Existence of a confinement phase in quantum electrodynamics

Kei-Ichi Kondo*
Department of Physics, Faculty of Science, Chiba University, Chiba 263, Japan
(Received 17 March 1998; published 16 September 1998)

We show that four-dimensional U(1) gauge theory in the continuum formulation has a confining phase (exhibiting the area law of the Wilson loop) in the strong coupling region above a critical coupling g_c. This result is obtained by taking into account topological nontrivial sectors in U(1) gauge theory. The derivation is based on the reformulation of gauge theory as a deformation of topological quantum field theory and a subsequent dimensional reduction of the D-dimensional topological quantum field theory to the (D-2)-dimensional nonlinear σ model. The topological quantum field theory part of four-dimensional U(1) gauge theory is exactly equivalent to the two-dimensional O(2) nonlinear σ model. The confining (r. Coulomb) phase of U(1) gauge theory corresponds to the high- (r. low-) temperature phase of the O(2) nonlinear σ model and the critical point g_c is determined by the Berezinskii-Kosterlitz-Thouless phase transition temperature. The quark (charge) confinement in the strong coupling phase is caused by vortex condensation. Thus the continuum gauge theory has direct correspondence to the compact formulation of lattice gauge theory.
[S0556-2821(98)05718-X]

PACS number(s): 12.38.Aw, 12.38.Lg

I. INTRODUCTION

In this paper we study the phase structure of the *continuum* Abelian U(1) gauge theory by including the effect due to the compactness of the U(1) group. The reason for taking compactness into account is as follows. From the viewpoint of unified field theory, the Abelian group should be embedded as a subgroup in the larger non-Abelian gauge group. In view of this, the Abelian group should be compact. Another import aspect of the compactness of the Abelian gauge group stems from the possibility of explaining the quantization of charge [1]. In noncompact QED there is no reason for charge quantization.

In this paper we show that four-dimensional U(1) gauge theory has a confinement phase in the strong coupling region $g>g_c$ due to the compactness (periodicity) leading to a nontrivial topological configuration. If we neglect the periodicity, we have a free U(1) gauge theory which has only one phase, the Coulomb phase, as expected. This work confirms the claim made by Polyakov [2,3]. However, the claim that the Abelian gauge theory has a confinement phase sounds strange from the conventional wisdom based on the continuum Abelian gauge theory. We clarify the meaning of this statement in what follows.

More than twenty years ago, it was pointed out by many authors that four-dimensional SU(2) non-Abelian gauge theory bears many similarities with a two-dimensional O(3) nonlinear σ model (NLSM). Both theories possess asymptotic freedom, a multiinstanton (and antiinstanton) solution, dynamical mass generation and scale invariance (i.e., no intrinsic scale parameter), see Ref. [4].

These similarities can be seen also in the lattice regularized versions of these models, between spin models and lattice gauge theories [5]. Naively the scaling limit of the classical O(3) Heisenberg model is the O(3) NLSM, whereas that of SU(2) lattice gauge theory is the SU(2) gauge theory. One can take the scaling limit of lattice theory at a second order phase transition point. Hence the scaling limit is taken by approaching the critical point $T\to T_c$ (or $g\to g_c$) as the lattice spacing a goes to zero, $a\to 0$, in such a way that the physical quantities remain finite.

In the two-dimensional classical O(3) Heisenberg model, the two-point correlation function decays exponentially at any finite temperature. This corresponds to the claim in four-dimensional lattice SU(2) gauge theory that the confinement phase survives as long as the coupling constant g is positive, even if $g\ll 1$. Both models have a phase transition at $T=0$ $(g=0)$, i.e., $T_c=0$ $(g_c=0)$ which is believed to be second order.

In a previous paper [4], it has been shown that these similarities between two models are not merely an accident; actually we have proved the exact equivalence between the (D-2)-dimensional O(3) NLSM and the D-dimensional topological quantum field theory (TQFT) obtained by removing the perturbative deformation (topological trivial sector) from D-dimensional SU(2) non-Abelian gauge theory $(D\geqslant 3)$. This proof is based on the idea of the dimensional reduction of Parisi and Sourlas [6]. The case of $D=4$ is the most interesting case of physical reality.

What can we say in the Abelian case? For this, recall the fact that the two-dimensional O(2) NLSM or XY model undergoes a phase transition without the appearance of spontaneous magnetization. This absence of an order parameter in two dimensions is consistent with the Coleman-Mermin-Wagner (CMW) theorem [7]. The low-temperature phase $(T<T_c)$ contains massless spin waves. On the other hand, the high-temperature phase $(T>T_c)$ is completely disordered. For this phase transition, the periodicity of the angular variable φ is quite essential. The model has topological singularities, called vortices. These vortices condense at high temperature and disorder the correlation function [8]. This phase transition is called the Berezinskii-Kosterlitz-Thouless (BKT) transition [8]. The vortex part is equivalent to the

*Email address: kondo@cuphd.nd.chiba-u.ac.jp

neutral Coulomb gas and sine-Gordon theory, see Refs. [9–11]. Although the existence of the BKT transition is rather subtle, it was rigorously proved by Fröhlich and Spencer [12].

