Threshold expansion and dimensionally regularized NRQCD

Harald W. Grießhammer*
Nuclear Theory Group, Department of Physics, University of Washington, Box 351 560, Seattle, Washington 98195-1560

(Received 16 January 1998; published 5 October 1998)

A Lagrangian and a set of Feynman rules are presented for non-relativistic QFT's with manifest power counting in the heavy particle velocity v. A regime is identified in which energies and momenta are of order Mv. It is neither identical to the ultrasoft regime corresponding to radiative processes with energies and momenta of order Mv^2, nor to the potential regime with on shell heavy particles and Coulomb binding. In this soft regime, massless particles are on shell, and heavy particle propagators become static. Examples show that it contributes to one- and two-loop corrections of scattering and production amplitudes near threshold. Hence, NRQFT agrees with the results of threshold expansion. A simple example also demonstrates the power of dimensional regularization in NRQFT. [S0556-2821(98)01021-2]

PACS number(s): 12.38.Bx, 12.39.Hg, 12.39.Jh

I. INTRODUCTION

Velocity power counting in non-relativistic quantum field theories (NRQFT) [1,2], especially in NRQCD and NRQED, and identification of the relevant energy and momentum regimes has proven more difficult than previously believed. In a recent article, Beneke and Smirnov [3] pointed out that the velocity rescaling rules proposed by Luke and Manohar for Coulomb interactions [4], and by Grinstein and Rothstein for bremsstrahlung processes [5], as united by Luke and Savage [6], and by Labelle's power counting scheme in time ordered perturbation theory [7], do not reproduce the correct behavior of the two gluon exchange contribution to Coulomb scattering between non-relativistic particles near threshold. This has cast some doubt on whether NRQCD, especially in its dimensionally regularized version [6], can be formulated using a self-consistent low energy Lagrangian. The aim of this paper is to demonstrate that a Lagrangian establishing explicit velocity power counting exists, and to show that this Lagrangian reproduces the results in Ref. [3].

This article presents the ideas to resolve the puzzle, postponing some more formal arguments, calculations and derivations to a future, longer publication [8] which will also deal with gauge theories and exemplary calculations in NRQCD. It is organized as follows: In Sec. II, the relevant regimes of NRQFT are identified. A simple example demonstrates the usefulness of dimensional regularization in enabling explicit velocity power counting. Sec. III proposes the rescaling rules necessary for a Lagrangian with manifest velocity power counting. The Feynman rules are given. Simple examples in Sec. IV establish further the necessity of the new, soft regime introduced in Sec. II. A Summary and outlook conclude the paper, together with the Appendix on split dimensional regularization [9].

II. IDEA OF DIMENSIONALLY REGULARIZED NRQFT

For the sake of simplicity, let us—following [3]—deal with the Lagrangian

$$\mathcal{L}=(\partial_\mu\Phi_{\rm R})^\dagger(\partial^\mu\Phi_{\rm R})-M^2\Phi_{\rm R}^\dagger\Phi_{\rm R} +\tfrac{1}{2}(\partial_\mu A)(\partial^\mu A)-2Mg\Phi_{\rm R}^\dagger\Phi_{\rm R}A \tag{2.1}$$

of a heavy, complex scalar field $\Phi_{\rm R}$ with mass M coupled to a massless, real scalar A. The coupling constant g has been chosen dimensionless. $\Phi_{\rm R}$ will be referred to as the ''quark'' and A as the ''gluon'' in a slight but clarifying abuse of language. In NRQFT, excitations with four-momenta bigger than M are integrated out, giving rise to four-point interactions between quarks. The first terms of the non-relativistically reduced Lagrangian read

$$\mathcal{L}_{\rm NRQFT}=\Phi^\dagger\left(i\partial_0+\frac{\vec{\partial}^2}{2M}-gc_1A\right)\Phi +\tfrac{1}{2}(\partial_\mu A)(\partial^\mu A)+c_2(\Phi^\dagger\Phi)^2+\dots, \tag{2.2}$$

where the non-relativistic quark field is $\Phi=\sqrt{2M}e^{iMt}\Phi_{\rm R}$ and the coefficients c_i are to be determined by matching relativistic and non-relativistic scattering amplitudes. To lowest order, $c_1=1$ and $c_2=-g^4/24\pi^2M^2$. The non-relativistic propagators are

$$\Phi:\frac{i}{T-\dfrac{\vec{p}^2}{2M}+i\epsilon},\quad A:\frac{i}{k^2+i\epsilon}, \tag{2.3}$$

where $T=p_0-M=\vec{p}^2/2M+\dots$ is the kinetic energy of the quark.

When a Coulombic bound state of two quarks exists, the two typical energy and momentum scales in the non-relativistic system are the bound state energy Mv^2 and the relative momentum of the two quarks Mv (i.e., the inverse size of the bound state) [2]. Here, $v=\beta\gamma\ll1$ is the relativistic generalization of the relative particle velocity. Cuts and poles in scattering amplitudes close to threshold stem from bound states and on-shell propagation of particles in intermediate states. They give rise to infrared divergences and in general dominate contributions to scattering amplitudes.

*Email address: hgrie@phys.washington.edu

With the two scales at hand, and energies and momenta being of either scale, three regimes are identified in which either Φ or A in Eq. (2.3) is on shell:

$$\text{soft regime:}\quad A_{\rm s}:k_0\sim|\vec{k}|\sim Mv,$$
$$\text{potential regime:}\quad \Phi_{\rm p}:T\sim Mv^2,\quad |\vec{p}|\sim Mv, \tag{2.4}$$
$$\text{ultrasoft regime:}\quad A_{\rm u}:k_0\sim|\vec{k}|\sim Mv^2.$$

Ultrasoft gluons $A_{\rm u}$ are emitted as bremsstrahlung or from excited states in the bound system. Soft gluons $A_{\rm s}$ do not describe bremsstrahlung: Because in- and outgoing quarks $\Phi_{\rm p}$ are close to their mass shell, they have an energy of order Mv^2. Therefore, overall energy conservation forbids all processes with outgoing soft gluons but without ingoing ones, and vice versa, as their energy is of order Mv.

