-computed, so there are startup and storage costs, as well. I used the stabilized biconjugate gradient (biCGstab) algorithm for matrix inversion [18].

The two actions which were tested most extensively both have a hypercubic kinetic term and APE-blocked links with N=7 and $\alpha=0.3$. Action A has a full Pauli term. It is the best approximation to a FP action I found. It used the very expensive gauge action of Ref. [17]. Action C has only the clover term but is otherwise identical. The particular choice



FIG. 12. A scaling test for new actions (squares action C, octagons action A) vs Wilson actions on lattices of fixed physical size (diamonds) and larger volumes (crosses), and the nonperturbatively improved (bursts) and tadpole improved (fancy crosses) clover actions. Data are interpolated to $\pi/\rho = 0.84$ (a), 0.80 (b), 0.75 (c) and 0.70 (d).

TABLE III. Table of best-fit masses, action C, $\beta = 3.092 (aT_c = 1/2)$.

am _q	PS	V	Ν	Δ
0.30	1.883(8)	2.066(8)	3.332(17)	3.450(21)
0.15	1.623(9)	1.861(10)	3.002(20)	3.156(25)
0.10	1.535(9)	1.796(11)	2.891(22)	3.061(27)
0.05	1.446(9)	1.733(12)	2.781(24)	2.967(30)
0.00	1.363(9)	1.672(14)	2.670(27)	2.875(33)
-0.05	1.251(10)	1.575(20)	2.526(37)	2.770(46)
-0.10	1.159(10)	1.551(20)	2.432(39)	2.687(43)
-0.15	1.039(12)	1.491(26)	2.303(54)	2.591(49)
-0.20	0.899(16)	1.429(35)	2.147(70)	2.505(64)

TABLE IV. Table of best-fit masses, action C, $\beta = 3.50 (aT_c)$ = 1/3).

am_q	PS	V	Ν	Δ
0.15	1.156(7)	1.363(11)	2.077(21)	2.258(28)
0.10	1.066(15)	1.328(20)	2.090(33)	2.277(51)
0.05	0.935(8)	1.218(15)	1.816(30)	2.049(31)
0.00	0.825(9)	1.153(19)	1.688(38)	1.959(41)
-0.05	0.689(10)	1.124(18)	1.603(27)	1.941(33)

TABLE V. Table of best-fit masses, action C, $\beta = 3.70 (aT_c)$ = 1/4).

am_q	PS	V	Ν	Δ
0.15	0.943(4)	1.118(9)	1.722(10)	1.853(14)
0.10	0.836(4)	1.044(11)	1.592(12)	1.742(17)
0.07	0.768(5)	0.997(12)	1.514(14)	1.678(19)
0.03	0.672(5)	0.934(15)	1.410(18)	1.598(22)

TABLE VI. Table of best-fit masses, action A, $\beta = 2.38 (aT_c)$ = 1/2).

am_q	PS	V	Ν	Δ
0.15	1.816(11)	1.972(18)	3.274(36)	3.355(44)
0.10	1.727(11)	1.903(19)	3.161(37)	3.309(49)
0.05	1.637(11)	1.835(21)	3.047(39)	3.237(47)
0.00	1.546(11)	1.769(22)	2.930(43)	3.156(51)
-0.10	1.358(10)	1.582(20)	2.596(44)	2.825(43)
-0.15	1.246(10)	1.502(24)	2.452(44)	2.718(49)
-0.20	1.123(11)	1.417(29)	2.297(45)	2.605(57)
-0.25	0.982(13)	1.322(37)	2.155(42)	2.483(71)

TABLE VII. Table of best-fit masses, action A, $\beta = 2.85 (aT_c = 1/3)$.

am_q	PS	V	N	Δ
0.10	1.179(8)	1.353(11)	2.145(19)	2.245(24)
0.05	1.072(8)	1.275(12)	2.010(21)	2.115(27)
0.00	0.956(9)	1.194(13)	1.870(24)	1.989(32)
-0.05	0.829(10)	1.113(16)	1.720(28)	1.854(40)

of fattening comes from performing the minimum in Eq. (2.9). My scaling tests of it used the new gauge action presented in the Appendix.

I tested several other actions. All the variants of tadpole improvement I studied had large mass renormalizations. A pure hypercube action with no Pauli term had a good dispersion relation at the coarsest lattice spacing, but its hyperfine splittings were basically identical to those of the Wilson action.

