

## Modeling the glueball spectrum by a closed bosonic membrane

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We use an analogy between the Yang-Mills theory Hamiltonian and the matrix model description of the closed bosonic membrane theory to calculate the spectrum of glueballs in the large  $N_c$  limit. Some features of the Yang-Mills theory vacuum, such as the screening of the topological charge and vacuum topological susceptibility, are discussed. We show that the topological susceptibility has different properties depending on whether it is calculated in the weak coupling or strong coupling regimes of the theory. A mechanism of the formation of the pseudoscalar glueball state within pure Yang-Mills theory is proposed and studied. [S0556-2821(98)03019-7]

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### INTRODUCTION

Gluodynamics [1], being the asymptotically free theory [2] of colored [3] massless vector particles, is believed to underline the dynamics of strong interactions. Because of asymptotic freedom the theory is well studied at short distances; however, long distance phenomena deserve to be understood much better.

The theory predicts glueballs [4], the nonperturbative bound states composed of pure glue [5]. That prediction was confirmed some time ago by “observing” glueballs in lattice QCD simulations [6,7]. In addition to that, there are experimental signatures of resonances which strongly resemble properties of glueballs (for a recent analysis of these issues see Ref. [8]).

Studying glueballs one might hope to learn more about the complicated ground state structure of non-Abelian Yang-Mills (YM) theory. The main question one might wonder about is the mechanism of the formation of glueball states in YM theory. Those states appear to be heavy in comparison with the lightest hadrons and range, depending on the spin-parity structure, within the mass interval 1.5–2.3 GeV [6,7]. Thus, the naive picture of the glueball as a system of two massless gluons which interchange virtual perturbative gluons does not seem to be appropriate.

In this work we are looking for qualitative features responsible for the process of formation of pseudoscalar glueballs. A possible mechanism will be proposed. As an outcome we calculate the spectrum of lightest pseudoscalar glueballs. The results are in agreement with predictions of lattice calculations [6]. The paper deals with pure YM theory, no light fermions are included. A brief discussion of full QCD is given at the end of the work.

Our study relies on the existence of the  $\theta$  term in pure YM theory. We define the topological charge density operator as  $Q \equiv (1/32\pi^2)G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a$ , with  $G_{\mu\nu}^a$  being the non-Abelian gauge field strength tensor and the dual tensor is normalized as  $\tilde{G}_{\mu\nu}^a = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}^a$ . Because of instantons [9], the non-Abelian gauge theory possesses a complicated vacuum struc-

ture [10]. That is, there is an infinite number of degenerate vacuum states labeled by some topological invariant, the winding number or topological charge. Instantons, being defined in Euclidean space, provide that quasiclassical tunneling processes happen between the different vacua. Thus, the true ground state of the theory is a superposition of the vacua with different topological charges. The superposition can be provided in the path integral formulation by adding to the action the  $\theta$  term  $\Delta S_\theta \equiv \theta \int d^4x Q(x)$  [10]. However, such a modification is not a harmless procedure. The  $\theta$  term in QCD leads to an induced neutron electric dipole moment. Experimental bounds on that quantity restrict the value of the  $\theta$  parameter to be unnaturally small, less than the billion'th part of the unity, and give rise the famous strong  $CP$  problem [11].

The picture outlined above implies that the integral of the topological charge density  $\int d^4x Q(x)$ , being the topological charge in Euclidean space, is quantized if the instanton boundary conditions are imposed on gauge fields.<sup>1</sup> Thus, the whole scenario of the superposition of the different vacua by means of the  $\theta$  term relies on quasiclassical arguments. In general, one expects that the quasi-classical approximation is justified in a weak coupling regime only [12]. What the  $\theta$  term leads to in a strong coupling approximation where quasiclassical arguments cease to be valid, is not clear *a priori*.

It was argued by Witten [13] that in the confining phase of the theory noninteracting instanton boundary conditions should not be relevant. The key observation was that gauge fields with instanton boundary conditions do not yield the area law for the Wilson loop. Thus, configurations with those boundary conditions, as any configurations tending to a pure gauge at infinity, fail to satisfy the confinement criterion [14]. As a result, in the strong coupling approximation of the confining theory one should rather encounter some smeared distributions of interacting topological charges as opposed to the noninteracting instanton system with quantized topologi-

<sup>1</sup>Under those boundary condition one means that the vector potential  $A_\mu$  tends to a pure gauge configuration at spatial infinity  $A_\mu \rightarrow U^{-1}(x)\partial_\mu U(x)$ , with  $U(x)$  being an element of the  $SU(N_c)$  gauge group.

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cal charge [13]. This statement finds support in recent models of the YM vacuum. Properties of hadrons are correctly described by the model in which instantons and antiinstantons are coupled in moleculelike (or even more complicated) entities [15]. This kind of instanton clustering, indicating on strong correlations between them, was also observed in some recent lattice QCD studies [16].

Interactions between instantons, if sufficiently strong, lead to the screening of the topological charge at finite distances [17,18], in analogy with the well-known phenomenon in plasma physics.

A quantitative study of the screening phenomenon from fundamental principles is a cumbersome task. However, as we mentioned already, both hadron phenomenology and lattice simulations seem to support that picture. Below, in Sec. I we present some arguments (other than the ones mentioned above) indicating that the screening of the topological charge should really exist. We show (Sec. II) that the three-form composite field, which is Hodge dual to the Chern-Simons current and is known to propagate the Coulomb type interaction, should be relevant for the description of the screening phenomenon.

In Sec. II we present a possible mechanism of the formation of a pseudoscalar glueball. The main ingredients needed for that mechanism to be realized in a confining theory are the screening of the topological susceptibility and the presence of the  $\theta$  angle in the theory.

Having this part set, we discuss in Sec. III an analogy between the spectrum of the YM Hamiltonian and that of a closed bosonic membrane with the topology of a sphere. We use that analogy to calculate the spectrum of glueballs. In fact, we derive a matching condition between the spectrum of YM theory in the large  $N_c$  limit and the matrix quantum mechanics formulation of the closed bosonic membrane theory. Then, studying the spectrum of a spherical closed membrane and using the matching condition we calculate the spectrum of the YM Hamiltonian. That gives the prediction for masses of lightest glueballs. We also show that the scenario discussed in this work is realized only when the  $\theta$  parameter is a macroscopic number, i.e., a number of order of the unity or so. More accurate estimates are given in Sec. III C. We briefly discuss how the strong  $CP$  violation, being present in pure YM theory, might still not be observable in full QCD with light quarks.

Discussions in the present paper are based on the results obtained by Lüscher [19,20,21], by Goldstone and Hoppe (see Ref. [22]), and by de Wit, Hoppe, and Nicolai [23]. Where it is possible we present below brief summaries of those results.

## I. TOPOLOGICAL SUSCEPTIBILITY

In this section we study properties of the correlator of the vacuum topological susceptibility. We work in Euclidean space-time assuming that the theory is defined in a compact Euclidean four-volume  $V \equiv l^3 \times \tau$ , with  $l$  being the linear size of the volume and  $\tau$  stands for Euclidean time. The correlator of the vacuum topological susceptibility  $\chi(V)$  can be written as follows:

$$\chi(V) \equiv \int_V \langle 0 | T Q(x) Q(0) | 0 \rangle d^4x, \quad (1)$$

where  $Q$  is the topological charge density operator defined in the previous section.<sup>2</sup>

The function  $\chi(V)$  is determined by the nonperturbative sector of Yang-Mills theory. Calculation of  $\chi(V)$  in that respect is a matter of modeling of the vacuum structure of Yang-Mills theory. This in its turn is a complicated task. Below we show however, that one can still study qualitative features of the volume dependence of the correlator of the vacuum topological susceptibility for small and for large values of the volume.<sup>3</sup> Let us first define, following [19], what one could call a small volume or large volume limit.

There is a dynamically generated mass scale in YM theory  $\Lambda_{\text{YM}}$ . The reciprocal quantity of  $\Lambda_{\text{YM}}$  sets the characteristic correlation length for the model. Let us denote that length by  $\zeta \equiv \Lambda_{\text{YM}}^{-1}$ . Restricting for simplicity the YM beta function to the next-to-leading order approximation, the expression for  $\zeta$  can be written as follows:<sup>4</sup>

$$\zeta = \mu^{-1} (\alpha_s)^{\beta_1/2\beta_0^2} \exp\left(\frac{2\pi}{\beta_0 \alpha_s}\right),$$

where  $\mu$  is the renormalization scale,  $\alpha_s \equiv \alpha_s(\mu^2/\Lambda_{\text{YM}}^2)$  is the scale-dependent strong coupling constant, and  $\beta_0$  and  $\beta_1$  are the first two scheme independent coefficients of the beta function  $\beta_0 = 11N_c/3$ ,  $\beta_1 = 34N_c^2/3$ . The expression for  $\zeta$  is renormalization group invariant in the corresponding order.

Let us now introduce the following two limits. One can define the value of the volume element  $V$  to be small if the correlation length  $\zeta$  is much larger than  $V^{1/4}$ , i.e.,  $\zeta \gg V^{1/4}$  [19]. The large volume limit in that case would refer to a volume element satisfying the condition  $\zeta \ll V^{1/4}$ .

One can show that the two limits defined above correspond respectively to the weak coupling and strong coupling regimes of the theory. In order to see this let us keep the product of the renormalization scale  $\mu$  and the value of  $V^{1/4}$  fixed, say,  $V\mu^4 = 1$ . This condition sets the scale  $\mu$  as an infrared cutoff. Then, the expression for the correlation length takes the form

$$\frac{\zeta}{V^{1/4}} = (\alpha_s)^{\beta_1/2\beta_0^2} \exp\left(\frac{2\pi}{\beta_0 \alpha_s}\right).$$

Thus, the small volume approximation given by the condition  $\zeta \gg V^{1/4}$  corresponds to the weak coupling regime, i.e.,

<sup>2</sup>The correlator (1) is in general a divergent Green's function. These divergences can be removed by means of standard procedures which are discussed in the Appendix.

<sup>3</sup>We vary  $l$  keeping  $\tau$  fixed. Thus, we actually study the three-volume dependence of  $\chi(V)$ .

<sup>4</sup>Since we are interested in the magnitude of this quantity the explicit  $\theta$  dependence in this expression is dropped.

$\alpha_s \ll 1$ . For instance, if one sets  $\zeta/V^{1/4} \approx 10 \gg 1$ , the corresponding value of the coupling constant<sup>5</sup> is  $\alpha_s \approx 0.2$ .

Let us now turn to the discussion of the large volume limit defined by the condition  $\zeta \ll V^{1/4}$ . This limit is equivalent to the small  $\mu$  approximation. However, for small values of  $\mu$  the running coupling constant  $\alpha_s$  is a big number. Hence, the large volume limit corresponds to the strong coupling regime of the theory. Regretfully, we can not estimate (as we did in the case of the small coupling constant) how large the coupling constant should actually be. The approximation which is used defining  $\zeta$  breaks down for large values of the coupling. Though one could present the definition of  $\zeta$  for any orders of perturbation theory (see, for example, [24]), that definition would contain the exact form of the beta function  $\beta(\alpha_s)$  which is known only perturbatively. So, the all order formula also becomes inappropriate for practical calculations in the strong coupling approximation.