The list of particles is not yet complete. In a bound system, one needs gluons which change the quark momenta but keep them close to their mass shell:

$$A_{\rm p}:k_0\sim Mv^2,\quad |\vec{k}|\sim Mv. \tag{2.5}$$

So far, only potential gluons and quarks, and ultrasoft gluons had been identified in the literature of power counting in NRQFT [4,5,7]. Because the soft regime was overlooked cast doubts on the completeness of NRQFT after Beneke and Smirnov [3] demonstrated its relevance near threshold in explicit one- and two-loop calculations. In this article, the fields representing a non-relativistic quark or gluon came naturally by identifying all possible particle poles in the non-relativistic propagators, given the two scales at hand.

When a soft gluon $A_{\rm s}$ couples to a potential quark $\Phi_{\rm p}$, the outgoing quark is far off its mass shell and carries energy and momentum of order Mv. Therefore, consistency requires the existence of quarks in the soft regime as well,

$$\Phi_{\rm s}:T\sim|\vec{p}|\sim Mv. \tag{2.6}$$

As the potential quark has a much smaller energy than either of the soft particles, it cannot—by the uncertainty relation—resolve the precise time at which the soft quark emits or absorbs the soft gluon. So, we expect a ''temporal'' multipole expansion to be associated with this vertex. In general, the coupling between particles of different regimes will not be point-like but will contain multipole expansions for the particle belonging to the weaker kinematic regime. For the coupling of potential quarks to ultrasoft gluons, this has been observed in Refs. [5, 7].

Propagators will also be different from regime to regime: for soft quarks, $\vec{p}^{\,2}/2M$ is negligible against the kinetic energy T, so that the soft quark propagator may be expanded in powers of $\vec{p}^{\,2}/2M$, and $\Phi_{\rm s}$ is expected to become static to lowest order. As the energy of potential gluons is much smaller than their momentum, the $A_{\rm p}$-propagator is expected to become instantaneous.

With these five fields $\Phi_{\rm s},\Phi_{\rm p},A_{\rm s},A_{\rm p},A_{\rm u}$ representing quarks and gluons in the three different non-relativistic regimes, soft, potential and ultrasoft, NRQFT becomes self-consistent. The application of these ideas to NRQCD with the inclusion of fermions and gauge particles is straightforward and will be summarized in a future publication [8]. An ultrasoft quark (which would have a static propagator) is not relevant for this paper. It is hence not considered, as is a fourth (''exceptional'') regime in which momenta are of the order Mv^2 and energies of the order Mv or any regime in which one of the scales is set by M. They do not derive from poles in propagators, and hence will be relevant only under ''exceptional'' circumstances. A future publication [8] has to prove that the particle content outlined is not only consistent but complete.

It is worth noticing that the particles of the soft regime can neither be mimicked by potential gluon exchange, nor by contact terms generated by integrating out the ultraviolet modes: fields in the soft regime have momenta of the same order as the momenta of the potential regime, but much higher energies. Therefore, seen from the potential scale they describe instantaneous but non-local interactions, as pointed out in [3]. Integrating out the scale Mv, one arrives at soft gluons and quarks as point-like multi-quark interactions in the ultrasoft regime. The physics of potential quarks and gluons will still have to be described by spatially local, but non-instantaneous interactions. The resulting theory—baptized potential NRQCD by Pineda and Soto [10]—can be derived from NRQCD as presented here by integrating out the fields $\Phi_{\rm s}$, $A_{\rm s}$ and $A_{\rm p}$. Therefore, there is no overlap between interactions and particles in different regimes.

In order to clarify this point, and before investigating the interactions of the various regimes further, the following example will demonstrate the power of dimensional regularization in NRQFT. It also highlights some points which simplify the discussion of the following sections. The integral corresponding to a one-dimensional loop

$$\begin{aligned} I(a,b)&\equiv\int dk\,\frac{1}{k^2-a^2+i\epsilon}\,\frac{1}{k^2-b^2+i\epsilon}\\ &=\frac{i\pi}{ab(a+b)} \end{aligned}\tag{2.7}$$

is easily calculated using contour integration. Assuming $v^2\equiv a^2/b^2\ll1$, the dominating contributions come from the regions where $|k|$ is close to a (''smaller regime'') or b (''larger regime''). Then, one can approximate the integral by

$$I(a,b)\approx\left[\int_{|k|\sim a}+\int_{|k|\sim b}\right]dk\,\frac{1}{k^2-a^2+i\epsilon}\,\frac{1}{k^2-b^2+i\epsilon}. \tag{2.8}$$

In the first integral, k is small against b, so that a Taylor expansion in $k/b\sim v$ in that regime is applicable and yields

$$\frac{-1}{b^2}\sum_{n=0}^{\infty}\int_{|k|\sim a}dk\,\frac{1}{k^2-a^2+i\epsilon}\,\frac{k^{2n}}{b^{2n}}. \tag{2.9}$$

If k^2 becomes comparable to b^2, the expansion breaks down, so that the approximated integral cannot be solved by contour integration. In general, the (arbitrary) borders of the integration regimes (the ''cutoffs'') will enter in the result, and lead to divergences as they are taken to infinity because of contributions from regions where $|k|\sim b \gg a$. A cutoff regularization may hence jeopardize power counting in v.