In lattice formulation, it is well known that all of these properties in two-dimensional Abelian spin models have correspondences in the Abelian gauge theory in four dimensions. The vortices in two dimensions are closely related to magnetic monopoles in four dimensions, see Refs. [13–15]. The condensation of closed loops of magnetic monopole currents leads to quark (charge) confinement in the strongly coupled phase of Abelian gauge theory, since electric flux cannot easily penetrate such a medium (which we call the dual Meissner effect). It is worth remarking that the dual superconductor vacuum of quantum chromodynamics (QCD) has been derived recently without any *ad hoc* assumption [16] from QCD in the continuum formulation.

In lattice gauge theory, charge confinement in the sense of area decay of the Wilson loop is derived in the strong coupling region by using the strong coupling expansion [5,17]. Quite remarkably, the quark (charge) confinement in lattice gauge theory occurs irrespective of the details of the gauge group, as long as it is compact (discrete [18,19] or continuous), even for the Abelian gauge group. However, one expects that U(1) lattice gauge theory in four dimensions $[\mathrm{U}(1)_4]$ has a Coulomb phase in the weak coupling region, which was proved rigorously by Guth [20] and Fröhlich and Spencer [21]. Therefore U(1) lattice gauge theory undergoes a phase transition at a finite nonzero coupling g_c. In continuum gauge theory, such a nontrivial phase structure was suggested to occur due to topological nontrivial configurations by Polyakov [2,1]. Actually he has shown the confinement phase in three-dimensional U(1) gauge theory for arbitrary gauge coupling, in agreement with the lattice analysis. In four dimensions, he claimed that the weak coupling U(1) gauge theory does not confine. Accordingly, it is expected that U(1) gauge theory in four dimensions has two phases, confinement and deconfinement (Coulomb) phases, whereas only one phase, i.e., the confinement phase, exists in three dimensions.

In this paper, we show that continuum four-dimensional U(1) gauge theory has two phases, a (strong coupling) confinement phase and a (weak coupling) Coulomb phase, which have direct correspondence with the high-temperature and low-temperature phases in O(2) NLSM, respectively. The phase transition point corresponds to the BKT transition in the XY model. Therefore, the phase transition point g_c is determined by the BKT transition temperature T_c. This is one of the main results of this paper. This result is obtained as a specific case of the previous paper [4].

Therefore, in the strong coupling phase ($g>g_c$), continuum U(1) gauge theory confines quarks and the gauge field becomes massive, in agreement with the result of lattice gauge theory. In the weak coupling phase, on the other hand, quarks are liberated and the gauge field remains massless. In the weak coupling phase ($g<g_c$), the β function of the renormalization group [22] is identically zero and $0<g<g_c$ is the line of fixed points, if we neglect the perturbative deformations. For a review of lattice gauge theory, see, e.g,. Refs. [23–25].

The plan of this paper is as follows. In Sec. II, we give the reformulation of Abelian gauge theory as a deformation of a topological quantum field theory. In Sec. III, using the reformulation of Sec. II, we evaluate the Wilson loop expectation value in four-dimensional Abelian gauge theory. In the final section we discuss the renormalization group properties and the extension of our scheme to other dimensional cases. More details about the interplay between the Abelian and non-Abelian cases are given in a forthcoming paper.

II. ABELIAN GAUGE THEORY AS A DEFORMATION OF TQFT AND DIMENSIONAL REDUCTION

Now we reformulate quantum electrodynamics (QED) as a deformation of topological quantum field theory. It is obtained as a special case of the non-Abelian gauge theory given in a previous paper [4].

A. Decomposition into perturbative and topological nontrivial sectors

QED on the D-dimensional space-time is defined by the action

$$S_{\mathrm{QED}}^{\mathrm{tot}}=\int d^Dx(\mathcal{L}_{\mathrm{QED}}[a_\mu,\psi]+\mathcal{L}_{\mathrm{GF}}), \tag{2.1}$$

$$\mathcal{L}_{\mathrm{QED}}[a_\mu,\psi]:=-\frac{1}{4}f_{\mu\nu}f_{\mu\nu}+\bar{\psi}(i\gamma^\mu D_\mu[a]-m)\psi, \tag{2.2}$$

where

$$f_{\mu\nu}(x):=\partial_\mu a_\nu(x)-\partial_\nu a_\mu(x), \tag{2.3}$$

$$D_\mu[a]:=\partial_\mu-iga_\mu. \tag{2.4}$$

The gauge transformation of the U(1) gauge field $a_\mu(x)$ and the fermion field ψ is defined by