Below we show, at least qualitatively, that the topological susceptibility as a function of  $V$  has different behavior depending whether it is calculated in the weak or strong coupling regimes. In the weak coupling phase it is an increasing function of the argument, and on the contrary, in the strong coupling regime the function decreases with the argument monotonically.

#### A. The weak coupling approximation

Let us start with the small volume or weak coupling approximation. The non-Abelian gauge theory provides a good description of physics in that domain. Excitations with a zero topological charge do not contribute to the value of  $\chi(V)$  defined in Eq. (1). Only nontrivial topological configurations of gauge fields are to be taken into account. In the weak coupling regime the YM vacuum can be approximated by noninteracting, well-separated instantons [25]. In that approximation instantons can be treated as pointlike objects. The expression for the topological charge density can be written as follows:

$$Q(x) = \sum_i q_i \delta^{(4)}(x - x_i),$$

where  $q_i$  denotes the topological charge for a configuration localized at the point  $x_i$ . Assuming that instantons do not interact with one another we derive

$$\langle 0 | T Q(x) Q(0) | 0 \rangle = \frac{1}{V} \sum_{q=-\infty}^{+\infty} q^2 P_q(V) \delta^{(4)}(x), \quad (2)$$

where the index  $i$  in the definition of the topological charge was omitted. The quantity  $P_q$  denotes the probability for a nonabelian gauge field configuration to have a topological

charge equal to  $q$ . These probabilities are exponentially suppressed for nonzero  $q$  and one expects that the infinite series in Eq. (2) converges.<sup>6</sup>

Substituting Eq. (2) into Eq. (1) one derives

$$\chi(V) = \frac{1}{V} \sum_{q=-\infty}^{+\infty} q^2 P_q(V). \quad (3)$$

In the approximation we set above the following relation is valid:

$$P_q(V) = [P_1(V)]^{|q|},$$

where  $P_1(V)$  denotes the one instanton contribution. Substituting this relation into Eq. (3) and performing the summation of the infinite series one gets the following expression for the topological susceptibility:

$$\chi(V) = \frac{2P_1(V)}{V} \frac{1+P_1(V)}{[1-P_1(V)]^3}. \quad (4)$$

Thus, the small volume behavior of the function  $\chi(V)$  is approximately defined by Eq. (4). The expression for  $P_1(V)$  can be calculated in the one loop approximation using the well known results of Ref. [26],

$$P_1(V) = \text{const} \times \exp\left(-\frac{2\pi}{\alpha_s(V^{1/2}\Lambda_{\text{YM}}^2)}\right),$$

where the following expression for the strong coupling constant is supposed to be used:

$$\alpha_s(V^{1/2}\Lambda_{\text{YM}}^2) = -\frac{4\pi}{\beta_0 \ln(V^{1/2}\Lambda_{\text{YM}}^2)} + \dots$$

The result for  $P_1(V)$  is

$$P_1(V) = \text{const} \times \left(\frac{V}{\zeta^4}\right)^{\beta_0/4} \times \text{logarithms},$$

where the logarithmic corrections appear in the next-to-leading approximation. Hence, as  $V \rightarrow 0$  the ratio  $P_1(V)/V$  also tends to zero. As a consequence, in the small volume limit  $\lim_{V \rightarrow 0} \chi(V) \rightarrow 0$ . Moreover, based on the relations given above one concludes that for small volume elements the quantity  $\chi(V)$  is a monotonically increasing function of the argument  $V$ . This property should hold as the condition  $\zeta \gg V^{1/4}$  is satisfied.

Suppose now that the quantity  $V^{1/4}$  becomes comparable in magnitude with  $\zeta$  so that the weak coupling approximation breaks down. As a result, the pointlike noninteracting instanton approximation ceases to be valid. Interactions between instantons start to play a crucial role providing the screening of the topological charge [18].

<sup>5</sup>If one approximates  $\Lambda_{\text{YM}} \approx (100-200)$  MeV, then the small volume limit refers to  $V^{1/4} \ll (1-2)$  fm.

<sup>6</sup>The partition function  $Z(\theta)$  in that case can be approximated as  $Z(\theta) = \sum_q P_q \exp(i\theta q)$ .

Let us assume for a moment that one neglects instanton interactions even for large values of the volume and let us study what happens in this unrealistic case keeping in mind that the interaction effects are going to be included later. Doing so one is dealing with an ideal gas of instantons placed in a large volume. The approximate calculation of the partition function with noninteracting instantons in the thermodynamic limit yields the following Gaussian distribution function for  $P_q(V)$  [27]:

$$P_q(V \gg \zeta^4) \approx \frac{1}{\sqrt{2\pi Vd}} \exp\left(-\frac{q^2}{2Vd}\right),$$

where  $d$  is a not yet defined positive constant. We substitute this expression into Eq. (3) and perform the summation of the infinite series in the large volume limit. The final expression can be found using the following relation:

$$\sum_{q=-\infty}^{+\infty} e^{-\pi b q^2} = \frac{1}{\sqrt{b}} \sum_{q=-\infty}^{+\infty} e^{-\pi q^2/b},$$

where  $b$  is an arbitrary positive number [28]. As a result one gets

$$\lim_{V \gg \zeta^4} \chi(V) = d. \quad (5)$$

Let us summarize briefly what we learned about the volume dependence of the topological susceptibility  $\chi(V)$ . In the zero volume approximation the topological susceptibility was zero. Increasing the value of  $V$ , so that the weak coupling approximation still holds, the function  $\chi(V)$  increases monotonically. If one goes further and neglects the interaction between instantons even in the large volume (strong coupling) approximation, one finds that the function  $\chi(V)$  reaches its asymptotic value<sup>7</sup> denoted above by  $\chi(V \gg \zeta^4) = d$ . However, as we stressed earlier, interactions between instantons play a crucial role in the strong coupling approximation. In the next section we show that the topological susceptibility becomes a decreasing function of the argument for large values of  $V$  when the effects of finite distance correlations between topological charges are taken into account.

### B. The strong coupling approximation

Let us consider the large volume or strong coupling limit. In that limit the theory is in a confining phase. Instantons are interacting strongly.<sup>8</sup> Those interactions become responsible for formation of spin zero glueball states [29]. A description in terms of colored variables is not a good approximation anymore. The theory, however, can be defined by means of the low-energy effective action containing colorless degrees

of freedom. The explicit form of that effective action for pure YM theory is not known. In general, the action can be written as

$$S_{\text{eff}} = \int d^4x \mathcal{L}(G_n, \nabla G_n, \nabla^2 G_n, \dots),$$

where  $G_n$ 's stand for glueball fields.

Below we study properties of the correlator of the vacuum topological susceptibility in the effective theory. We denote this quantity by  $\chi_{\text{eff}}(V)$ . The correlator in Eq. (1) is saturated by the set of intermediate glueball states

$$\begin{aligned} \langle 0 | T Q(x) Q(0) | 0 \rangle &= d \delta^{(4)}(x) + \sum_n \langle 0 | Q | n \rangle \\ &\quad \times \langle n | Q | 0 \rangle D_F(m_n | x|), \end{aligned} \quad (6)$$

where  $m_n$  is the mass of the  $n$ th intermediate physical state and  $D_F(m_n | x|)$  stands for the Euclidean  $x$ -space Feynman propagator of a scalar massive particle

$$D_F(m_n | x|) = \frac{m_n}{4\pi^2 |x|} K_1(m_n | x|),$$

with  $K_1(m_n | x|)$  being the Bessel function of an imaginary argument.

The parameter  $d$  given in Eq. (6) is a positive number. It was introduced in the preceding section and in the simplest case of a dilute instanton gas approximation corresponds to the value of the topological susceptibility in the large volume limit. From the point of view of the effective theory we deal with, the parameter  $d$  is a momentum independent subtraction coefficient in the dispersion relation for  $\chi_{\text{eff}}$  written in momentum space.<sup>9</sup> In Eq. (6) we implicitly assumed that the volume element is sufficiently large so that the YM topological susceptibility occurring as the first term on the right-hand side (RHS) equals to its asymptotic value  $d$ .

Strictly speaking, there are additional continuum contributions on the RHS of Eq. (6). They account for possible many-particle intermediate states. Those contributions are studied in the Appendix. We just mention here that the continuum contributions do not affect the physical picture we are going to discuss in this subsection.

One notices that Eq. (2), which includes only noninteracting instanton effects, reflects the lack of finite distance correlations between topological charge densities, i.e., the RHS of Eq. (2) is zero for any nonzero value of  $x$ . This would not be the case if instanton interactions were taken into account. We also saw that the insertion of the intermediate glueball states into Eq. (2) yields the expression (6) with finite distance correlations occurring on its RHS. Thus, one argues

<sup>7</sup>These properties were originally studied in Ref. [19] considering YM theory on a four-sphere  $S^4$ .

<sup>8</sup>It is not even clear whether it makes sense to talk about a configuration with a definite topological charge in this case [13].

<sup>9</sup>There is another subtraction term in Eq. (6). It is proportional to the second derivative of the Dirac delta function. This term, being integrated in Eq. (1) gives a vanishing contribution and does not appear in the definition of  $\chi(V)$ . A detailed discussion is given in the Appendix.

that the strong correlations between instantons, which are responsible for finite distance effects, are phenomenologically included in Eq. (6) as the intermediate glueball states are taken into account. The argument above becomes more sensible if one recalls that interactions between instantons are responsible for the formation of those intermediate glueballs [29].

Let us now define the matrix elements occurring in Eq. (6). The operator of the topological charge density  $Q$  is an antihermitian operator in Euclidean space. Taking this into account one introduces the following parametrization for the matrix elements:

$$\langle 0|Q|n\rangle = -if_n m_n^2, \quad \langle 0|Q|n\rangle\langle n|Q|0\rangle = -f_n^2 m_n^4,$$

where  $f_n$  can be thought of as a decay constant of the corresponding  $n$ th glueball state (in analogy with the pion decay constant  $f_\pi$ ). If one substitutes these definitions back into Eq. (6) the following expression emerges:

$$\langle 0|TQ(x)Q(0)|0\rangle = d\delta^{(4)}(x) - \sum_n f_n^2 m_n^4 \frac{m_n}{4\pi^2|x|} K_1(m_n|x|). \quad (7)$$

Having this relation established let us study what happens with the correlator of the topological susceptibility [Eq. (1)]. Substituting Eq. (7) into Eq. (1) we find

$$\chi_{\text{eff}}(V) = d - \sum_n f_n^2 m_n^2 \mathcal{G}_n(V), \quad (8)$$

where

$$\mathcal{G}_n(V) \equiv m_n^2 \int_V D_F(m_n|x|) d^4x.$$

The function  $\mathcal{G}_n(V)$  determines the volume dependence of the topological susceptibility for large values of  $V$ . This function has a simple behavior.<sup>10</sup> The straightforward calculation yields

$$\mathcal{G}_n(0) = 0, \quad \mathcal{G}_n(\infty) = 1.$$

In general,  $\mathcal{G}_n(V)$  is a monotonically increasing function of the argument. Its value increases rapidly from zero at  $V=0$  to almost its asymptotic value at some finite  $V$ . Then, increasing very slowly, the function approaches the unity as  $V \rightarrow \infty$ . Relying on these properties one derives

$$\lim_{V \rightarrow \infty} \chi_{\text{eff}}(V) = d - \sum_n f_n^2 m_n^2. \quad (9)$$

<sup>10</sup>For a spherically symmetric volume element with the radius  $R$  one can calculate that  $\mathcal{G}_n(V) = 1 - m_n^2 R^2 K_2(m_n R)/2$ .

