Critical dynamics of symmetry breaking: Quenches, dissipation, and cosmology

Pablo Laguna
Theoretical Astrophysics, MS-B288, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
and Department of Astronomy & Astrophysics and CGPG, Penn State University, University Park, Pennsylvania 16802

Wojciech Hubert Zurek
Theoretical Astrophysics, MS-B288, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
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Symmetry-breaking phase transitions may leave behind topological defects with a density dependence on the quench rate. We investigate the dynamics of such quenches for the one-dimensional, Landau-Ginzburg case and show that the density of kinks, n, scales differently with the quench time scale, τ_Q, depending on whether the dynamics in the vicinity of the critical point is overdamped ($n \propto \tau_Q^{-1/4}$) or underdamped ($n \propto \tau_Q^{-1/3}$). Either of these cases may be relevant to the early Universe, and we derive bounds on the initial density of topological defects in cosmological phase transitions. [S0556-2821(98)01820-7]

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The dynamics of symmetry breaking phase transformations is of interest in cosmology, condensed matter, and high energy physics. The recent surge of interdisciplinary interest has been fueled by the experiments involving creation of topological defects during rapid second order phase transitions in superfluids [3–6] and other systems [7], in a setting reminiscent of similar cosmological processes [1,8].

In the cosmological context, topological defects such as cosmic strings may have played a role in seeding structure formation [1,9]. In high energy physics, accelerator experiments may allow one to probe restoration of some of the symmetries (e.g. chiral), which were originally broken early in the history of the Universe. The signature of whether such restoration has occurred will come from the fluctuation of decay products [10,11], determined by the relevant dynamics of the order parameter during a quench. In superfluids, the very same process controls the production of topological defects. The interest in experimental exploration of the critical dynamics is therefore well justified by its wide-ranging applications, which may also come to include in the near future the creation of vortices in Bose-Einstein condensates [12].

These processes span many orders of magnitude in spatial and energetic scales. Yet, a large class of them is well-approximated by the Landau-Ginzburg theory. Therefore, the dynamics of the order parameter φ is governed by an equation of the form [13]

$$\ddot{\varphi}+\eta\dot{\varphi}-c^2\nabla^2\varphi+[\beta\varphi^3-m^2\epsilon(t)\varphi]/2=\vartheta(t,x). \qquad (1)$$

Above, η characterizes viscosity, while c,β and m are constant coefficients, and $\epsilon(t)$ is the time-dependent relative temperature, assumed to vary with time as $\epsilon=t/\tau_Q$, where τ_Q is the quench time scale. The term $\vartheta(t,x)$ is noise characterized by its spatial and temporal correlations, as well as by its amplitude θ. We assume: $\langle\vartheta(x,t),\vartheta(x',t')\rangle=2\eta\theta\delta(x'-x)\delta(t'-t)$. Equation (1) can be expressed in ''natural'' units $t\rightarrow t/m, x\rightarrow xc/m, \eta\rightarrow\eta m, \varphi\rightarrow\varphi m/\sqrt{\beta}$ and $\theta\rightarrow\theta m^3 c/\beta$, which leads to

$$\ddot{\varphi}+\eta\dot{\varphi}-\nabla^2\varphi+(\varphi^3-\epsilon\varphi)/2=\vartheta. \qquad (2)$$

Thus, in the vicinity of the second-order phase transition, an enormous range of ''bare'' parameters can be reduced to two: the ''renormalized'' damping rate η and the noise temperature θ. The quench adds a dependence of the consequences of critical dynamics on the quench rate $\dot{\epsilon}=\tau_Q^{-1}$. The aim of our study is to investigate the dependence of the critical dynamics on the value of η and τ_Q, under the assumption that θ is sufficiently small, so the probability of thermally activated symmetry restoration is negligible after the quench. In this paper we take $\theta=0.1$. We focus on creation of kinks — zero-dimensional topological defects in 1D systems — in the course of rapid quenches.

