Smallest Dirac eigenvalue distribution from random matrix theory

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We derive the hole probability and the distribution of the smallest eigenvalue of chiral Hermitian random matrices corresponding to Dirac operators coupled to massive quarks in QCD. They are expressed in terms of the QCD partition function in the mesoscopic regime. Their universality is explicitly related to that of the microscopic massive Bessel kernel. [S0556-2821(98)03118-X]

There has long been the attractive idea that the low-energy physics of a complex system can be described by a simple effective theory which respects the global symmetries of the original system. As an example, the quantum spectral statistics of a classically chaotic system is believed to be described by a random matrix theory belonging to the same universality class as the former [1]. One new manifestation of essentially the same idea is the recent observation that QCD Dirac operator spectra on the scale $\lambda = O(1/V_4)$ (where V_4 is the space-time volume) measured in lattice Monte Carlo simulations [2] are in excellent agreement with the predictions from those large-N random matrix theories [3,4] that share the same global symmetries as QCD. The suitably rescaled (microscopic) spectral correlation functions thus seem to provide exact finite-size scaling functions for QCD in a finite volume. Very recently, the microscopic spectral correlators have been calculated from random matrix theories that include the effect of fermion determinants with masses $m \simeq O(1/V_4)$ [5–7] (see also [8]). When λ and m are measured in units of the mean level spacing at zero virtuality, all the random matrix predictions turn out to be universal, i.e., insensitive to the details of the random matrix potential [4–6]. Although the question of whether or not QCD is included in the same universality class cannot be answered by demonstrating the existence of the wide range of universality within random matrix theories, it provides strong support for the former.

From the field-theoretic point of view [9] it would be most surprising if these observables would not also be computable solely within the framework of finite-volume generating functionals (partition functions) for the order parameter $\langle \bar{\psi}\psi \rangle$. If not, large-$N$ random matrix theory, which in principle is foreign to the pertinent field theory language, would seem to be a new ingredient required to describe the observed spectral correlators. It has recently been shown that a description entirely in terms of finite-volume partition functions is indeed also possible [10].

In order to confirm by numerical simulations that the low-lying spectra of QCD Dirac operators can be described alternatively by large-N random matrix theories, it is in practice most convenient to measure the distribution of the smallest eigenvalue [2] and compare that to the random matrix prediction [11,7]. Since the smallest eigenvalue distribution largely consists of the first peak of the microscopic spectral density (see Fig. 1, $\zeta \equiv N\lambda$), we expect it to be universal. In fact, the proven universality of the massive kernels [5,6] of (chiral) unitary ensembles of random matrices guarantees the universality of the hole probability $E(s)$, i.e., the probability that the interval $[-s,s]$ is free of eigenvalues, and of the smallest eigenvalue distribution $P(s) = -E'(s)$ because it is related to the kernel via the Fredholm determinant formula [12],

$$E(s) = \det(1 - \hat{K}). \qquad (1)$$

Here, $\hat{K}$ is an integral operator whose kernel is the microscopic massive Bessel kernel [Eq. (17) below or its nonchiral counterpart] over the interval $[-s,s]$. While other universal statistical quantities such as the number variance $\Sigma^2(s)$ and the spectral rigidity $\Delta_3(s)$ of the eigenvalues in the interval $[-s,s]$ are directly related to the kernel by integral transforms, the techniques required to compute (1) from the kernel are rather involved. In the case of the universal (mass-
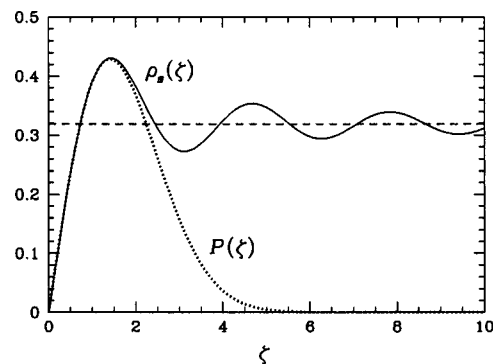

FIG. 1. Microscopic spectral density [solid line, normalized as $\rho_s(\zeta \to \infty) = 1/\pi$] and smallest eigenvalue distribution (dotted line) for the quenched chiral unitary ensemble.