B. Spectroscopy

Lattice volumes were $4^3 \times 16$ at $aT_c = 1/2$ (excessively long in the time direction, in retrospect), $6^3 \times 16$ at aT_c = 1/3, and $8^3 \times 16$ (dangerously short) and $8^3 \times 24$ (safer) at $aT_c = 1/4$.

The data set for action A consists of 80 lattices at $aT_c = 1/2$, 50 lattices at $aT_c = 1/3$ and 36 $8^3 \times 16$ lattices at $aT_c = 1/4$. The data set for action C consists of 80 lattices at $aT_c = 1/2$ and $aT_c = 1/3$ and 60 $8^3 \times 24$ lattices at $aT_c = 1/4$.

The spectroscopy measurement is entirely straightforward. I gauge fixed to a Coulomb gauge and used a Gaussian independent particle source wave function $\psi(r)$ $= \exp(-\gamma r^2)$ with $\gamma = 1$, 0.5, and 0.25 at $aT_c = 2,3,4$. I used pointlike sinks projected onto low momentum states. I used naive currents ($\bar{\psi}\gamma_5\psi$, etc.) for interpolating fields. The spectra appeared to be asymptotic [as shown by good (correlated) fits to a single exponential] beginning at t=2 (at $aT_c=1/2$), 3–5 (at $aT_c=1/3$) and 5–7 (at $aT_c=1/4$) and the best fits were selected using the old HEMCGC criterion [19].

My fiducial for comparison, simply because there are extensive data sets, is Wilson-action quenched spectroscopy. I have tried to restrict the data I used for comparison to lattices with the proper physical volume. I constructed my own $aT_c = 1/2$ and $aT_c = 1/3$ Wilson data sets ($\beta = 5.1$ and 5.54) since I could not find any results for these. I also ran off 40

TABLE VIII. Table of best-fit masses, action A, $\beta = 3.05 (aT_c)$ = 1/4).

am_q	PS	V	Ν	Δ
0.15	1.100(5)	1.228(8)	1.903(20)	1.998(27)
0.10	0.993(6)	1.146(9)	1.783(18)	1.875(30)
0.05	0.880(6)	1.066(11)	1.640(20)	1.788(26)
0.00	0.758(7)	1.000(12)	1.504(25)	1.662(29)



FIG. 13. Dispersion relation for heavy hadrons at $aT_c = 1/2$ ($a \approx 0.36$ fm) from action C. The curves are the continuum dispersion relation for the appropriate (measured) hadron mass.

Wilson lattices at $\beta = 5.7 (N_t = 4)$ to measure a dispersion relation. At that coupling my masses were within a standard deviation of the much superior data set of Butler *et al.* [20].

To compare with the more standard improved actions, I also performed a fiducial study using the clover action (Wilson fermion action plus on-site clover term) in a background of Wilson action gauge fields. I tadpole-improved the action

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using $u_0 = (\text{Tr } U_p/3)^{1/4}$ where U_p is the average plaquette. Data are at $aT_c = 1/2$ and $aT_c = 1/3$, where $u_0 = 0.802$ and 0.844. This data set was 180 and 60 lattices at the two couplings.

I show first plots of m_{ρ}/T_c and m_N/T_c vs m_{π}/T_c with Wilson fermions. These plots are scaling tests by themselves, or one can interpolate in the curves to fixed values of m_{π}/T_c (equivalent to fixed quark mass) and plot the variation in the observable vs aT_c . Figure 4 shows the rho mass and Fig. 5 shows the nucleon mass. Notice that the rho mass has the worst scaling violations of the three particles.

I can roughly estimate the critical bare quark mass (at which the pion is massless) by linearly extrapolating m_{π}^2 to zero in m_0 . Figure 6 shows the squared pion mass vs bare quark mass for action A, at β_c for $N_t=2$, 3, 4, and the same plot, but for action C, is shown in Fig. 7. Both actions have small bare mass renormalization. This is important from the point of view of principle because a true FP action would have no additive mass renormalization. It is important in practice because we only really know the kinetic parameters by solving the RG equation for positive bare mass; they must be extrapolated in some artistic way if one needs to go to negative bare mass.