Dimensional regularization overcomes this problem in a natural and elegant way: If one treats (2.9) as a d-dimensional integral with $d\to 1$ only at the end of the calculation, the exact result will emerge as a power series in $v=a/b$. First, one extends the integration regime from the neighborhood of $|a|$ to the whole d-dimensional space. Then, one calculates the integral order by order in the expansion, still treating $k^2/b^2\sim v^2$ as formally small. Rewriting

$$k^{2n}=\sum_{m=0}^{n}\binom{n}{m}a^{2m}(k^2-a^2)^{n-m}, \tag{2.10}$$

only the $(m=n)$-term contributes thanks to the fact that dimensionally regularized integrals vanish when no intrinsic scale is present:

$$\int \frac{d^dk}{(2\pi)^d}k^{\alpha}=0. \tag{2.11}$$

The result

$$\frac{i\pi}{ab^2}\sum_{n=0}^{\infty}\frac{a^{2n}}{b^{2n}}\left(=\frac{i\pi}{a}\frac{1}{b^2-a^2}\right) \tag{2.12}$$

is exactly the contribution one obtains in the contour integration from the pole at $|k|=a$. Albeit the integral was expanded over the whole space, dimensional regularization missed the poles at $\pm b$ after expansion.

The integration about $|k|\sim b$ is treated likewise by expansion and term-by-term dimensional regularization. Adding this contribution

$$\frac{-i\pi}{b^3}\sum_{n=0}^{\infty}\frac{a^{2n}}{b^{2n}}\left(=\frac{-i\pi}{b}\frac{1}{b^2-a^2}\right) \tag{2.13}$$

to Eq. (2.12), one obtains term by term the Taylor expansion of the exact result (2.7) in the small parameter $v=a/b$. Each of the two regularized integrals sees only the pole in either of the regimes $|k|\sim a$ or $|k|\sim b$. Indeed, the overlap of the two regimes is zero in dimensional regularization, even for arbitrary v. But then, the expansion in the two different regimes can be terminated only at the cost of low accuracy.

One could therefore have started with the definition of two different integration variables, one formally living in the smaller regime with $|K_a|\sim a\sim vb$, the other formally living in the larger regime, $|K_b|\sim b$:

$$\int dk \to \int d^dK_a+\int d^dK_b. \tag{2.14}$$

The momentum k is represented in each of the kinematic regimes by either K_a or K_b. The integrands *must* then be expanded in a formal way *as if* $|K_a|\sim a\sim vb$ and $|K_b|\sim b$. Otherwise, poles are double counted. If one wants to calculate to a certain order in v, the expansion in the different variables K_a/b and a/K_b is just terminated at the appropriate order. Treated thus, no double counting of the poles can occur. Coming back to the three different regimes of NRQFT (2.4), there will therefore be no double counting between any pair of domains.

Finally, note that the limit $a\to 0$ is not smooth: for $a=0$, dimensional regularization of Eq. (2.9) is zero because of the absence of a scale (2.11). A pinch singularity encountered in contour integration at $k=\pm i\epsilon$ behaves hence like a pole of second order in dimensional regularization and is discarded, see also the Appendix.

By induction, the arguments presented here can be extended to prove that for any convergent one-dimensional integral containing several scales, Laurent expansion about each saddle point and dimensional regularization gives the same result as contour integration. A formal proof of the validity of threshold expansion does not presently exist for the case of multi-dimensional and divergent integrals, but Beneke and Smirnov [3] could reproduce the correct structures of known non-trivial two-loop integrals using threshold expansion, which is highly suggestive that such a proof can be given. This claim is supported by the observation that threshold expansion is very similar to the asymptotic expansion of dimensionally regularized integrals in the limit of loop momenta going to infinity, for which such a proof exists [11].[1]

III. RESCALING RULES, LAGRANGIAN AND FEYNMAN RULES

In order to establish explicit velocity power counting in the NRQFT Lagrangian, one rescales the space-time coordinates such that typical momenta in either regime become dimensionless, as first proposed in [4] for the potential regime, and in [5] for the ultrasoft regime:

$$\begin{aligned}
&\text{soft:} && t=(Mv)^{-1}T_{\rm s}, && \vec{x}=(Mv)^{-1}\vec{X}_{\rm s},\\
&\text{potential:} && t=(Mv^2)^{-1}T_{\rm u}, && \vec{x}=(Mv)^{-1}\vec{X}_{\rm s},\\
&\text{ultrasoft:} && t=(Mv^2)^{-1}T_{\rm u}, && \vec{x}=(Mv^2)^{-1}\vec{X}_{\rm u}.
\end{aligned} \tag{3.1}$$

In order for the propagator terms in the NRQFT Lagrangian to be properly normalized, one has to set for the representatives of the gluons in the three regimes

[1] I am indebted to M. Beneke for conversation on this point.

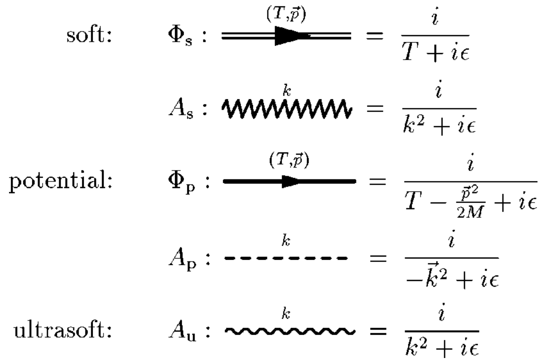

FIG. 1. Feynman rules for propagators.

$$\text{soft:}\quad A_s(\vec{x},t)=(Mv)\mathcal{A}_s(\vec{X}_s,T_s),$$

$$\text{potential:}\quad A_p(\vec{x},t)=(Mv^{3/2})\mathcal{A}_p(\vec{X}_s,T_u),\tag{3.2}$$

$$\text{ultrasoft:}\quad A_u(\vec{x},t)=(Mv^2)\mathcal{A}_u(\vec{X}_u,T_u),$$

and for the quark representatives

$$\text{soft:}\quad \Phi_s(\vec{x},t)=(Mv)^{3/2}\phi_s(\vec{X}_s,T_s),$$

$$\text{potential:}\quad \Phi_p(\vec{x},t)=(Mv)^{3/2}\phi_p(\vec{X}_s,T_u).\tag{3.3}$$

