

$\pi^+ \pi^-$ atom in chiral perturbation theory

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The hadronic $\pi^+ \pi^-$ atom is studied in the relativistic perturbative approach based on the Bethe-Salpeter equation. The general expression for the atom lifetime is derived. Lowest-order corrections to the relativistic Deser-type formula for the atom lifetime are evaluated within chiral perturbation theory. The lifetime of the $\pi^+ \pi^-$ atom in the two-loop order of chiral perturbation theory is predicted to be $\tau_1 = (3.03 \pm 0.10) \times 10^{-15}$ s. [S0556-2821(98)07919-3]

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I. INTRODUCTION

The pion-pion scattering amplitude at low energies forms one of the basic building blocks in the hierarchy of strong interaction processes. It serves as a useful probe for the investigation of the effect of chiral symmetry breaking since in the chiral limit the pion interactions vanish at the threshold. According to common belief, low-energy interactions of pions are described within chiral perturbation theory (ChPT) [1,2] which exploits the full content of global QCD symmetries. The $\pi\pi \rightarrow \pi\pi$ amplitude in ChPT is obtained as an expansion in quark masses and external pion momenta. The predictions of ChPT are sensitive to the magnitude of the quark condensate $\langle 0 | \bar{q}q | 0 \rangle$. In the standard scheme [1] with a ‘‘large’’ condensate, the S -wave $\pi\pi$ scattering lengths are predicted to be $a_0^0 = 0.217$ and $a_0^0 - a_0^2 = 0.258$ in units of the inverse charged pion mass [3]. Calculations within the generalized ChPT with a small quark condensate which contains more parameters lead to a most likely value of $a_0^0 = 0.27$ [2]. Despite a significant difference between these numbers, both results for the scattering length a_0^0 are compatible with the experimental value $a_0^0 = 0.26 \pm 0.05$ [4]. Consequently, a precise measurement of $\pi\pi$ scattering lengths will be an excellent test of ChPT. In particular, an experiment of that sort would provide important information about the behavior of the quark condensate in the chiral limit which in its turn is related to the properties of the QCD gluon vacuum.

An experimental study of the $\pi\pi$ scattering process is a very difficult task mainly due to the absence of a pion target. Indirect information extracted from the available data for, e.g., the process $\pi N \rightarrow \pi\pi N$ [5] produces large error bars when extrapolated to the two-pion threshold. The study of

K_{e4} decay, which is preferable for determining of the parameters of $\pi\pi$ interaction near threshold, is complicated because of a very small branching ratio of this process [4] (for a review of the recent status of $\pi\pi$ experiments see, e.g., Ref. [6]). In view of the above-mentioned experimental uncertainty in the determination of the $\pi\pi$ data near threshold, the forthcoming high-precision measurement of the $\pi^+ \pi^-$ atom lifetime by the DIRAC Collaboration at CERN (project no. PS212) has attracted much attention since it will allow direct determination of the difference $a_0^0 - a_0^2$ and thus will provide an excellent probe for the predictions of ChPT. The possibility of the observation of such atoms was argued in Ref. [7]. The first estimation of the lifetime of an atom formed by π^+ and π^- in the ground $1S$ state $\tau_1 = 2.9_{-2.1}^{+\infty} \times 10^{-15}$ s was given in Ref. [8]. The expected high-precision experimental data from the DIRAC experiment call for a refined theoretical treatment of this sort of bound system.

Since the characteristic average momenta in hadronic atoms are of an order of a few MeV, these systems are highly nonrelativistic. With the use of this fact the nonrelativistic Deser formula was derived in Refs. [9,10]. For the particular case of the $\pi^+ \pi^-$ atom the formula reads

$$\tau_n^{-1} = \frac{16\pi}{9} \left(\frac{2\Delta m_\pi}{m_\pi} \right)^{1/2} (a_0^0 - a_0^2)^2 |\Psi_n(0)|^2, \quad (1)$$

which relates the lifetime τ_n of the atom in the n -excited state to the value of the Coulomb wave function (WF) of the atom at the origin $\Psi_n(0)$ and the difference of the S -wave $\pi\pi$ scattering lengths with total isospin $I=0$ and $I=2$. Using the standard ChPT two-loop values of the scattering lengths given above, for the pionium ground-state lifetime

from Eq. (1) one obtains $\tau_1 = 3.20 \times 10^{-15}$ s. Note that in Eq. (1) one assumes isospin symmetry when expressing the scattering length for the reaction $\pi^+ \pi^- \rightarrow \pi^0 \pi^0$ in terms of scattering lengths with a definite total isospin, though the factor Δm_π on the right-hand side (RHS) of Eq. (1) indicates the necessity of taking into account the isospin-breaking effects in the theoretical description of the decay process. Note also that Deser-type formulas for the energy level displacement and lifetime are now widely used for the theoretical analysis of the experimental data for other hydrogenlike bound systems, such as pionic hydrogen [11], pionic deuterium [12], etc.

It is well known that Deser-type formulas for hadronic atoms in nonrelativistic scattering theory are valid up to the electromagnetic corrections to hadron scattering processes, if the mass difference between the charged and neutral components of the isotopic multiplet is assumed to be of a purely electromagnetic origin. Put differently, the clear-cut factorization of strong and electromagnetic interactions in the hadronic atom observables, which is explicit in Eq. (1), is valid up to (small) electromagnetic effects. In Ref. [13] a regular approach was constructed for the evaluation of these corrections. The ideas of this approach have been successfully applied to the study of the properties of a $\pi^+ \pi^-$ atom in Refs. [14,15] where the coupled-channel scattering problem with $(\pi^+ \pi^-)$ and $(\pi^0 \pi^0)$ free pairs in the asymptotic states has been considered. Nonrelativistic scattering theory has been used for the investigation of hadronic atoms also in Refs. [16,17]. Note, however, that the very ‘‘narrow’’ and ‘‘deep’’ phenomenological $\pi\pi$ potentials used in Ref. [17] lead to instability in the calculated observables of the bound state with respect to a small variation of input parameters. The strong enhancement of the potential in the vicinity of the origin within the inverse scattering theory approach, used by the authors of Ref. [17], stems mainly from the particular parametrization of the $\pi\pi$ scattering phase shift in the high-energy domain, where theoretical calculations of this quantity cannot be performed.

To summarize, the lowest-order Deser formula (1) and its counterpart for the energy-level displacement of an atom due to strong interactions are valid irrespective of the concrete choice of the strong interaction potential between hadrons. The magnitude and, even, the sign of corrections to it, however, strongly depend on this choice. For a theoretical analysis of the high-precision experimental data expected from the DIRAC experiment, a model-independent evaluation of these corrections is needed, based on the underlying (chiral) Lagrangian dynamics of hadrons, rather than the nonrelativistic potential picture of strong interactions.

The problem of a relativistic description of hadronic atoms is much richer in content than the same problem in the nonrelativistic scattering theory formulation. Many new effects which were absent, or were mimicked in the potential treatment, now arise naturally from the beginning (e.g., the ‘‘vacuum polarization’’ and ‘‘finite size’’ corrections which are borrowed in the potential picture from field theory). Namely, the problem of the evaluation of the atom lifetime on the basis of the underlying strong interaction dynamics was addressed in Refs. [18,19]. Reference [20] deals with the

so-called retardation correction in the ponium lifetime. In Ref. [21] the radiative corrections to $\pi\pi$ scattering lengths have been evaluated, which induce the corresponding $O(\alpha)$ correction in the ponium lifetime. Recently, a paper by Jalilouli and Sazdjian appeared [22] which is aimed at a consistent description of the properties of the $\pi^+ \pi^-$ atom on the basis of the 3D bound-state equation obtained in the framework of constraint theory, with the underlying strong $\pi\pi$ interactions described by ChPT. The authors have calculated corrections to the ponium lifetime coming from the mass difference $m_{\pi^\pm} - m_{\pi^0}$, as well as corrections from second-order perturbation theory and electromagnetic radiative corrections. In Ref. [23] the correction due to vacuum polarization to the ponium lifetime was calculated. Below we shall present a detailed comparison of our results with those given in Refs. [22,23].

Our previous papers [24,25] were aimed at a consistent field-theoretical treatment of $\pi^+ \pi^-$ atom observables on the basis of the Bethe-Salpeter (BS) equation. Namely, in Ref. [24] we derived the relativistic analogue of the Deser formula (1) for the ponium lifetime and evaluated the correction to it, coming from the displacement of the bound-state pole by strong interactions, referred to hereafter as the ‘‘strong correction.’’ In Ref. [25] we presented a systematic perturbative approach based on the BS equation for the calculation of hadronic atom observables. In this paper we give a closed expression, containing all first-order corrections to the ponium lifetime, and evaluate part of them. Namely, apart from the ‘‘strong correction,’’ which is reproduced here, we calculate the correction due to the exchange of Coulomb photon ladders, corresponding to the ‘‘second-order perturbation theory’’ correction from Ref. [22], and the relativistic correction to the bound-state WF.

The purpose of the present paper is to give a detailed description of the systematic perturbative approach to the hadronic atom characteristics, based on the BS bound-state equation. This approach is by construction free of any double-counting problems. Within this approach we collect together and calculate or give an estimate of all lowest-order corrections to the Deser-type relativistic formula for the ponium lifetime. The underlying strong $\pi\pi$ interactions within our approach are described in ChPT. Consequently, the results of the present calculations of the lifetime of the $\pi^+ \pi^-$ atom form a self-consistent basis for the verification of the predictions of ChPT in the DIRAC experiment. Further, in the present paper we discuss in detail the links to other approaches used for the description of the $\pi^+ \pi^-$ atom and, in particular, that from Refs. [22,23].

The layout of the present paper is follows: In Sec. II, we present a detailed description of the perturbative approach to the $\pi^+ \pi^-$ atom characteristics. In this section, we give a closed expression for the first-order correction to the ponium lifetime. In Sec. III, we give an evaluation, term by term, of various first-order corrections to the Deser formula (1). Section IV contains our conclusions.

II. PERTURBATIVE BETHE-SALPETER APPROACH TO THE $\pi^+ \pi^-$ ATOM

The evaluation of corrections to the Deser-type formulas for hadronic atom observables cannot be confined solely to

the evaluation of corrections to the ponium WF or to scattering lengths in Eq. (1). One has to develop a consistent perturbative scheme for the calculation of atom observables, which in the lowest-order approximation should yield the Deser-type relations for these quantities. Our approach is based on the field-theoretical BS equation with the kernel constructed from the underlying Lagrangian of ChPT.

Below we shall briefly discuss the basic ideas and assumptions of the approach which employs the following physical picture. The formation of a $\pi^+ \pi^-$ atom proceeds mainly due to the static Coulomb potential whereas strong interactions are responsible for its decay. The atom is described by the exact WF which obeys the field-theoretical BS equation. For our purposes we split this kernel into a Coulomb piece and the remainder, the latter considered in our scheme as a perturbation. Then, the exact BS WF is related to the relativistic Coulomb WF in the perturbation theory. The crucial point of our approach is that with the use of the above relation the observables of an atom (lifetime and binding energy) in every perturbative order are expressed in terms of the Coulomb WF. In the leading order of a perturbative expansion we reproduce the Deser formulas for atom observables [9]. The next-to-leading term in the perturbative expansion produces all lowest-order corrections to the Deser formulas.

Let us now pass to the description of the perturbative BS approach to $\pi^+ \pi^-$ atom observables. The existence of a quasistable $\pi^+ \pi^-$ atom ($\tau \sim 10^{-15}$ s) corresponds to the bound-state pole in the four-point Green function for the transition $\pi^+ \pi^- \rightarrow \pi^+ \pi^-$ at a complex value of the c.m. energy $P^2 = \bar{M}^2 = M^2 - iM\Gamma$. Here M denotes the ‘‘mass’’ of an atom, and Γ stands for the decay width. Hereafter all formulas are restricted to the c.m. system (c.m.s.) of an atom.

The BS WF of an atom χ_{BS} for $P^2 \rightarrow \bar{M}^2$ obeys the exact BS equation (Fig. 1a)

$$G_2^{-1}(P;p)\chi_{BS}(P;p) = \int \frac{d^4q}{(2\pi)^4} V_{BS}(P;p,q)\chi_{BS}(P;q). \quad (2)$$

Here $G_2(P;p) = D(\frac{1}{2}P+p)D(\frac{1}{2}P-p)$ is the two-pion Green function where $D(k)$ stands for the dressed pion propagator. Further, V_{BS} denotes the BS equation kernel, which is a sum of all four-point one-particle-irreducible diagrams with amputated external legs.

It is appropriate to ‘‘transfer’’ the self-energy insertions in the charged pion external legs to the RHS of the BS equation. This can be easily achieved if one defines $\chi(P;p) = G_0(P;p)G_2^{-1}(P;p)\chi_{BS}(P;p)$ and $V(P;p,q) = V_{BS}(P;p,q)G_2(P;q)G_0^{-1}(P;q)$, where $G_0(P;p) = i[(\frac{1}{2}P+p)^2 - m_\pi^2]^{-1} \times i[(\frac{1}{2}P-p)^2 - m_\pi^2]^{-1}$ is the free two-particle Green function and $m_\pi = m_{\pi^\pm}$ denotes the charged pion mass. The diagrammatic expansion of the new kernel V is given in Fig. 1b. In addition to the diagrams included in the ‘‘true’’ kernel V_{BS} , it contains the self-energy diagrams in *outgoing* external pion legs, i.e., only *half* of the possible insertions in external legs. Note that this property of the new

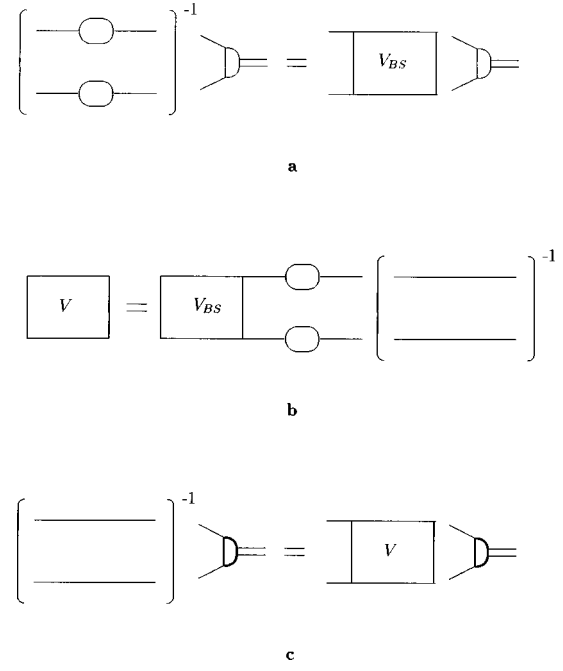


FIG. 1. Diagrammatic representation of the Bethe-Salpeter equation for the $\pi^+ \pi^-$ atom WF. Initial equation (a) through the redefinition of the kernel (b) and the WF is transformed into equation (c). The new kernel V contains the self-energy insertions in the *outgoing* external lines only [see (b)].

kernel V is crucial for proving the gauge invariance of bound-state characteristics, as well as for demonstrating the cancellation of infrared singularities in these characteristics (see below). The BS equation for the new WF χ depicted in Fig. 1c is given by

$$G_0^{-1}(P;p)\chi(P;p) = \int \frac{d^4q}{(2\pi)^4} V(P;p,q)\chi(P;q). \quad (3)$$

The kernel V contains the instantaneous Coulomb part V_C which, in a complete analogy with the positronium case, is responsible for the formation of the bound state composed of π^+ and π^- . We are willing to develop the perturbative expansion of the atom observables in the ‘‘remainder’’ of the potential denoted by $V' = V - V_C$. For this purpose we give first a complete solution of the ‘‘unperturbed’’ problem, with the kernel containing only the instantaneous Coulomb part.