Another way to estimate the critical bare mass is to use the partially conserved axial-vector current (PCAC) relation

 $\nabla_{\mu} \cdot \langle \bar{\psi} \gamma_5 \psi(0) \bar{\psi} \gamma_5 \gamma_{\mu} \psi(x) \rangle = 2m_a \langle \bar{\psi} \gamma_5 \psi(0) \bar{\psi} \gamma_5 \psi(x) \rangle.$

(4.2)



FIG. 14. Squared speed of light vs hadron mass in units of T_c , for (a) pseudoscalars, (b) vectors and (c) protons, from action C. Octagons, crosses, and diamonds label $aT_c = 1/2$, 1/3 and 1/4.



FIG. 15. Squared speed of light vs hadron mass in units of T_c , for (a) pseudoscalars, (b) vectors and (c) protons, from the Wilson and clover actions. Octagons, crosses, and diamonds label $aT_c = 1/2$, 1/3 and 1/4 for the Wilson action, and for the clover action the labels are pluses for $aT_c = 1/4$ and fancy crosses for $aT_c = 1/3$.

If I convert to lattice operators, sum over spatial slices, and measure distance in the t direction, this becomes

$$Z_{A} \frac{\partial}{\partial t} \sum_{x,y,z} \left\langle \bar{\psi} \gamma_{5} \psi(0) \bar{\psi} \gamma_{5} \gamma_{0} \psi(x) \right\rangle$$
$$= 2am_{q} Z_{P} \sum_{x,y,z} \left\langle \bar{\psi} \gamma_{5} \psi(0) \bar{\psi} \gamma_{5} \psi(x) \right\rangle.$$
(4.3)

I follow [21] by fitting the pseudoscalar source-pseudoscalar sink to

$$P(t) = Z[\exp(-m_{\pi}t) + \exp(-m_{\pi}(N_t - t))] \qquad (4.4)$$

and the pseudoscalar source-axial sink to

$$A(t) = \frac{Z_P}{Z_A} \frac{2m_q}{m_{\pi}} Z[\exp((m_{\pi}t) - \exp((-m_{\pi}(N_t - t)))]$$
(4.5)

to extract m_q . I only use the naive (pointlike) currents. I do not know the Z-factors, but for finding the value of m_0^c that does not matter. Extrapolating m_{π}^2 or m_q linearly in m_0 ignores all the well-known problems associated with extracting quark masses from lattice data [22], but the procedure is perfectly adequate to distinguish a small quark mass from a large one. The quark masses are shown in Figs. 6 and 7. I find for action A that $m_0^c = -0.42$, -0.20, and -0.14 at $aT_c = 1/2$, 1/3, and 1/4, respectively. For action C, the corresponding numbers are $m_0^c = -0.36$, -0.16, and -0.095. These numbers should be compared to the analogous quantities for Wilson fermions, using $m_0^c = 1/(2\kappa_c) - 4$: -1.58 at $\beta = 5.1 (aT_c = 1/2)$, -1.04 at $\beta = 5.7 (aT_c = 4)$, and still -0.70 at $\beta = 6.3$ [12]. In the latter case tadpole improved perturbation theory can explain most of the mass shift.

For a scaling test I compare m_{ρ}/T_c and m_N/T_c vs. m_{π}/T_c for actions A and C in Figs. 8, 9, 10, and 11.

I can compare scaling violations in hyperfine splittings by interpolating the data to fixed π/ρ mass ratios and plotting the N/ρ mass ratio vs $m_\rho a$. I do this at four π/ρ mass ratios, 0.84, 0.80, 0.75 and 0.70, in Fig. 12. In these figure the diamonds are Wilson action data in lattices of fixed physical size (4³ at β =5.1, 6³ at β =5.54, 8³ at β =5.7 [20], 16³ at β =6.0 [23] 24³ at β =6.3 [24]) and the crosses are data in various larger lattices: 16³ and 24³ at β =5.7 and 32³ at β =6.17 [20], 24³ at β =6.0 [23]. When they are present the data points from larger lattices illustrate the danger of performing scaling tests with data from different volumes. The bursts are from the nonperturbatively improved clover action of Ref. [25] and the fancy crosses are the TI clover action. The other plotting symbols show the test actions A and C.

I give tables of masses from action C in Tables III, IV, and V, and for action A in Tables VI, VII, and VIII.



FIG. 16. Variation of m_{ρ}/T_c and m_N/T_c vs aT_c at fixed $m_{\pi}/T_c = 2.75$ [(a) and (b)] and Eq. (3.15) [(c) and (d)], for actions A (octagons), C (squares), Wilson action (diamonds), and TI clover action (fancy crosses).