The rescaled free Lagrangian in the three regions reads then

$$\text{soft:}\quad d^3X_s dT_s\left[\phi_s^\dagger\left(i\partial_0+\frac{v}{2}\vec{\partial}^2\right)\phi_s+\tfrac{1}{2}(\partial_\mu\mathcal{A}_s)(\partial^\mu\mathcal{A}_s)\right],\tag{3.4}$$

$$\text{potential:}\quad d^3X_s dT_u[\phi_p^\dagger(i\partial_0+\tfrac{1}{2}\vec{\partial}^2)\phi_p + \tfrac{1}{2}(\mathcal{A}_p\vec{\partial}^2\mathcal{A}_p-v^2\mathcal{A}_p\partial_0^2\mathcal{A}_p)],\tag{3.5}$$

$$\text{ultrasoft:}\quad d^3X_u dT_u\tfrac{1}{2}(\partial_\mu\mathcal{A}_u)(\partial^\mu\mathcal{A}_u).\tag{3.6}$$

Here, as in the following, the positions of the fields have been left out whenever they coincide with the variables of the volume element. Derivatives are to be taken with respect to the rescaled variables of the volume element. The (un-rescaled) propagators are depicted in Fig. 1. As expected, the soft quark becomes static, resembling the quark propagator in heavy quark effective theory, and the potential gluon becomes instantaneous. In order to maintain velocity power counting, corrections of order v or higher must be treated as insertions as in the example, Eq. (2.7). Insertions are represented by the (un-rescaled) Feynman rules of Fig. 2.

Except for the physical gluons A_s and A_u, there is no distinction between Feynman and retarded propagators in

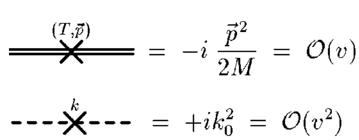

FIG. 2. Feynman rules for insertions.

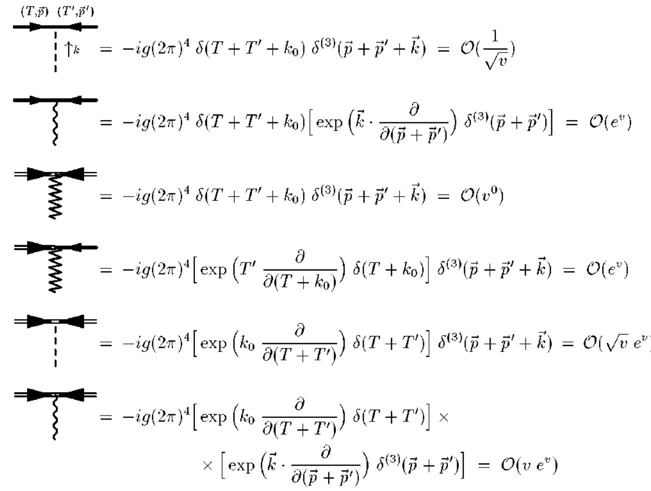

FIG. 3. Feynman rules for vertices.

NRQFT: Antiparticle propagation has been eliminated by the field transformation from the relativistic to the non-relativistic Lagrangian, and both propagators have maximal support for on-shell particles, the Feynman propagator outside the light cone vanishing like e^{-M}. Feynman's perturbation theory becomes more convenient than the time-ordered formalism, as less diagrams have to be calculated.

Finally, the interaction part of the Lagrangian reads [neglecting for the moment the Φ^4 vertex in Eq. (2.2)]

$$\text{soft:}\quad d^3X_s\, dT_s(-g)[(\mathcal{A}_s+\sqrt{v}\mathcal{A}_p(\vec{X}_s,vT_s) + v\mathcal{A}_u(v\vec{X}_s,vT_s))\phi_s^\dagger\phi_s + (\mathcal{A}_s\phi_s^\dagger\phi_p(\vec{X}_s,vT_s)+\text{H.c.})]\tag{3.7}$$

$$\text{potential:}\quad d^3X_s dT_u(-g)\left[\frac{1}{\sqrt{v}}\mathcal{A}_p+\mathcal{A}_u(v\vec{X}_s,T_u)\right]\phi_p^\dagger\phi_p.\tag{3.8}$$

Note that the scaling regime of the volume element is set by the particle with the highest momentum and energy. Vertices like $\mathcal{A}_s\phi_p^\dagger\phi_p$ cannot occur as energy and momentum must be conserved within each regime to the order in v one works. Among the fields introduced, these are the only interactions within and between different regimes allowed. One sees that technically, the multipole expansion comes from the different scaling of $\vec{x}$ and t in the three regimes. It is also interesting to note that there is no choice but to assign one and the same coupling strength g to each interaction. Different couplings for one vertex in different regimes are not allowed. This is to be expected, as the example (2.14) demonstrated that the fields in the various regimes are representatives of one and the same non-relativistic particle, whose interactions are fixed by the non-relativistic Lagrangian (2.2).

The interaction Feynman rules are depicted in Fig. 3. The exponents representing the multipole expansion have to be expanded to the desired order in v. Double counting is prevented by the fact that in addition to most of the propagators, all vertices are distinct because of different multipole expansions.

Using the equations of motion, the temporal multipole expansion may be re-written such that energy becomes conserved at the vertex. Now, both soft and potential or ultrasoft energies are present in the propagators, making it necessary to expand it in ultrasoft and potential energies. An example would be to restate the $\Phi_{\rm s}A_{\rm p}\Phi_{\rm s}$-vertex as

$$-ig(2\pi)^4\delta(T+T'+k_{\rm p,0})\\ \times\delta^{(3)}(\vec{p}+\vec{p}\,'+\vec{k})=\mathcal{O}(\sqrt{v}),\qquad(3.9)$$

and the soft propagator as containing insertions $\mathcal{O}(v)$ for potential energies $k_{\rm p}$:

$$\frac{i}{T+i\epsilon}\sum_{n=0}^{\infty}\left(\frac{-k_{\rm p,0}}{T}\right)^n.\qquad(3.10)$$

The same holds of course for the momentum-non-conserving vertices.