We choose the instantaneous Coulomb part of the potential, according to the Barbieri-Remiddi prescription [26], to be [25]

$$V_C(\vec{p}, \vec{q}) = [w(\vec{p})]^{1/2} \frac{4im_\pi e^2}{(\vec{p} - \vec{q})^2} [w(\vec{q})]^{1/2}, \quad (4)$$

$$w(\vec{p}) = (m_\pi^2 + \vec{p}^2)^{1/2}.$$

Note that the particular choice of the Barbieri-Remiddi kernel (4) is only the matter of convenience and the final results are not affected by this choice (below we shall demonstrate this property of the perturbative expansion explicitly). How-

ever, choosing Eq. (4) as an initial kernel, one can take advantage of the fact that the BS equation with a kernel of this sort is exactly solvable, with the properly normalized ground-state solution given by [25]

$$\begin{aligned}\psi_C(p) &= iG_0(M^*;p)4[w(\vec{p})]^{1/2}\frac{4\pi\alpha m_\pi\phi_0}{\vec{p}^2+\gamma^2}, \\ \bar{\psi}_C(p) &= \psi_C(p),\end{aligned}\quad (5)$$

where $\gamma = \frac{1}{2}m_\pi\alpha$, $\pi\phi_0^2 = \gamma^3$, and $(M^*)^2 = m_\pi^2(4 - \alpha^2)$ is the eigenvalue corresponding to the unperturbed ground-state solution. The c.m.s. momentum in the free Green function G_0 in Eq. (5) has the components $(M^*, \vec{0})$.

The normalization condition for the Coulomb WF reads as

$$\langle \psi_C | N(M^*) | \psi_C \rangle = 1,$$

$$N(M^*;p,q) = (2\pi)^4 \delta^{(4)}(p-q) \frac{i}{2M^*} \frac{\partial}{\partial M^*} G_0^{-1}(M^*;p), \quad (6)$$

and the scalar product in momentum space is defined by the integral over $d^4q/(2\pi)^4$. We shall use this shorthand notation hereafter.

The exact solution for the Green function, corresponding to the nonrelativistic Coulomb problem, was given by Schwinger [27]: Using this result, one can obtain the solution for the 4D Coulomb Green function corresponding to the kernel (4) [25]:

$$\begin{aligned}G_C(P^*;p,q) &= (2\pi)^4 \delta^{(4)}(p-q) G_0(P^*;p) + G_0(P^*;p) \\ &\times [w(\vec{p})w(\vec{q})]^{1/2} T_C(E^*;p,\vec{q}) G_0(P^*;q),\end{aligned}\quad (7)$$

where

$$\begin{aligned}T_C(E^*;p,\vec{q}) &= 16i\pi m_\pi\alpha \left[\frac{1}{(\vec{p}-\vec{q})^2} + \int_0^1 \frac{\nu d\rho \rho^{-\nu}}{D(\rho;E^*;p,\vec{q})} \right], \\ D(\rho;E^*;p,\vec{q}) &= (\vec{p}-\vec{q})^2 - \frac{m_\pi}{4E^*} \left(E^* - \frac{\vec{p}^2}{m_\pi} \right) \left(E^* - \frac{\vec{q}^2}{m_\pi} \right) (1-\rho)^2, \\ \nu &= \alpha \left(\frac{m_\pi}{-4E^*} \right)^{1/2}, \quad E^* = \frac{(P^*)^2 - 4m_\pi^2}{4m_\pi}.\end{aligned}\quad (8)$$

The first and second terms on the RHS of Eq. (8) correspond to the exchange of one and multiple Coulomb photons, respectively. In the vicinity of the bound-state pole $(P^*)^2 \rightarrow (M^*)^2$, $\nu \rightarrow 1$, and the integral on the RHS of Eq. (8) diverges as $\int_0^1 d\rho/\rho$. Extracting this divergent piece, which corresponds to the bound-state pole in the Green function, one can write

$$G_C(P^*;p,q) = i \frac{\psi_C(M^*;p)\bar{\psi}_C(M^*;q)}{(P^*)^2 - (M^*)^2 + i0} + G_R(P^*;p,q), \quad (9)$$

where in the vicinity of the bound-state pole the regular part of the Coulomb Green function takes the form

$$\begin{aligned}G_R(M^*;p,q) &= (2\pi)^4 \delta^{(4)}(p-q) G_0(M^*;p) \\ &+ i[w(\vec{p})w(\vec{q})]^{1/2} \\ &\times \left[\bar{\Phi}(\vec{p},\vec{q}) - S(\vec{p})S(\vec{q}) \frac{8}{M^*} \frac{\partial}{\partial M^*} \right] \\ &\times G_0(M^*;p) G_0(M^*;q),\end{aligned}$$

$$\begin{aligned}\bar{\Phi}(\vec{p},\vec{q}) &= 16\pi m_\pi\alpha \left[\frac{1}{(\vec{p}-\vec{q})^2} + I_R(\vec{p},\vec{q}) \right] \\ &+ \frac{1}{(m_\pi\alpha)^2} S(\vec{p})S(\vec{q})\bar{R}(\vec{p},\vec{q}),\end{aligned}$$

$$S(\vec{p}) = \frac{4\pi m_\pi\alpha\phi_0}{\vec{p}^2 + \gamma^2},$$

$$\bar{R}(\vec{p},\vec{q}) = 20 - \left(\frac{8}{\pi m_\pi\alpha} \right)^{1/2} [S(\vec{p}) + S(\vec{q})],$$

$$\begin{aligned}I_R(\vec{p},\vec{q}) &= \int_0^1 \frac{d\rho}{\rho} \left[D^{-1} \left(\rho; -\frac{1}{4}m_\pi\alpha^2; \vec{p},\vec{q} \right) \right. \\ &\left. - D^{-1} \left(0; -\frac{1}{4}m_\pi\alpha^2; \vec{p},\vec{q} \right) \right].\end{aligned}\quad (10)$$

The solution $\chi(P;p)$ of the exact BS equation (3) can be expressed via the unperturbed solution $\psi_C(M^*;p)$ by the following limiting procedure [24,25]:

$$\langle \chi | = C \langle \psi_C | G_C^{-1}(P^*) G(P), \quad (P^*)^2 \rightarrow (M^*)^2, \quad P^2 \rightarrow \bar{M}^2, \quad (11)$$

where C denotes the normalization constant. Note that this relation is the relativistic generalization of the well-known nonrelativistic formula

$$\langle \chi | = \lim_{\eta \rightarrow 0(+)} i\eta \langle \psi_0 | \frac{1}{E - H + i\eta}, \quad (12)$$

which connects eigenvectors of the total Hamiltonian H with the unperturbed eigenvectors (see, e.g., [28]).

The result in Eq. (11) depends on the details of the limiting procedure. This equation makes sense if the quantities $(P^*)^2 - (M^*)^2$ and $P^2 - \bar{M}^2$ are assumed to be infinitesimal variables of equal strength. In Refs. [24,25] we have assumed the prescription $(P^*)^2 = (M^*)^2 + \lambda$, $P^2 = \bar{M}^2 + \lambda$, $\lambda \rightarrow 0$. Note that we can employ this prescription without loss of generality, since the change of the direction in the

$((P^*)^2, P^2)$ plane along which this limiting procedure is performed affects only the normalization constant C . Further, the validity of Eq. (11) can be trivially checked by extracting the bound-state pole in $G(P)$ and using the BS equation for $\langle \psi_C |$.

Let us now introduce the relativistic generalization of the projector operator onto the states orthogonal to the ground-state solution:

$$Q = 1 - N(M^*) |\psi_C\rangle \langle \psi_C|. \quad (13)$$

Then, with the use of the Hilbert identity it is easy to demonstrate that Eq. (11) can be rewritten as follows:

$$\begin{aligned} \langle \chi | &= \langle \chi | N(M^*) |\psi_C\rangle \langle \psi_C | \\ &\times \{1 + V'(P) [G_0^{-1}(P^*) - V_C - QV'(P)]^{-1} Q\} \\ &- \langle \chi | \Delta G_0^{-1} [G_0^{-1}(P^*) - V_C - QV'(P)]^{-1} Q, \end{aligned} \quad (14)$$

where $\Delta G_0^{-1} = G_0^{-1}(P) - G_0^{-1}(P^*)$ and the limiting procedure is implicit. In the derivation of Eq. (14) we have used

$$\langle \psi_C | G_C^{-1}(P^*) [G_0^{-1}(P^*) - V_C - QV'(P)]^{-1} = 0, \quad (15)$$

which stems from the fact that the inverse operator on the LHS of this equation does not have a pole in this limit. With the limiting prescription chosen above the normalization constant equals [24]

$$-C^{-1} = \langle \chi | N(M^*) |\psi_C\rangle. \quad (16)$$

Equation (14) can be solved with respect to $\langle \chi |$, resulting in

$$\langle \chi | = -C^{-1} \langle \psi_C | [1 + V'(P) G_V Q] [1 + \Delta G_0^{-1} G_V Q]^{-1}, \quad (17)$$

where the operator $G_V Q$ obeys the equation

$$G_V Q = G_R(P^*) Q + G_R(P^*) Q V'(P) G_V Q \quad (18)$$

and G_R stands for the regular part of the Coulomb Green function. It can be easily demonstrated that Eq. (17) can be rewritten as

$$\langle \chi | = -C^{-1} \langle \psi_C | \{1 + [\Delta G_0^{-1} - V'(P)] G_R Q\}^{-1}. \quad (19)$$

Substituting this solution into the complete BS equation (3) and using the BS equation for the function $|\psi_C\rangle$, we arrive at the final relation

$$\begin{aligned} \langle \psi_C | \{1 + [\Delta G_0^{-1} - V'(P)] G_R Q\}^{-1} \\ \times [\Delta G_0^{-1} - V'(P)] |\psi_C\rangle = 0, \end{aligned} \quad (20)$$

which provides the basis for the perturbative expansion of bound-state observables (the energy of the atomic level and decay width). Namely, the only unknown quantity on the LHS of Eq. (20) is the bound-state total four-momentum P ,

which enters parametrically into this expression. Expanding the LHS of Eq. (20) in perturbation theory up to a given order, one can then determine bound-state observables with a required accuracy. Note also that Eq. (20) is a complex equation, and in every perturbative order it provides two real equations for determining the energy level shift and decay width.

Equation (20), however, still contains the BS kernel, and does not contain the $\pi\pi$ scattering amplitudes. Below we carry out first-order perturbative calculations and demonstrate explicitly that only these amplitudes appear in the final result. For this purpose let us note first that the quantity $G_R Q$ in Eq. (20) is given by formulas similar to Eqs. (10), with $\tilde{\Phi}$ and \tilde{R} replaced by Φ and R , respectively, and

$$R(\vec{p}, \vec{q}) = \tilde{R}(\vec{p}, \vec{q}) + 5 + \text{higher orders in } \alpha \quad (21)$$

[this can be demonstrated by straightforward calculations, using Eqs. (5), (6), (10), and (13)]. We isolate the free part in $G_R Q$ by writing $G_R Q = G_0(M^*) + \delta G$.

Let us now turn to the perturbation kernel $V'(P)$. This potential can be decomposed into the following parts.

(1) A purely strong part, which is isotopically invariant. This part survives when electromagnetic interactions are ‘‘turned off’’ the theory.

(2) The part which is responsible for the $m_{\pi^\pm} - m_{\pi^0}$ electromagnetic mass difference.

(3) Remaining electromagnetic effects, including the exchanges of virtual photons.

Parts (1) and (2) are regarded to be more important for the following reasons. The first term includes strong interactions which govern the decay of a pionium. The second term makes this decay kinematically allowed. Consequently, it seems to be natural to consider them together, denoting the corresponding potential as $V_{12} = V_1 + V_2$. The T matrix corresponding to summation of the potential V_{12} in all orders is given by $T_{12}(P) = V_{12}(P) + V_{12}(P) G_0(P) T_{12}(P)$. The rest of the potential is referred to as $V_3 = V' - V_{12}$ and is treated perturbatively.

We would like to emphasize once more that this splitting is rather convention dependent and is dictated by convenience considerations. In practice it is convenient to include into parts (1) and (2), as much terms as possible. It is obvious, however, that the final results do not depend on the prescription chosen for that splitting.