Evolution generated by Eq. (2) is overdamped when $\eta\dot{\varphi}>\ddot{\varphi}$. In this regime, the relaxation time $\tau_{\dot{\varphi}}\simeq|\varphi/\dot{\varphi}|$ scales with the relative temperature ϵ as: $\tau_{\dot{\varphi}}\simeq\eta\tau_o^2|\epsilon|^{-1}\simeq\eta\tau_Q\tau_o^2|t|^{-1}$, in the units of Eq. (1) with $\tau_o=m^{-1}$ the dynamical time scale. In accord with [2], one expects the initial size $\hat{\xi}$ of the pieces of the new broken symmetry phase to be set at the time $\hat{t}$, when the time to (from) the phase transition is comparable to the relaxation time scale, and the freeze-out of the field evolution occurs; that is, its state cannot keep up with the change of the thermodynamic parameters as a result of critical slowing down. This freeze-out condition, $\tau_{\dot{\varphi}}(\hat{t}_{\dot{\varphi}})=\hat{t}_{\dot{\varphi}}$ yields

$$\hat{t}_{\dot{\varphi}}\simeq\tau_o(\eta\tau_Q)^{1/2} \qquad (3)$$

$$\hat{\epsilon}_{\dot{\varphi}}\simeq\left(\frac{\eta\tau_o^2}{\tau_Q}\right)^{1/2}. \qquad (4)$$

The correlation length $\hat{\xi}$, which sets the stage for the defect formation [1], is then:

$$\hat{\xi}_{\dot{\varphi}}\simeq\frac{\xi_o}{|\hat{\epsilon}_{\dot{\varphi}}|^{1/2}}\simeq\xi_o\left(\frac{\tau_Q}{\eta\tau_o^2}\right)^{1/4}, \qquad (5)$$

where $\xi_o=c/m$ characterizes the low temperature ($\epsilon=0$) healing length.

To test these arguments [2], we have recently carried out a numerical study of defect formation [14], showing that the density of kinks formed in a quench indeed varies, in the overdamped regime, as

$$n_{\dot{\varphi}} \simeq \frac{1}{f\hat{\xi}_{\dot{\varphi}}} \propto \left(\frac{\eta\tau_o^2}{\tau_Q}\right)^{1/4} \tag{6}$$

with $f \simeq 8$. A similar study was independently carried out by Lythe [15], who has estimated $f = 2\pi|\ln\theta|^{1/4}$ when $\theta \ll 1$.

Our purpose here is to extend these studies from the regime where damping dominates (which is most relevant in condensed matter applications) to the range where the evolution is underdamped (as may be the case in cosmology). For details of the numerical technique, see Ref. [14].

In the underdamped case, $\ddot{\varphi}$ will dominate, and the order parameter reacts to the quench-induced changes in the effective potential on the time scale $\tau_{\ddot{\varphi}} \simeq |\varphi/\ddot{\varphi}|^{1/2}$. Thus, $\tau_{\ddot{\varphi}} \simeq \tau_o|\epsilon|^{-1/2}$. The freeze-out condition, $\tau_{\ddot{\varphi}}(\hat{t}_{\ddot{\varphi}}) = \hat{t}_{\ddot{\varphi}}$ yields in this underdamped regime:

$$\hat{t}_{\ddot{\varphi}} \simeq \tau_o(\tau_Q/\tau_o)^{1/3}; \tag{7}$$

$$\hat{\epsilon}_{\ddot{\varphi}} \simeq (\tau_o/\tau_Q)^{2/3}. \tag{8}$$

Consequently, the scaling of the characteristic correlation length with the quench rate τ_Q^{-1} is expected to change to

$$\hat{\xi}_{\ddot{\varphi}} \simeq \frac{\xi_o}{|\hat{\epsilon}_{\ddot{\varphi}}|^{1/2}} \simeq \xi_o\left(\frac{\tau_Q}{\tau_o}\right)^{1/3}. \tag{9}$$

Furthermore, the density of the number of kinks is given in this case by

$$n_{\ddot{\varphi}} \simeq \frac{1}{f\hat{\xi}_{\ddot{\varphi}}} \propto \left(\frac{\tau_o}{\tau_Q}\right)^{1/3}, \tag{10}$$

although f may now be different.

We can therefore draw two related conclusions: (i) In the overdamped regime, the density of kinks should scale with $\eta^{1/4}$, and should become viscosity independent in the underdamped case. (ii) Power-law dependence of the density of kinks with the quench time scale should change from $\propto \tau_Q^{-1/4}$ in the overdamped case to $\propto \tau_Q^{-1/3}$ in the underdamped case. The overdamped scalings should apply when the evolution is dominated by the first derivative ($\eta\dot{\varphi} > \ddot{\varphi}$, i.e. $\eta/\tau_{\dot{\varphi}} > 1/\tau_{\ddot{\varphi}}^2$) at the instant when topological defects ''freeze-out.'' This will happen for: $|\hat{\epsilon}_{\dot{\varphi}}| > |\hat{\epsilon}_{\ddot{\varphi}}|$, or — using Eqs. (4) and (8) — when