C. Dispersion relations

There are two ways to look at dispersion relations. The simplest is to plot E(p) the energy of the state produced with spatial momentum \vec{p} , as a function of $|\vec{p}|$. The result for action C at bare mass 0.15 is compared to the free dispersion relation at $aT_c = 1/2$ in Fig. 13. All of the test actions I have studied have good dispersion relations even at $aT_c = 1/2$. I believe that is a generic feature of the hypercubic kinetic term.

The signal to noise ratio for the nonzero momentum meson channels dies away at large t like $\exp[-(E(p) - m_{\pi})t]$. This means that large statistics are required to go to small quark mass or to high \vec{p} . However, it is possible to extract a fitted $c_{eff}^2 = (E(p)^2 - m^2)/p^2$ for the lowest nonzero momentum mode, for larger masses. This was done by performing a 4-parameter correlated fit to a pair of single exponentials, one for the $\vec{p} = (0,0,0)$ mode and the other the \vec{p} = (1,0,0) mode. I compare my results from action C and

TABLE IX. Couplings of the few-parameter approximate FP action. $c_0 = -2.517$.

operator	<i>c</i> ₁	<i>c</i> ₂	<i>c</i> ₃	<i>c</i> ₄
C _{plaq}	3.248	- 1.580	.1257	.0576
C _{6-link}	2810	.0051	.00490	0096

from the Wilson and clover actions in Figs. 14 and 15. Hadron masses are again scaled by T_c to allow the display of several lattice spacings at once. Action C has $c_{eff}^2 \approx 1$ for all observed hadrons even at $p = \pi T_c$ (860 MeV/c) at aT_c = 1/2.

D. Summary

It appears that these actions are members of a family of actions which show improved scaling, even at $\beta_c(N_t=2)$,

TABLE X. Critical couplings at finite volume and extrapolated to infinite volume for the FP action with parameters in Table I.

volume	$N_t = 2$	$N_t = 3$	$N_t = 4$
4 ³	3.025(25)		
6 ³	3.06(1)	3.47(1)	
8 ³	3.08(1)	3.49(1)	3.67(1)
10^{3}	3.085(5)	3.50(1)	3.69(1)
12^{3}			3.69(1)
infinite	3.092(7)	3.50(1)	3.70(1)
T_c / Λ	8.67	8.96	8.33
$a^2\sigma$	0.56(5)	.302(16)	.164(3)
$\sqrt{\sigma}/T_c$	1.50(7)	1.65(2)	1.62(2)
r_0/a	1.3(2)	2.19(2)	2.93(1)
r_0T_c	.65(10)	.730(7)	.733(3)



FIG. 17. The potential of the approximate FP action vs distance scaled with T_c . Octagons show data for $aT_c = 1/2$, diamonds for $aT_c = 1/3$, and squares for $aT_c = 1/4$.

about 0.36 fm lattice spacing. While action A is a better approximation to a FP action [based on the solution of Eq. (2.9)], it does not perform any better than action C, and is much slower. The spectroscopy seems to be insensitive to the choice of gauge action.

The hypercubic actions have much better dispersion relations than either the clover or Wilson action. They share this improvement with the D234 family of actions [26] and with the Hamber-Wu [27] action as tested in Ref. [28].

However, the hypercubic actions tested here seem to produce about the same level of improvement in hyperfine splittings as the clover action, at heavier quark masses. Leaving out the Pauli term gives noticeable scaling violations with a too-large N/ρ ratio; probably one needs to keep some kind of explicit Pauli or clover term in the lattice action to boost the hyperfine splittings.

The hyperfine splittings show worse scaling violations than the dispersion relation. Controlling and approximating the quark anomalous magnetic moment is the most difficult part of the construction of a FP action, and that may be the source of the scale violations.

Of course, there is still the possibility that all the actions tested here are missing some other common physics ingredient, which is responsible for scaling of the hyperfine splittings.

Since the ρ meson is the particle which shows the largest scaling violations, the best way to quantify improvement is by taking "sections" of the m_{ρ}/T_c and m_N/T_c vs m_{π}/T_c plots and displaying them vs aT_c at fixed m_{π}/T_c in Fig. 16. The improved actions at $aT_c = 1/2$ seem to show the same level of scaling violations as Wilson data at a lattice spacing a factor of 3 smaller. The smaller lattice spacing data seem to pick up about a factor of two improvement in lattice spacing, although the uncertainty in the data is larger. Quenched calculations are thought to scale in difficulty like $1/a^6$; the cost of action C is about a factor of 20 compared to the Wilson action.