We perform the perturbative expansion of the basic equation (20) in V_3 and δG up to the first nontrivial order. Meanwhile we expand ΔG_0^{-1} in a Taylor series in the variable $\delta M = \bar{M} - M^*$ and substitute

$$\begin{aligned} \bar{M} &= M^* + \Delta E^{(1)} + \Delta E^{(2)} - \frac{i}{2} \Gamma^{(1)} - \frac{i}{2} \Gamma^{(2)} \\ &+ \frac{1}{8M^*} (\Gamma^{(1)})^2 + \dots \end{aligned} \quad (22)$$

Expressing everywhere V_{12} in terms of T_{12} , we finally arrive at the identity

$$\begin{aligned}
0 = & -2iM^* \delta M - \langle \psi_C | T_{12} | \psi_C \rangle + \delta M \langle \psi_C | (G_0^{-1})' G_0 T_{12} | \psi_C \rangle + \frac{1}{2} (\delta M)^2 \langle \psi_C | (1 + T_{12} G_0) (G_0^{-1})'' + (G_0^{-1})'' G_0 T_{12} | \psi_C \rangle \\
& + \langle \psi_C | (\delta M (G_0^{-1})' - T_{12}) \delta G T_{12} | \psi_C \rangle - \langle \psi_C | (1 + T_{12} G_0) V_3 (1 + G_0 T_{12}) | \psi_C \rangle,
\end{aligned} \tag{23}$$

where $G_0 = G_0(M^*)$ and the prime stands for the differentiation with respect to M^* .

Equation (23) contains all first-order corrections to the pionium lifetime. Below we examine this relation term by term. Note that we employ the commonly accepted ‘‘local’’ approximation; i.e., we assume that the quantity T_{12} does not depend on relative momenta. The origin of this approximation can be traced to the ‘‘sharpness’’ of the Coulomb WF of an atom which in momentum space has the characteristic range $\gamma \sim 1$ MeV, much smaller than typical hadronic scales.

III. RELATIVISTIC DESER-TYPE FORMULAS WITH LOWEST-ORDER CORRECTIONS

In the lowest-order calculations only the first two terms on the RHS of Eq. (23) contribute. Further, one can assume $\psi_C(0) = \int d^4k / (2\pi)^4 \psi_C(M^*; k) = m_\pi^{-1/2} \phi_0$ in this approximation. Then, taking the real and imaginary parts of Eq. (23), we arrive at

$$\Delta E^{(1)} = \text{Re} \left(\frac{iT_{12}}{2M^* m_\pi} \phi_0^2 \right), \quad -\frac{1}{2} \Gamma^{(1)} = \text{Im} \left(\frac{iT_{12}}{2M^* m_\pi} \phi_0^2 \right). \tag{24}$$

Further, we can write

$$\text{Re} (iT_{12}) = 16\pi \mathcal{T}_{\pi^+ \pi^- \rightarrow \pi^+ \pi^-} (4m_\pi^2; \vec{0}, \vec{0}),$$

$$\begin{aligned}
\text{Im}(iT_{12}) = & -16\pi \left(\frac{\Delta m_\pi}{2m_\pi} \right)^{1/2} \left(1 - \frac{\Delta m_\pi}{2m_\pi} \right)^{1/2} \\
& \times |\mathcal{T}_{\pi^+ \pi^- \rightarrow \pi^0 \pi^0} (4m_\pi^2; \vec{0}, \vec{q}_0)|^2,
\end{aligned} \tag{25}$$

where $\mathcal{T}(s; \vec{p}, \vec{q})$ denote the (dimensionless) S -wave $\pi\pi$ scattering amplitudes and \vec{q}_0 is the relative momentum of the $\pi^0 \pi^0$ pair at the threshold $s = 4m_\pi^2$, with the magnitude given by the relation $m_\pi^2 = m_{\pi^0}^2 + \vec{q}_0^2$.

We would like to emphasize that the second relation in Eq. (25) differs from an analogous relation given in Ref. [22], though the starting equations (24) in both papers coincide. In Ref. [22] the magnitude of the three-momentum both for $\pi^+ \pi^-$ and $\pi^0 \pi^0$ pairs was set equal to 0. Consequently, neutral pions in the final state turned out to be off shell. Contrary to Ref. [22] we deduce from the Cutcoski’s rule that neutral pions in Eq. (25) are on shell. Note that this discrepancy with Ref. [22] leads to *different* predictions for the corrections, e.g., due to the mass difference $m_{\pi^\pm} - m_{\pi^0}$ in one loop (see below).

Substituting Eq. (25) into Eq. (24), we reproduce the lowest-order relativistic Deser-type formulas for the energy-level shift and lifetime of the pionium:

$$\Delta E = \frac{4\pi}{m_\pi^2} \mathcal{T}_{\pi^+ \pi^- \rightarrow \pi^+ \pi^-} \phi_0^2,$$

$$\Gamma = \frac{16\pi}{m_\pi^2} \left(\frac{2\Delta m_\pi}{m_\pi} \right)^{1/2} \left(1 - \frac{\Delta m_\pi}{2m_\pi} \right)^{1/2} |\mathcal{T}_{\pi^+ \pi^- \rightarrow \pi^0 \pi^0}|^2 \phi_0^2. \tag{26}$$

A. Relativistic correction to the pionium WF

The correction in the pionium lifetime coming from this effect is contained in the second term of Eq. (23). Namely, up to $O(\alpha)$ terms

$$\begin{aligned}
\psi_C(0) = & \int \frac{d^4k}{(2\pi)^4} \psi_C(M^*; k) = \frac{\phi_0}{m_\pi^{1/2}} (1 - C_0 \alpha), \\
C_0 = & 0.381 \dots
\end{aligned} \tag{27}$$

(details can be found in Appendix A). Thus, up to order $O(\alpha)$ in the local approximation the second term in Eq. (23) reads as $m_\pi^{-1} T_{12} \phi_0^2 (1 - 2C_0 \alpha)$. The term proportional to $C_0 \alpha$ in this expression induces the corresponding correction in the pionium lifetime. Note that the value of this correction is determined by the expression of the unperturbed solution ψ_C , Eq. (5), and hence depends on the particular choice of the instantaneous Coulomb part of the potential [for our case the Barbieri-Remiddi prescription (4)]. We shall see, however, that in the full expression for the correction to the atom lifetime the term proportional to $C_0 \alpha$ disappears, indicating that the final results do not depend on a particular choice of the zeroth-order kernel.

B. Correction due to the displacement of the bound-state pole by strong interactions

This correction is induced by the third and fourth terms in Eq. (23). The calculation of this sort of integral is carried out in a straightforward way (see Ref. [24]). Below we give the result of these calculations:

$$\begin{aligned} \langle \psi_C | (G_0^{-1})' G_0 T_{12} | \psi_C \rangle &= T_{12} \psi_C(0) \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}_C(M^*; p) \\ &\quad \times [G_0^{-1}(M^*; p)]' G_0(M^*; p) \\ &= \frac{i}{\alpha^2} \frac{i T_{12} \phi_0^2}{m_\pi^2} + \dots, \end{aligned} \quad (28)$$

$$\begin{aligned} \langle \psi_C | (G_0^{-1})'' | \psi_C \rangle &= \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}_C(M^*; p) \\ &\quad \times [G_0^{-1}(M^*; p)]'' \psi_C(M^*; p) \\ &= \frac{10i}{\alpha^2} + \dots, \end{aligned} \quad (29)$$

$$\begin{aligned} \langle \psi_C | T_{12} G_0 (G_0^{-1})'' | \psi_C \rangle &= \psi_C(0) T_{12} \int \frac{d^4 p}{(2\pi)^4} G_0(M^*; p) \\ &\quad \times [G_0^{-1}(M^*; p)]'' \psi_C(M^*; p) \\ &= -\frac{2i}{\alpha^2} \frac{i T_{12} \phi_0^2}{m_\pi^3} + \dots \\ &= \langle \psi_C | (G_0^{-1})'' G_0 T_{12} | \psi_C \rangle, \end{aligned} \quad (30)$$

and ellipses stand for higher-order terms in α . From Eq. (24) it is easy to see that the real and imaginary parts of integral (30) are down by small factors $\Delta E^{(1)}/m_\pi$ and $\Gamma^{(1)}/m_\pi$ as compared to integral (28). Therefore we shall neglect Eq. (30) in our calculations.

Note that Eq. (28) contains the second derivative of the inverse free Green function G_0 with respect to the bound-state mass. Hence this is a true relativistic correction arising from the BS treatment of the bound-state problem since in

the nonrelativistic case the free inverse Green function is linear in the bound-state energy (cf. Ref. [24])

C. Correction due to the exchange of Coulomb photons

This correction stems from the fifth term of Eq. (23). The calculation of the corresponding integrals is considered in Appendix A. Below we give the results:

$$\begin{aligned} \langle \psi_C | (G_0^{-1})' \delta G T_{12} | \psi_C \rangle &= T_{12} \psi_C(0) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \bar{\psi}_C(M^*; p) \\ &\quad \times [G_0^{-1}(M^*; p)]' \delta G(p, q) \\ &= -\frac{i}{\alpha^2} \frac{i T_{12} \phi_0^2}{m_\pi^2} + \dots, \end{aligned} \quad (31)$$

$$\begin{aligned} \langle \psi_C | T_{12} \delta G T_{12} | \psi_C \rangle &= (T_{12})^2 [\psi_C(0)]^2 \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \delta G(p, q) \\ &= \frac{i\alpha}{16\pi m_\pi} (\ln\alpha - 2.694) (T_{12})^2 \phi_0^2 + \dots, \end{aligned} \quad (32)$$

where the term nonanalytic in the fine structure constant [containing $\ln\alpha$ in Eq. (32)] comes from the infrared-singular one-photon-exchange piece in the Coulomb Green function [Eqs. (10), (21)].

With the calculated integrals and the lowest-order relations (24), it is a simple algebraic task to derive, from Eqs. (22) and (23), the first-order correction to the ponium decay width [25]:

$$\begin{aligned} \Gamma^{(2)} &= \Gamma^{(1)} \left\{ \underbrace{\left(-\frac{9}{8} \frac{\Delta E^{(1)}}{E_1} \right)}_{\text{strong}} + \underbrace{(-2C_0\alpha)}_{\text{relativistic w.f.}} + \underbrace{(1/2 + 2.694 - \ln\alpha) \frac{\Delta E^{(1)}}{E_1}}_{\text{Coulomb photon exchanges}} \right. \\ &\quad \left. - \left(M^* \Gamma^{(1)} \right)^{-1} \text{Re} \langle \psi_C | (1 + T_{12} G_0) V_3 (1 + G_0 T_{12}) | \psi_C \rangle \right\} \end{aligned} \quad (33)$$

where E_1 stands for the energy of the unperturbed ground-state level $E_1 = -m_\pi \alpha^2/4$.

Comparing Eq. (33) with the corresponding expression given in Ref. [22] (referred to as the ‘‘second-order strong correction’’), it is easy to see that our expression for the contribution of Coulomb photon exchanges contains an additional $\ln\alpha$ term. The origin for this disagreement can be easily established. Namely, the authors of Ref. [22] include an additional contribution from the so-called ‘‘constraint dia-

gram’’ in this term. This diagram cancels explicitly the one-Coulomb-photon-exchange term, leaving only multiphoton exchanges. We have checked that, having merely discarded this term, after our calculations we come to a result numerically very close to that given in Ref. [22]. The true result, however, cannot depend on the formalism used for the description of a bound state, either the BS equations or equations of the 3D constraint theory. So one can ask whether a contribution regarded as the counterpart of the ‘‘constraint

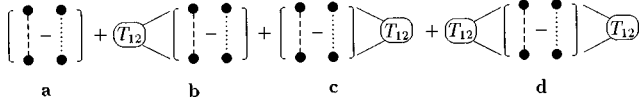


FIG. 2. Matrix element describing the “residual” photon exchange. The imaginary part of diagram (a) vanishes at the bound-state energy. Diagram (d), which is of second order in the “strong” amplitude T_{12} , is neglected in the present approximation. The dashed line denotes the virtual photon propagator, and dots correspond to the instantaneous virtual photon exchange.

diagram” exists in the BS framework for bound states. The answer is yes; this diagram is contained in the “electromagnetic kernel” V_3 in Eq. (33). The reason why we include this diagram in V_3 rather than in the “second-order correction” is simple: this diagram is accompanied by the diagram of virtual photon exchange (see Fig. 2d) neglected in Ref. [22]. The latter diagram also produces the $\ln\alpha$ term which exactly cancels the corresponding term from the “constraint diagram” (below we shall discuss this in more detail). The remainder is regular in the fine structure constant. Consequently, we find it more natural at the present stage to omit both these contributions on a equal footing, rather than to retain only one of them, namely, the “constraint diagram.”

We would like to mention here that the sign and magnitude of the nonanalytic term appear to be exactly the same as in the nonrelativistic treatment of the pionium, indicating that, as one could expect from the beginning, the “electromagnetic kernel” produces the corrections which are analytic in α (at least in the lowest order). In the scattering theory, when the electromagnetic corrections are taken into account, the expression for the decay width corresponding to the second relation from Eq. (24), $\text{Im}i T_{12}$, is replaced by $\text{Im}i T_{cc}$, where T_{cc} denotes the scattering amplitude of charged particles at the threshold in the presence of the Coulomb potential [14,15]. Accordingly, it leads to the replacement of $\text{Im} a_{had}$ by $\text{Im} a_{cc}$, where a_{had} and a_{cc} denote the “hadronic” and exact scattering lengths of charged particles. However, assuming that the hadronic potential has a finite range denoted by R , the following relation between a_{had} and a_{cc} can be established [13,28]:

$$\frac{1}{a_{cc}} = \frac{1}{a_{had}} - \frac{2}{r_B} \ln\left(\frac{2R}{r_B}\right) + \frac{1}{r_B} \left(\text{series in powers of } \frac{2R}{r_B} \right), \quad (34)$$

where r_B is the Bohr radius of the pionium which is the inverse proportional to α . So up to logarithmic terms,

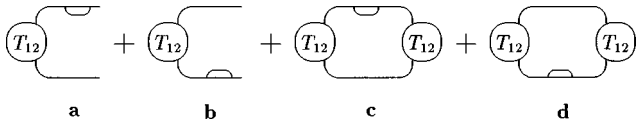


FIG. 3. Matrix element corresponding to the self-energy correction in the external pion legs [Eq. (43)].

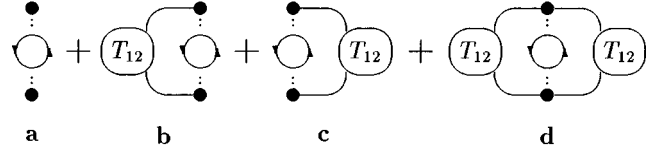


FIG. 4. Vacuum polarization correction.

$$\begin{aligned} \text{Im} a_{cc} &= \text{Im} a_{had} \left(1 - \frac{4\text{Re} a_{had}}{r_B} \ln\alpha \right) \\ &= \text{Im} a_{had} \left(1 - \frac{\Delta E^{(1)}}{E_1} \ln\alpha \right), \end{aligned} \quad (35)$$

where in the second relation we have used the Deser formula for the atom energy-level displacement. Consequently, the decay width is modified according to $\Gamma \rightarrow \Gamma(1 - \Delta E^{(1)} \ln\alpha / E_1)$ [cf. Eq. (33)].