Thus, we see that the topological susceptibility gets additional positive subtraction terms in the effective theory<sup>11</sup> (the sum on the RHS).

Finally, using Eq. (8) and the monotonicity of the function  $\mathcal{G}_n(V)$  one concludes that the topological susceptibility decreases from its value defined at relatively small volumes to its value reached in the large volume limit. Physically this can be thought of as following. Suppose we set a sequence of subvolumes enclosing some topological charge distribution  $V_1 < V_2 < V_3 < \dots < \infty$ . The result of our discussion is that  $\chi_{\text{eff}}(V_1) > \chi_{\text{eff}}(V_2) > \chi_{\text{eff}}(V_3) > \dots > \chi_{\text{eff}}(\infty)$ . Thus, for smaller volumes one gets larger values of the topological susceptibility. One should remember, however, that this picture emerges in the confining phase of the theory, i.e., when  $V_1 \gg \xi^4$  and effective degrees of freedom are colorless excitations. The interpretation in terms of colored gluons does not make sense in that region because of the lack of asymptotic *in* and *out* states for those excitations.

Recalling that the behavior of the function  $\mathcal{G}_n(V)$  is governed by exponents of the type  $e^{(-m_n R)}$ , one concludes that the effective size at which  $\chi_{\text{eff}}(V)$  gets substantially suppressed is defined by the Compton wavelength of the lightest  $0^{-+}$  glueball state [Eq. (8)]. In accordance with lattice calculations the lightest pseudoscalar glueball of pure YM theory is expected to have mass approximately equal to 2.3 GeV [6]. Thus, the effective suppression length scale is  $L = 1/m_{G_0} \approx 0.09$  fm.

We complete this subsection by listing the main qualitative conclusions of the discussion presented above.

(i) In the domain of asymptotic freedom, where YM theory is defined most accurately, the topological susceptibility is an increasing function of the argument.

(ii) In contrast, in the phase where composite, colorless excitations are formed the topological susceptibility decreases monotonically.

(iii) The suppression length of the topological susceptibility is defined by the inverse mass of the lightest  $0^{-+}$  glueball

<sup>11</sup>In general, the quantity on the RHS of Eq. (9) is a nonzero number. However, it was argued in Ref. [17] that the topological susceptibility might completely be screened in the infinite volume limit if instanton interactions are sufficiently strong, i.e.,  $\chi_{\text{eff}}(\infty)$  would equal to zero in that case. This condition would yield a relation between the quantity  $d$  and parameters of glueballs. Imposing the condition  $\chi_{\text{eff}}(\infty) = 0$  [17] one derives  $d = \sum_n f_n^2 m_n^2$ . This relation is the analog of the Witten-Veneziano formula [30,31] for the  $\eta'$  meson mass (if one considers full QCD and combines the relation derived above with the Witten-Veneziano formula one necessarily needs to take into account the fact that the value of  $d$  depends on whether it is calculated in pure YM theory or in full QCD). Thus, the relation  $d = \sum_n f_n^2 m_n^2$  is a phenomenological criterion of the validity of the proposal of Ref. [17]. That relation can be tested in lattice QCD studies by measuring  $d$  in noninteracting instanton gas picture of pure YM theory and also by studying masses and decay constants of the whole tower of pseudoscalar glueball states. For the mass and decay constant of the lightest glueball QCD sum rule results can also be used [32,33].

state and equals approximately 0.09 fm. This length is less than the effective radius of the  $0^{-+}$  glueball itself (approximately 0.7–1.0 fm [29,34]).

An underlying nonperturbative mechanism which is responsible for the formation of the  $0^{-+}$  glueball state, most likely, should also be responsible for the suppression of the topological susceptibility and vice versa. However, the dynamical reason underlying these properties is not captured by the qualitative discussion of this section. The question why can this happen will be addressed below.

Let us notice that the properties listed above are in analogy with what happens in the  $(2+1)$ -dimensional Polyakov [35] model. In the case of the Polyakov model those features can be derived in a rather model-independent way [35,36].

## II. THE $\theta$ ANGLE AND FORMATION OF GLUEBALLS

In this section we study how glueballs can be formed in the vacuum of YM theory. In order to address this question let us first recall how quark containing hadrons are formed in QCD [37,38]. It was found in Ref. [25] that nonperturbative fluctuations lower the value of the vacuum energy density in QCD: If the ground state energy density for a perturbative vacuum was zero, then instantons lower it yielding a negative value [25]. When colored quarks are submerged in that vacuum the QCD ground state responds to the insertion of the quarks by suppressing the instanton density in a small domain around the quarks [37]. In other words, quarks, being submerged in the YM vacuum, yield a positive energy density which in the domain around the quarks partially compensates the existed negative ground state energy density. The size of that domain is determined by the dynamics of nonperturbative QCD [37]. Hence, if one takes the value of the vacuum energy density inside the quark containing domain and subtracts the value of the vacuum energy density outside the domain one would be left with a positive energy density excess in the interior of the domain.

Having a positive energy excess inside of some region means that there should be an inward pressure acting on each small volume element of that domain. In other words, the outside region with the negative energy density produces a pressure on the quark containing domain tending to squeeze its volume down to zero. The quark confinement emerges in this picture as an effect of the complicated structure of the QCD ground state. This serves as a derivation of the bag model for hadrons [39,40] from fundamental principles of nonperturbative QCD [37].

In this section we show that the same phenomenon might occur in pure YM theory. The crucial difference from the previous case is the existence of a purely gluonic domain with a positive energy density excess. That positive energy density can be provided by the  $\theta$  term. We show below that the positive energy density in the interior of the domain is proportional to the value of  $\theta^2$  multiplied by the value of the topological susceptibility. Since the topological susceptibility is screened outside of some region, this naturally yields a compact region of space with a positive vacuum energy density excess inside. We show that this domain can hadronize forming a YM glueball state.

Let us start with the action for Yang-Mills theory with the  $CP$  violating  $\theta$  term. In this section we work in Minkowski space-time.

We decompose the action  $S$  into the usual  $CP$  even  $S^{(+)}$ , and  $CP$  odd  $S^{(-)}$ , parts.  $S = S^{(+)} + S^{(-)}$ , where

$$S^{(+)} = \int d^4x \left( -\frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a \right),$$

$$S^{(-)} = \theta \int d^4x \left( \frac{1}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a \right). \quad (10)$$

The total energy density of this system  $\mathcal{E}$  is the sum of the energy densities of the  $CP$  even part  $\mathcal{E}^{(+)}$  and  $CP$  odd part  $\mathcal{E}^{(-)}$ , i.e.,  $\mathcal{E} = \mathcal{E}^{(+)} + \mathcal{E}^{(-)}$ .

Let us consider the  $CP$  even part of the action. As we mentioned above, nonperturbative contributions yield a negative vacuum energy density [25]. The total energy density of the  $CP$  even part is a sum of the negative ground state energy density  $\mathcal{E}_{\text{vac}}^{(+)}$  and the energy density of perturbations about that ground state  $\mathcal{E}_{\text{pert}}^{(+)}$

$$\mathcal{E}^{(+)} = \mathcal{E}_{\text{vac}}^{(+)} + \mathcal{E}_{\text{pert}}^{(+)} \quad (11)$$

Suppose we start with no perturbations being excited, i.e., put  $\mathcal{E}_{\text{pert}}^{(+)} = 0$ . Then,  $\mathcal{E}^{(+)} = \mathcal{E}_{\text{vac}}^{(+)} = \frac{1}{4} \langle 0 | \Theta_{\mu\mu}^{(+)} | 0 \rangle = (\beta(\alpha_s)/4\alpha_s^2) \langle 0 | (1/32\pi^2) G_{\rho\tau}^2 | 0 \rangle \simeq -(0.250 \text{ GeV})^4$  [41]. Here,  $\Theta_{\mu\mu}^{(+)}$  stands for the anomalous trace of the energy-momentum tensor corresponding to  $S^{(+)}$  and perturbative contributions to the gluon condensate are subtracted.

Let us now address the question what is the contribution of the  $CP$  odd part of the action to the total energy density of the whole system given in Eq. (10). It is convenient to introduce a new variable by rewriting the expression for the topological charge density<sup>12</sup>  $Q$  in terms of a four-form field  $F^{\mu\nu\alpha\beta}$

$$Q = \frac{1}{4!} \varepsilon_{\mu\nu\alpha\beta} F^{\mu\nu\alpha\beta},$$

where the four-form field  $F^{\mu\nu\alpha\beta}$  is the field strength for the three-form potential  $C_{\mu\nu\alpha}$

$$F_{\mu\nu\alpha\beta} = \partial_\mu C_{\nu\alpha\beta} - \partial_\nu C_{\mu\alpha\beta} - \partial_\alpha C_{\nu\mu\beta} - \partial_\beta C_{\nu\alpha\mu}.$$

The  $C_{\mu\nu\alpha}$  field itself is defined as a composite operator of colored gluon fields  $A_\mu^a$

$$C_{\mu\nu\alpha} = \frac{1}{16\pi^2} (A_\mu^a \bar{\partial}_\nu A_\alpha^a - A_\nu^a \bar{\partial}_\mu A_\alpha^a - A_\alpha^a \bar{\partial}_\nu A_\mu^a + 2 f_{abc} A_\mu^a A_\nu^b A_\alpha^c),$$

<sup>12</sup>Though in Minkowski space-time  $Q$  does not have the meaning of the topological charge density and, moreover, differs from Euclidean definition of the topological charge by  $i$ , we formally keep that name and letter for simplicity.

with  $f_{abc}$  being structure constants of the corresponding  $SU(N_c)$  gauge group. The right-left derivative in this expression acts as  $A\bar{\partial}B \equiv A(\partial B) - (\partial A)B$ .

The topological charge density can also be expressed through the Chern-Simons current  $K_\mu$  as  $Q = \partial_\mu K_\mu$ . Using this expression one can deduce the relation between the Chern-Simons current and the three-form potential  $C_{\nu\alpha\beta}$ , these two quantities are Hodge dual to each other  $K^\mu = (1/3!) \varepsilon^{\mu\nu\alpha\beta} C_{\nu\alpha\beta}$ .