In the renormalization group approach, there is therefore only one relevant coupling (i.e. only one which dominates at zero velocity) at tree level. As expected, it is the $\Phi_{\rm p}\Phi_{\rm p}A_{\rm p}$ coupling providing the binding. The $\Phi_{\rm s}\Phi_{\rm s}A_{\rm p}$-, $\Phi_{\rm s}\Phi_{\rm s}A_{\rm u}$-couplings and both insertions (Fig. 2) are irrelevant. The marginal couplings $\Phi_{\rm p}\Phi_{\rm p}A_{\rm u}$, $\Phi_{\rm s}\Phi_{\rm s}A_{\rm s}$ and $\Phi_{\rm s}\Phi_{\rm p}A_{\rm s}$ are irrelevant in gauge theories in carefully chosen gauges like the Coulomb gauge. This point will be elaborated upon in the future [8].

The velocity power counting is not yet complete. As one sees from the volume element used in Eq. (3.7), the vertex rules for the soft regime count powers of v with respect to the soft regime. One hence retrieves the velocity power counting of heavy quark effective theory [12,13] (HQET), in which the interactions between one heavy (and hence static) and one or several light quarks are described. Usually, HQET counts inverse powers of mass in the Lagrangian, but because in the soft regime $Mv\sim$const, the two approaches are actually equivalent. HQET becomes a sub-set of NRQCD, complemented by interactions between soft (HQET) and potential or ultrasoft particles.

In NRQCD with two potential quarks as initial and final states, the soft regime can occur only inside loops, as noted above. Therefore, the power counting in the soft sub-graph has to be transferred to the potential regime. Because soft loop momenta scale like $[d^4k_{\rm s}]\sim v^4$, while potential ones like $[d^4k_{\rm p}]\sim v^5$, each largest sub-graph which contains only soft quarks and no potential ones (a ‘‘soft blob’’) is enhanced by an additional factor $1/v$. Couplings between soft quarks and any gluons inside a blob take place in the soft regime and hence are counted according to the rules of that regime. Because each soft blob contributes at least four orders of g, but only one inverse power of $v\sim g^2$, power counting is preserved. These velocity power counting rules in loops are verified in explicit calculations of the exemplary graphs (see also below), but a rigorous derivation is left for a future publication [8].

With rescaling, multipole expansion and loop counting, the velocity power counting rules are established, and one can now proceed to check the validity of the proposed Lagrangian, matching NRQFT to the relativistic theory in the examples given by Beneke and Smirnov [3].

IV. MODEL CALCULATIONS

The first example is the lowest order correction to the two quark production graph. Without proof, it will be used that in dimensional regularization, one can match NRQFT and the relativistic theory graph by graph, so that not the whole scattering amplitude has to be considered [3]. The collection of graphs to be matched to the relativistic diagram is depicted in Fig. 4. Here and in the following, hard (ultraviolet) contributions will not be shown explicitly. They are taken care of by the four-quark interaction of the non-relativistic Lagrangian (2.2) and renormalization of the external currents [6].

The energy and momentum routing has been chosen to be the one of the non-relativistic center of mass system, with $2T$ the total kinetic energy, and $y=-(\vec{p})^2\propto-v^2$ the relative four-momentum squared of the outgoing quarks as indicator for the thresholdness of the process considered. Thanks again to dimensional regularization, any other assignment can be chosen and reproduces the result.

The vanishing of the ultrasoft gluon exchange diagram and the value of the potential gluon exchange diagram have already been calculated in [6]. The soft exchange diagram vanishes, so that no new contribution is obtained. It is not even necessary to specify how soft quarks couple to external sources: If energy is conserved at the production vertex, the integral to be calculated is

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{T+k_0+i\epsilon}\frac{1}{T-k_0+i\epsilon}\frac{1}{k_0^2-\vec{k}^2}.\qquad(4.1)$$

Because the gluon is soft, $T\ll k_0$, and the quark propagators must be expanded in $T/k_0\sim v$, giving zero to any order as no scale is present in the dimensionally regularized integral. If

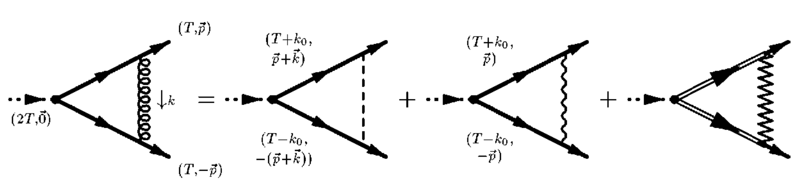

FIG. 4. Matching of the $\mathcal{O}(g^2)$ correction to two quark production off an external current to lowest order in v in each of the three regimes.

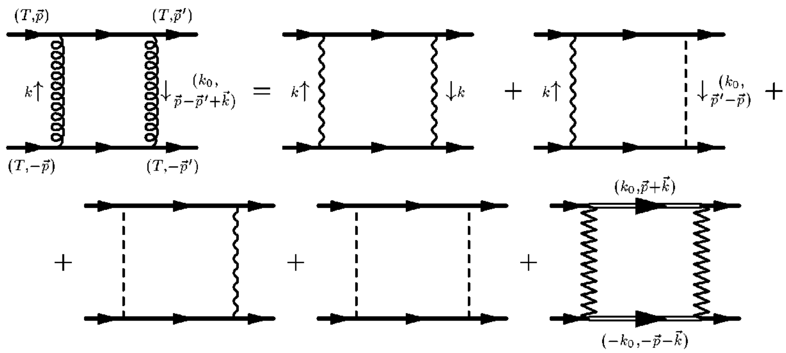

FIG. 5. Planar $\mathcal{O}(g^4)$ contributions to Coulomb scattering. The four-point interaction and insertion diagrams are not displayed.

energy is not conserved at the production vertex, the soft quark propagator is $1/\pm k_0$, and the contribution vanishes again. Therefore, there is no coupling of soft subgraphs to external sources to any order in v. Soft quarks in external lines are far off their mass shell and hence violate the assumptions underlying threshold expansion and NRQFT. In general, we conclude that soft quarks are present only in internal lines, and that the first non-vanishing contribution from the soft regime for the production vertex occurs not earlier than at $\mathcal{O}(g^4)$.