In addition, we would like to note that the logarithmic term, which, just as in our approach, is of the second order in strong interactions [$\sim (T_{12})^2$], was found by Roig and Swift [29]. The authors of Ref. [29] have studied the electromagnetic radiative corrections to the $\pi\pi$ scattering and discovered the term proportional to $\ln p$ in the amplitude. When substituted into bound-state integrals, after rescaling, as usual, the integration momenta by $p \rightarrow \gamma p$, this term produces the $\ln\alpha$ correction, as in Eq. (33).

So far our treatment of the pionium decay width has been incomplete. Now we turn to the calculation of corrections induced by the last term in Eq. (23) containing the “electromagnetic” kernel V_3 .

D. Mass shift and radiative corrections

Into the kernel V_3 we include the following diagrams: the diagram with the “residual” photon exchange (i.e. the virtual photon exchange minus Coulomb potential), Fig. 2, the self-energy corrections to outgoing pion legs, Fig. 3, the vacuum polarization diagram, Fig. 4, and the vertex corrections, Fig. 5. The contributions containing low-energy constants and tadpole terms are included in T_{12} . Thus, we take here advantage of the arbitrariness in splitting the potential

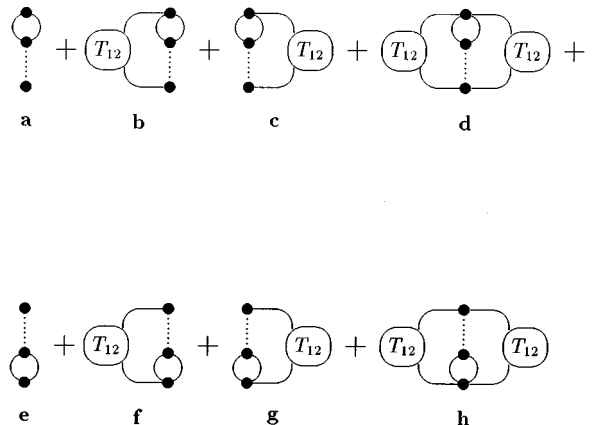


FIG. 5. Vertex correction.

and include all terms in T_{12} which are *a priori* known to have a smooth momentum dependence on the bound-state scale γ . Only the potentially ‘‘dangerous’’ terms which contain the photon propagator with vanishing mass are to be treated with the bound-state equation.

In this section we are concerned with the first two terms in V_3 . They read as follows.

(1) Residual photon exchange (Fig. 2):

$$V_\gamma - V_C = ie^2(P+p+q)_\mu(P-p-q)_\nu D^{\mu\nu}(p-q) - V_C(\vec{p}, \vec{q}), \quad (36)$$

where $D^{\mu\nu}$ denotes the photon propagator. The calculations are most easily carried in the Coulomb gauge with

$$D^{00}(\vec{k}) = -\frac{1}{\vec{k}^2}, \quad D^{0i}(\vec{k}) = D^{i0}(\vec{k}) = 0, \\ D^{ij}(\vec{k}) = -\left(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\right) \frac{1}{k^2 + i0}. \quad (37)$$

The corresponding matrix element equals

$$\text{Re}\langle\psi_C|(V_\gamma - V_C) + T_{12}G_0(V_\gamma - V_C) + (V_\gamma - V_C)G_0T_{12} + T_{12}G_0(V_\gamma - V_C)G_0T_{12}|\psi_C\rangle.$$

It is easy to observe that $\text{Re}\langle\psi_C|(V_\gamma - V_C)|\psi_C\rangle = 0$ at the bound-state energy. Consequently, the first term in the matrix element vanishes. The following two terms are equal for symmetry considerations. Further, we completely neglect the fourth term in this matrix element (Fig. 2d). This term, as we have mentioned before, contains the contribution of ‘‘constraint diagram’’ given by V_C . However, it is obvious that the D^{00} component in the expression of V_γ which has exactly the same infrared singular behavior as the Coulomb potential leads to the same $\ln\alpha$ nonanalytic term in the lifetime. In the expression of V_3 these nonanalytic terms cancel and the remainder is analytic in α (at least in the lowest order). For this reason we find it safer to neglect the combination $V_\gamma - V_C$ rather than V_γ alone, as in Ref. [22].

The remaining term in the expression of the matrix element is exactly of the form of the ‘‘retardation correction’’ discussed in Ref. [20]. To demonstrate this, we note that $|\delta\psi_\gamma\rangle = G_0(V_\gamma - V_C)|\psi_C\rangle$ gives the first-order perturbative correction to the bound-state WF due to the retardation effect (i.e., the difference between V_γ and V_C). The correction induced in the pionium lifetime due to this effect is given by

$$\Gamma \rightarrow \Gamma \left(1 - \frac{1}{M^* \Gamma^{(1)}} \text{Re}[2\psi_C(0)T_{12}\delta\psi_\gamma(0)] \right) \\ = \Gamma \left(1 + \frac{2\delta\psi_\gamma(0)}{\psi_C(0)} \right), \quad (38)$$

where the second relation was derived with the use of Eq. (24). From Eq. (38) it is apparent that the kernel $V_\gamma - V_C$ is responsible for the retardation correction in the pionium lifetime.

Finally, the matrix element we are looking for can be written in the following form:

$$\mathcal{M}_\gamma = \text{Re} \left[2\psi_C(0)T_{12} \int \frac{d^4q}{(2\pi)^4} (\Lambda_\gamma - \Lambda_C) \psi_C(M^*; q) \right], \quad (39)$$

where

$$\Lambda_\gamma = -ie^2 \int \frac{d^4p}{(2\pi)^4} G_0(M^*; p) (P+p+q)_\mu (P-p-q)_\nu \\ \times D^{\mu\nu}(p-q), \\ \Lambda_C = - \int \frac{d^4p}{(2\pi)^4} G_0(M^*; p) V_C(\vec{p}, \vec{q}). \quad (40)$$

(2) The kernel corresponding to the insertion of self-energy graphs into the outgoing pion legs (Fig. 3):

$$V_\Sigma = e^2 V_{12}(P; p, q) \left(\frac{\tilde{\Pi}\left(\frac{P}{2} + q\right)}{\left(\frac{P}{2} + q\right)^2 - m_\pi^2} + \frac{\tilde{\Pi}\left(\frac{P}{2} - q\right)}{\left(\frac{P}{2} - q\right)^2 - m_\pi^2} \right), \quad (41)$$

where

$$\Pi(l) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(l-k)^2 - m_\pi^2} (2l-k)_\mu (2l-k)_\nu D^{\mu\nu}(k), \\ \tilde{\Pi}(l) = \Pi(l) - \Pi(l^2 = m_\pi^2), \\ Z(l) = \frac{\tilde{\Pi}(l)}{l^2 - m_\pi^2}, \quad Z_\pm(l) = Z\left(\frac{P}{2} \pm l\right). \quad (42)$$

The corresponding matrix element can be rewritten as

$$\mathcal{M}_\Sigma = \langle\psi_C|e^2 T_{12}(Z_+ + Z_-)(1 + G_0 T_{12})|\psi_C\rangle. \quad (43)$$

We neglect here the term which is of second order in the strong interaction amplitude. Then, the sum of matrix elements reads as

$$\mathcal{M}_\gamma + \mathcal{M}_\Sigma = \text{Re} \left[2\psi_C(0)T_{12} \int \frac{d^4q}{(2\pi)^4} \left(\Lambda_\gamma - \Lambda_C - \frac{e^2}{2} Z_+(q) - \frac{e^2}{2} Z_-(q) \right) \psi_C(M^*; q) \right]. \quad (44)$$

We would like to emphasize here that the coefficient $\frac{1}{2}$ which emerges naturally in front of the self-energy term on the RHS of Eq. (44) ensures that the correction to the decay

width calculated from Eq. (44) is gauge invariant. Moreover, it is well known that in this particular combination of vertex and self-energy diagrams the ‘‘photon mass’’ disappears in the calculated width without inclusion of the ‘‘soft photon emission’’ terms, which seem to be rather awkward in the context of the bound-state problem. In its turn, the origin of the coefficient $\frac{1}{2}$ can be traced back to Eq. (3), where, as was mentioned before, the kernel V contains only half of all possible self-energy graphs attached to the external pion legs. Note also that in the S -matrix elements the origin of emerging the coefficient $\frac{1}{2}$ is quite different. In the latter case one takes into account all self-energy graphs in the external legs. However, restricting these matrix elements on mass shell one encounters an expression of the type $0/0$ which can be tackled, introducing an explicit ‘‘smearing function’’ in the initial Lagrangian. Passing then to the limit when the ‘‘smear-

ing function’’ tends to unity, one discovers the coefficient $\frac{1}{2}$ which multiplies the contribution from the self-energy diagram (for a detailed discussion see e.g., [30]). Consequently, though the coefficient $\frac{1}{2}$ emerges in the bound state and the scattering problems from different sources, one ends up with the same expression in these two cases, being gauge invariant and infrared finite at threshold.

The integral on the RHS of Eq. (44) can be easily evaluated in the Coulomb gauge, bearing in mind that the final result is gauge invariant.

In the calculation of the contribution from Λ_γ one can use the fact that the D^{ij} component of the photon propagator in the Coulomb gauge contributes only in order $O(\alpha^2 \ln \alpha)$ in the decay width and thus can be neglected. This considerably simplifies the calculations. The result reads as (details can be found in Appendix B)

$$\begin{aligned} \int \frac{d^4 q}{(2\pi)^4} \Lambda_\gamma \psi_C(M^*; q) &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} G_0(M^*; p) \frac{ie^2[(M^*)^2 - (p_0 + q_0)^2]}{(\vec{p} - \vec{q})^2} G_0(M^*; q) 4i[w(\vec{q})]^{1/2} \frac{4\pi\alpha m_\pi \phi_0}{\vec{q}^2 + \gamma^2} \\ &= \frac{\phi_0}{m_\pi^{1/2}} \left(1 + \frac{\alpha}{4\pi} N_\epsilon - \frac{3\alpha}{2\pi} \right) + \dots, \end{aligned} \quad (45)$$

where the dimensional regularization was used to handle the ultraviolet divergences, and

$$N_\epsilon = \frac{1}{2-n/2} + \Gamma'(1) + \ln 4\pi - \ln \left(\frac{m_\pi^2}{\mu^2} \right),$$

with n being the dimension of space and μ the mass scale used in dimensional regularization.

The contribution containing Λ_C can be trivially carried out:

$$\begin{aligned} \int \frac{d^4 q}{(2\pi)^4} \Lambda_C \psi_C(M^*; q) &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} G_0(M^*; p) \frac{4im_\pi e^2[w(\vec{p})w(\vec{q})]^{1/2}}{(\vec{p} - \vec{q})^2} G_0(M^*; q) 4i[w(\vec{q})]^{1/2} \frac{4\pi\alpha m_\pi \phi_0}{\vec{q}^2 + \gamma^2} \\ &= m_\pi e^2 \int \frac{d^3 \vec{p}}{(2\pi)^2} \frac{d^3 \vec{q}}{(2\pi)^2} \frac{1}{w(\vec{p})^{1/2}} \frac{1}{\vec{p}^2 + \gamma^2} \frac{1}{(\vec{p} - \vec{q})^2} \frac{4\pi\alpha m_\pi \phi_0}{(\vec{q}^2 + \gamma^2)^2} = \frac{\phi_0}{m_\pi^{1/2}} (1 - C_0 \alpha) + \dots \end{aligned} \quad (46)$$

The pion self-energy graph (42) calculated within the dimensional regularization scheme in the Coulomb gauge is given by

$$\Pi(l) = - \left(\frac{3m_\pi^2}{16\pi^2} N_\epsilon + \frac{7m_\pi^2}{16\pi^2} \right) - \Delta \left(\frac{1}{8\pi^2} N_\epsilon + \frac{1}{4\pi^2} \right) + \left[- \frac{\Delta}{8\pi^2} L - \frac{\vec{l}^2}{6\pi^2} \frac{\Delta}{m_\pi^2 + \Delta} \ln \left(- \frac{\Delta}{m_\pi^2} \right) - \frac{(\vec{l}^2)^2}{3\pi^2} [I(\vec{l}^2; \Delta) - I(\vec{l}^2; 0)] \right], \quad (47)$$

where $\Delta = l^2 - m_\pi^2 + i0$ and

$$L = 2 + \frac{w(\vec{l})}{|\vec{l}|} \ln \left(\frac{w(\vec{l}) - |\vec{l}|}{w(\vec{l}) + |\vec{l}|} \right), \quad L = O(\vec{l}^2) \text{ at small } |\vec{l}|,$$

$$I(\vec{l}^2, \Delta) = \int_0^1 dx \int_0^1 du \frac{xu^4}{x(1-u^2)\vec{l}^2 + xm_\pi^2 - (1-x)\Delta}. \quad (48)$$

Note that the terms in square brackets on the RHS of Eq. (47) are of higher order in $|\vec{l}|$ and/or Δ as compared to the second term in the round brackets. Consequently, in the calculations in the lowest order in α the term in square brackets can be dropped, since it contributes only in order $\alpha^2 \ln \alpha$. Then we immediately obtain

$$Z_\pm(l) = - \frac{1}{8\pi^2} N_\epsilon - \frac{1}{4\pi^2} + \dots \quad (49)$$

and

$$\mathcal{M}_\Sigma = \text{Re} \left[2[\psi_C(0)]^2 T_{12}(-e^2) \left(-\frac{1}{8\pi^2} N_\epsilon - \frac{1}{4\pi^2} \right) \right]. \quad (50)$$

Putting things together, we finally obtain

$$\mathcal{M}_\gamma + \mathcal{M}_\Sigma = 2\text{Re} T_{12} \frac{\phi_0^2}{m_\pi} \left(\frac{3\alpha}{4\pi} N_\epsilon + C_0 \alpha - \frac{\alpha}{2\pi} \right), \quad (51)$$

which induces the corresponding first-order correction in the $\pi^+ \pi^-$ atom decay width:

$$\Gamma \rightarrow \Gamma \left(\frac{3\alpha}{2\pi} N_\epsilon + 2C_0 \alpha - \frac{\alpha}{\pi} \right). \quad (52)$$

The above expression is of course ultraviolet divergent. It is well known that, along with the diagrams contributing to this expression, one should consider the four-pion Lagrangians containing (divergent) low-energy constants in order to cure this ultraviolet divergence. This will be done below. We would like to mention here that the term $2C_0\alpha$ from this expression cancels with a similar term coming from the atom WF in Eq. (33), and the final result for the decay width does not depend on the initial approximation chosen for the Coulomb WF of an atom (as it should be). Further, the term $-\alpha/\pi$ exactly coincides with the result given in Ref. [22] obtained from the same set of diagrams in an arbitrary covariant gauge. This provides an independent check of the gauge invariance of our result also for noncovariant gauges (and, in particular, for the Coulomb gauge).