My data by themselves do not suggest a unique way to extrapolate to a=0. FP actions are classically perfect with no a^n scaling violations for any n. With a Pauli term normalized so that $m_B = m_0$, any hypercube action is formally $O(a^2)$, due to the same cancellation of graphs as for the usual clover action [29]. With no Pauli term the action is only O(a), just like the usual Wilson action.

V. CONCLUSIONS

The important ingredients of these actions which contribute to their improved scaling behavior are the hypercubic kinetic term and the lattice anomalous magnetic moment term. The very small additive renormalization of the bare lattice mass is due to the use of fermion-gauge field couplings which are insensitive to the short-distance fluctuations of the gauge fields.

The specific implementation of these ideas in the actions I have tested involves many arbitrary choices. I believe that essentially every choice I made for a particular parametrization could be replaced by another choice, which would give an action which would have the same quality of scaling violations. Some changes should add some additional good features. For example, it might be possible to find a parametrization of a fat link which would lend itself to simple perturbation theory calculations.

There are several obvious extensions of this work. The first involves the kinetic term: It would be useful to find a parametrization of the kinetic term which extends from zero mass to a very large mass. It would also be interesting to develop a fat link parametrization of the action which could be efficiently incorporated into one of the standard algorithms for dynamical fermions. Next, is there a better parametrization of the anomalous magnetic moment term which might improve scaling?

Fixed point actions have many desirable formal properties [10]: they include scale invariant instanton solutions, the index theorem, an absence of exceptional configurations, and a particular implementation of chiral symmetry [30,31]. These properties may not be present in an action which is a bad approximation to a FP action. I do not know how well they are satisfied by these actions (other than the apparent small renormalization of the quark mass).

Indeed, the particular choice of a free field action which I made was motivated only by the locality and spectral properties of the free action. No attempt was made to optimize the chiral properties of the approximate action. This is clearly the outstanding problem for future study.

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APPENDIX: PURE GAUGE ACTIONS

This work used a new few-parameter approximation to an FP action for SU(3) gauge theory. Using it, isolated instanton configurations have constant actions to within 1.5 per cent. Like the action of Ref. [16], it is a superposition of powers of the plaquette and the perimeter-6 "twisted" link (x,y,z, -x, -y, -z). Like the action of Ref. [17] it includes a constant term. It is designed to be used for couplings such that the lattice spacing is $aT_c \approx 1/3$ or 1/4 to 1/8 or so. Explicitly

$$S(V) = c_0 + \frac{1}{N_c} \sum_C c_1(C) (N_c - \text{Tr}(V_C)) + c_2(C) (N_c - \text{Tr}(V_C))^2 + \dots$$
(A1)

with coefficients tabulated in Table IX.

This action costs about a factor of 7 times the usual Wilson plaquette action to simulate.

I have measured the critical couplings for the deconfinement transition at $aT_c = 1/2$, 1/3, and 1/4. The critical couplings on the measured spatial volumes and my extrapolation to infinite volume are shown in Table X. I have also measured the string tension from Wilson loops at these values of the coupling, on 8⁴ lattices (at $\beta = 3.092$) and 12⁴ lattices for the other couplings. The data was fitted to a static potential V(r) of the form

$$V(r) = V_0 + \sigma r - E/r \tag{A2}$$

using the techniques of Ref. [32]. The fit to the largest lattice spacing data is very difficult. The signal from large r is not good, and there is very little left of the Coulomb part of the potential due to the coarseness of the lattice. Nevertheless, I present the string tension and the Sommer [33] parameter r_0 $(r_0^2 dV(r_0)/dr = -1.65)$ in Table X. I see scaling within errors for both these parameters (vs T_c) at $aT_c = 1/3$ and 1/4. There is a ten per cent scaling violation at $aT_c = 1/2$. The asymptotic value inferred from large scale Wilson simulations [34] is $\sqrt{\sigma}/T_c = 1.600(11)$.

Finally, I show a plot of $V(r)/T_c$ vs rT_c for the three lattice spacings in Fig. 17. The overall vertical shift in the potentials is not physical, but it allows the reader to separate the different data sets by eye.

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