less) sine kernel describing the bulk of unitary ensembles, its eigenmodes are given by the prolate spheroidal functions whose eigenvalues are known. However, the massive generalization of these eigenmodes is difficult to construct. The alternative technique of determining $E(s)$ as a τ function of a system of integrable partial differential equations [13] does not simplify the situation either. Consequently, the explicit calculation for the massive case has been done so far solely for the chiral Gaussian unitary ensemble [7]. The result contains an explicit factor of $\exp(-\zeta^2/4)$ which at first sight is due to the Gaussian potential. Since these results should be universal, the identical factor should also arise from a generic random matrix potential. How this happens is not immediately obvious and will be addressed in the sequel. We shall provide a concise method to circumvent the difficulty in explicitly evaluating (1) and to efficiently calculate $E(s)$ for the massive chiral unitary ensembles with generic potentials.

We define the partition function of the chiral unitary ensemble

$$Z^{(\alpha)}(\{m\}) = \int dM \; e^{-N\mathrm{tr}\, v(M)} \prod_{f=1}^{\alpha} \; \det(M + im_f), \quad (2)$$

where M is a $2N \times 2N$ block Hermitian matrix

$$M = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}, \quad (3)$$

dM is the Haar measure of W, and $v(M)$ is an even analytic function. The matrix M models the Dirac operator for $SU(N_c \geq 3)$ four-dimensional QCD (QCD$_4$) in the Weyl basis, and N can be identified with the spacetime volume V_4. The integer α corresponds to the number of flavors. The case of nonzero topological charge ν can be treated by introducing $|\nu|$ massless flavors [3]. One then has $\alpha = N_f + |\nu|$, where N_f is the number of (massive) flavors and $m_{N_f+1}, \ldots, m_{N_f+|\nu|} = 0$.

The (unnormalized) probability of having no eigenvalues in the interval $[-s, s]$ is given by

$$E^{(\alpha)}(s; \{m\}) = \int_{|\text{eigenvalues}| \geq s} dM e^{-N\mathrm{tr}\, v(M)} \prod_{f=1}^{\alpha} \; \det(M + im_f). \quad (4)$$

It is convenient to change the picture from M to an $N \times N$ positive definite Hermitian matrix $H = W^\dagger W$,

$$E^{(\alpha)}(s; \{m\}) = \int_{\text{eigenvalues} \geq s^2} dH e^{-N\mathrm{tr}\, V(H)}$$

$$\times \prod_f \; \det(H + m_f^2)$$

$$= \int_{\text{eigenvalues} \geq 0} dH e^{-N\mathrm{tr}\, V(H+s^2)}$$

$$\times \prod_f \; \det(H + s^2 + m_f^2), \quad (5)$$

where $V(z^2) \equiv 2v(z)$. Hereafter we write

$$m' = \sqrt{m^2 + s^2}. \quad (6)$$

We can express (5) in terms of an expectation value $\langle \cdots \rangle_{m'}$ with respect to the measure $e^{-N\mathrm{tr}\, V(H)} \prod_f \det(H + m_f'^2)$,

$$E^{(\alpha)}(s; \{m\}) = \frac{\displaystyle\int_0^\infty dH \; e^{-N\mathrm{tr}\, V(H+s^2)} \prod_f \; \det(H + m_f'^2)}{\displaystyle\int_0^\infty dH \; e^{-N\mathrm{tr}\, V(H)} \prod_f \; \det(H + m_f'^2)}$$

$$\times \int_0^\infty dH \; e^{-N\mathrm{tr}\, V(H)} \prod_f \; \det(H + m_f'^2)$$

$$= \langle e^{-N\mathrm{tr}\, (V(H+s^2)-V(H))} \rangle_{m'} Z^{(\alpha)}(\{m'\}). \quad (7)$$

In the following we shall show that the two factors in Eq. (7) are piecewise universal in the limit $N \to \infty$ with $\zeta = Ns$ and $\mu = Nm$ fixed finite.