Below we would like to discuss briefly the connection of our result with the ‘retardation correction’ given in Ref. [20]. As we have mentioned above, the matrix element \mathcal{M}_γ gives exactly what is called the ‘retardation correction.’ Note, however, that our result differs somewhat from that of Ref. [20]. Namely, in this paper the virtual photon-exchange diagram corresponds to the Wick-Cutcoski model, whereas we have used the pion-photon vertex which emerges in scalar electrodynamics. The ultraviolet divergence which occurs in our result is a consequence of the choice of the pion-photon vertex. Thus, strictly speaking, the present results, and the results of Ref. [20], refer to different physical models, and cannot be directly compared. Further, as we have

seen, the contribution from \mathcal{M}_γ alone is gauge dependent and should be combined with the self-energy diagrams to yield a gauge-invariant result. Moreover, in gauges other than the Coulomb gauge, individual contributions from \mathcal{M}_γ and \mathcal{M}_Σ contain a nonanalytic $\ln\alpha$ dependence which cancels in the sum. Reference [20] which mimicked the Feynman gauge calculations, does not contain a nonanalytic term.

Let us now evaluate the contributions from local four-pion Lagrangians. As we have mentioned before, it is appropriate to include these terms, which are smooth functions of external momenta, into the definition of T_{12} . According to this convention, the transition amplitude $\mathcal{T}_{\pi^+ \pi^- \rightarrow \pi^0 \pi^0}$ from Eq. (26) can be written as $\mathcal{T}_{\pi^+ \pi^- \rightarrow \pi^0 \pi^0} = \mathcal{T}_1 + \mathcal{T}_2$, where \mathcal{T}_1 denotes the isotopically symmetric ‘strong’ $\pi\pi$ scattering amplitude with the mass of the isotriplet taken equal to the charged pion mass, and \mathcal{T}_2 includes the effect of isospin breaking as well as terms with low-energy constants from the four-pion Lagrangians. From Eqs. (24) and (52) we come to the expression

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} = \left(\frac{3\alpha}{2\pi} N_\epsilon + 2C_0 \alpha - \frac{\alpha}{\pi} + 2\frac{\mathcal{T}_2}{\mathcal{T}_1} \right), \quad (53)$$

which displays only the electromagnetic and mass shift corrections.

As we noted before, in Ref. [22] the amplitude \mathcal{T}_2 was evaluated at the *off-mass-shell* point for π^0 mesons. However, as we see from Eq. (24), the amplitude emerging here is restricted to being on mass shell for all external particles, and we use this prescription hereafter. Moreover, an explicit expression for this amplitude calculated within ChPT has become recently available [31], and in the following we can use the expression given in Ref. [31] as granted. All that we have to do is to extract from the amplitude of Ref. [31] the terms which we have already taken into account through the bound-state equation (vertex and self-energy corrections, i.e., only the ones which are taken into account in the model of Roig and Swift [29]).

The calculations in Ref. [31] were carried out in the Feynman gauge. However, as we mentioned before, the combination $\Lambda_\gamma - (e^2/2)Z_+ - (e^2/2)Z_-$ we are concerned with is gauge invariant, and we can safely use the results of Ref. [31]. Thus, we can identify [see Eqs. (4.9)–(4.11) and (4.19) from Ref. [31]]

$$-2e^2(s-2m_\pi^2)G_{+-\gamma}(s) - e^2\bar{J}_{+-}(s) = \Lambda_\gamma - \frac{e^2}{2}Z_+ - \frac{e^2}{2}Z_- - \frac{3e^2}{16\pi^2}N_\epsilon - \frac{e^2}{8\pi^2}\ln\left(\frac{m_\pi^2}{\lambda^2}\right) - \frac{e^2}{4\pi^2},$$

where [31]

$$G_{+-\gamma}(s) = -i \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - \lambda^2)(q^2 - 2q \cdot p_+)(q^2 + 2q \cdot p_-)}, \quad p_\pm = \frac{P}{2} \pm p,$$

$$J_{\alpha\beta}(l^2) = -i \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2 - m_\alpha^2)[(q-l)^2 - m_\beta^2]}, \quad \bar{J}_{\alpha\beta}(s) = J_{\alpha\beta}(s) - J_{\alpha\beta}(0), \quad (54)$$

and we have introduced the photon ‘‘mass’’ λ to regularize the infrared-divergent integrals.

The amplitude $\mathcal{T}_{\pi^+\pi^-\rightarrow\pi^0\pi^0}$ can be easily found, subtracting $\Lambda_{\gamma^-(e^2/2)Z_+-(e^2/2)Z_-}$ defined by Eq. (54) from the total amplitude given in Ref. [31]:

$$-32\pi\mathcal{T}_{\pi^+\pi^-\rightarrow\pi^0\pi^0} = -\frac{s-m_{\pi^0}^2}{F^2} - B_R(s,t,u) - C_R(s,t,u), \quad (55)$$

with [31]

$$B_R(s,t,u) = \frac{s-m_{\pi^0}^2}{F^4} \left[\frac{m_{\pi^0}^2}{2} \bar{J}_{00}(s) + \left(\frac{s}{2} + 2\Delta_\pi \right) \bar{J}_{+-}(s) \right] + \frac{1}{12F^4} \left[3 \left(t - 2m_\pi^2 + \frac{\Delta_\pi^2}{t} \right)^2 + \frac{\lambda(t, m_\pi^2, m_{\pi^0}^2)}{t^2} (\Delta_\pi^2 + t(s-u)) \right] \bar{J}_{+0}(t) \\ + \frac{1}{12F^4} \left[3 \left(u - 2m_\pi^2 + \frac{\Delta_\pi^2}{t} \right)^2 + \frac{\lambda(u, m_\pi^2, m_{\pi^0}^2)}{u^2} (\Delta_\pi^2 + u(s-t)) \right] \bar{J}_{+0}(u), \quad (56)$$

$$C_R(s,t,u) = \frac{s-m_{\pi^0}^2}{32\pi^2 F^4} \left(-\frac{\Sigma_\pi}{3} - 4\Delta_\pi - \frac{L_\pi}{\Delta_\pi} (4m_\pi^4 - 7m_{\pi^0}^2 m_\pi^2 + 5m_{\pi^0}^4) + e^2 F^2 (-6 - 6N_\epsilon + \mathcal{K}_1^{\pm 0}) \right) \\ - \frac{m_{\pi^0}^2}{32\pi^2 F^4} \left(\frac{m_\pi^2}{3} - \frac{10m_{\pi^0}^2}{9} - \frac{L_\pi}{\Delta_\pi} (2m_\pi^2 - m_{\pi^0}^2) + m_{\pi^0}^2 \bar{I}_3 + e^2 F^2 \mathcal{K}_2^{\pm 0} \right) \\ + \frac{m_\pi^4}{24\pi^2 F^4} \left(\frac{1}{3} + L_\pi \right) - \frac{\Delta_\pi}{96\pi^4 F^4} \left(\frac{1}{t} + \frac{1}{u} \right) (\Sigma_\pi \Delta_\pi - 2m_{\pi^0}^2 m_\pi^2 L_\pi) \\ - \frac{1}{48\pi^2 F^4} \left\{ \frac{1}{6} (11s^2 - t^2 - u^2) + \frac{L_\pi}{\Delta_\pi} \left[\left(m_\pi^2 - \frac{3}{2} m_{\pi^0}^2 \right) s^2 + m_{\pi^0}^2 (t^2 + u^2) \right] \right\} \\ + \frac{1}{48\pi^2 F^4} \bar{I}_1 (s - 2m_{\pi^0}^2) (s - 2m_\pi^2) + \frac{1}{48\pi^2 F^4} \bar{I}_2 [(t - \Sigma_\pi)^2 + (u - \Sigma_\pi)^2]. \quad (57)$$

Here

$$-\frac{s-m_{\pi^0}^2}{F^2} \rightarrow -\frac{s-m_{\pi^0}^2}{F^2} \left(1 - \frac{3\alpha}{4\pi} N_\epsilon \right). \quad (59)$$

$$\mathcal{K}_1^{\pm 0} = \left(3 + \frac{4Z}{9} \right) \bar{k}_1 + \frac{32Z}{9} \bar{k}_2 + 3\bar{k}_3 + 4Z\bar{k}_4 - 6L_\pi,$$

$$\mathcal{K}_2^{\pm 0} = 8Z\bar{k}_2 + 3\bar{k}_3 + 4Z\bar{k}_4 - 2(1 + 8Z)\bar{k}_6 - (1 - 8Z)\bar{k}_8,$$

$$\Sigma_\pi = m_\pi^2 + m_{\pi^0}^2, \quad \Delta_\pi = m_\pi^2 - m_{\pi^0}^2,$$

$$L_\pi = \ln \left(\frac{m_\pi^2}{m_{\pi^0}^2} \right), \quad Z = \frac{\Delta_\pi}{2e^2 F^2}, \quad (58)$$

and \bar{I}_i , \bar{k}_i denote the finite, renormalization-scale-independent low-energy constants [31]. Note that in the expression for C_R the term containing the photon mass has been explicitly cancelled (cf. Ref. [31]). The term proportional to N_ϵ in this expression exactly cancels the ultraviolet divergence which appears in Eq. (52). To demonstrate this, we note that in the presence of this term only the lowest-order scattering amplitude is modified as [see Eq. (57)]

The modification in the $\pi^+\pi^-$ atom decay width is twice as large, and this cancels the term $(3\alpha/2\pi)N_\epsilon$ in Eq. (52). Consequently, we can merely discard the ultraviolet-divergent quantities simultaneously in Eqs. (52) and (57).

It is convenient to expand the rest of the $\pi\pi$ scattering amplitude, which is ultraviolet and infrared stable, in powers of Δ_π near threshold. Below we present the result of this expansion, retaining only the terms proportional to Δ_π and $e^2 F^2$ which are sufficient up to the accuracy required. Denoting the corresponding amplitude by $\mathcal{T}_{\pi^+\pi^-\rightarrow\pi^0\pi^0}^{(0)}$, we find

$$-32\pi\mathcal{T}_{\pi^+\pi^-\rightarrow\pi^0\pi^0}^{(0)} = \left[-\frac{3m_\pi^2}{F^2} - \frac{m_\pi^4}{32\pi^2 F^4} \left(11 + \frac{8}{3}\bar{I}_1 + \frac{16}{3}\bar{I}_2 - \bar{I}_3 + 12\bar{I}_4 \right) \right]$$

$$\begin{aligned}
 & + \Delta_\pi \left[-\frac{1}{F_\pi^2} - \frac{m_\pi^2}{48\pi^2 F_\pi^4} (1 + 4\bar{T}_1 + 3\bar{T}_3 - 12\bar{T}_4) \right] \\
 & - \frac{e^2 m_\pi^2}{32\pi^2 F_\pi^2} (-18 + 3\mathcal{K}_1^{\pm 0} - \mathcal{K}_2^{\pm 0}) + \dots, \quad (60)
 \end{aligned}$$

where the charged pion decay constant F_π is related to the parameter F entering into the Lagrangian through [31]

$$F = F_\pi \left(1 - \frac{m_{\pi^0}^2}{16\pi^2 F_\pi^2 \bar{T}_4} \right). \quad (61)$$

The low-energy constants \bar{T}_i and \bar{k}_i in this equation are fixed on the renormalization scale $\mu^2 = m_{\pi^0}^2$ according to [31]

$$\begin{aligned}
 l_i^r(\mu) &= \frac{\eta_i}{32\pi^2} \left[\bar{l}_i + \ln \left(\frac{m_{\pi^0}^2}{\mu^2} \right) \right], \\
 k_i^r(\mu) &= \frac{\sigma_i}{32\pi^2} \left[\bar{k}_i + \ln \left(\frac{m_{\pi^0}^2}{\mu^2} \right) \right], \quad (62)
 \end{aligned}$$

where η_i and σ_i are constants. However, to make a comparison with the calculations carried out in the isotopically symmetric case, it is necessary to bring the normalization scale to m_π^2 . This induces a change in the second term of Eq. (60):

$$\begin{aligned}
 & \frac{m_\pi^2}{48\pi^2 F_\pi^4} (1 + 4\bar{T}_1 + 3\bar{T}_3 - 12\bar{T}_4) \\
 & \rightarrow \frac{m_\pi^2}{48\pi^2 F_\pi^4} (1 + 4\bar{T}_1 + 3\bar{T}_3 - 12\bar{T}_4) + \frac{19m_\pi^2}{32\pi^2 F_\pi^4}. \quad (63)
 \end{aligned}$$

After this rescaling the first term gives the isotopically symmetric ‘‘strong’’ amplitude \mathcal{T}_1 and the remaining part corresponds to \mathcal{T}_2 . Using then Eqs. (53) and (60), it is easy to ‘‘read off’’ the first-order mass shift and radiative corrections in the $\pi^+ \pi^-$ atom decay width:

$$\begin{aligned}
 \delta_M &= \frac{2\Delta_\pi}{3m_\pi^2} \left[1 + \frac{m_\pi^2}{48\pi^2 F_\pi^2} (1 + 4\bar{T}_1 + 3\bar{T}_3 - 12\bar{T}_4) + \frac{19m_\pi^2}{32\pi^2 F_\pi^2} \right. \\
 & \left. - \frac{m_\pi^2}{96\pi^2 F_\pi^2} \left(11 + \frac{8}{3}\bar{T}_1 + \frac{16}{3}\bar{T}_2 - \bar{T}_3 + 12\bar{T}_4 \right) \right], \quad (64) \\
 \bar{\delta}_{em} &= 2C_0\alpha - \frac{\alpha}{\pi} + \frac{\alpha}{12\pi} (-18 + 3\mathcal{K}_1^{\pm 0} - \mathcal{K}_2^{\pm 0}). \quad (65)
 \end{aligned}$$

It is interesting to note that the Deser-type formula with account of the first-order mass shift and radiative corrections can be rewritten in a simple and transparent way. Namely, it is well known that the scattering amplitude of charged particles develops a pole at threshold, which corresponds to the

long-range Coulomb interactions in the initial state. Thus, at the threshold we can write [31]

$$\text{Re}A^{+-,00}(s,t,u) = -\frac{4m_\pi^2 - m_{\pi^0}^2}{F_\pi^2} \frac{e^2}{16} \frac{m_\pi}{q} + \text{Re}A_{thr}^{+-,00} \dots, \quad (66)$$

where q is the c.m.s. relative three-momentum of charged pions.