$$a_\mu(x)\to a_\mu^U(x):=a_\mu(x)+\frac{i}{g}U(x)\partial_\mu U^\dagger(x),\quad U(x)\in U(1), \tag{2.5}$$

$$\psi(x)\to\psi^U(x):=U(x)\psi(x). \tag{2.6}$$

The gauge-fixing term $\mathcal{L}_{\mathrm{GF}}$ is given by

$$\mathcal{L}_{\mathrm{GF}}:=-i\delta_B G_{\mathrm{gf}}[a_\mu,C,\bar{C},\phi], \tag{2.7}$$

using the nilpotent Becchi-Rouet-Stora-Tyupin (BRST) transformation δ_B,

$$\delta_B a_\mu(x)=\partial_\mu C(x),$$

$$\delta_B C(x)=0,$$

$$\delta_B \bar{C}(x)=i\phi(x),$$

$$\delta_B \phi(x)=0,$$

$$\delta_B \psi(x)=igC(x)\psi(x),$$

$$\delta_B \bar{\psi}(x)=-igC(x)\bar{\psi}(x), \tag{2.8}$$

where ϕ is the Lagrange multiplier field.

The partition function of QED with the source term

$$S_J[a_\mu,C,\bar{C},\phi,\psi,\bar{\psi}] := \int d^D x (J^\mu a_\mu + J_c C + J_{\bar{c}} \bar{C} + J_\phi \phi + \bar{\eta}\psi + \eta\bar{\psi}) \tag{2.9}$$

is given by

$$Z_{\rm QED}[J] := \int [da_\mu][dC][d\bar{C}][d\phi][d\psi][d\bar{\psi}] \times \exp\{iS_{\rm QED}^{\rm tot} + iS_J\}. \tag{2.10}$$

To reformulate QED as a deformation of topological quantum field theory according to Ref. [4], we first regard the U(1) gauge field a_μ and the fermion field ψ as the gauge transformation of the U(1) gauge fields v_μ and Ψ:

$$a_\mu(x) := v_\mu(x) + \omega_\mu(x), \quad \omega_\mu(x) := \frac{i}{g} U(x)\partial_\mu U^\dagger(x). \tag{2.11}$$

$$\psi(x) := U(x)\Psi(x). \tag{2.12}$$

Here[1] v_μ and Ψ are identified with the field variables in the perturbative (topological trivial) sector $(Q=0)$, whereas ω_μ belongs to the topological nontrivial sector $(Q\neq 0)$.

Furthermore we introduce the new ghost field γ, antighost field $\bar{\gamma}$, and the Lagrange multiplier field β in the perturbative sector. They are subject to a new BRST transformation $\tilde{\delta}_B$:

$$\tilde{\delta}_B v_\mu(x) = \partial_\mu \gamma(x),$$

$$\tilde{\delta}_B \gamma(x) = 0,$$

$$\tilde{\delta}_B \bar{\gamma}(x) = i\beta(x),$$

$$\tilde{\delta}_B \beta(x) = 0,$$

$$\tilde{\delta}_B \Psi(x) = ig\gamma(x)\Psi(x),$$

$$\tilde{\delta}_B \bar{\Psi}(x) = -ig\gamma(x)\bar{\Psi}(x). \tag{2.13}$$

[1]The decomposition of a_μ, $a_\mu = v_\mu + \omega_\mu$, corresponds to the superposition of two independent configuration, $\varphi = \varphi_{\rm SW} + \varphi_V$ (spin waves and vortex parts) in the XY model.

Then the partition function of QED is rewritten as

$$Z_{\rm QED}[J] = \int [dU][dC][d\bar{C}][d\phi] \times \int [dv_\mu][d\gamma][d\bar{\gamma}][d\beta][d\Psi][d\bar{\Psi}] \times \exp\Big\{ i\int d^D x(-i\delta_B G_{\rm gf}[\omega_\mu + v_\mu, C, \bar{C}, \phi]) + i\int d^D x(\mathcal{L}_{\rm QED}[v,\Psi] - i\tilde{\delta}_B \tilde{G}_{\rm gf}[v_\mu,\gamma,\bar{\gamma},\beta]) + iS_J[\omega_\mu + v_\mu, C, \bar{C}, \phi, U\Psi, \bar{\Psi}U^\dagger]\Big\}, \tag{2.14}$$

where $\tilde{G}_{\rm gf}$ is a gauge-fixing functional for the perturbative (topological trivial) sector.