The first soft non-zero contribution comes actually from the two gluon direct exchange diagram of Fig. 5 calculated by Beneke and Smirnov [3] using threshold expansion. The Mandelstam variable $t=-(\vec{p}-\vec{p}\,')^2$ describes the momentum transfer in the center of mass system. The ultraviolet behavior of this graph is mimicked in NRQFT by a four-fermion exchange given by the vertex $ic_2=-ig^4/24\pi^2M^2=\mathcal{O}(t^0,y^0)$ of the Lagrangian (2.2), which using the rescaling rules is seen to be $\mathcal{O}(v)$.

The Feynman rules in Fig. 3 show that the $A_\mathrm{u}A_\mathrm{u}$-diagram is of order e^v with a leading loop integral contribution (similar to Eq. (32) in [3])

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k_0^2-\vec{k}^2}\,\frac{1}{k_0^2-\vec{k}^2}\,\frac{1}{T+k_0-\dfrac{\vec{p}^{\,2}}{2M}}\,\frac{1}{T-k_0-\dfrac{\vec{p}^{\,2}}{2M}}. \tag{4.2}$$

The diagram is expected to be zero to all orders since the ultrasoft gluons do not change the quark momenta and therefore the scattering takes place only in the forward direction, $\vec{p}=\vec{p}\,'$. Upon employing the on-shell condition for potential quarks, $T=\vec{p}^{\,2}/2M$ to leading order, it indeed vanishes as no scale is present. Since $T-\vec{p}^{\,2}/2M\sim Mv^4\ll|\vec{k}|\sim Mv$ (and $k_0\sim Mv^2$) in the potential regime, this is a legitimate expansion. The $A_\mathrm{u}A_\mathrm{p}$ and $A_\mathrm{p}A_\mathrm{u}$ contributions ($\mathcal{O}[(1/v)e^v]$) are zero for the same reason. The lowest order contribution to the $A_\mathrm{p}A_\mathrm{p}$ graph ($\mathcal{O}[(1/v^2)]$) is

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\vec{k}^2-i\epsilon}\,\frac{1}{(\vec{p}-\vec{p}\,'+\vec{k})^2-i\epsilon}\,\frac{1}{T+k_0-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon}\,\frac{1}{T-k_0-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon}. \tag{4.3}$$

In light of the discussion at the end of Sec. II, it is most consistent to perform the k_0 integration by dimensional regularization, using $\int[d^dk/(2\pi)^d]=\int[d^\sigma k_0/(2\pi)^\sigma][d^{d-\sigma}\vec{k}/(2\pi)^{d-\sigma}]$, $\sigma\to 1$ ([14], Chap. 4.1). Split dimensional regularization was introduced by Leibbrandt and Williams [9] to cure the problems arising from pinch singularities in non-covariant gauges. The Appendix shows that in the case at hand, it has the same effect as closing the k_0-contour and picking the quark propagator poles prior to using dimensional regularization in $d-1$ Euclidean dimensions. To achieve $\mathcal{O}(v^1)$ accuracy, one must also consider one insertion (Fig. 2) at the potential gluon lines, giving rise to a contribution

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{1}{\vec{k}^2-i\epsilon}\,\frac{1}{(\vec{p}-\vec{p}\,'+\vec{k})^2-i\epsilon}\,k_0^2\left[\frac{1}{\vec{k}^2-i\epsilon}+\frac{1}{(\vec{p}-\vec{p}\,'+\vec{k})^2-i\epsilon}\right]\frac{1}{T+k_0-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon}\,\frac{1}{T-k_0-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon}. \tag{4.4}$$

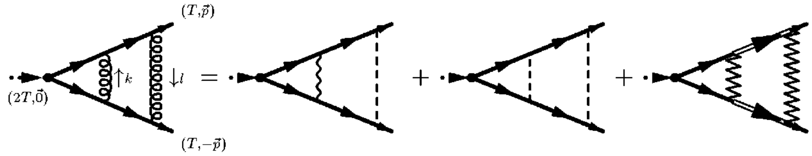

FIG. 6. The non-vanishing contributions to the planar fourth order correction to two quark production. Diagrams with insertions or four-point interactions not displayed.

The k_0 integration is naively linearly divergent, and hence closing the contour is not straightforward. As the Appendix demonstrates, split dimensional regularization circumvents this problem. The sum of both contributions (4.3), (4.4),

$$\frac{i}{8\pi t}\,\frac{M+T}{\sqrt{y}}\left(\frac{2}{4-d}-\gamma_{\mathrm{E}}-\ln\frac{-t}{4\pi\mu^2}\right), \tag{4.5}$$

agrees with Eq. (31) in [3], when one keeps in mind that in that reference, heavy particle external lines were normalized relativistically, while a non-relativistic normalization was chosen here. Also, this article uses the minimal subtraction (MS) rather than the modified (MS) scheme. Near threshold, the scale is set by the total threshold energy $4\pi\mu^2=4(M+T)^2$.