Then, the following simple expression, valid in the lowest-order approximation, is obtained for the $\pi^+ \pi^-$ atom decay width:

$$\begin{aligned}
 \Gamma &= \frac{1}{64\pi m_\pi^2} \left(\frac{2\Delta m_\pi}{m_\pi} \right)^{1/2} \left(1 - \frac{\Delta m_\pi}{2m_\pi} \right)^{1/2} (\text{Re}A_{thr}^{+-,00})^2 \phi_0^2 \\
 & \times \left[1 + \left(-\frac{9}{8} \frac{\Delta E^{(1)}}{E_1} \right) + (-2C_0\alpha) + (+2C_0\alpha) \right. \\
 & \left. + (1/2 + 2.694 - \ln\alpha) \frac{\Delta E^{(1)}}{E_1} + \delta \right], \quad (67)
 \end{aligned}$$

where the electromagnetic and mass shift corrections, being *excluded* from δ , are completely taken into account in $\text{Re}A_{thr}^{+-,00}$. Thus, the correction factor, δ displayed in Eq. (67), includes contributions from other sources, e.g., vacuum polarization, finite size corrections, etc. Further, Eq. (67) demonstrates explicitly the cancellation of $2C_0\alpha$ terms which depend on a particular choice of the initial approximation for the relativistic Coulomb WF

It is worth noting that the quantity $\text{Re}A_{thr}^{+-,00}$ is *not* proportional to the conventionally defined $\pi\pi$ scattering lengths which acquire an additional finite contribution due to the emission of real soft photons [see Eq. (5.17) of Ref. [31]]. Thus Eq. (67) demonstrates that in the presence of long-range Coulomb force the pole-removed real part of the scattering amplitude rather than the scattering length enters into the expression of the first-order corrected Deser-type formula for the decay width.

Below we shall briefly discuss the comparison of our results with those obtained in Refs. [22,21]. As we have noted before, the main difference between our work and Ref. [22], where the corrections to the pionium decay width are also evaluated in ChPT, consists in the fact that we argue the necessity of a different *kinematic* prescription in the calculation of the $\pi\pi$ scattering amplitude entering into the Deser-type formula. It is a completely on-mass-shell amplitude which naturally emerges in our calculations with neutral pions having small, but nonzero relative momentum \vec{q}_0 in the final state. By contrast, in Ref. [22] both charged and neutral pions have zero relative momenta, and, consequently, neutral pions in the final state are slightly off shell. Different kinematic prescriptions lead to different predictions for the pionium decay rate in Ref. [22] and in the present work. Further, in Ref. [21] the radiative corrections to the $\pi\pi$ scattering amplitude were evaluated in the Roig-Swift model, with an explicit cutoff. The cutoff parameter was chosen to be equal to the ρ -meson mass. The author also

presents calculations carried out in the ρ -meson dominance model where the integrals are ultraviolet convergent and the explicit cutoff is not needed. Thus, a direct comparison of the results of Ref. [21] with the calculations carried out on the basis of ChPT is not possible. Note, however, that in Ref. [21] the contribution from real photon radiation is also included in the scattering amplitude.

The mass shift and electromagnetic corrections have been evaluated in the framework of the nonrelativistic scattering theory approach [14,15]. Strong $\pi\pi$ interactions in this approach are described by energy-independent local potentials. It turns out that the effect of mass splitting on the pionium lifetime is opposite in sign as compared to the case of field-theoretical calculations. It is obvious that the sign and the magnitude of the mass shift correction depend on the choice of the ‘reference mass’ corresponding to the case of an isotopically symmetric world. In particular, if the reference mass is chosen equal to the charged pion mass [15], then this effect in the lifetime turns out to be negative ($\sim -7.4\%$). If one chooses the neutral pion mass to be the reference mass [14], then this effect changes its sign ($\sim +7.6\%$). We observe the different situation in our calculations based on the chiral Lagrangian. Thus one can conclude that the energy- and mass-independent local potentials used in Refs. [14,15] might not provide an adequate description of the $\pi^+\pi^-$ system in the nonrelativistic limit in dealing with the sophisticated issue of the isospin-breaking effect in $\pi\pi$ interactions.

E. Correction due to vacuum polarization

The vacuum polarization due to the virtual electron-positron pair contributes in order α^2 to the pionium decay width. However, this effect is amplified since a small electron mass m_e is present in the denominator.

In the instantaneous approximation the photon propagator is modified by the vacuum polarization effect as follows [18]:

$$\frac{1}{\vec{k}^2} \rightarrow \frac{1}{\vec{k}^2} + \frac{\alpha}{3\pi} I_{vac}(-\vec{k}^2), \quad (68)$$

where

$$I_{vac}(-\vec{k}^2) = \int_{4m_e^2}^{\infty} \frac{\rho(s) ds}{s + \vec{k}^2},$$

$$\rho(s) = \frac{1}{s} \left(1 + \frac{2m_e^2}{s} \right) \left(1 - \frac{4m_e^2}{s} \right)^{1/2}. \quad (69)$$

The perturbation potential which is responsible for the vacuum polarization effect is given by

$$V_{vac} = \frac{16i}{3} m_\pi^2 \alpha^2 I_{vac}[-(\vec{p}-\vec{q})^2], \quad (70)$$

and the corresponding matrix element from Eq. (33) is equal to

$$\mathcal{M}_{vac} = \text{Re} \langle \psi_C | V_{vac} + T_{12} G_0 V_{vac} + V_{vac} G_0 T_{12} + T_{12} G_0 V_{vac} G_0 T_{12} | \psi_C \rangle. \quad (71)$$

The first term in this expression vanishes at the bound-state energy which is below the elastic threshold. As in the calculation of electromagnetic radiative corrections, we neglect the fourth term (Fig. 4d). Thus, the matrix element can be written as follows:

$$\mathcal{M}_{vac} = \text{Re} \left[\frac{32}{3} m_\pi^2 \alpha^2 i T_{12} \psi_C(0) \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} G_0(M^*; p) I_{vac}[-(\vec{p}-\vec{q})^2] \psi_C(M^*; q) \right]. \quad (72)$$

With the use of Eq. (69) and after integrating over relative energy variables the integral in Eq. (72) takes the form

$$\int_{4m_e^2}^{\infty} ds \rho(s) \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{-i\pi\alpha m_\pi \phi_0}{w(\vec{p}) [w(\vec{q})]^{1/2} (\vec{p}^2 + \gamma^2) (\vec{q}^2 + \gamma^2)^2 [s + (\vec{p}-\vec{q})^2]}. \quad (73)$$

In the calculation of the integral over $d^3 \vec{q}$ we can replace the smooth factor $[w(\vec{q})]^{1/2}$ by its value at $\vec{q}=0$. Then

$$\int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{[w(\vec{q})]^{1/2}} \frac{1}{(\vec{q}^2 + \gamma^2)^2 [s + (\vec{p}-\vec{q})^2]} = \frac{1}{8\pi\gamma m_\pi^{1/2}} \frac{1}{\vec{p}^2 + (\gamma + \sqrt{s})^2} + \dots \quad (74)$$

The remaining integral over $d^3 \vec{p}$ can be computed analytically. Rescaling the integration variable s in Eq. (73), we finally arrive at

$$\mathcal{M}_{vac} = \frac{3}{16} \alpha^2 \frac{\phi_0^2}{m_e} \text{Re} T_{12} b_0, \quad (75)$$

where

$$b_0 = \int_1^\infty \frac{ds(s-1)^{1/2}}{s^2} \left(1 + \frac{1}{2s}\right) f_{vac}(s) \left[\int_1^\infty \frac{ds(s-1)^{1/2}}{s^2} \left(1 + \frac{1}{2s}\right) \right]^{-1} = 0.6865 \dots, \quad (76)$$

$$f_{vac}(s) = \frac{2\bar{m}_\pi}{\pi(2\bar{\gamma} + \sqrt{s})} \left[\theta(\bar{m}_\pi - \bar{\gamma} - \sqrt{s}) \frac{\bar{\gamma} + \sqrt{s}}{[\bar{m}_\pi^2 - (\bar{\gamma} + \sqrt{s})^2]^{1/2}} \operatorname{arctg} \frac{[\bar{m}_\pi^2 - (\bar{\gamma} + \sqrt{s})^2]^{1/2}}{\bar{\gamma} + \sqrt{s}} \right. \\ \left. + \theta(\sqrt{s} - \bar{m}_\pi + \bar{\gamma}) \frac{\bar{\gamma} + \sqrt{s}}{[(\bar{\gamma} + \sqrt{s})^2 - \bar{m}_\pi^2]^{1/2}} \ln \left(\frac{\bar{\gamma} + \sqrt{s} + [(\bar{\gamma} + \sqrt{s})^2 - \bar{m}_\pi^2]^{1/2}}{\bar{\gamma} + \sqrt{s} - [(\bar{\gamma} + \sqrt{s})^2 - \bar{m}_\pi^2]^{1/2}} \right) \right. \\ \left. - \frac{\bar{\gamma}}{(\bar{m}_\pi^2 - \bar{\gamma}^2)^{1/2}} \operatorname{arctg} \frac{(\bar{m}_\pi^2 - \bar{\gamma}^2)^{1/2}}{\bar{\gamma}} \right], \quad (77)$$

and $\bar{m}_\pi = m_\pi / (2m_e)$, $\bar{\gamma} = \gamma / (2m_e)$.

Using the relativistic Deser formula in the lowest-order approximation (24), it is easy to observe that Eq. (75) leads to the following modification in the $\pi^+ \pi^-$ atom decay width due to the vacuum polarization effect:

$$\Gamma \rightarrow \Gamma \left(1 + \frac{3}{16} \alpha^2 \frac{m_\pi}{m_e} b_0 \right). \quad (78)$$

Note that in Ref. [18] the vacuum polarization correction to the pionium lifetime was calculated only with account of discrete spectrum transitions. Thus, our result is a generalization of that from Ref. [18].

The nonrelativistic QED (NRQED) based calculation of the vacuum polarization effect in the pionium lifetime has become available recently [23]. We find that the analytic expression of the so-called 0-Coulomb term in Ref. [23] coincides with our result up to the relativistic kinematic factor in the WF whose presence is due to the choice of Barbieri-Remiddi kernel. Numerically the effect of this factor, which contributes in higher orders in α , is very small. In Ref. [23] the results for 1-Coulomb and multi-Coulomb contributions are also given. These contributions, which formally are of higher order in α , would emerge in our calculations as second-order perturbative corrections to the pionium lifetime.

F. Finite size correction

In the presence of a pion loop the pion electromagnetic form factor given by the vector-pseudoscalar-pseudoscalar (VPP) vertex in ChPT is modified. According to Ref. [1],

$$F_V(t) = 1 + \frac{1}{6F_\pi^2} (t - 4m_\pi^2) \bar{J}_{+-}(t) + \frac{t}{96\pi^2 F_\pi^2} \left(\bar{I}_6 - \frac{1}{3} \right) \\ = 1 + \delta F_V(t). \quad (79)$$

In the instantaneous approximation the perturbation potential is given by (see Fig. 5)

$$V_F = 8ie^2 m_\pi^2 \frac{\delta F_V[-(\vec{p} - \vec{q})^2]}{(\vec{p} - \vec{q})^2} \quad (80)$$

and the corresponding matrix element reads as

$$\mathcal{M}_F = \langle \psi_C | V_F + T_{12} G_0 V_F + V_F G_0 T_{12} + T_{12} G_0 V_F G_0 T_{12} | \psi_C \rangle. \quad (81)$$

Again, the first term in this matrix element vanishes at the bound-state energy, and we neglect the fourth term. The remainder is then given by

$$\mathcal{M}_F = \operatorname{Re} \left[2\psi_C(0) T_{12} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} G_0(M^*; p) \right. \\ \left. \times \frac{8ie^2 m_\pi^2 \delta F_V[-(\vec{p} - \vec{q})^2]}{(\vec{p} - \vec{q})^2} \psi_C(M^*; q) \right]. \quad (82)$$

The integral over $d^4 p$ diverges in the ultraviolet region. This stems from the fact that the diagram depicted in Fig. 6a is ultraviolet divergent as a whole, though the subdivergence in the VPP vertex has already been removed by an appropriate counterterm depicted in Fig. 6b [this counterterm is implicit in Eqs. (79) and (82)]. Thus, a higher-order counterterm shown in Fig. 6c is needed to cancel the overall divergence in Fig. 6a [and, consequently, in Eq. (82)]. It is obvious that this divergence is removed analogously to that from Sec. III D, and we shall not further dwell upon this question.