Let us rewrite the  $CP$  odd part of the action in terms of the three-form potential  $C_{\nu\alpha\beta}$ . For the further convenience the integration over space-time will be restricted to a finite, not yet specified domain denoted by  $\mathcal{M}$

$$\begin{aligned} S^{(-)} &= \theta \int_{\mathcal{M}} Q d^4x = -\frac{\theta}{4!} \int_{\mathcal{M}} F_{\mu\nu\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta \\ &\equiv -\theta \int_{\mathcal{M}} F, \end{aligned}$$

where the following differential four-form was introduced:

$$F \equiv \frac{1}{4!} F_{\mu\nu\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\alpha \wedge dx^\beta.$$

In terms of differential forms and an exterior derivative  $d$  the equations above formally simplify.<sup>13</sup> Indeed,  $F = dC$ , where  $C \equiv (1/3!) C_{\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta$  and the expression for  $S^{(-)}$  reads as

$$\begin{aligned} S^{(-)} &= -\theta \int_{\mathcal{M}} F = -\theta \int_{\partial\mathcal{M}} C \\ &= -\frac{\theta}{3!} \int_{\partial\mathcal{M}} C_{\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta. \end{aligned} \quad (12)$$

In the last equation we used the Stokes theorem assuming that the boundary  $\partial\mathcal{M}$  enclosing the domain  $\mathcal{M}$  is an orientable smooth surface. Speaking in terms of the  $C_{\nu\alpha\beta}$  field, the nonzero value of the  $\theta$  angle corresponds in Minkowski space to the nonzero coupling of the  $C_{\nu\alpha\beta}$  field to the boundary manifold  $\partial\mathcal{M}$ . That coupling is gauge invariant, although the  $C_{\nu\alpha\beta}$  field itself is not a gauge invariant quantity. Indeed, if the gauge transformation parameter is denoted by  $\Lambda^a$ , then the three-form field transforms as  $C_{\nu\alpha\beta} \rightarrow C_{\nu\alpha\beta} + \partial_\nu \Lambda_{\alpha\beta} - \partial_\alpha \Lambda_{\nu\beta} - \partial_\beta \Lambda_{\alpha\nu}$ , where  $\Lambda_{\alpha\beta} \propto A_\alpha^a \partial_\beta \Lambda^a - A_\beta^a \partial_\alpha \Lambda^a$ . However, it is easy to check that the expression for the field strength  $F_{\mu\nu\alpha\beta}$  is gauge invariant. Since the coupling of the three-form field to the boundary can be expressed in terms of the  $F_{\mu\nu\alpha\beta}$  field [as in Eq. (12)], then that coupling is also gauge invariant and can lead to some physically observable results. The same conclusion could be drawn without referring to the field strength. The gauge variation of the last

<sup>13</sup>We apologize for using the same letter  $d$  for an exterior derivative utilized in this section and the quantity  $d$ , which has to do with the instanton charge density, defined in the previous section.

expression in Eq. (12) is zero for any smooth closed surface which does not enclose any field singularities.

It was noticed some time ago [20] that the  $C_{\nu\alpha\beta}$  field propagates in the bulk of the domain  $\mathcal{M}$  if the topological susceptibility is nonzero in that domain. This becomes more evident if one recalls the notion of the Kogut-Susskind (KS) pole [42] in the correlator of two Chern-Simons currents. We briefly present those arguments.

Consider the correlator of the vacuum topological susceptibility at a nonzero momentum. The topological charge density  $Q$  is the derivative of the Chern-Simons current  $Q = \partial_\mu K_\mu$ . One can substitute this definition back into the expression for the correlator of the topological susceptibility. In that way one discovers that  $\chi$  is defined as the zero momentum limit of the correlator of two Chern-Simons currents multiplied by two momenta<sup>14</sup>

$$\chi = i \lim_{q \rightarrow 0} q^\mu q^\nu \int e^{iqx} \langle 0 | T K_\mu(x) K_\nu(0) | 0 \rangle d^4x. \quad (13)$$

The only way for this expression to be nonzero is to claim that the correlator of two Chern-Simons currents develops a pole as the momentum goes to zero. This is called the Kogut-Susskind pole [42].

Knowing that the correlator of two Chern-Simons currents has a pole, one can use the relation between the Chern-Simons current and the three-form  $C_{\nu\alpha\beta}$  field and conclude that the  $C_{\nu\alpha\beta}$  field also has a nonzero Coulomb propagator [20]. Thus, the  $C_{\nu\alpha\beta}$  field behaves as a massless collective excitation transferring a long-range interaction [20].

Let us summarize briefly the results of the discussion given above. Following Ref. [20] we established that the three-form field  $C_{\nu\alpha\beta}$  propagates in the bulk transferring a long-range Coulomb interaction. The exact propagator of this field is of the Coulomb type and is proportional to the value of the vacuum topological susceptibility.

We also saw that the  $CP$  odd term in the action of Yang-Mills theory can be expressed as a coupling of the three-form composite field  $C_{\nu\alpha\beta}$  to the boundary manifold. Hence, the three-form field  $C_{\nu\alpha\beta}$  being a free field in the bulk actually does couple to the boundary surface.

All the properties mentioned above can be summarized in the following effective action for the  $C_{\nu\alpha\beta}$  field:

<sup>14</sup>The multiplier  $-i$  appears in the definition of the topological susceptibility in Minkowski space-time. There are some delicate issues regarding the definition of the correlator of the vacuum topological susceptibility. If one defines  $\chi$  as a second derivative of the partition function with respect to the  $\theta$  angle, then some contact term appears in that expression [30]. Likewise, a special care is needed while treating the covariant  $T$  product in Eqs. (1) and (13) when this last is taken to be the definition of  $\chi$  [43]. One should add a contact term (given in the Appendix of Ref. [30]) to the RHS of Eq. (1) in order to define  $\chi$  as a second derivative of the vacuum energy with respect to the  $\theta$  angle [30,43] [this contact term can effectively be included in Eq. (6) by redefining the value of the positive constant  $d$ ].

$$S_{\text{eff}}^{(-)} = -\frac{1}{2 \times 4! \chi(V_{\mathcal{M}})} \int_{\mathcal{M}} F_{\mu\nu\alpha\beta}^2 d^4x - \frac{\theta}{3!} \int_{\partial\mathcal{M}} C_{\nu\alpha\beta} dx^\nu \wedge dx^\alpha \wedge dx^\beta. \quad (14)$$

The first term in this expression yields the correct Coulomb propagator for the three-form  $C_{\nu\alpha\beta}$  field. The second term is just the usual  $CP$  odd  $\theta$  term of the initial YM action.  $V_{\mathcal{M}}$  denotes the three-volume of the domain  $\mathcal{M}$ .

Our next step is to study the effective action given<sup>15</sup> in Eq. (14). In particular, we will calculate the ground state energy of the system using the effective action (14). Before we turn to that calculation let us mention that Maxwell's equations for a free four-form potential  $F_{\mu\nu\alpha\beta}$  yield only a constant solution in  $(3+1)$ -dimensional space-time [44]. The reason is the following. A four-form potential has only one independent degree of freedom in four-dimensional space-time, let us call it  $\Sigma$ . Then, the four Maxwell's equations written in terms of the  $\Sigma$  field ensure that this field is independent of all four space-time coordinates. Hence, the solution can only be a space-time constant. Thus, the  $F_{\mu\nu\alpha\beta}$  field propagates no dynamical degrees of freedom.<sup>16</sup> However, this field can be responsible for the existence of a positive vacuum energy density in different models of quantum field theory (see Ref. [45]). Thus, studying classical equations of motion for the  $F_{\mu\nu\alpha\beta}$  field one can determine the value of the ground state energy given by configurations of  $F_{\mu\nu\alpha\beta}$ . We are going to solve explicitly classical equations of motion for the effective action (14). Then, the energy density associated with those solutions will be calculated.

Let us start with the equations of motion. Taking the variation of the action (14) with respect to the  $C_{\nu\alpha\beta}$  field one gets

$$\partial^\mu F_{\mu\nu\alpha\beta}(z) = \theta \chi(V_{\mathcal{M}}) \int_{\partial\mathcal{M}} \delta^{(4)}(z-x) dx_\nu \wedge dx_\alpha \wedge dx_\beta.$$

It has been shown in Ref. [44] that this kind of equations can be solved exactly in four-dimensional space-time. The solution is the sum of a particular solution of the inhomogeneous equation and a general solution of the homogeneous equation

$$F_{\mu\nu\alpha\beta}(z) = \theta \chi(V_{\mathcal{M}}) \int_{\mathcal{M}} \delta^{(4)}(z-x) dx_\mu \wedge dx_\nu \wedge dx_\alpha \wedge dx_\beta + h \varepsilon_{\mu\nu\alpha\beta}.$$

<sup>15</sup>One should notice that the action (14) is not an effective action in the Wilsonian sense. It is rather related to the generating functional of one-particle-irreducible diagrams of the composite field. The effective action in Eq. (14) is not to be quantized and loop diagrams of that action are not to be taken into account in calculating higher order Green's functions. The analogous effective action for the  $CP$  even part of the theory was constructed in Refs. [46, 47].

<sup>16</sup>The exception is when that field couples to other fields.

The integration constant  $h$ , if nonzero, induces an additional  $CP$  violation beyond the existed  $\theta$  angle [45]. Since we would like to avoid to have an extra  $CP$  violating term we set  $h=0$ . Simplifying the previous equation one finds that the classical solution is a nonzero constant tensor density inside of the domain  $\mathcal{M}$

$$F_{\mu\nu\alpha\beta} = -\theta \chi(V_{\mathcal{M}}) \varepsilon_{\mu\nu\alpha\beta}, \quad (15)$$

and is zero,  $F_{\mu\nu\alpha\beta}=0$ , outside of  $\mathcal{M}$ .

As a next step let us compute the vacuum energy associated with the solution given in Eq. (15). The density of the energy-momentum tensor for the  $CP$  odd sector of the theory can be written as

$$\Theta_{\mu\nu}^{(-)} = -\frac{1}{3! \chi(V_{\mathcal{M}})} \left( F_{\mu\alpha\beta\tau} F_\nu^{\alpha\beta\tau} - \frac{1}{8} g_{\mu\nu} F_{\rho\alpha\beta\tau}^2 \right).$$

Using the expression (15) one calculates the corresponding energy density<sup>17</sup>  $\mathcal{E}^{(-)}$

$$\mathcal{E}^{(-)} = \frac{1}{2} \theta^2 \chi(V_{\mathcal{M}}).$$

Since the  $F_{\mu\nu\alpha\beta}$  field does not propagate dynamical degrees of freedom the expression above is the total energy density of the system given by the action (14). The crucial thing about this energy density is that it is a positive quantity proportional to  $\theta^2$  multiplied by the value of the topological susceptibility<sup>18</sup>

We learned in the preceding section that the magnitude of the topological susceptibility depends on the value of the subvolume in which it is calculated, and also, most importantly, it is screened by nonperturbative effects of YM theory outside of some finite subvolume element. So far we treated the domain  $\mathcal{M}$  as some arbitrary volume. Let us now suppose that  $\mathcal{M}$  is the subvolume outside of which the topological susceptibility is screened.<sup>19</sup> Thus, the topological susceptibility is given by the quantity  $d$  inside of the volume  $\mathcal{M}$  and by the difference  $d - \sum_n f_n^2 m_n^2$  outside of the volume. As

<sup>17</sup>One can use either  $\mathcal{E} = \frac{1}{4} \Theta_{\mu}^{\mu}$  or  $\mathcal{E} = \Theta_{00}$ .

<sup>18</sup>One might wonder whether the same result is obtained if one treats  $\theta$  not as a constant multiplying  $Q$  in the Lagrangian, but as the phase that the states acquire under a topologically nontrivial gauge transformations. These two ways of presenting the  $\theta$  dependence are equivalent. Thus, results of our discussion should be equivalent in both cases. The key observation is that if  $\theta$  is not entering the Lagrangian, the arbitrary integration constant in Eq. (15) has to be nonzero. It should be chosen in a way that would guarantee a proper  $\theta$  dependence of the VEV of the topological charge density. That would leave the results of our discussion without modifications.