$$(\eta\tau_o)^3 > (\tau_o/\tau_Q). \tag{11}$$

We identify kinks as zeros of the order parameter. This can be justified only well after the phase transition, when φ has locally settled into the broken symmetry state. Kinks annihilate, and their number slowly decreases with time. Previously, in the overdamped regime, we were able to confirm the predicted [2] dependence of the initial number of kinks on τ_Q from the numerical data by using a fairly straightforward procedure of simply counting zeros at a fixed value of t/τ_Q [14]. The nature of that dependence did not change dramatically even when the counting of kinks was taking at a constant value of t (although a change on the slope as well as evidence of the saturation in the number of kinks for small τ_Q were noted). But the nature of critical dynamics and especially the annihilation rate depend on η, which we shall vary by several orders of magnitude. To compare ''initial densities'' of kinks now, we therefore need more objective procedures independent of the time at which the kinks are counted. We have done this by using whole runs of kink densities (such as the ones shown in Fig. 1) to model annihilation either as a power-law $N \simeq N_o(t/\tau_Q)^{-b}$, or as an exponential, $N \simeq N_o\exp(-at/\tau_Q)$, with N the number of zeros of the order parameter. The actual dependences are usually sufficiently similar to a straight line that both of these procedures yield comparable initial kink numbers N_o.

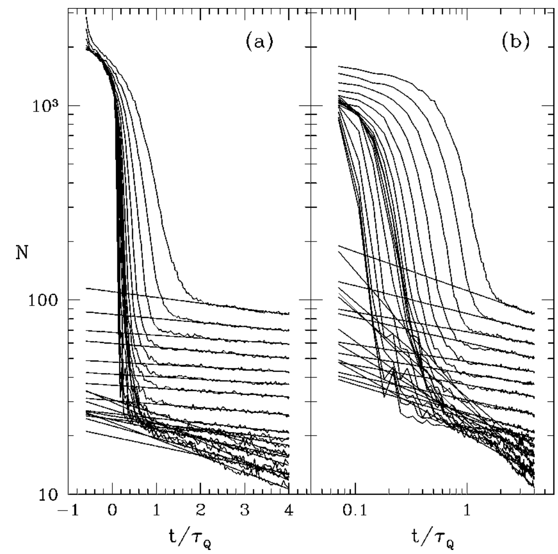

FIG. 1. Number of kinks N as a function of time for simulations with different viscosity parameters η, but with the same quench time scale $\tau_Q = 256$. Two models for the annihilation rate are shown, (a) exponential $N = N_o\exp(-at/\tau_Q)$ and (b) power-law $N = N_o(t/\tau_Q)^{-b}$. Note the increase of the annihilation rates for small η.

The dependence of the initial number of kinks N_o on the damping coefficient η for three different quench rates ($\tau_Q = 128,\ 256,$ and 512) is shown in Fig. 2. In the regime of large viscosities, critical dynamics is overdamped, in accord with Eq. (11), and leads to the power-law dependence $\propto \eta^{1/4}$, Eq. (6). The spacing between the three lines is also roughly consistent with the one anticipated from that equation. As the damping rate decreases below the value estimated from Eq. (11), namely $\eta \lesssim 0.1$, the number of kinks becomes essentially independent of η. Moreover, the spacing between the (now approximately horizontal) lines of constant τ_Q is consistent with the underdamped case $N_o \propto \tau_Q^{-1/3}$, Eq. (10).

We should note, however, that for $\eta \lesssim 10^{-2}$, the number of kinks are small and the annihilation is more efficient. Consequently, our results in this range are less reliable. In a