B. Gauge fixing

The Lorentz gauge is given by

$$F[a] := \partial_\mu a^\mu = 0. \tag{2.15}$$

The most familiar choice of $G_{\rm gf}$

$$G_{\rm gf} = \bar{C}\left(\partial_\mu a^\mu + \frac{\alpha}{2}\phi\right) \tag{2.16}$$

yields the familiar form of the gauge-fixing term

$$\mathcal{L}_{\rm GF} := -i\delta_B G_{\rm gf}[a_\mu, C, \bar{C}, \phi] = \phi\partial_\mu a^\mu + i\bar{C}\partial^\mu\partial_\mu C + \frac{\alpha}{2}\phi^2. \tag{2.17}$$

In this paper we propose to use the choice

$$G_{\rm gf}^{U(1)} = -\bar{\delta}_B\left(\frac{1}{2}a_\mu^2 + iC\bar{C}\right), \tag{2.18}$$

where $\bar{\delta}_B$ is the anti-BRST transformation [4],

$$\bar{\delta}_B a_\mu(x) = \partial_\mu \bar{C}(x),$$

$$\bar{\delta}_B C(x) = i\bar{\phi}(x),$$

$$\bar{\delta}_B \bar{C}(x) = 0,$$

$$\bar{\delta}_B \bar{\phi}(x) = 0,$$

$$\bar{\delta}_B \psi(x) = \bar{C}(x)\psi(x),$$

$$\phi(x) + \bar{\phi}(x) = 0, \tag{2.19}$$

where $\bar{\phi}$ is defined in the last equation.

Apart from a total derivative term, this choice yields

$$\mathcal{L}_{\rm GF} = i\delta_B \bar{\delta}_B \left(\frac{1}{2} a_\mu^2 + iC\bar{C} \right) = -i\delta_B [\bar{C}(\partial_\mu a^\mu - \phi)]. \tag{2.20}$$

Therefore, the choice (2.18) corresponds in Eq. (2.17) to the choice of the gauge-fixing parameter

$$\alpha = -2, \tag{2.21}$$

which has appeared also in the non-Abelian case [4]. The above choice for $G_{\rm gf}^{\rm U(1)}$ yields the decomposition

$$\mathcal{L}_{\rm GF} = -i\delta_B G_{\rm gf}^{\rm U(1)}[\omega_\mu + v_\mu, C, \bar{C}, \phi] \tag{2.22}$$

$$= i\delta_B \bar{\delta}_B \left(\frac{1}{2}(\omega_\mu + v_\mu)^2 + iC\bar{C} \right)$$

$$\mathcal{L}_{\rm TQFT} = \mathcal{L}_{\rm TQFT} + iv_\mu \delta_B \bar{\delta}_B \omega_\mu, \tag{2.23}$$

where we have defined

$$\mathcal{L}_{\rm TQFT} := i\delta_B \bar{\delta}_B \left(\frac{1}{2}\omega_\mu^2 + iC\bar{C} \right). \tag{2.24}$$

Here we have used that the action of δ_B is trivial in the perturbative sector,

$$\delta_B v_\mu = 0 = \bar{\delta}_B v_\mu, \tag{2.25}$$

while

$$\delta_B \omega_\mu = \partial_\mu C, \quad \bar{\delta}_B \omega_\mu = \partial_\mu \bar{C}. \tag{2.26}$$

C. Deformation of topological quantum field theory

Finally, the partition function of QED is cast into the form

$$Z_{\rm QED}[J] := \int [dU][dC][d\bar{C}][d\phi] \times \exp\Big\{ iS_{\rm TQFT}[\omega_\mu, C, \bar{C}, \phi] + i\int d^Dx [J^\mu \omega_\mu + J_c C + J_{\bar{c}} \bar{C} + J_\phi \phi] + iW[U; J^\mu, \bar{\eta}, \eta] \Big\}, \tag{2.27}$$

where $W[U;J^\mu,\bar{\eta},\eta]$ is the generating functional of QED in the perturbative sector (PQED) given by

$$e^{iW[U;J^\mu,\bar{\eta},\eta]} := \int [dv_\mu][d\gamma][d\bar{\gamma}][d\beta][d\Psi][d\bar{\Psi}] \times \exp\Big\{ iS_{\rm PQED}[v,\Psi,\gamma,\bar{\gamma},\beta] + i\int d^Dx [v_\mu \mathcal{J}_\mu + \bar{\eta} U \Psi + \eta \bar{\Psi} U^\dagger] \Big\}, \tag{2.28}$$

$$S_{\rm PQED}[v,\Psi,\gamma,\bar{\gamma},\beta] := \int d^Dx [\mathcal{L}_{\rm QED}[v,\Psi] - i\tilde{\delta}_B \tilde{G}_{\rm gf}(v_\mu,\gamma,\bar{\gamma},\beta)], \tag{2.29}$$

$$\mathcal{J}_\mu := J_\mu + i\delta_B \bar{\delta}_B \omega_\mu. \tag{2.30}$$

The correlation functions of the original (fundamental) field $a_\mu, \psi, \bar{\psi}$ are obtained by differentiating $Z_{\rm QED}[J]$ with respect to the corresponding source $J_\mu, \bar{\eta}, \eta$.