<sup>19</sup>In this discussion we assume that the domain has a more or less definite boundary, or in other words, that there is a narrow interval where the topological susceptibility drops in magnitude from the value  $d$  to  $d - \sum_n f_n^2 m_n^2$ . That assumption does not seem to be far from the reality if one recalls the behavior of the function  $\mathcal{G}_n(V)$  (see Sec. I).



we mentioned above, the difference  $d - \sum_n f_n^2 m_n^2$  turns to zero if interactions between instantons are sufficiently strong [17]. However, this is not guaranteed in general. Trying to deal with the most general case, we assume here that  $d - \sum_n f_n^2 m_n^2$  is some number not necessarily equal to zero. Clearly, our result presented below will also be applicable to the case when the topological susceptibility is zero outside of the volume  $\mathcal{M}$  and  $d = \sum_n f_n^2 m_n^2$ . In accordance with our previous calculations, the vacuum energy density inside of the domain  $\mathcal{M}$  is  $\mathcal{E}_{\text{inside}} = \mathcal{E}_{\text{vac}}^{(+)} + \frac{1}{2} \theta^2 d$ . Let us now turn to the vacuum energy density outside of the volume  $\mathcal{M}$ . As we assumed, the topological susceptibility is not necessarily zero outside of  $\mathcal{M}$ . Hence, the three-form field can propagate in that region too and  $F_{\mu\nu\alpha\beta} \neq 0$  outside of  $\mathcal{M}$  in the general case. As a result, there exists some nonzero vacuum energy density outside of  $\mathcal{M}$ . In analogy with the previous case one derives the following expression for the vacuum energy density  $\mathcal{E}_{\text{outside}} = \mathcal{E}_{\text{vac}}^{(+)} + \frac{1}{2} \theta^2 (d - \sum_n f_n^2 m_n^2)$ . This value is less than the energy density inside of the volume  $\mathcal{M}$ . Thus, there is a positive energy density excess inside of  $\mathcal{M}$ . The expression for the corresponding energy excess in the subvolume takes the form

$$\Delta E = \frac{1}{2} \theta^2 \Delta \chi V_{\mathcal{M}}. \quad (16)$$

Here  $\Delta \chi = \chi_{\text{inside}} - \chi_{\text{outside}} = \sum_n f_n^2 m_n^2$  is the difference between the topological susceptibility defined inside and outside of the subvolume. As we mentioned, Eq. (16) is also valid in the particular case when the topological susceptibility equals to zero outside of the volume  $\mathcal{M}$ . In that case  $\Delta \chi = \sum_n f_n^2 m_n^2 = d$  and the energy difference in Eq. (16) coincides with the energy of the  $CP$  odd part inside of the volume  $\mathcal{M}$ .

There are two basic questions to be elucidated here. First of all why would any distinguished subvolume exist in the YM vacuum? The reason, as we already mentioned above, is the screening of the topological susceptibility. This naturally provides finite domains in the YM vacuum with the positive vacuum energy excess in accordance with Eq. (16).

The second question is what happens with this finite volume if it is allowed to flow freely to a stationary state. The system will tend to minimize the energy given in Eq. (16). The expansion is not an energetically allowed process. An alternative possibility for the system is to squeeze its volume down as much as it is possible. In that case the RHS of Eq. (16) would be decreasing. In other words, there should be an inward pressure acting on the system and tending to squeeze its volume down. That pressure is due to the positive difference between the energy densities in the interior and exterior of the domain. Hence, one concludes that the system will tend to minimize its energy by squeezing its volume down, or decreasing  $V_{\mathcal{M}}$  in Eq. (16).

At first glance such a system is unstable and should collapse to a point. However, that would be a wrong conclusion. The point is that we did not yet take into account perturbations of YM fields which should get excited inside of  $V_{\mathcal{M}}$  while the system is shrinking its volume down. Those exci-

tations could stabilize the system. The energy of those excitations, identified in Eq. (11) as  $\mathcal{E}_{\text{pert}}^{(+)}$ , provides an additional contribution to the total energy. Anticipating the results of the next section we present the expression for the total energy inside of the domain  $\mathcal{M}$

$$\Delta E_{\text{total}} = \frac{1}{2} \theta^2 \Delta \chi V_{\mathcal{M}} + \frac{\text{(positive number)}}{V_{\mathcal{M}}^{1/3}}, \quad (17)$$

where the first term is related to the  $CP$  odd part of the initial action and the second one is the contribution of perturbations of the  $CP$  even part.

The structure of this equation allows one to minimize the quantity  $\Delta E_{\text{total}}(V_{\mathcal{M}})$  with respect to  $V_{\mathcal{M}}$  and find an optimal value of the three-volume occupied by the system. We treat this physical system as a model for a pure YM glueball state. The spectrum and some properties of that system are studied below.

### III. YANG-MILLS VS CLOSED MEMBRANE SPECTRUM

It was shown in the previous section that the screened topological susceptibility leads to a positive energy density excess inside of some finite volume. The system tends to minimize that volume. The compression of the volume continues until some YM states are exited inside of that domain. Those states have nonzero energy, i.e.,  $\langle \mathcal{H}_{\text{YM}} \rangle \neq 0$ , where  $\mathcal{H}_{\text{YM}}$  is the Hamiltonian density of YM theory (the  $CP$  even part). We mentioned already that the phenomenon described above is related to the fact that  $\chi$  is screened. On the other hand, we saw that the effects responsible for the screening of the topological charge should also be responsible for the formation of the  $0^{-+}$  glueball state. Hence, it is reasonable to identify those finite volume YM excitations with physical  $0^{-+}$  glueball states.

In this section we study the spectrum of the physical YM Hamiltonian in a finite volume. Under some approximations, elucidated below, the spectrum of YM theory resembles that of a closed bosonic membrane with the topology of a sphere or torus [22,23]. One can use that analogy to derive the relation between the spectrum of YM theory and that of a closed bosonic membrane. Using that relation and calculating the spectrum for a closed spherical bosonic membrane we predict masses for two lightest pseudoscalar YM glueball states.

#### A. Studying the spectrum of Yang-Mills theory

Let us start with the physical Hamiltonian density of YM theory. In order to stress the approximations we make we present the brief discussion of the Hamiltonian formalism of the theory (for detailed discussions see Refs. [48,49,50]).

One starts with the Lagrangian density of pure YM theory in Minkowski space-time

$$\mathcal{L} = - \frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a.$$

The canonically conjugate momentum is defined as the derivative of the Lagrangian density with respect to the time derivative of the canonical coordinate and is given by

$$P_i^a = -\frac{1}{g^2} G_{0i}^a.$$

The Hamiltonian density takes the form

$$\mathcal{H}_0 = \frac{g^2}{2} P_i^a P_i^a + \frac{1}{4g^2} G_{ij}^a G_{ij}^a + A_0^a (D_i P_i)^a, \quad i, j = 1, 2, 3.$$

This is not the physical Hamiltonian density yet. There are extra degrees of freedom in this expression. The existence of those extra variables is related to the gauge invariance of the theory.

The Lagrangian density does not contain time derivatives of the  $A_0$  field. As a result, the following primary constraint appears  $P_0^a = 0$ .

Introduce [48] the so called ‘‘total’’ Hamiltonian  $H_T(t) \equiv \int d^3x (\mathcal{H}_0 + \lambda^a P_0^a)$ , where  $\lambda^a(x)$  denotes a Lagrange multiplier. Time evolution of a physical quantity is given by the Poisson brackets of  $H_T$  and the quantity itself. Thus, one needs to set conventions for the Poisson brackets. For any two (bosonic) functionals  $A$  and  $B$  we use the following expression:

$$\{A, B\} \equiv \int d^3z \left( \frac{\delta A}{\delta q(z)} \frac{\delta B}{\delta p(z)} - \frac{\delta B}{\delta q(z)} \frac{\delta A}{\delta p(z)} \right),$$

where  $q$  and  $p$  denote canonical coordinates and momenta respectively. Using this definition one finds that the conservation of the primary constraint  $\{P_0^a(x, t), H_T(t)\} = 0$ , leads one to the secondary constraint in the form of the Gauss’s law  $D_i^{ab} P_i^b = 0$ . One can also check that the conservation of the Gauss’s law is identically satisfied and no further constraints are produced at this stage.

We are going to work in the axial gauge  $A_3^a = 0$ . Requiring the conservation of the gauge condition  $\{A_3^a(x, t), H_T(t)\} = 0$ , one derives the additional secondary constraint  $g^2 P_3^a - \partial_3 A_0^a = 0$ . Finally, the conservation of that constraint leads to the equation for determination of the Lagrange multiplier  $\partial_3 \lambda^a(x) + \partial_3 \partial_i A_i^a(x) - f^{abc} A_0^b \partial_3 A_0^c = 0$ . Thus, the whole system of gauge conditions and constraints can be summarized as

$$\begin{aligned} \Phi_1^a &= P_0^a, & \Phi_2^a &= D_i^{ab} P_i^b, \\ \Phi_3^a &= A_3^a, & \Phi_4^a &= g^2 P_3^a - \partial_3 A_0^a. \end{aligned}$$

The physical Hamiltonian in the axial gauge can be written in terms of the following physical variables  $P_m^a$  and  $A_m^a$ , where  $m = 1, 2$  [50]. In general, the straightforward procedure implies the elimination of all nonphysical variables by solving (wherever it is possible) constraint equations and substituting those expressions back into the formula for the Hamiltonian. In most cases the result is a complicated nonlocal expression for the Hamiltonian. There is a formally simpler

way to follow, however. One can solve only some part of the constraint equations keeping the rest of the constraints unsolved and allowing some of nonphysical variables to be present in the Hamiltonian. Then, the physical system is defined by that Hamiltonian accompanied by unsolved constraint equations imposed on the physical states of the Fock space. For our purposes we found it convenient to follow that way. Using the conditions  $\Phi_1^a = 0$ ,  $\Phi_3^a = 0$ , and  $\Phi_4^a = 0$  the Hamiltonian density can be rewritten as

$$\begin{aligned} \mathcal{H}_{\text{YM}} &= \frac{g^2}{2} P_m^a P_m^a + \frac{1}{4g^2} G_{mn}^a G_{mn}^a \\ &+ \frac{1}{2g^2} (\partial_3 A_0^a)^2 + \frac{1}{2g^2} (\partial_3 A_m^a)^2, \end{aligned} \quad (18)$$

where  $m, n = 1, 2$  and the expression contains the physical variables  $A_m^a$  and  $P_m^a$  along with the nonphysical  $A_0^a$ . The constraint which is still left relates  $A_0^a$  to the physical variables

$$\partial_3^2 A_0^a + g^2 (D_m P_m)^a = 0. \quad (19)$$

Thus, the system is defined by the Hamiltonian density (18) and the constraint (19).

Let us now turn to the discussion of the spectrum of the system (18),(19) which is placed in a finite three-volume denoted by  $V_{\mathcal{M}}$ . Calculating the spectrum we are going to keep only ‘‘slow’’ modes, i.e., the modes with zero momenta but a nonzero energy. All the ‘‘fast’’ modes with nonzero momenta can be thought of as being integrated out. The net result of the corrections due to the fast modes is just a perturbative splitting of the energy levels determined by the slow modes<sup>20</sup> (for a detailed discussion see Ref. [21]). Adopting that approximation one can drop all the spatial derivatives in the expression for the Hamiltonian and constraint equation assuming that all the canonical variables depend on the time variable only.