Due to the large-N factorization of macroscopic correlation functions of $U(N)$-invariant operators $\mathcal{O}, \mathcal{O}'$ [14],

$$\langle \mathcal{O}\mathcal{O}' \rangle = \langle \mathcal{O} \rangle \langle \mathcal{O}' \rangle + O\left(\frac{1}{N^2}\right), \quad (8)$$

the first factor of Eq. (7) is, in the large-N limit, approximated by

$$\sim e^{-N^2 \langle (1/N)\mathrm{tr}\, (V(H+s^2)-V(H)) \rangle_{m'}}$$

$$\sim e^{-N^2 (s^2 \langle (1/N)\mathrm{tr}\, V'(H) \rangle + O(s^4))}. \quad (9)$$

Since the fermion determinant does not contribute to the macroscopic correlator in the large-N limit, we have dropped the suffix m'.

Now we change the picture back to M, whose macroscopic spectral density is $\rho(z) \equiv \langle (1/2N)\mathrm{tr}\, \delta(z - M) \rangle = R(z^2)\sqrt{a^2 - z^2}$, where $[-a, a]$ is the support of the spectrum of M and $R(z^2)$ is an analytic function which depends on the details of $v(M)$ [4]. We obtain

$$Q \equiv \left\langle \frac{1}{N}\mathrm{tr}\, V'(H) \right\rangle = \left\langle \frac{1}{2N}\mathrm{tr}\, \frac{v'(M)}{M} \right\rangle = \int_{-a}^a dz \; \rho(z)\frac{v'(z)}{z}. \quad (10)$$

In terms of $\rho(z)$, the leading [of order $O(N^2)$] part of the action in Eq. (2) is written as

$$S = \int_{-a}^a dz \; \rho(z)2v(z) - \int_{-a}^a dzdw \; \rho(z)\rho(w)2P\ln|z-w|. \quad (11)$$

The second term is the exponentiated Vandermonde determinant. By substituting the large-N saddle-point equation [15] $\delta S/\delta\rho(z) = 0$, i.e.,

$$v'(z) - 2\int_{-a}^a dw \; \rho(w)P\frac{1}{z-w} = 0, \quad (12)$$

into Eq. (10), we have

$$Q = 2 \int_{-a}^{a} dz\, dw\, \rho(z)\rho(w) P \frac{1}{z} P \frac{1}{z-w}. \qquad (13)$$

Using the identity $1/(z \pm i\epsilon) = P(1/z) \mp i\pi\delta(z)$ and taking into account that $\rho(z)$ is even, we finally obtain

$$Q = \int_{-a}^{a} dz\, dw\, \rho(z)\rho(w)\pi\delta(z)\pi\delta(z-w) = (\pi\rho(0))^2. \qquad (14)$$

That is,

$$\langle e^{-N\mathrm{tr}\,(V(H+s^2)-V(H))} \rangle_{s,m'}\big|_{s=\zeta/N} \xrightarrow{N\to\infty} e^{-(\pi\rho(0)\zeta)^2} \qquad (15)$$

which, after the rescaling $\zeta \to \zeta/(2\pi\rho(0))$, universally reads $\exp(-\zeta^2/4)$. This answers the question raised in the introduction how the Gaussian factor arises from a generic potential.

We denote the microscopic limit of the partition function $Z^{(\alpha)}(\{m\}=\{\mu/N\})$ by $\mathcal{Z}^{(\alpha)}(\{\mu\})$. It is related to the microscopic kernel by the master formula [10]

$$K^{(\alpha)}(\zeta_1,\zeta_2;\mu_1,\dots,\mu_\alpha) = C\sqrt{|\zeta_1\zeta_2|} \prod_f \sqrt{(\zeta_1^2+\mu_f^2)(\zeta_2^2+\mu_f^2)} \frac{\mathcal{Z}^{(\alpha+2)}(i\zeta_1,i\zeta_2,\mu_1,\dots,\mu_\alpha)}{\mathcal{Z}^{(\alpha)}(\mu_1,\dots,\mu_\alpha)}, \qquad (16)$$

where C is a normalization constant. This formula is reminiscent of the very definition of the partition function and the kernel [16] and is valid in the large-N limit, regardless of whether microscopic or macroscopic eigenvalue and mass variables are kept finite.