All the field configurations are classified according to the integer-valued topological charge Q which is specified later. The above reformulation of gauge theory is the decomposition of the original theory into the topological trivial sector with $Q=0$ and topological nontrivial sector with $Q\neq 0$. This corresponds to the decomposition of the XY model into a spin wave part ($Q=0$) and a vortex part ($Q\neq 0$), where Q is given by the winding number of the vortex solution. However, the XY model is not a gauge theory and does not have any local gauge invariance.

The integration over the fields $(U,C,\bar{C},\phi)$ in TQFT should be treated nonperturbatively by taking into account the topological nontrivial configurations. The deformation $W[U;J^\mu,\bar{\eta},\eta]$ from the TQFT should be calculated according to the ordinary perturbation theory in the coupling constant g. The perturbative expansion around the TQFT means the integration over the new fields $(v_\mu,\gamma,\bar{\gamma},\beta)$ based on the perturbative expansion in powers of the coupling constant g.

D. Dimensional reduction to O(2) NLSM

Following the argument given in Ref. [4] based on the Parisi-Sourlas dimensional reduction, it turns out that the D-dimensional TQFT [as the topological nontrivial sector of D-dimensional U(1) Abelian gauge theory] with an action

$$S_{\rm TQFT}[\omega_\mu, C, \bar{C}, \phi] = \int d^Dx\, i\delta_B \bar{\delta}_B \left(\frac{1}{2}\omega_\mu(x)\omega_\mu(x) + iC(x)\bar{C}(x) \right) \tag{2.31}$$

is equivalent to the (D-2)-dimensional O(2) NLSM with the action

$$S_{\rm O(2)NLSM}[U] := 2\pi \int d^{D-2}z \frac{1}{2}\omega_\mu(z)\omega_\mu(z), \quad \left(\omega_\mu(z) := \frac{i}{g} U(z)\partial_\mu U^\dagger(z) \right), \tag{2.32}$$

$$= \int d^{D-2}z \frac{\pi}{g^2} \partial_\mu U(z) \partial_\mu U^\dagger(z). \tag{2.33}$$

Dimensional reduction is due to a fact that the action (2.31) has a hidden supersymmetry and can be rewritten in the OSp(D/2) symmetric form in the superspace formulation, see Ref. [4].

III. QUARK CONFINEMENT IN ABELIAN GAUGE THEORY

Now we calculate the Wilson loop expectation in U(1) gauge theory based on the reformulation given in the previous section. In what follows we move to the Euclidean formulation.

A. Dimensional reduction of the Wilson loop

We define the Wilson loop operator for the closed loop C by

$$W_C[a]=\exp\left(iq \oint_C a_\mu(x)dx^\mu \right), \tag{3.1}$$

where q is a test charge. In Abelian gauge theory, the Wilson loop factorizes,

$$W_C[a]=\exp\left(iq \oint_C \omega_\mu(x)dx^\mu \right)\exp\left(iq \oint_C v_\mu(x)dx^\mu \right) =: W_C[\omega]W_C[v]. \tag{3.2}$$

For the $Q=0$ sector, we choose the gauge-fixing function

$$\tilde{G}_{\rm gf}=\bar{\gamma}\left(\partial_\mu v^\mu + \frac{\xi}{2}\beta \right), \tag{3.3}$$

with a gauge-fixing parameter ξ. For U(1) gauge theory with the action (omitting matter fields)

$$S_{pU(1)}[v,\gamma,\bar{\gamma},\beta] := \frac{1}{4g^2}(\partial_\mu v_\nu - \partial_\mu v_\nu)^2 - i\tilde{\delta}_B \tilde{G}_{\rm gf}, \tag{3.4}$$

the perturbative part is given by

$$e^{iW[\omega,J,0,0]} = \int [dv_\mu][d\gamma][d\bar{\gamma}][d\beta] \times \exp\left\{ -S_{pU(1)}[v,\gamma,\bar{\gamma},\beta] + i\int d^Dx v_\mu \mathcal{J}_\mu \right\}. \tag{3.5}$$

Integrating out the fields $\gamma,\bar{\gamma},\beta$ yields

$$e^{iW[\omega,J,0,0]} = \int [dv_\mu]\exp\left\{ -S[v] - \int d^Dx\, i v_\mu(x) \times [J_\mu(x) + i\delta_B\bar{\delta}_B\omega_\mu(x)] \right\}, \tag{3.6}$$

where

$$S[v] := \int d^Dx \left[\frac{1}{4g^2}(\partial_\mu v_\nu - \partial_\mu v_\nu)^2 + \frac{1}{2\xi}(\partial_\mu v^\mu)^2 \right]. \tag{3.7}$$