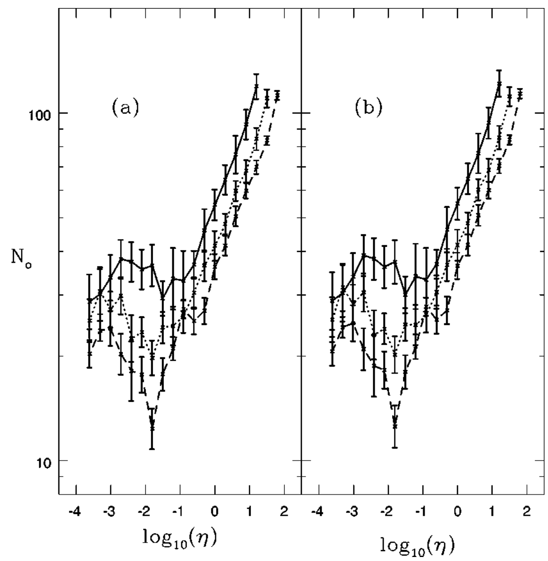

FIG. 2. Initial number of kinks N_o as a function of the damping rate η for a fixed quench rate time scales (top to bottom $\tau_Q=$ 128, 256 and 512). Both exponential (a) and power-law (b) model results (see Fig. 1) are shown (and are essentially identical). For $\eta > 0.1$, $N_o \propto \eta^g$, where N_o is obtained from the fittings in Fig 1. Best fits yield $g = (0.27 \pm 0.035,\ 0.25 \pm 0.029,\ 0.27 \pm 0.011)$ for $\tau_Q = (128,\ 256,\ 512)$, respectively.

sense, Fig. 1 indicates a more systematic (and probably more consistent with the theoretical expectations) trend with the damping rate η than our estimates of the number of kinks plotted in Fig. 2. Most of the scatter in Fig. 2 stems from our inability to model annihilation rate in a consistent fashion over a broad range of parameters, rather than from the ''raw'' data shown in Fig. 1.

These conclusions concerning the transition from overdamped to underdamped behavior are strengthened by comparing families of simulations corresponding to the same η but for varying τ_Q (see Figs. 3 and 4). As before, we consider two methods for obtaining the initial number of kinks N_o from the time-dependent data, Fig. 3. In the range of long quench time scales they produce similar (but not identical) power-laws.

According to condition (11), the $\eta = 5$ and 1 cases in Fig. 4 are, for the values of τ_Q under consideration $(2 \leqslant \tau_Q \leqslant 4098)$, entirely within the overdamped regime. For these two cases, we find power-laws $\sim \tau_Q^{-1/4}$, consistent with Eq. (6). On the other hand, for the $\eta = 1/5$ case in accord with Eq. (11), a transition between overdamped and underdamped regimes should occur at $\tau_Q \sim 125$. We find (see Fig. 4) indication of a change in the power-law dependence, from $N_o \propto \tau_Q^{-1/4}$ to $\propto \tau_Q^{-1/3}$ as η decreases.

One obvious case of breakdown of power-laws, Eqs. (6) and (10), occurs when the condition $\hat{\epsilon} \ll 1$ of applicability of the theory of Ref. [2] is not satisfied [16]. In that case, the predicted initial separation of kinks would be comparable to (or even smaller than) the zero-temperature healing length ξ_o. This would of course result in a rapid initial annihilation, so that the density of defects would be set by the annihilation process rather than by the critical dynamics. We have seen

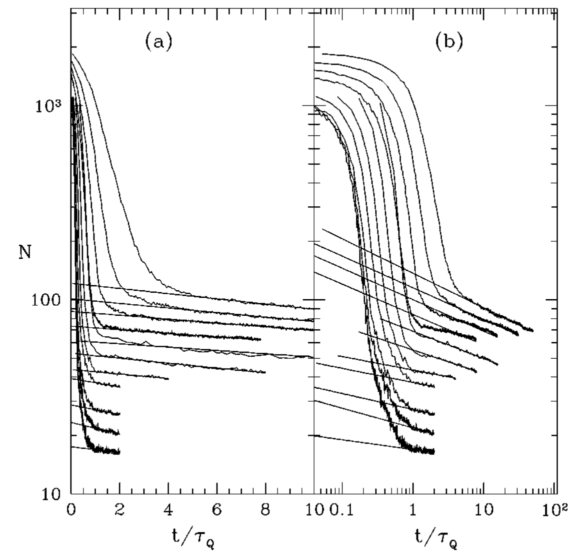

FIG. 3. Number of kinks N as a function of time for a fixed viscosity $\eta = 1$ but different quench times cales τ_Q. As with Fig. 1, (a) corresponds to an exponential fit and (b) to a power-law fit.

evidence for this behavior for sufficiently small τ_Q in Ref. [14].

Equations (6) and (10) for the initial density of topological defects can be used in the cosmological setting. Phase transitions are likely to occur in the radiation dominated era, when the temperature T of the plasma and the Hubble time t_H since the big bang are tied with the equation $T^2 t_H =$ constant. This immediately yields quench time scales $\tau_Q = 2t_H = H^{-1}$, where H is the Hubble parameter.