Using the technique of orthogonal polynomials and rescaling $\zeta \to \zeta/(2\pi\rho(0)), \mu \to \mu/(2\pi\rho(0))$, the left-hand side is shown to be universally given by [5]

$$K^{(\alpha)}(\zeta_1,\zeta_2;\{\mu\}) = C \frac{\sqrt{|\zeta_1\,\zeta_2|}}{\zeta_1^2-\zeta_2^2} \frac{\det_{1\leqslant i,j\leqslant\alpha+2} B_{ij}(\zeta_1,\zeta_2;\{\mu\})}{\prod_f \sqrt{(\zeta_1^2+\mu_f^2)(\zeta_2^2+\mu_f^2)}\, \det_{1\leqslant i,j\leqslant\alpha} A_{ij}(\{\mu\})}, \qquad (17)$$

$$A_{ij} = \mu_i^{j-1} I_{j-1}(\mu_i), \qquad B_{ij} = \begin{cases} A_{ij} & (i=1,\dots,\alpha), \\ (-\zeta_{1,2})^{j-1} J_{j-1}(\zeta_{1,2}) & (i=\alpha+1,\alpha+2), \end{cases}$$

where J and I denote the Bessel function of real and imaginary argument, respectively. Therefore, after continuing $\zeta \to -i\mu$, we can derive $\mathcal{Z}^{(2n)}(\{\mu\})$ iteratively using Eq. (16), starting from $\mathcal{Z}^{(0)} = 1$. $\mathcal{Z}^{(2n-1)}(\{\mu\})$ can be obtained by decoupling one of the masses in $\mathcal{Z}^{(2n)}(\{\mu\})$ by sending it to infinity. In this way, one obtains universally

$$\mathcal{Z}^{(\alpha)}(\{\mu\}) = \frac{\det_{1\leqslant i,j\leqslant\alpha} A_{ij}(\{\mu\})}{\det_{1\leqslant i,j\leqslant\alpha}\mu_i^{2(j-1)}}, \qquad (18)$$

which is identical to the finite-volume QCD partition function as calculated from the chiral Lagrangian in the ''mesoscopic'' scaling limit [9,17].

Both factors in Eq. (7) being universal, the normalized hole probability

$$E^{(\alpha)}(\zeta;\{\mu\}) = e^{-\zeta^2/4} \frac{\mathcal{Z}^{(\alpha)}(\{\sqrt{\mu^2+\zeta^2}\})}{\mathcal{Z}^{(\alpha)}(\{\mu\})} \qquad (19)$$

as well as the smallest eigenvalue distribution [7]

$$P^{(\alpha)}(\zeta;\{\mu\}) = -\frac{\partial}{\partial\zeta} E^{(\alpha)}(\zeta;\{\mu\})$$

$$= \frac{\zeta}{2} e^{-\zeta^2/4} \frac{\det_{1\leqslant i,j\leqslant\alpha} C_{ij}(\{\sqrt{\mu^2+\zeta^2}\})}{\det_{1\leqslant i,j\leqslant\alpha} A_{ij}(\{\mu\})} \qquad (20)$$

with $C_{ij}(\{\mu\}) = \mu_i^{j-1} I_{j+1}(\mu_i)$ are universal. As explained above, the general case of nonzero topological charge ν is obtained by introducing $|\nu|$ massless flavors ($\alpha = N_f + |\nu|$).

Except for the Gaussian prefactor the expressions (19) and (20) are explicitly given in terms of a finite-volume field theory partition function in the mesoscopic scaling regime. This indicates that also these quantities can be derived directly from field theory in the mesoscopic scaling regime, without the bypass through random matrix theory [10].

The hole probabilities for the chiral orthogonal ($\beta=1$) and symplectic ($\beta=4$) ensembles are also related to the unitary kernel K by formulas analogous to Eq. (1),

$$E_1(s) = \det^{1/2}(1-\hat{K}_1), \qquad (21a)$$

$$E_4(s) = \det^{1/2}(1-\hat{K}_4). \qquad (21b)$$

Here, $\hat{K}_1$ and $\hat{K}_4$ are 2x2 matrix kernels given in terms of K [18,19]. Equations (21) as well as the relationships between the kernels hold also in the presence of the fermion determinants. Hence, the smallest eigenvalue distributions for these ensembles are also guaranteed to be universal. However, explicit expressions for $E_1(s)$ and $E_4(s)$ are not known except for the massless chiral Gaussian/Laguerre case [20,21]. To obtain these quantities for the corresponding massive ensembles, our method presented here needs modification be-

cause the mapping $M \mapsto H$ involves a nontrivial Jacobian, $(\det H)^{\beta/2-1}$. This point will be discussed elsewhere.

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