The soft gluon part is to lowest order [$\mathcal{O}(v^{-1})$ because of one soft blob] given by

$$\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k_0^2-\vec{k}^2+i\epsilon}\,\frac{1}{k_0^2-(\vec{p}-\vec{p}\,'+\vec{k})^2+i\epsilon}\,\frac{1}{k_0+i\epsilon}\,\frac{1}{-k_0+i\epsilon}, \tag{4.6}$$

which corresponds to Eq. (33) in [3]. Now, split dimensional regularization must be used if no *ad hoc* prescription for the pinch singularity at $k_0=0$ is to be invoked, see the Appendix. That the pinch is accounted for by potential gluon exchange and hence must be discarded, agrees with the intuitive argument that zero four-momentum scattering in QED is mediated by a potential only, and no retardation or radiation effects occur. On the other hand, the model Lagrangian contains three marginal couplings as seen at the end of Sec. III, which may give finite contributions as energies and momenta of the scattered particles go to zero. Although the prescription and the result from split dimensional regularization coincide in the present case as demonstrated at the end of Sec. II and in the Appendix, this may not hold in general. The result to $\mathcal{O}(v^1)$ exhibits another collinear divergence.

$$\frac{-i}{4\pi^2 t}\left(\frac{2}{4-d}-\gamma_{\mathrm{E}}-\ln\frac{-t}{4\pi\mu^2}\right)
+\frac{i}{24\pi^2M^2}\left[1+\frac{2y}{t}\left(\frac{2}{4-d}-\gamma_{\mathrm{E}}-\ln\frac{-t}{4\pi\mu^2}\right)\right], \tag{4.7}$$

and agrees with the one given by Beneke and Smirnov [3], following Eq. (36). The second term comes from insertions and multipole expansions to achieve $\mathcal{O}(v)$ accuracy.

It is easy to see that the power counting proposed works. As expected, the potential diagram is $\sqrt{y}\propto v$ stronger that the leading soft contribution, and $t\sqrt{y}\propto v^3$ stronger than the four-fermion interaction.

In conclusion, the proposed NRQFT Lagrangian reproduces the result for the planar graph of the relativistic theory *only if* the soft gluon and the soft quark are accounted for: the four-fermion contact interaction produces just a $1/M^2$-term, graphs containing ultrasoft gluons were absent, and the potential gluon (4.5) gave no $\mathcal{O}(y^0)$ contribution. This shows the necessity of soft quarks and gluons. The coupling strength of the $\Phi_{\mathrm{s}}A_{\mathrm{s}}\Phi_{\mathrm{p}}$ vertex is also seen to be identical to the other vertex coupling strengths, g.

The planar fourth order correction to two quark production (Fig. 6) was also compared to the result of [3], and is correctly accounted for when the Feynman rules proposed above are used to $\mathcal{O}(v^1)$.

V. CONCLUSIONS AND OUTLOOK

The objective of this paper was a simple presentation of the ideas behind explicit velocity power counting in dimensionally regularized NRQFT. It started with the identification of three different regimes of scale for on-shell particles in NRQFT from the poles in the non-relativistic propagators. This leads in a natural way to the existence of a new quark field and a new gluon field in the soft scaling regime $E\sim|\vec{p}|\sim Mv$. In it, quarks are static and gluons on shell, and HQET becomes a sub-set of NRQCD. Neither of the five fields in the three regimes should be thought of as ‘‘physical particles.’’ Rather, they represent the ‘‘true’’ quark and gluon in the respective regimes as the infrared-relevant degrees of freedom. None of the regimes overlap. An NRQFT Lagrangian has been proposed which leads to the correct behavior of scattering and production amplitudes. It establishes explicit velocity power counting which is preserved to all orders in perturbation theory. The reason for the existence of such a Lagrangian, once dimensional regularization is chosen to complete the theory, was elaborated upon in a simple example: the non-commutativity of the expansion in small parameters with dimensionally regularized integrals.

Due to the similarity between the calculation of the examples in the work presented here and in [3], one may get the impression that the Lagrangian presented is only a simple re-formulation of the threshold expansion. Partially, this is true, and a future publication [8] will indeed show the equivalence of the two approaches to all orders in the threshold and coupling expansion. A list of other topics to be addressed there contains the straightforward generalization to NRQCD; a proof whether the particle content outlined above is not only consistent but complete, i.e. that no new fields (e.g. an ultrasoft quark) or ''exceptional'' regimes arise; an investigation of the influence of soft quarks and gluons on bound state calculations in NRQED and NRQCD; a full list of the various couplings between the different regimes and an exploitation of their relevance for physical processes. The formal reason why double counting between different regimes and especially between soft and ultrasoft gluons does not occur, a derivation of the way soft quarks couple to external sources, and the role of soft gluons in Compton scattering deserve further attention, too.

I would like to stress that the diagrammatic threshold expansion derived here allows for a more automatic and intuitive approach and makes it easier to determine the order in $\sqrt{-y}\propto v$ to which a certain graph contributes. On the other hand, the NRQFT Lagrangian can easily be applied to bound state problems. As the threshold expansion of Beneke and Smirnov starts in a relativistic setting, it may formally be harder to treat bound states there. Indeed, I believe that even if one may not be able to prove the conjectures of the one starting from the other, both approaches will profit from each other in the wedlock of NRQFT and threshold expansion.

ACKNOWLEDGMENTS

It is my pleasure to thank J.-W. Chen, D. B. Kaplan and M. J. Savage for stimulating discussions. The work was supported in part by a Department of Energy grant DE-FG03-97ER41014.

APPENDIX: SOME DETAILS ON SPLIT DIMENSIONAL REGULARIZATION

This appendix presents the part of the calculations in the examples of Sec. IV which makes use of split dimensional regularization as introduced by Leibbrandt and Williams [9]. In its results, split dimensional regularization agrees with other methods to compute loop integrals in non-covariant gauges, such as the non-principal value prescription [15], but two features make it especially attractive: It treats the temporal and spatial components of the loop integrations on an equal footing, and no recipes are necessary. Rather, it uses the fact that, like in ordinary integration, the axioms of dimensional regularization ([14], Chap. 4.1) allow the splitting of the integration into two separate integrals:

$$\int \frac{d^d k}{(2\pi)^d} = \int \frac{d^\sigma k_0}{(2\pi)^\sigma}\,\frac{d^{d-\sigma}\vec{k}}{(2\pi)^{d-\sigma}}. \tag{A1}$$

Both integrations can be performed consecutively, and the limit $\sigma\to 1$ can—if finite—be taken immediately, because the integration over the spatial components of the loop momentum in Eq. (A1) is still regularized in $d-1$ dimensions. Finally, the limit $d\to 4$ is taken at the end of the calculation.