Damping rate is the other important parameter set by cosmology. In the radiation-dominated epoch $\eta = 3H + \gamma$,

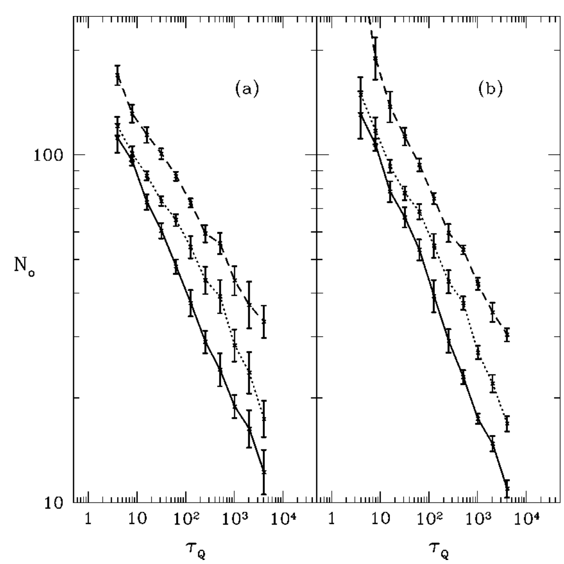

FIG. 4. Dependence of the initial number of kinks N_o on the quench time scale τ_Q for values of the damping rate $\eta = 5$ (top), 1 (middle) and 1/5 (bottom). Case (a) is obtained from an exponential fitting to the decay of number of kinks and (b) from a power-law fit. Fittings to $N_o \propto \tau_Q^{-g}$ yield (from top to bottom) $g = (0.23 \pm 0.010,\ 0.26 \pm 0.011,\ 0.33 \pm 0.011)$ in case (a) and $g = (0.28 \pm 0.010,\ 0.30 \pm 0.011,\ 0.36 \pm 0.010)$ in case (b).

where $3H$ is the effective viscosity caused by the Hubble expansion, while γ is damping due to the coupling with the other degrees of freedom. Early on, the ''Hubble viscosity'' may even dominate.

The nature of the critical dynamics in the immediate vicinity of the phase transition is decided by the inequality (11), which now reads

$$\left(\frac{3H+\gamma}{m}\right)^3 > \frac{H}{m}. \tag{12}$$

In the overdamped case

$$\hat{t} \simeq \tau_o (3+\gamma/H)^{1/2} \tag{13}$$

$$\hat{\epsilon} \simeq \tau_o H (3+\gamma/H)^{1/2}. \tag{14}$$

This in turn leads to

$$\hat{\xi} \simeq (\xi_o H^{-1})^{1/2} (3+\gamma/H)^{-1/4}, \tag{15}$$

where we have set $c=1$, so $\xi_o = m^{-1}$. Thus the density of topological defects is principally set by the geometric average of the characteristic length scale of the order parameter (ξ_o) and the size of the horizon (H^{-1}). For small γ/H, damping is dominated by Hubble expansion. In that regime, $H/m > 1$ (or $H^{-1} < \xi_o$) would be required for the critical dynamics to be overdamped. This would lead to $\hat{\xi} \gtrsim H^{-1}$, which in effect implies that in this case the density of defects is set by the size of the horizon.

When the critical dynamics is underdamped,

$$\hat{t} \simeq \tau_o (\tau_o H)^{-1/3}, \tag{16}$$

$$\hat{\epsilon} \simeq (\tau_o H)^{2/3}. \tag{17}$$

This immediately leads to

$$\hat{\xi} \simeq \xi_o (H^{-1}/\xi_o)^{1/3} \tag{18}$$

The characteristic distance $\hat{\xi}$ is then bigger than the healing length ξ_o by the third root of the size of the horizon at the time of the transition measured in the units of ξ_o.

We conclude by noting that the dependence of the number of kinks on the viscosity parameter corroborates anticipated existence of the two regimes in the critical dynamics, each with a distinct scaling of the relevant characteristic time scale with the relative temperature ϵ. Overdamped regime produces kinks with separations $\hat{\xi} \propto (\tau_Q/\eta)^{1/4}$, while in the underdamped case $\hat{\xi} \propto \tau_Q^{1/3}$ and is independent of viscosity. The subsequent annihilation rate of the kinks strongly depends on viscosity, and is much more rapid in the underdamped case. The location of the borderline between the two is consistent with the considerations of [2,16].

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[13] The order parameter φ will, in general, have internal symmetries which will influence its dynamics (i.e., by deciding the existence and character of the topological defects). It may also be coupled to a gauge field, which will influence its dynamics (although it is thought to be subdominant in the phase-ordering process following the quench [1]).
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