To simplify the calculations in the relevant low- t region, instead of Eq. (79) we use the well-known monopole parametrization

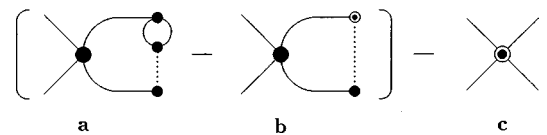


FIG. 6. Cancellation of the divergences which are present in the expression for the vertex correction. The divergence in the vertex subdiagram is cancelled by the counterterm depicted in (b) whereas the remaining overall divergence in diagram (a) is cancelled by the counterterm given in (c).

$$\delta F_V(t) = \frac{t}{\mu_V^{-2} - t}, \quad \mu_V^{-2} = \frac{1}{6} \langle r^2 \rangle_V^\pi, \quad (83)$$

with the same $\langle r^2 \rangle_V^\pi$ as in Eq. (79). The integral in Eq. (82) is then convergent and can be easily evaluated, resulting in

$$\begin{aligned} & -2e^2 m_\pi^2 \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \\ & \times \frac{4\pi\alpha m_\pi \phi_0}{w(\vec{p}) [w(\vec{q})]^{1/2} (\vec{p}^2 + \gamma^2) (\vec{q}^2 + \gamma^2) [\mu_V^2 + (\vec{p} - \vec{q})^2]}. \end{aligned} \quad (84)$$

With the use of Eq. (74) and the inequalities $\mu_V \gg m_\pi \gg \gamma$ the integration of Eq. (84) gives

$$\begin{aligned} & -2e^2 m_\pi^2 \frac{\phi_0}{m_\pi^{1/2}} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{w(\vec{p}) (\vec{p}^2 + \gamma^2) (\vec{p}^2 + \mu_V^2)} \\ & = \frac{\phi_0}{m_\pi^{1/2}} \frac{e^2 m_\pi^2}{2\pi^2 \mu_V^2} \ln \frac{4\mu_V^2}{m_\pi^2} + \dots \end{aligned} \quad (85)$$

and

$$\mathcal{M}_F = -\frac{\phi_0^2}{m_\pi} \text{Re} T_{12} \frac{4\alpha}{\pi} \frac{m_\pi^2}{\mu_V^2} \ln \frac{4\mu_V^2}{m_\pi^2} + \dots \quad (86)$$

The modification of the $\pi^+ \pi^-$ atom decay width due to the finite size effect is given by

$$\Gamma \rightarrow \Gamma \left(1 - \frac{4\alpha}{\pi} \frac{m_\pi^2}{\mu_V^2} \ln \frac{4\mu_V^2}{m_\pi^2} \right). \quad (87)$$

IV. NUMERICAL RESULTS AND DISCUSSION

In this section we present the numerical results on the lowest-order corrections to the pionium decay width. To this end we combine various corrections obtained in the previous section. As we have seen, the correction due to the relativistic modification of the Coulomb WF cancels with the corresponding piece in electromagnetic radiative corrections. In the final result we cancel these corrections explicitly.

Below we give a list of analytic results on the lowest-order corrections to the $\pi^+ \pi^-$ atom decay width:

$$\begin{aligned} \Gamma & = \frac{16\pi}{9} \left(\frac{2\Delta m_\pi}{m_\pi} \right)^{1/2} \left(1 - \frac{\Delta m_\pi}{2m_\pi} \right)^{1/2} (a_0^0 - a_0^2)^2 \\ & \times \phi_0^2 (1 + \delta_S + \delta_C + \delta_M + \delta_{em} + \delta_{vac} + \delta_F), \end{aligned} \quad (88)$$

where a_0^0 and a_0^2 denote the $\pi\pi$ scattering lengths in the isospin-symmetric case, with the charged pion mass taken to be the common mass of the pion isotriplet.

δ_S is the correction due to the displacement of the bound-state pole by strong interactions [see Eq. (33)]:

$$\delta_S = -\frac{9}{8} \frac{\Delta E^{(1)}}{E_1} = -5.47 \times 10^{-3} m_\pi (2a_0^0 + a_0^2). \quad (89)$$

δ_C corresponds to the correction due to the Coulomb photon exchanges [Eq. (33)]

$$\delta_C = (1/2 + 2.694 - \ln \alpha) \frac{\Delta E^{(1)}}{E_1} = 3.95 \times 10^{-2} m_\pi (2a_0^0 + a_0^2). \quad (90)$$

δ_M stands for the correction due to the $m_{\pi^\pm} - m_{\pi^0}$ mass difference [Eq. (64)], and δ_{em} corresponds to the electromagnetic corrections without $2C_0\alpha$ term [Eq. (65)]. The quantity δ_{vac} denotes the correction due to the vacuum polarization effect [Eq. (78)]:

$$\delta_{vac} = \frac{3}{16} \alpha^2 \frac{m_\pi}{m_e} \times 0.6865. \quad (91)$$

δ_F corresponds to the finite size correction [Eq. (87)]:

$$\delta_F = \frac{2\alpha}{3\pi} m_\pi^2 \langle r^2 \rangle_V^\pi \ln \left(\frac{1}{24} m_\pi^2 \langle r^2 \rangle_V^\pi \right). \quad (92)$$

To make a numerical estimation of the above-listed corrections, one has to substitute the values of the low-energy constants into these expressions. For the constants \bar{l}_i we take the numerical values from Ref. [1]: $\bar{l}_1 = -2.3 \pm 2.7$, $\bar{l}_2 = 6.0 \pm 1.3$, $\bar{l}_3 = 2.9 \pm 2.4$, $\bar{l}_4 = 4.3 \pm 0.9$. The constants \bar{k}_i are more difficult to estimate. In our paper we use the values from Ref. [31] based on a rough estimate at the scale coinciding with the ρ -meson mass, $|k_i^i(m_\rho)| \leq 1/(16\pi^2)$. This estimate yields $(e^2 F_\pi^2/m_\pi^2) \mathcal{K}_1^{\pm 0} = 1.8 \pm 0.9$, $(e^2 F_\pi^2/m_\pi^2) \mathcal{K}_2^{\pm 0} = 0.5 \pm 2.2$ [31]. Large error bars in the low-energy constants $\mathcal{K}_1^{\pm 0}$, $\mathcal{K}_2^{\pm 0}$, in turn, do not allow one to calculate the electromagnetic radiative correction to the atom decay width with high accuracy. Other input parameters in our calculations are the S -wave $\pi\pi$ scattering lengths $a_0^0 = 0.217 m_\pi^{-1}$, $a_0^2 = -0.041 m_\pi^{-1}$ calculated in ChPT and the electromagnetic (em) charge radius of the pion, $\langle r^2 \rangle_V^\pi = 0.439$ fm [1].

Substituting the above values of the input parameters into the expressions for various corrections to the decay width we obtain the results collected in Table I. The lifetime of the pionium ground state is predicted to be

$$\tau_1 = (3.03 \pm 0.10) \times 10^{-15} \text{ s}. \quad (93)$$

For comparison, in Table I we give the results of theoretical calculations of the corrections to the pionium lifetime within different approaches. As we observe from Table I, the largest correction in the decay width is caused by the mass splitting effect in accordance with the result of Refs. [14,15,22]. Note that in Ref. [22] the mass shift corrections coming from direct quark-photon coupling ($q\gamma$) and electromagnetic insertions in the internal pion propagators (em) are given separately. It is obvious that only the total effect of the mass shift and electromagnetic radiative corrections can be compared in different field-theoretical approaches whereas individually these contributions are rather convention depen-

TABLE I. Corrections to the $\pi^+ \pi^-$ atom decay width. In the right column the results of theoretical calculations in other approaches are given.

Effect	Value	Correction (in %)	Correction (in %)
Strong	δ_S	-0.22	
Coulomb photon exchange	δ_C	+1.55	+0.4 [22], +1.45 [15]
Mass shift	δ_M	$+2.99 \pm 0.77$	$+6.4(q\gamma) + 0.3(\text{em})$ [22], -7.4 [15]
Electromagnetic radiative	δ_{em}	$+1.73 \pm 2.31$	-0.1 [22], -0.25 [21]
Vacuum polarization	δ_{vac}	+0.19	+0.19(0-C), +0.31(tot) [23]
Finite size	δ_F	-0.16	
Total	δ_{tot}	$+6.1 \pm 3.1$	

dent. Our result for the total effect has the same sign as the result of Ref. [22], but is different in magnitude. One reason for this difference is traced back to different kinematic prescriptions used for the $\pi\pi$ scattering amplitude in the Deser formula (see the discussion in the text).¹ Another reason is that in Ref. [22] different numerical values of the low-energy constants \bar{k}_i are used. The sign of the mass splitting effect obtained in the nonrelativistic scattering theory approach [15] turns out to be opposite as compared to our result, and is of the same order of magnitude. In our opinion, owing to the derivative character of pion couplings in the chiral Lagrangian, a possible reason for this discrepancy might be an explicit energy independence of strong potentials used in calculations in Ref. [15], rather than the specific choice of the shape of the potential. In the future we plan to investigate this problem in more detail.

Our result for the correction due to the exchange of Coulomb photons agrees with the result obtained in potential scattering theory [15] (pure Coulomb correction to the scattering lengths) and disagrees with the result from Ref. [22] (second-order strong correction). Numerically the largest part in this effect comes from the nonanalytic $\ln\alpha$ piece which is exactly the same in our approach and in the potential theory, and is absent in Ref. [22].

As was mentioned above, our result for the vacuum polarization effect completely agrees with the zero-Coulomb (0-C) piece of the result given in Ref. [23]. In this paper additional contributions coming from one-Coulomb and many-Coulomb pieces are given. In our approach these contributions arise in second-order perturbation theory.

Our last remark concerns the effect of the $m_d - m_u$ mass splitting on the pionium decay width. It is well known that in one-loop order this leads only to a shift in the neutral pion mass [1]. Since in our calculations of the on-shell $\pi\pi$ scattering amplitude we use the physical values of the pion masses, with the mass difference caused both by $m_d - m_u \neq 0$ and electromagnetic corrections, the resulting mass splitting correction includes both these effects.²

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APPENDIX A

In this appendix, we present the calculation of various integrals which appear in the first-order correction terms. We start from the evaluation of the relativistic Coulomb WF at the origin [Eq. (27)]. By carrying out the p_0 integration with the use of the Cauchy theorem, this quantity can be written as

$$\begin{aligned} \psi_C(0) &= \int \frac{d^4 p}{(2\pi)^4} \psi_C(M^*; p) = \frac{\phi_0}{m_\pi^{1/2}} \int \frac{d^3 \vec{p}}{(2\pi)^3} \left(\frac{m_\pi}{w(\vec{p})} \right)^{1/2} \frac{4\pi\alpha m_\pi}{(\vec{p}^2 + \gamma^2)^2} \\ &= \frac{\phi_0}{m_\pi^{1/2}} \left(1 - \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{\vec{p}^2}{[w(\vec{p})]^{1/2} \{m_\pi^{1/2} + [w(\vec{p})]^{1/2}\} [m_\pi + w(\vec{p})]} \frac{4\pi\alpha m_\pi}{(\vec{p}^2 + \gamma^2)^2} \right). \end{aligned} \quad (\text{A1})$$

In the lowest-order approximation in α we replace the factor $(\vec{p}^2 + \gamma^2)^2$ in the denominator by $(\vec{p}^2)^2$ and obtain

¹In recent calculations by the same authors the discrepancy due to the different kinematic prescriptions is removed (H. Sazdjian, private communication).

²We are indebted to Professor J. Gasser for clarifying discussions on this problem.

$$\psi_C(0) = \frac{\phi_0}{m_\pi^{1/2}}(1 - C_0\alpha) + \dots, \quad (\text{A2})$$

where

$$C_0 = \frac{2}{\pi} \int_0^\infty \frac{dp}{(1+p^2)^{1/4} [1+(1+p^2)^{1/4}] [1+(1+p^2)^{1/2}]} = 0.381 \dots \quad (\text{A3})$$

Next we turn to the calculation of the integral which is present in Eq. (31). This integral contains the Green function $\delta G(p, q)$ which corresponds to the exchanged Coulomb photon ladders and, according to Eqs. (10), (21), is given by

$$\begin{aligned} \delta G(p, q) &= i [w(\vec{p})w(\vec{q})]^{1/2} \left[\Phi(\vec{p}, \vec{q}) - S(\vec{p})S(\vec{q}) \frac{8}{M^*} \frac{\partial}{\partial M^*} \right] G_0(M^*; p) G_0(M^*; q), \\ \Phi(\vec{p}, \vec{q}) &= 16\pi m_\pi \alpha \left[\frac{1}{(\vec{p} - \vec{q})^2} + I_R(\vec{p}, \vec{q}) \right] + \frac{1}{(m_\pi \alpha)^2} S(\vec{p})S(\vec{q})R(\vec{p}, \vec{q}), \\ R(\vec{p}, \vec{q}) &= 25 - \left(\frac{8}{\pi m_\pi \alpha} \right)^{1/2} [S(\vec{p}) + S(\vec{q})] + \dots, \end{aligned} \quad (\text{A4})$$

where ellipses stand for higher-order terms in α .