Equation (4.3) contains the simplest k_0 sub-integral:

$$\int \frac{d^\sigma k_0}{(2\pi)^\sigma}\,\frac{1}{k_0+T-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon}\,\frac{1}{-k_0+T-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon}. \tag{A2}$$

Using standard formulae for dimensional regularization in Euclidean space ([16], Appendix B), the result is finite as $\sigma\to 1$:

$$\begin{aligned}\int \frac{d^\sigma k_0}{(2\pi)^\sigma}\,&\frac{1}{\left[T-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon\right]^2-k_0^2}\\ &=-\frac{\Gamma\left[1-\dfrac{\sigma}{2}\right]}{(4\pi)^{\sigma/2}\Gamma[1]}\left(-\left[T-\frac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon\right]^2\right)^{\sigma/2-1}\\ &\to-\frac{i}{2}\left(T-\frac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon\right)^{-1}\quad\text{as}\quad\sigma\to 1.\end{aligned} \tag{A3}$$

It is no surprise that closing the contour produces the same result, because for any finite integral, the answer of all regularization methods have to coincide. The integral over the spatial components of the loop momentum is now straightforward.

The potential gluon diagram with one insertion at a gluon leg (4.4) yields a split dimensional integral which diverges linearly in k_0, so that naive contour integration is not legitimate.

$$\begin{aligned}\int \frac{d^\sigma k_0}{(2\pi)^\sigma}\,&\frac{k_0^2}{\left[T-\dfrac{(\vec{k}+\vec{p})^2}{2M}+i\epsilon\right]^2-k_0^2}\\ &\to-\frac{i}{2}\left[T-\frac{(\vec{k}+\vec{p})^2}{2M}\right]\quad\text{as}\quad\sigma\to 1.\end{aligned} \tag{A4}$$

To arrive at this result, the numerator was re-written as $[k_0^2 - (T-(\vec{k}+\vec{p})^2/2M)^2]+[T-(\vec{k}+\vec{p})^2/2M]^2$. Its first term cancels the denominator, yielding an integral without scale which therefore vanishes in dimensional regularization. The second term has been calculated in Eq. (A3). The integral over the spatial components of the loop momentum provides again no complications, leading to Eq. (4.5).

Finally, it was already shown at the end of Sec. II that dimensional regularization discards pinch singularities encountered in contour integrations. This is validated again by looking at the split dimensional integral for k_0 in the soft gluon contribution (4.6),

$$\int \frac{d^\sigma k_0}{(2\pi)^\sigma}\frac{1}{k_0^2-a^2}\frac{1}{k_0^2-b^2}\frac{1}{-k_0^2}, \tag{A5}$$

where $a^2 \equiv \vec{k}^2 - i\epsilon$ and $b^2 \equiv (\vec{p}-\vec{p}\,'+\vec{k})^2 - i\epsilon$. After combining denominators, the resulting integral is simple:

$$-2\frac{\Gamma\left[3-\frac{\sigma}{2}\right]}{(4\pi)^{\sigma/2}\Gamma[3]}\int_0^1 dx\ dy\ x(-a^2(1-x)-b^2xy)^{\sigma/2-3}$$

$$\to \frac{i}{2}\frac{1}{a^2-b^2}\left(\frac{1}{a^3}-\frac{1}{b^3}\right) \quad \text{as} \quad \sigma\to 1. \tag{A6}$$

This agrees with the result of Beneke and Smirnov ([3], Eq. (34)) who use contour integration and drop the contribution from the pinch singularity. The integral over the spatial components of the loop momentum provides again no unfamiliar complications, leading to Eq. (4.7).

[1] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B**, 437 (1986).
[2] G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995); **55**, 5853 (1997).
[3] M. Beneke and V. A. Smirnov, Nucl. Phys. **B522**, 321 (1998).
[4] M. Luke and A. V. Manohar, Phys. Rev. D **55**, 4129 (1997).
[5] B. Grinstein and I. Z. Rothstein, Phys. Rev. D **57**, 78 (1998).
[6] M. Luke and M. J. Savage, Phys. Rev. D **57**, 413 (1998).
[7] P. Labelle, Phys. Rev. D (to be published), Report No. MCGILL-96-33, hep-ph/9608491.
[8] H. W. Grießhammer (in preparation); see also in *Proceedings of the Workshop ''Nuclear Physics With Effective Field Theories*,'' Pasadena, California, 1998, edited by R. Seki, U. van Kolck and M. J. Savage (World Scientific, Singapore, in press), NT@UW-98-12, hep-ph/9804251.
[9] G. Leibbrandt and J. Williams, Nucl. Phys. **B475**, 469 (1996).
[10] A. Pineda and J. Soto, Nucl. Phys. B (Proc. Suppl.) **64**, 428 (1998).
[11] V. A. Smirnov, *Renormalization and Asymptotic Expansion*, Progress in Physics (Birkhäuser, Basel, Switzerland, 1991), Vol. 14.
[12] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989).
[13] N. Isgur and M. B. Wise, Phys. Lett. B **237**, 527 (1990).
[14] J. Collins, *Renormalization*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 1984).
[15] K.-C. Lee and S.-L. Nyeo, J. Math. Phys. **35**, 2210 (1994).
[16] P. Ramond, *Field Theory: A Modern Primer Frontiers in Physics* (Addison-Wesley, Reading, MA, 1990), Vol. 74.