For the calculation of the Wilson loop expectation, J^μ is taken to be the current along the closed loop C such that

$$\int d^Dx v_\mu(x) J^\mu(x) = q \oint v_\mu(x) dx^\mu. \tag{3.8}$$

It is easy to see that

$$e^{iW[\omega,J,0,0]} = \langle W_C[v] e^{(v_\mu,\delta_B\bar{\delta}_B\omega_\mu)} \rangle_{p\rm U(1)} \times \int [dv_\mu] e^{-S[v]+(v_\mu,\delta_B\bar{\delta}_B\omega_\mu)}, \tag{3.9}$$

where

$$(v_\mu,\delta_B\bar{\delta}_B\omega_\mu) := \int d^Dx v_\mu(x)\delta_B\bar{\delta}_B\omega_\mu(x). \tag{3.10}$$

The Wilson loop expectation is rewritten as

$$\langle W_C[a]\rangle_{\rm U(1)} = \frac{\langle W_C[\omega] e^{iW[\omega,J,0,0]}\rangle_{\rm TQFT}}{\langle e^{iW[\omega,0,0,0]}\rangle_{\rm TQFT}}, \tag{3.11}$$

where

$$e^{iW[\omega,0,0,0]} = \langle e^{(v_\mu,\delta_B\bar{\delta}_B\omega_\mu)}\rangle_{p\rm U(1)} \int [dv_\mu] e^{-S[v]+(v_\mu,\delta_B\bar{\delta}_B\omega_\mu)}. \tag{3.12}$$

Expanding the exponential

$$e^{(v_\mu,\delta_B\bar{\delta}_B\omega_\mu)} = e^{\int d^Dx \delta_B\bar{\delta}_B(v_\mu\omega_\mu)(x)} = e^{\int d^Dx \{Q_B, \bar{\delta}_B(v_\mu\omega_\mu)(x)\}} \tag{3.13}$$

into a power series and using a fact that the vacuum of TQFT obeys

$$Q_B|0\rangle_{\rm TQFT} = 0, \tag{3.14}$$

we find that this term does not contribute to the expectation value (3.11). Therefore, the Wilson loop expectation is completely separated into the topological part ($Q \neq 0$) and the perturbative part ($Q=0$),

$$\langle W_C[a]\rangle_{U(1)} = \langle W_C[\omega]\langle W_C[v]\rangle_{p\rm U(1)}\rangle_{\rm TQFT} = \langle W_C[\omega]\rangle_{\rm TQFT}\langle W_C[v]\rangle_{p\rm U(1)}. \tag{3.15}$$

This corresponds to the result Eq. (10) of Polyakov [2]. This property does not hold in the non-Abelian case, which makes the systematic calculation rather difficult.

In order to use the dimensional reduction for calculating the topological part, we choose the loop C so that C is contained in the $(D-2)$-dimensional space. For $D=4$, the loop C must be planar. Then the dimensional reduction of the topological part leads to the equivalence of the Wilson loop expectation value between the D-dimensional U(1) TQFT and (D-2)-dimensional O(2) NLSM,

$$\langle W_C[\omega(z)]\rangle_{\mathrm{TQFT}_D}=\langle W_C[\omega(z)]\rangle_{\mathrm{O(2)NLSM}_{D-2}}, \quad z\in R^{D-2}. \tag{3.16}$$

if the Wilson loop has its support on the (D-2)-dimensional space on which the NLSM is defined. Hence, the calculation of the Wilson loop in the four-dimensional U(1) gauge theory is reduced to those in the two-dimensional O(2) NLSM and the four-dimensional perturbative U(1) gauge theory

$$\langle W_C[a]\rangle_{U(1)_D}=\langle W_C[\omega]\rangle_{\mathrm{O(2)NLSM}_{D-2}}\langle W_C[v]\rangle_{p\mathrm{U(1)}_D}. \tag{3.17}$$

For large rectangular loop with sides R,T, the static potential is obtained by

$$V(R)=\lim_{T\to\infty}\frac{-1}{T}\ln\langle W_C[a]\rangle. \tag{3.18}$$

In four dimensions $(D=4)$, it is well known [23] that for large Wilson loop $\langle W_C[v]\rangle_{p\mathrm{U(1)}_4}$ gives the Coulomb potential,

$$V(R)=-\frac{g^2}{4\pi}\frac{1}{R}+\mathrm{const.} \tag{3.19}$$

For a derivation, see, e.g., the Appendix of Ref. [26].