Substituting this expression in the integral from Eq. (31) and carrying out p_0, q_0 integrations, we obtain

$$\begin{aligned} I_1 &= \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \bar{\psi}_C(M^*; p) [G_0^{-1}(M^*; p)]' \delta G(p, q) \\ &= \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{4\pi\alpha m_\pi \phi_0}{(\vec{p}^2 + \gamma^2)^3} \left[\Phi(\vec{p}, \vec{q}) - S(\vec{p})S(\vec{q}) \left(\frac{6}{\vec{p}^2 + \gamma^2} + \frac{4}{\vec{q}^2 + \gamma^2} \right) \right] \frac{-M^*}{8[w(\vec{q})]^{1/2}} \frac{1}{\vec{q}^2 + \gamma^2}, \end{aligned} \quad (\text{A5})$$

where we have used

$$\int \frac{dp_0}{2\pi i} [G_0(M^*; p)]' [G_0^{-1}(M^*; p)]' G_0(M^*; p) = \frac{3(M^*)^2}{32w(\vec{p})(\vec{p}^2 + \gamma^2)^3} + \dots \quad (\text{A6})$$

In the calculation of 3D integrals containing the function $\Phi(\vec{p}, \vec{q})$, we use

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{(\vec{p} - \vec{q})^2} \frac{1}{(\vec{p}^2 + \gamma^2)^3} = \frac{1}{4\pi\alpha^3 m_\pi^3} \frac{1}{(\vec{q}^2 + \gamma^2)} + \frac{1}{8\pi\alpha m_\pi} \frac{1}{(\vec{q}^2 + \gamma^2)^2} \quad (\text{A7})$$

and

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{I_R(\vec{p}, \vec{q})}{(\vec{p}^2 + \gamma^2)^3} = \int_0^1 \frac{d\rho}{\rho} [\mathcal{J}'_1(\rho; \vec{q}) - \mathcal{J}'_1(\rho; \vec{q})], \quad (\text{A8})$$

where

$$\mathcal{J}'_1(\rho; \vec{q}) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{(\vec{p}^2 + \gamma^2)^3} \frac{1}{(\vec{p} - \vec{q})^2 \rho + m_\pi^{-2} \alpha^{-2} (\vec{p}^2 + \gamma^2) (\vec{q}^2 + \gamma^2) (1 - \rho)^2}. \quad (\text{A9})$$

The integration over $d^3 \vec{p}$ can be carried out with the use of Feynman parametrization. We obtain

$$\mathcal{J}'_1(\rho; \vec{q}) = \frac{3}{8\pi\alpha^3 m_\pi^3} \frac{1}{(\vec{q}^2 + \gamma^2)} \int_0^1 \frac{dx(1-x)^2}{d_-^{3/2} d_+^{5/2}} + \frac{3}{32\pi\alpha m_\pi} \frac{1}{(\vec{q}^2 + \gamma^2)^2} \int_0^1 \frac{dx(1-x)^2}{d_-^{5/2} d_+^{5/2}},$$

$$d_{\pm} = 1 - x + \frac{x}{4}(1 \pm \rho)^2, \quad (\text{A10})$$

and

$$\mathcal{J}'_1(\rho; \vec{q}) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{1}{(\vec{p}^2 + \gamma^2)^3} \frac{1}{m_{\pi}^{-2} \alpha^{-2} (\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} = \frac{1}{2\pi \alpha^3 m_{\pi}^3} \frac{1}{(\vec{q}^2 + \gamma^2)}. \quad (\text{A11})$$

Substituting Eqs. (A11) and (A10) into Eq. (A8) and carrying out the integration over dx and $d\rho$, we finally obtain

$$\int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{I_R(\vec{p}, \vec{q})}{(\vec{p}^2 + \gamma^2)^3} = -\frac{1}{4\pi \alpha^3 m_{\pi}^3} \frac{1}{(\vec{q}^2 + \gamma^2)} + \frac{1}{8\pi \alpha m_{\pi}} \frac{1}{(\vec{q}^2 + \gamma^2)^2}. \quad (\text{A12})$$

With the use of Eqs. (A4), (A7), and (A12) the integration in Eq. (A5) is trivially carried out, resulting in

$$I_1 = \frac{\phi_0}{m_{\pi}^{1/2}} \frac{1}{\alpha^2 m_{\pi}^2} + \dots \quad (\text{A13})$$

Substituting this result back in Eq. (31), we readily obtain the final result given in this equation.

Next we turn to the calculation of the integral which is present in Eq. (32)

$$I_2 = \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \delta G(p, q) = I'_2 + I''_2, \quad (\text{A14})$$

where I'_2 and I''_2 correspond to the ‘‘nonderivative’’ and ‘‘derivative’’ terms in Eq. (A4). Carrying out the integration over the relative energies with the use of the Cauchy theorem, I'_2 can be written as

$$I'_2 = -\frac{i}{16} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{d^3 \vec{q}}{(2\pi)^3} \frac{1}{[w(\vec{p})w(\vec{q})]^{1/2}} \frac{1}{(\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} \Phi(\vec{p}, \vec{q}). \quad (\text{A15})$$

I'_2 receives a contribution from (a) one-photon exchange, (b) multiphoton exchanges concentrated in $I_R(\vec{p}, \vec{q})$, and (c) the rest, proportional to the function R [see Eq. (A4)]. Below we shall evaluate these contributions separately.

(a) One-Coulomb-photon exchange:

$$I'_{2a} = \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{-i\pi \alpha m_{\pi}}{[w(\vec{p})w(\vec{q})]^{1/2}} \frac{1}{(\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} \frac{1}{(\vec{p} - \vec{q})^2} = -\frac{i\alpha m_{\pi}}{4} \int d^3 r \frac{\varphi^2(r)}{r}, \quad (\text{A16})$$

where

$$\varphi(r) = \int \frac{d^3 \vec{p}}{(2\pi)^3} e^{-i\vec{p}\vec{r}} \frac{1}{[w(\vec{p})]^{1/2}(\vec{p}^2 + \gamma^2)}. \quad (\text{A17})$$

Using exponential parametrization, the integration over $d^3 \vec{p}$ in Eq. (A17) can be carried out, resulting in

$$\varphi(r) = \frac{1}{8\pi^{3/2} \Gamma\left(\frac{1}{4}\right)} \int_0^{\infty} du u^{-5/4} \int_0^1 dx x^{-3/4} \exp\left[-\frac{\vec{r}^2}{4u} - u(1-x)\gamma^2 - uxm_{\pi}^2\right]. \quad (\text{A18})$$

Substituting Eq. (A18) into Eq. (A16) and integrating, we obtain

$$I'_{2a} = -\frac{i\alpha m_{\pi}}{32\pi^{3/2} \Gamma^2\left(\frac{1}{4}\right)} \int_0^1 d\tau \int_0^1 dx_1 \int_0^1 dx_2 \frac{\tau^{-1/4}(1-\tau)^{-1/4} x_1^{-3/4} x_2^{-3/4}}{\{\gamma^2 + (m_{\pi}^2 - \gamma^2)[\tau x_1 + (1-\tau)x_2]\}^{1/2}}. \quad (\text{A19})$$

Note that one cannot directly assume here $\gamma=0$ in the denominator, since the integral over the Feynman parameters diverges in this limit. In order to overcome this difficulty, we split the integration area into two domains according to

$$\int_0^1 dx_1 \int_0^1 dx_2 f(x_1, x_2) = \int_0^1 \rho d\rho \int_0^1 dt f(\rho t, \rho(1-t)) + \int_1^2 \rho d\rho \int_{1-1/\rho}^{1/\rho} dt f(\rho t, \rho(1-t)). \quad (\text{A20})$$

Performing explicitly the integration over $d\rho$ in the first domain, we obtain

$$\begin{aligned} I'_{2a}(1) &= - \frac{i\alpha m_\pi}{16\pi^{3/2}\Gamma^2\left(\frac{1}{4}\right)(m_\pi^2 - \gamma^2)^{1/2}} \int_0^1 dt \int_0^1 d\tau \frac{\tau^{-1/4}(1-\tau)^{-1/4}t^{-3/4}(1-t)^{-3/4}}{[\tau t + (1-\tau)(1-t)]^{1/2}} \\ &\quad \times \ln [\gamma^{-1}((m_\pi^2 - \gamma^2)^{1/2}[\tau t + (1-\tau)(1-t)]^{1/2} + \{\gamma^2 + (m_\pi^2 - \gamma^2)[\tau t + (1-\tau)(1-t)]\}^{1/2})] \\ &= \frac{i\alpha \ln \alpha}{16\pi} - \frac{i\alpha}{16\pi} \left[2\ln 2 + \frac{c_1}{2\pi^{1/2}\Gamma^2(1/4)} \right] + \dots, \end{aligned} \quad (\text{A21})$$

where ellipses stand for the higher order terms in α and

$$c_1 = \int_0^1 dt \int_0^1 d\tau \frac{\tau^{-1/4}(1-\tau)^{-1/4}t^{-3/4}(1-t)^{-3/4}}{[\tau t + (1-\tau)(1-t)]^{1/2}} \ln[\tau t + (1-\tau)(1-t)] = -40.374 \dots \quad (\text{A22})$$

The second integral converges when $\gamma \rightarrow 0$ in the denominator, resulting in

$$I'_{2a}(2) = - \frac{i\alpha c_2}{4\pi^{3/2}\Gamma^2\left(\frac{1}{4}\right)}, \quad (\text{A23})$$

where

$$c_2 = \frac{1}{4} \int_0^1 d\tau \tau^{-1/4}(1-\tau)^{-1/4} \int_1^2 d\rho \ln \rho (\rho-1)^{-3/4} [\tau + (1-\tau)(\rho-1)]^{-1/2} = 0.288 \dots \quad (\text{A24})$$

(b) Multiphoton exchanges: In this contribution we can safely replace the smooth factor in the denominator $[w(\vec{p})w(\vec{q})]^{1/2} \rightarrow m_\pi$. Then

$$I_{2b} = -i\pi\alpha \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} \frac{I_R(\vec{p}, \vec{q})}{(\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} = -i\pi\alpha \int_0^1 \frac{d\rho}{\rho} \mathcal{J}'_{2b}(\rho), \quad (\text{A25})$$

where, according to Eq. (10),

$$\begin{aligned} \mathcal{J}'_{2b}(\rho) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{(\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} \\ &\quad \times \left[\frac{1}{(\vec{p}-\vec{q})^2 \rho + m_\pi^{-2} \alpha^{-2} (\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)(1-\rho)^2} - \frac{1}{m_\pi^{-2} \alpha^{-2} (\vec{p}^2 + \gamma^2)(\vec{q}^2 + \gamma^2)} \right]. \end{aligned} \quad (\text{A26})$$

Introducing Feynman parameters and carrying out the momentum integration, we obtain

$$\mathcal{J}'_{2b}(\rho) = \frac{1}{16\pi^2} \left[\frac{1}{2} \int_0^1 \frac{dx}{d_- d_+^{1/2} (d_-^{1/2} + d_+^{1/2})} - 1 \right] \quad (\text{A27})$$

and

$$I'_{2b} = - \frac{i\alpha}{16\pi}. \quad (\text{A28})$$

(c) Factorizing integrals: The integral containing the function R is evaluated in the straightforward manner, resulting in

$$I'_{2c} = -\frac{17i\alpha}{128\pi} + \dots \quad (\text{A29})$$

The ‘‘derivative’’ term I''_2 can be easily calculated. The integration in the variables p and q again factorizes, and we have

$$I''_2 = \frac{i\alpha}{16\pi} + \dots \quad (\text{A30})$$

Putting all together, we finally arrive at the result

$$I_2 = \frac{i\alpha}{16\pi} \ln\alpha + \frac{i\alpha}{16\pi} \left[-\frac{17}{8} - 2\ln 2 - \frac{c_1}{2\pi^{1/2}\Gamma^2\left(\frac{1}{4}\right)} - \frac{4c_2}{\pi^{1/2}\Gamma^2\left(\frac{1}{4}\right)} \right] = \frac{i\alpha}{16\pi} (\ln\alpha - 2.694), \quad (\text{A31})$$

which is substituted in Eq. (32).

APPENDIX B

In this appendix we shall present the calculation of the integrals appearing in the electromagnetic radiative corrections [Eq. (45)]:

$$\int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} G_0(M^*; p) \frac{ie^2[(M^*)^2 - (p_0 + q_0)^2]}{(\vec{p} - \vec{q})^2} G_0(M^*; q) 4i[w(\vec{q})]^{1/2} \frac{4\pi\alpha m_\pi \phi_0}{\vec{q}^2 + \gamma^2} = \tilde{I}_1 + \tilde{I}_2. \quad (\text{B1})$$

Integrating over the relative energy variables, the first term is rewritten in the form

$$\begin{aligned} \tilde{I}_1 &= \frac{e^2(M^*)^2}{4} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{w(\vec{p})(\vec{p}^2 + \gamma^2)} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{[w(\vec{q})]^{1/2}(\vec{p} - \vec{q})^2} \frac{4\pi\alpha m_\pi \phi_0}{(\vec{q}^2 + \gamma^2)^2} \\ &= \frac{e^2\phi_0(M^*)^2}{4m_\pi^{1/2}} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{w(\vec{p})(\vec{p}^2 + \gamma^2)^2} + \dots \end{aligned} \quad (\text{B2})$$

Using the same trick as in Eqs. (A1)–(A3), we can write

$$\tilde{I}_1 = \frac{\phi_0}{m_\pi^{1/2}} \left(1 - \frac{e^2}{2\pi^2} \int_0^\infty \frac{dp}{(1+p^2)^{1/2}[1+(1+p^2)^{1/2}]} + \dots \right) = \frac{\phi_0}{m_\pi^{1/2}} \left(1 - \frac{2\alpha}{\pi} \right) + \dots \quad (\text{B3})$$

In the calculations of \tilde{I}_2 the term containing $2p_0q_0$ vanishes since it is odd in p_0 and q_0 . Thus one can write $\tilde{I}_2 = \tilde{I}'_2 + \tilde{I}''_2$ where \tilde{I}'_2 and \tilde{I}''_2 contain p_0^2 and q_0^2 , respectively.

\tilde{I}'_2 is ultraviolet divergent. Introducing dimensional regularization, we can write

$$\begin{aligned} \int \frac{d^4p}{(2\pi)^4} G_0(M^*; p) \frac{p_0^2}{(\vec{p} - \vec{q})^2} \rightarrow -(\mu^2)^{2-n/2} \int \frac{d^n p}{(2\pi)^n} \frac{p_0^2}{\left[\left(\frac{P}{2} + p \right)^2 - m_\pi^2 \right] \left[\left(\frac{P}{2} - p \right)^2 - m_\pi^2 \right] (\vec{p} - \vec{q})^2} \\ = \frac{i}{16\pi^2} \left(N_\epsilon + 4 + \frac{w(\vec{q})}{|\vec{q}|} \ln \frac{w(\vec{q}) - |\vec{q}|}{w(\vec{q}) + |\vec{q}|} \right) \end{aligned} \quad (\text{B4})$$

and

$$\tilde{I}'_2 = \frac{\alpha}{4\pi} (N_\epsilon + 2) \frac{\phi_0}{m_\pi^{1/2}}. \quad (\text{B5})$$

\tilde{I}''_2 does not contain the ultraviolet divergence. Integrating over relative energy variables, we obtain

$$\tilde{T}_2'' = \frac{e^2}{4} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{w(\vec{p})[w(\vec{q})]^{1/2}} \frac{1}{(\vec{p}-\vec{q})^2(\vec{p}^2+\gamma^2)} \frac{4\pi\alpha m_\pi\phi_0}{(\vec{q}^2+\gamma^2)}. \quad (\text{B6})$$

It is easy to see that \tilde{T}_2'' leads to a modification of the $\pi^+\pi^-$ decay width in the order $\alpha^2\ln\alpha$ and thus can be safely neglected. The final result reads [cf. with Eq. (45)]

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 = \frac{\phi_0}{m_\pi^{1/2}} \left(1 + \frac{\alpha}{4\pi} N_\epsilon - \frac{3\alpha}{2\pi} \right). \quad (\text{B7})$$

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