Let us turn to the Hamiltonian instead of the Hamiltonian density. Dropping all the spatial derivatives one writes down

$$H_{\text{YM}} = \frac{g^2 V_{\mathcal{M}}}{2} P_m^a P_m^a + \frac{V_{\mathcal{M}}}{4g^2} (f^{abc} A_m^b A_n^c)^2.$$

It is convenient to perform the following rescaling of the canonical variables

$$A_m \rightarrow \frac{g^{2/3}}{V_{\mathcal{M}}^{1/3}} A_m \quad \text{and} \quad P_m \rightarrow \frac{1}{g^{2/3} V_{\mathcal{M}}^{2/3}} P_m.$$

<sup>20</sup>The crucial point in this discussion is that the spectrum is calculated in a small volume limit. As we mentioned in Sec. I, this corresponds to the weak coupling approximation. As a result, corrections due to the fast modes are of order  $g^{2/3}/4\pi = (\alpha_s/16\pi^2)^{1/3}$  [21] and can be neglected in the leading approximation.

The new, rescaled variables are dimensionless. In terms of these variables the expression for the Hamiltonian takes the form

$$H_{\text{YM}} = \frac{g^{2/3}}{V_{\mathcal{M}}^{1/3}} \left[ \frac{1}{2} P_m^a P_m^a + \frac{1}{4} (f^{abc} A_m^b A_n^c)^2 \right] \quad (20)$$

and the constraint equation is given as follows:

$$f^{abc} A_m^b P_m^c = 0, \quad m = 1, 2. \quad (21)$$

This is the system which defines the spectrum.<sup>21</sup> The first thing to notice is that the potential in Eq. (20) has flat directions. Thus, one would expect a continuous spectrum without a mass gap. However, it was proved in Ref. [51] (see also Ref. [21]) that in the quantum theory, contrary to the naive classical expectation, the operator defined in Eq. (20) has only discrete positive eigenvalues. As a result, the following expression for the spectrum emerges:

$$E_{\text{pert}}^{(+)} \equiv \langle H_{\text{YM}} \rangle = \mathcal{E}_{\text{pert}}^{(+)} V_{\mathcal{M}} = \frac{g^{2/3} \times (\text{positive number})}{V_{\mathcal{M}}^{1/3}}.$$

This expression was used earlier in Eq. (17). The exact calculation of the positive numbers occurring in the expression above is a complicated problem of YM theory. However, as it will be shown below, one can use some analogies and calculate the spectrum explicitly. We turn now to that discussion.

### B. The membrane matrix model

It was shown some time ago [22,23] that the Hamiltonian of a closed bosonic membrane in the light-cone gauge can be reduced to the form given in Eq. (20).

The variables substituting the gauge fields in that case occur as coefficients of the harmonic expansion of the spatial coordinates on the membrane world surface. The two Hamiltonians, one for the membrane and the other one given in Eq. (20) formally look similar.

The YM theory constraint (21) also has an analogue in the case of the closed membrane theory. The constraint in that case is related to the residual reparametrization invariance of the membrane action which is still left in the light-cone gauge.

Below we discuss briefly the membrane action and the way it reduces to the form given in Eq. (20). Then we deduce the matching condition relating the spectrum of the closed bosonic membrane to the spectrum of YM theory. The matching condition allows one to obtain the spectrum of YM theory by calculating the spectrum of the closed bosonic membrane.

We present below only the basic features of the membrane Hamiltonian construction in the light-cone gauge. For

details we refer to the original papers [22,23].

The membrane action in flat Minkowski space-time can be written as

$$S_m = -T \int d^3\sigma \sqrt{|\det g_{ij}|}, \quad (22)$$

where  $T$  is the membrane tension, the constant with the dimensionality of mass cubed,  $\sigma_0, \sigma_1, \sigma_2$  are the coordinates on the membrane world volume,  $g_{ij}$  denote the components of the induced metric in the membrane world volume

$$g_{ij}(\sigma) \equiv \frac{\partial X^\mu(\sigma)}{\partial \sigma^i} \frac{\partial X_\mu(\sigma)}{\partial \sigma^j}, \quad (23)$$

where  $X_\mu$ ,  $\mu = 0, 1, 2, 3$ , are the space-time coordinates.

The membrane action is reparametrization invariant. Thus, in accordance with the Noether second theorem not all of the variables in the action are independent (as in gauge theories). One should carry out the gauge fixing procedure. It is convenient to introduce the light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}} (X^3 \pm X^0),$$

and choose the light-cone gauge

$$X^+(\sigma) = X^+(0) + \sigma_0.$$

The light-cone gauge does not completely fix the gauge freedom of the membrane action.<sup>22</sup> As a result, there still is a residual local invariance left. Hence, one should expect to have the Hamiltonian of the theory accompanied by a constraint equation. The detailed discussion and the construction of the Hamiltonian is given in Refs. [22, 23]. We present the final result here. The expressions for the mass squared operator and the constraint can be written as follows:

$$\frac{M^2}{2} = \left[ \frac{1}{2} \mathcal{P}_m^a \mathcal{P}_m^a + \frac{T^2}{4} (g^{abc} X_m^b X_n^c)^2 \right], \quad (24)$$

$$g^{abc} X_m^b \mathcal{P}_m^c = 0 \quad m = 1, 2.$$

The canonical coordinates and momenta are the functions of the time variable only. The coordinates  $X_m^a$  in this expression are the coefficients of the harmonic expansion of the space-time coordinates  $X_m$  on the surface of the membrane. For example, if the membrane has the topology of a sphere, then the harmonic expansion mentioned above is just the expansion of the space-time coordinates in the basis of spherical functions

$$X_m(\sigma) = \sum_{a=1}^{\infty} X_m^a Y^a(\sigma_1, \sigma_2), \quad a = 1, 2 \dots \infty,$$

<sup>21</sup>The operator in Eq. (20) acts on functionals of the canonical variable while the momentum operator is defined as  $P_m = -i(\delta/\delta A^m)$ .

<sup>22</sup>As opposed to the case of a string action where in the light-cone gauge no freedom is left [52].

where  $Y^a(\sigma)$ 's are the harmonic functions on the sphere.

If the membrane has the topology of a sphere or torus the harmonic functions  $Y^a(\sigma)$  form a representation of the Lie algebra of the  $SU(\infty)$  gauge group<sup>23</sup> [22,23]. Thus, the  $SU(\infty)$  gauge group appears due to the reparametrization invariance of the membrane action.<sup>24</sup>

The expression (24) resembles the Hamiltonian of the YM system in the approximation given in Eq. (20) and in the  $N_c \rightarrow \infty$  limit. The constraint equations in the two cases are also similar.

In order to make use of this analogy let us perform the following rescaling of the canonical variables<sup>25</sup>

$$\mathcal{P} \rightarrow T^{1/3} \mathcal{P} \quad \text{and} \quad X \rightarrow T^{-1/3} X.$$

The new canonical variables are dimensionless. The expression for the mass squared operator in terms of those variables takes the form

$$\frac{M^2}{2} = T^{2/3} \left[ \frac{1}{2} \mathcal{P}_m^a \mathcal{P}_m^a + \frac{1}{4} (g^{abc} X_m^b X_n^c)^2 \right]. \quad (25)$$

Thus, one concludes that the spectrum of a closed bosonic spherical membrane is determined by the same differential operator as the one for YM theory in the large  $N_c$  limit.<sup>26</sup> Matching this expression with Eq. (20) one finds the relation between the spectrum of YM theory in a finite volume and the spectrum of the closed bosonic membrane

$$E_n^{\text{YM}} = \frac{g^{2/3}}{V_3^{1/3}} \frac{M_n^2}{2T^{2/3}}, \quad (26)$$

where  $M_n$ 's are the mass eigenvalues defined by the operator given in Eq. (25). The complimentary constraint equations acting on the physical states ensure that the physical eigenfunctionals of the Hamiltonians in both Eqs. (20) and (25) are the functionals of "colorless" (gauge invariant) variables only. Indeed, in both cases the constraint equations (the Gauss's law and its membrane counterpart) serve as the generators of the "gauge" transformations of the initial system. Since those generators are supposed to annihilate any physical state (imposed as the Gauss's law annihilating a state), then all the physical eigenfunctionals should be gauge invariant.

<sup>23</sup>The  $SU(\infty)$  group (and its Lie algebra) should be understood as a limit of the  $SU(N)$  group at  $N \rightarrow \infty$ .

<sup>24</sup>The supersymmetric version of the membrane matrix model is used for the formulation of the M theory in the infinite momentum frame [53].

<sup>25</sup>One can check that the rescaling procedures we perform lead to dimensionless canonical coordinates and momenta which satisfy the commutation relation with the unity on its RHS.

<sup>26</sup>In both cases these differential operators act on the functionals of the canonical coordinates. Those functionals are annihilated by the constraint equations. The boundary conditions will be fixed later during the actual calculations.

In the next subsection we calculate  $M_n^2$  for a closed membrane with the topology of a sphere and using the matching condition (26) deduce the energy levels for the YM theory excitations identifying them with the pseudoscalar glueballs.

### C. Calculating the membrane spectrum

We start with a closed spherical membrane. The space-time coordinates on the membrane world surface are given as

$$X_\mu = [t, r(t) \sin \theta \cos \varphi, r(t) \sin \theta \sin \varphi, r(t) \cos \theta],$$

$$0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi,$$

where  $r(t)$  is the time-dependent radius of the membrane.

The induced metric on the membrane worldvolume has the following nonzero components:

$$g_{tt} = 1 - \dot{r}^2(t), \quad g_{\theta\theta} = -r^2(t), \quad g_{\varphi\varphi} = -r^2(t) \sin^2 \theta.$$

The action functional for the membrane takes the form

$$S = -T \int dt d\theta d\varphi r^2(t) \sin \theta \sqrt{1 - \dot{r}^2(t)}.$$

Thus, the Lagrangian can be written as follows:

$$L(t) = -4\pi T r^2(t) \sqrt{1 - \dot{r}^2(t)}.$$

Calculating the canonically conjugate momentum

$$P = \frac{\partial L(t)}{\partial \dot{r}} = 4\pi T \frac{r^2(t) \dot{r}(t)}{\sqrt{1 - \dot{r}^2(t)}},$$

one derives the Hamiltonian for the spherical membrane<sup>27</sup>

$$H = \sqrt{P^2 + 16\pi^2 T^2 r^4}.$$

As we mentioned above we are looking for the mass squared operator for the membrane [see Eq. (25)]. Thus, we need to solve the following Schrödinger equation

$$M_n^2 \Psi(r) = \left( -\frac{d^2}{dr^2} + 16\pi^2 T^2 r^4 \right) \Psi(r),$$

with the boundary conditions  $\Psi(\infty) = 0$  and  $\Psi(0) = 0$ .