In the following, we show that $\langle W_C[\omega]\rangle_{\mathrm{O(2)NLSM}_2}$ exhibits area law for strong coupling $g>g_c$ with a finite and nonzero value of a critical point g_c. This confinement-deconfinement transition corresponds exactly to the BKT transition.

B. O(2) NLSM and Wilson loop

Defining the angle variable $\varphi(z)$ for $U(z)\in U(1)$,

$$U(z)=e^{i\varphi(z)}, \tag{3.20}$$

we obtain

$$\omega_\mu(z)=\frac{i}{g}U(z)\partial_\mu U^\dagger(z)=\frac{1}{g}\partial_\mu\varphi(z). \tag{3.21}$$

Then the action of O(2) NLSM reads

$$S_{\mathrm{O(2)}}[U]=\int d^{D-2}z\,\pi\omega_\mu(z)\omega_\mu(z) =\frac{\beta}{2}\int d^{D-2}z\,\partial_\mu\varphi(z)\partial_\mu\varphi(z), \quad \beta:=\frac{2\pi}{g^2}. \tag{3.22}$$

The partition function is defined by

$$Z_{\mathrm{O(2)}}[U]=\int[dU]\exp(-S_{\mathrm{O(2)}}[U]), \quad [dU]=\prod_x\frac{d\varphi(x)}{2\pi}, \tag{3.23}$$

using the Haar measure dU on U(1). For the notation of the two-dimensional vector,

$$\mathbf{S}(x)=[\cos\varphi(x),\sin\varphi(x)], \tag{3.24}$$

the action reads

$$S_{\mathrm{O(2)}}[U]=\frac{\beta}{2}\int d^{D-2}z\,\partial_\mu\mathbf{S}(x)\cdot\partial_\mu\mathbf{S}(x). \tag{3.25}$$

Hence the O(2) NLSM is regarded as a continuum version of classical planar spin model. In the following, we identify β with the inverse temperature $1/T$. Hence the high- (low-) temperature of the spin model corresponds to strong (weak) coupling of the gauge theory.

It should be remarked that O(2) NLSM with the action (3.22) is not a free scalar field theory, since this theory is periodic in the angle variable φ (modulo 2π). Of course, if we neglect this periodicity and treat the variable φ as a noncompact variable $\varphi(x)\in(-\infty,+\infty)$, we have a trivial theory, i.e., free massless scalar field theory. In this case, the Contour integral is zero,

$$\oint_C\omega_\mu(z)dz^\mu=\frac{1}{g}\oint_C\partial_\mu\varphi(z)dz^\mu=0. \tag{3.26}$$

Hence, the Wilson loop $W_c[\omega]$ is trivial, $W_c[\omega]\equiv1$, and hence the total static quark potential comes from the the Wilson loop expectation $W_c[v]$ of perturbative U(1) gauge theory and is equal to the Coulomb potential. Thus we obtain the trivial result that the four-dimensional noncompact Abelian gauge theory fails to confine quarks (charges). However, the periodicity (or compactness) leads to topological nontrivial solutions which are seeds for confinement, as shown below.

In the following, we restrict our consideration to the $d:=D-2=2$ case. The extremum of the classical action (3.22) is obtained as a solution of the classical field equation

$$\nabla^2\varphi=0 \pmod{2\pi}. \tag{3.27}$$

The harmonic function φ is constant or has singularities. We require φ be constant at infinity and assume only isolated singularities around which φ varies by $\pm2\pi$ as one turns anticlockwise. Then the solution[2] is written as

$$\varphi(z)=\sum_i Q_i\arctan\frac{(z-z_i)_2}{(z-z_i)_1}=\sum_i Q_i\mathrm{Im}\ln(z-z_i), \quad z:=x_1+ix_2. \tag{3.28}$$

This denotes a sum of vortex excitations located at points $x_i\in R^2$ and of vorticity Q_i (integers). The solution has the alternative form

[2]The two-dimensional Laplace equation is equivalent to the Cauchy-Riemann equation. Hence the solution is given by the holomorphic function. If one avoids branch cuts, it is a meromorphic function.

$$\varphi(z)=\ln\prod_i \frac{(z-z_i^+)/|z-z_i^+|}{(z-z_i^-)/|z-z_i^-|}$$

$$=\sum_i \left[\ln\frac{(z-z_i^+)}{|z-z_i^+|}-\ln\frac{(z-z_i^-)}{|z-z_i^-|}\right]. \qquad (3.29)$$

This means that vortices of intensity ± 1 are centered at the points $z_i^\pm$ and that intensities of higher magnitude are obtained when several z_i^+(or z_i^-) coincide. Note that the angle φ is a multivalued function, but $e^{i\varphi}$ is well defined everywhere, except at the singular points.