It is useful to turn to the dimensionless variable  $z$  defined as

$$z \equiv r(16\pi^2 T^2)^{1/6}.$$

In terms of  $z$  the Schrödinger equation takes the form

$$\frac{M_n^2}{(16\pi^2 T^2)^{1/3}} \Psi(z) = \left( -\frac{d^2}{dz^2} + z^4 \right) \Psi(z). \quad (27)$$

<sup>27</sup>The Hamiltonian looks similar to the one for a relativistic particle with the time-dependent mass  $m(t) = 4\pi T r^2(t)$  and describes the pulsation of the spherical membrane.

The Schrödinger equation with the quartic potential has been extensively studied in the literature (for a review see Ref. [54]). The results of numerical calculations of the first ten eigenvalues can be found in Ref. [54]. Those calculations are usually done for the potential defined on the whole  $z$  axis. In our case  $z$  is defined on the positive semiaxis only. Thus, only the odd parity solutions are relevant for the present case. Those solutions have nodes at  $z=0$  and satisfy the boundary conditions  $\Psi(0)=0$  and  $\Psi(\infty)=0$ .

Here, we present only the first two parity odd eigenvalues of Eq. (27)

$$\frac{M_0^2}{(16\pi^2 T^2)^{1/3}} = 2.393644,$$

$$\frac{M_1^2}{(16\pi^2 T^2)^{1/3}} = 7.335730.$$

Using these expressions and the matching condition (26) one calculates the first two energy levels for the YM system in the finite volume

$$E_{\text{YM}}^0 = \frac{g^{2/3}}{V_{\mathcal{M}}^{1/3}} \frac{(16\pi^2)^{1/3}}{2} 2.393644, \quad (28)$$

$$E_{\text{YM}}^1 = \frac{g^{2/3}}{V_{\mathcal{M}}^{1/3}} \frac{(16\pi^2)^{1/3}}{2} 7.335730. \quad (29)$$

The numerical values for the energy levels are determined by the strong coupling constant  $g$  and also by the volume of the domain  $\mathcal{M}$ . The strong coupling constant is supposed to be taken at the scale appropriate for corresponding glueballs.

Let us take Eqs. (28) and (29) and substitute them into Eq. (17). This leads to the expression for the total energy inside of the finite volume we are discussing

$$\Delta E(V_{\mathcal{M}}) = \frac{1}{2} \theta^2 \Delta \chi V_{\mathcal{M}} + \frac{u_n^2}{V_{\mathcal{M}}^{1/3}}, \quad (30)$$

where in accordance with Eqs. (28),(29)

$$u_0^2 = g^{2/3} (2\pi^2)^{1/3} 2.393644$$

and

$$u_1^2 = g^{2/3} (2\pi^2)^{1/3} 7.335730. \quad (31)$$

The expression (30) can be minimized with respect to the value of the three-volume  $V_{\mathcal{M}}$ . We denote the optimal value for the volume by  $\bar{V}$ , hence  $d\Delta E(V_{\mathcal{M}})/dV_{\mathcal{M}}|_{\bar{V}}=0$ . Using this condition and taking the derivative of Eq. (30) one finds

$$\frac{1}{2} \theta^2 \Delta \chi \approx \frac{1}{3} \frac{u_n^2}{\bar{V}^{4/3}}, \quad (32)$$

and the value of the total energy for the optimal volume

$$\Delta E(\bar{V}_n) \equiv m_n \approx \frac{4}{3} \frac{u_n^2}{\bar{V}_n^{1/3}}. \quad (33)$$

Here, we denote by  $m_n$  the mass of the corresponding  $n$ th glueball and by  $\bar{V}_n$  the corresponding optimal value of the volume element. Thus, knowing the value of the strong coupling constant at the scale appropriate for the lightest glueballs (which is about 1.5–2.5 GeV) and also knowing the value of the effective size of the YM  $0^{-+}$  glueball state one can predict the value of its mass by means of Eq. (33).

We present below the results of calculations for three different values of the strong coupling constant  $\alpha_s$ . The reasonable estimate for the lightest pseudoscalar glueball radius is  $R_0=0.7-1.0$  fm [34,29]. The size of the second excited glueball state  $R_1$  is not known. However, using Eqs. (32) and (31) one can estimate that  $R_1 \approx 1.3R_0 = (0.9-1.3)$  fm. The results of numerical calculations of glueball masses for those values of the coupling constant and radii are presented below.

$\alpha_s=0.3:$

$$R_0=0.7 \text{ fm}, \quad m_0=2340 \text{ MeV}, \quad R_1=0.91 \text{ fm},$$

$$m_1=5520 \text{ MeV},$$

$$R_0=0.8 \text{ fm}, \quad m_0=2050 \text{ MeV}, \quad R_1=1.04 \text{ fm},$$

$$m_1=4830 \text{ MeV},$$

$$R_0=0.9 \text{ fm}, \quad m_0=1820 \text{ MeV}, \quad R_1=1.17 \text{ fm},$$

$$m_1=4300 \text{ MeV},$$

$$R_0=1.0 \text{ fm}, \quad m_0=1640 \text{ MeV}, \quad R_1=1.30 \text{ fm},$$

$$m_1=3870 \text{ MeV}.$$

$\alpha_s=0.35:$

$$R_0=0.7 \text{ fm}, \quad m_0=2470 \text{ MeV}, \quad R_1=0.91 \text{ fm},$$

$$m_1=5800 \text{ MeV},$$

$$R_0=0.8 \text{ fm}, \quad m_0=2160 \text{ MeV}, \quad R_1=1.04 \text{ fm},$$

$$m_1=5090 \text{ MeV},$$

$$R_0=0.9 \text{ fm}, \quad m_0=1920 \text{ MeV}, \quad R_1=1.17 \text{ fm},$$

$$m_1=4520 \text{ MeV},$$

$$R_0=1.0 \text{ fm}, \quad m_0=1730 \text{ MeV}, \quad R_1=1.30 \text{ fm},$$

$$m_1=4070 \text{ MeV}.$$

$\alpha_s=0.4:$

$$R_0=0.7 \text{ fm}, \quad m_0=2580 \text{ MeV}, \quad R_1=0.91 \text{ fm},$$

$$m_1=6080 \text{ MeV},$$

$$\begin{aligned}
R_0=0.8 \text{ fm}, \quad m_0=2260 \text{ MeV}, \quad R_1=1.04 \text{ fm}, \\
m_1=5320 \text{ MeV}, \\
R_0=0.9 \text{ fm}, \quad m_0=2010 \text{ MeV}, \quad R_1=1.17 \text{ fm}, \\
m_1=4730 \text{ MeV}, \\
R_0=1.0 \text{ fm}, \quad m_0=1805 \text{ MeV}, \quad R_1=1.30 \text{ fm}, \\
m_1=4260 \text{ MeV}.
\end{aligned}$$

These predictions can be compared with the result of the lattice calculation for the lightest pseudoscalar glueball mass  $m_0=2.3\pm 0.2$  GeV [6]. We should stress that the masses presented above give just the large  $N_c$  approximation to the actual values. We regard these numbers as reasonable estimates for the pseudoscalar glueball masses.

Let us now discuss an interesting consequence of Eq. (32). If one knew the effective size of the glueball and also the value of  $\Delta\chi$ , then one would be able [using Eq. (32)] to calculate the value of the  $\theta$  parameter

$$\theta^2 \approx \frac{2}{3} \frac{u_n^2}{\bar{V}_n^{4/3} \Delta\chi}.$$

In general, the value of  $\Delta\chi$  is not known. However, in order to get an order of magnitude estimate for  $\theta$  one can crudely approximate  $\Delta\chi$  by the lightest glueball contribution  $f_\delta^2 m_0^2$  multiplied by the number of  $0^{-+}$  glueballs in the spectrum of the model (let us call that number  $N$ ):  $\Delta\chi \approx N f_\delta^2 m_0^2 \approx N(200 \text{ MeV})^4$  [32,33]. Then, if  $\alpha_s=0.3$  and the lightest glueball radius  $R_0=0.8$  fm the  $\theta$  parameter should be equal to  $\theta \approx 6/\sqrt{N}$ . One can also estimate the magnitude of  $\theta$  for different values of the radius. Generically, if the value of  $N$  is not too large, the magnitude of  $\theta$  is of order of the unity or so.

Some comments are in order here. First of all the estimate for the  $\theta$  parameter presented above appears as a result of the physical picture of the glueball formation discussed in this work. However, the method of modeling the glueball spectrum by means of the membrane Hamiltonian does not depend on a particular mechanism of the formation of glueballs. Indeed, whatever the mechanism of the formation is, the glueball can always be regarded in some extent as a close domain of space where the YM excitations are confined and the spectrum of which is determined by the YM Hamiltonian given in Eq. (20).

The second comment concerns the strong  $CP$  problem. In this work we deal with pure YM theory. No light quark degrees of freedom were included. The large value of the  $\theta$  parameter that we derived should somehow be neutralized when quark degrees of freedom are taken into account.

There are some possibilities for that. We list below three of them.

In full QCD the parameter which defines the magnitude of the strong  $CP$  violation is the sum of the  $\theta$  angle used in this work and the phase of the determinant of the quark mass matrix,  $\arg \det M$ . It is possible that those two contributions

compensate each other and the strong  $CP$  violation, being present in pure YM theory, does not appear in full QCD. This might lead to an interesting pattern of mixing between the pseudoscalar glueball and the  $\eta'$  meson.

The second possibility is realized if one has a massless quark in the model. In that case the  $\theta$  dependence can be eliminated from the QCD Lagrangian by an appropriate chiral rotation of that quark field. What happens with the glueball state in full QCD remains to be studied.

Finally, one can argue (using the results of Refs. [18,17]) that in full QCD the  $\eta'$  meson, mediating interactions between topologically charged objects, provides a sufficient (from the experimental point of view) screening of the topological susceptibility even in the massive theory. In terms of Eq. (9) that can be understood by including the  $\eta'$  contribution on the RHS and deducing a Witten-Veneziano type relation. More detailed studies of full QCD are needed in order to determine which of the above scenarios (if any) can actually be realized.

## DISCUSSIONS

In this paper we studied some properties of the YM vacuum which should be responsible for the formation of the  $0^{-+}$  glueball states. The properties of the correlator of the vacuum topological susceptibility as a function of the volume element  $V$  are discussed. In the weak coupling (small volume) approximation it is an increasing function of the argument. Increasing the volume continuously the theory passes through a crossover region after which it should be regarded as a strongly correlated one. Above the crossover region the topological susceptibility becomes a rapidly decreasing function of the argument and reaches its asymptotic value (not necessarily zero) in the large volume limit. Thus, the value of the vacuum topological susceptibility is screened if the strong coupling regime of the theory is considered.

It is shown that the presence of the  $\theta$  angle in the theory along with the screening phenomenon can lead to the formation of a glueball state. An important ingredient of that scenario is the existence of the three-form composite field propagating the Coulomb-like interaction.

The spectrum of the YM Hamiltonian resembles in the zero momentum approximation the spectrum of a closed bosonic membrane. Using that analogy and calculating the spectrum of a closed bosonic membrane we estimate the masses of glueballs in the large  $N_c$  limit. The result for the lightest  $0^{-+}$  glueball is in agreement with the lattice prediction. We also predict the mass of the next-to-lightest glueball. This result can be checked in future lattice calculations. In general, our approach allows us to compute the mass of any heavier glueball state (if such a state exists). The method of calculation of the spectrum is in general independent of the mechanism by which glueballs are formed in YM theory and the YM vs the membrane Hamiltonian analogy utilized for that calculation can always be applied.

Notice that the large  $N_c$  arguments were not used while deriving Eq. (17). The large  $N_c$  approximation was adopted later on in order to calculate the ‘‘positive number’’ occur-

ring on the RHS of Eq. (17). Thus, the approach and equations presented in this work are not peculiar to the  $N_c \rightarrow \infty$  limit. They should rather have some wider range of validity beyond the large  $N_c$  approximation. For instance, the second term in Eq. (17) can be thought as a result of the uncertainty principle alone.

There are a number of interesting questions left out of the discussion in the present paper. First of all we did not discuss the fate of a scalar glueball. The effective Lagrangian approach to the  $0^{++}$  channel of pure YM theory was developed in Refs. [46,47]. One can apply the YM Hamiltonian vs the membrane Hamiltonian analogy to the calculation of the scalar glueball mass too. This last would correspond to the lowest parity-even solution of the Schrödinger equation (27). Hence, the scalar glueball would emerge to be lighter than the pseudoscalar one. This is in agreement with what is known from various lattice and theoretical studies [6,5]. However, the mechanism of the formation of the scalar glueball can not be captured by our analysis.

We did not discuss here how colored degrees of freedom are confined inside of a finite closed volume. It was rather assumed that QCD provides this property by some mechanism. Formally, it was assumed that the operator in the Gauss's law, being the generator of gauge transformations, should annihilate all the physical states. Thus, all those states are supposed to be colorless states by the construction. In various models of hadrons, confinement can be warranted by imposing some boundary conditions on fields, as in the case of the MIT bag model [39] or the model of Ref. [55], or by postulating some specific dielectric properties of the vacuum as in the case of the Friedberg-Lee model [56]. Some discussions of these issues from the point of view of QCD can be found in Ref. [37].

Finally, one needs to know what happens when quark degrees of freedom are also included in the theory. In that case the mixing between the  $\eta'$  meson and the glueball should play an important role (if those two states exist simultaneously). Our discussion of the three-form field in that respect becomes crucial. It is known that the  $\eta'$  meson couples to the topological charge density, hence it couples to the three-form potential too. Thus, one can naturally couple the  $\eta'$  meson to the glueball by means of the three-form field. These and other related questions will be addressed elsewhere.

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#### APPENDIX

In this appendix we consider the dispersion relation for the correlator of the vacuum topological susceptibility in momentum space. The space-time is assumed to be a Euclidean one. The correlator is defined as in Eq. (1), Sec. I. Before we

go further let us specify how singularities are handled in Eq. (1). The product of two operators of the topological charge density is singular as  $x \rightarrow 0$ . The leading perturbative singularity at  $x \rightarrow 0$  can be calculated:

$$TQ(x)Q(0) \propto \frac{1}{x^8} + \mathcal{O}\left(\frac{\ln x^2}{x^8}\right).$$

Upon integration in Eq. (1) this expression yields a divergent term. A simple way to handle the divergence is to allow a small momentum  $k$  to flow through the correlator treating  $\chi(V)$  as a zero momentum limit of the corresponding momentum-dependent renormalized Green's function

$$\begin{aligned} \chi(V) &= \lim_{k^2 \rightarrow 0} \chi^{\text{ren}}(k^2, V) \\ &= \lim_{k^2 \rightarrow 0} \left[ \int_V e^{ikx} \langle 0 | TQ(x)Q(0) | 0 \rangle d^4x \right]^{\text{ren}}, \end{aligned}$$

where  $k$  is the regularizing momentum. This relation implies that the limiting procedure is supposed to be carried out after the integration and renormalization of the divergent parts are already done in momentum space. In what follows we adopt this prescription.

Another type of divergence occurring in Eq. (1) is related to the  $x \rightarrow \infty$  limit. In that limit

$$\langle 0 | TQ(x)Q(0) | 0 \rangle \rightarrow \langle 0 | Q | 0 \rangle \langle 0 | Q | 0 \rangle.$$

Supposing that generically the vacuum expectation value (VEV) of the topological charge density might not be zero in a  $CP$  violating model, one gets the divergence in Eq. (1) as  $V \rightarrow \infty$ . In order to eliminate this divergence one can work with the subtracted correlator. This amounts to saying that the actual integrand in Eq. (1) is the function with the following subtraction:

$$\langle 0 | TQ(x)Q(0) | 0 \rangle - \langle 0 | Q | 0 \rangle \langle 0 | Q | 0 \rangle.$$

The subtracted function goes to zero in the  $x \rightarrow \infty$  limit. The coordinate-independent subtraction term does not affect our analysis and was dropped for simplicity in Sec. I. It will also be omitted below. In what follows we show that continuum contributions vanish in the limit  $k^2 \rightarrow 0$ .

The dispersion relation for the correlator of the vacuum topological susceptibility in momentum space can be written as

$$\chi(k^2) = \chi(0) + \chi'(0)k^2 + \frac{k^4}{\pi} \int_{m_{G_0}^2}^{\infty} \frac{\rho(s)ds}{s^2(s-k^2)}, \quad (\text{A1})$$

where  $\rho(s) \equiv \text{Im}\chi(s+i\epsilon)$ . The correlator at zero momentum is denoted by  $\chi(0)$ . The quantity  $\chi'(0)$  stands for the derivative of the correlator with respect to  $k^2$  at  $k^2=0$ .

In order to make the integral convergent, and also to account for the correct asymptotic behavior of the correlation function at  $k^2 \rightarrow \infty$ , we have introduced the subtraction terms in the dispersion relation (A1). The dispersion relation in the form given in Eq. (A1) is not convenient for our purposes. In

the limit  $k^2 \rightarrow 0$  it turns into a trivial identity. One needs to rewrite Eq. (A1) in a form similar to the one given in Eq. (6). For this purpose let us use the following identity:

$$\frac{k^4}{s^2(s-k^2)} = \frac{1}{s-k^2} - \frac{1}{s} - \frac{k^2}{s^2}.$$

Substituting this formula into Eq. (A1) one rewrites the dispersion relation in the following form:

$$\chi(k^2) = d_0 + b_0 k^2 + \frac{1}{\pi} \int_{m_{G_0}^2}^{\infty} \frac{\rho(s) ds}{s-k^2}, \quad (\text{A2})$$

where

$$d_0 \equiv \chi(0) - \frac{1}{\pi} \int_{m_{G_0}^2}^{\infty} \frac{\rho(s) ds}{s},$$

$$b_0 \equiv \chi'(0) - \frac{1}{\pi} \int_{m_{G_0}^2}^{\infty} \frac{\rho(s) ds}{s^2}.$$

The form of the relation given in Eq. (A2) is very formal one. The constants  $d_0$ ,  $b_0$ , and the integral on the RHS are divergent quantities. When these terms are put together all divergences cancel and the whole expression is finite. The divergences mentioned above are related to perturbative contributions to the spectral density. Thus, it is convenient to separate nonperturbative and perturbative terms. We found it useful to apply the decomposition usually adopted in QCD sum rule calculations [41]. One decomposes the expression for the spectral density

$$\rho(s) = \rho^{\text{np}}(s) + \rho^{\text{pt}}(s) \vartheta(s-s_0),$$

where the superscripts np and pt denote nonperturbative and perturbative terms, respectively. Here  $\vartheta$  denotes the step function. The constant  $s_0$  sets the continuum threshold (or the duality interval) [41] and by the definition  $s_0 > m_{G_0}^2$ . It is assumed in this approach that resonance contributions are defined by the nonperturbative part of the spectral density. One also supposes that due to asymptotic freedom continuum contributions above the continuum threshold can be approximated by leading perturbative terms [41].

Let us make the same formal decomposition for the quantities  $d_0$  and  $b_0$ :

$$d_0 = d + d^{\text{pt}}, \quad d^{\text{pt}} = -\frac{1}{\pi} \int_{s_0}^{\infty} \frac{\rho^{\text{pt}}(s) ds}{s},$$

$$b_0 = b + b^{\text{pt}}, \quad b^{\text{pt}} = -\frac{1}{\pi} \int_{s_0}^{\infty} \frac{\rho^{\text{pt}}(s) ds}{s^2}.$$

Here  $d$  and  $b$  are the quantities determined by the complicated vacuum structure of YM theory. As we mentioned al-

ready, in the weak coupling approximation with noninteracting instantons  $d$  is defined as the value of the topological susceptibility of a dilute instanton gas in the large volume limit of pure YM theory. The quantity  $d$  appears in Eqs. (5) and (6)–(8) in the text. Finally, using all the expressions given above one derives

$$\chi(k^2) = d + b k^2 + \frac{1}{\pi} \int_{m_{G_0}^2}^{\infty} \frac{\rho^{\text{np}}(s) ds}{s-k^2} + \frac{k^4}{\pi} \int_{s_0}^{\infty} \frac{\rho^{\text{pt}}(s) ds}{s^2(s-k^2)}. \quad (\text{A3})$$

We should notice here that Eqs. (A3) and (A1) differ from each other by some formal redefinitions. Moreover, Eq. (A3) is written adopting some particular scheme of separation between perturbative and nonperturbative contributions. That procedure is not unambiguous. In that respect, Eq. (A3) should be regarded as an expression defined within the framework of the particular prescription outlined above.

Now one can use the fact that the quantity  $k^2$  is a regularizing momentum. Thus, one can assume that  $k^2$  is very small, so that the condition  $s_0 \gg k^2$  is readily satisfied. The last integral on the RHS of Eq. (A3) can be expanded in a power series of the ratio  $k^2/s_0$  (since that integral is convergent). Performing the expansion, and then Fourier transforming Eq. (A3) with the weight  $1/(2\pi)^4$ , one derives the expression for the correlator  $\langle 0|TQ(x)Q(0)|0\rangle$  in the following form<sup>28</sup>:

$$\begin{aligned} \langle 0|TQ(x)Q(0)|0\rangle &= d \delta^{(4)}(x) - b \partial^2 \delta^{(4)}(x) \\ &+ \frac{1}{\pi} \int_{m_{G_0}^2}^{\infty} \rho^{\text{np}}(s) D_F(\sqrt{s}|x|) ds \\ &+ \frac{1}{\pi} \sum_{n=2}^{\infty} (-1)^n \int_{s_0}^{\infty} \frac{\rho^{\text{pt}}(s)}{s} \left(\frac{\partial^2}{s}\right)^n \\ &\times \delta^{(4)}(x) ds. \end{aligned} \quad (\text{A4})$$

Equation (A4) is a general form of the expression given in Eq. (6) in Sec. I. In order to reproduce the sum on the RHS of Eq. (6) one needs to make the following substitution in Eq. (A4):

$$\rho^{\text{np}}(s) = -\pi \sum_n f_n^2 m_n^4 \delta(s-m_n^2).$$

Some terms on the RHS of Eq. (A4) with derivatives of the Dirac delta function yield vanishing contributions upon integration in Eq. (1). For that reason those derivative containing terms were omitted in Eq. (6).

<sup>28</sup>We use the following normalization for the delta function:  $\delta^{(4)}(x) = [1/(2\pi)^4] \int_{-\infty}^{+\infty} e^{ikx} d^4k$  and  $\delta^{(4)}(k) = \int_{-\infty}^{+\infty} e^{-ikx} d^4x$ .



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