The contribution of the solution (3.28) to ω_μ is

$$\omega_\mu(z)=\frac{1}{g}\partial_\mu\varphi(z)=\frac{1}{g}\sum_i Q_i\epsilon_{\mu\nu}\frac{(z-z_i)_\nu}{(z-z_i)^2}$$

$$=\frac{1}{g}\sum_i Q_i\epsilon_{\mu\nu}\partial_\nu \ln|z-z_i|. \qquad (3.30)$$

Therefore, the integral of the one-form $\omega:=\omega_\mu dx^\mu$ along the closed loop C is

$$\oint_C \omega_\mu dx^\mu=\oint_C \omega=\frac{1}{g}\sum_i \int_0^{2\pi} d\Theta_i=\sum_i \frac{2\pi}{g}Q_i, \qquad (3.31)$$

where the sum runs over all the vortices inside the closed loop C and Θ_i is an angle around $z=z_i$,

$$\Theta_i(z):=\arctan\frac{(z-z_i)_2}{(z-z_i)_1}. \qquad (3.32)$$

Note that ω is a closed form, $d\omega=0$, but it is not an exact form, that is, a function (zero form) does not exist such that $\omega=df$ with f being defined everywhere in $R^2-\{0\}$. Domain of $f(z)=\Theta(z)$ is restricted to R^2-R_+, in other words, for one unit vortex at the origin

$$\omega_\mu=\epsilon_{\mu\nu}\frac{x_\nu}{x^2}-2\pi\theta(x_1)\delta(x_2)\delta_{\mu 2}. \qquad (3.33)$$

This is analogous to the case of the magnetic monopole in three dimensions where the magnetic field is given by

$$H_\mu=\frac{1}{2}\frac{x_\mu}{|x|^3}-2\pi\delta_{3\mu}\delta(x_1)\delta(x_2)\theta(x_3). \qquad (3.34)$$

The singular line of Eq. (3.33) in two dimensions does not contribute to the action, nor does the Dirac string (on the positive Z axis) in three dimensions.

In order to calculate the classical action for the singular configuration (3.28), we consider a disk of radius R_0, $D_i:=\{|z-z_i|<R_0;\ z\in R^2\}$ (centered on each singular point z_i) which is small with respect to the distances between vortices. Let $\mathcal{R}$ be the remaining domain of integration outside the vortices. The classical action consists of two parts, the self-energy (action) part of the vortices,

$$S^{(1)}=\frac{\beta}{2}\sum_i \int_{|z-z_i|<R_0} d^2z[\nabla\varphi(z)]^2, \qquad (3.35)$$

and the remaining part

$$S^{(2)}:=\frac{\beta}{2}\int_{\mathcal{R}} d^2z[\nabla\varphi(z)]^2$$

$$=-2\pi\beta\sum_{i\neq j} Q_iQ_j\ln|z_i-z_j|+\sum_i Q_i^2\pi\beta\ \ln 1/R_0. \qquad (3.36)$$

Summing over all vortex sectors leads to the partition function of the form

$$Z_C=\sum_{n=0}^{\infty}\frac{\zeta^n}{(n!)^2}\int\prod_{j=1}^{n} d^2z_j \times\exp\left[(2\pi)^2\beta\sum_{i,j} Q_iQ_j\Delta(z_i,z_j)\right],$$

$$\zeta:=e^{-S^{(1)}}, \qquad (3.37)$$

where ζ comes from the self-energy (action) part of vortices and Δ expresses the two-dimensional inverse Laplacian given by

$$\Delta(x,0)=\frac{1}{2\pi}\ln\frac{R}{|x|}. \qquad (3.38)$$

Therefore the partition function agrees with the two-dimensional neutral Coulomb gas (i.e., a gas of classical charged particles with a Coulomb interaction and globally neutral, $\Sigma_i Q_i=0$).

The transition temperature is estimated as follows. The contribution to the free energy from one vortex pair at distance r_{12} in a box of linear dimension L is

$$F\sim\ln\int_{|z_1-z_2|>R_0} d^2z_1d^2z_2\exp(-2\pi\beta\ \ln|z_1-z_2|/R_0)$$

$$\sim\ln[L^4\exp(-2\pi\beta\ \ln L/R_0)]$$

$$\sim(4-2\pi\beta)\ln L. \qquad (3.39)$$

The vortices always arise in pairs of opposite Coulomb charges to yield finite energy configurations and each pair forms an elementary dipole (Coulomb dipole gas). If $2\pi\beta>4$ (the low-temperature or weak coupling phase), the contribution from the vortex pair is negligible in the limit $L\to\infty$. In this phase, charges are bound and one has a dielectric medium. As $\beta\to\infty$, few vortices are present and their correlation decreases rapidly with the relative distance. This describes a dielectric medium of neutral bound states. In this regime, the correlation $\langle U(z)U(z')\rangle$ decays polynomially,

$$\langle U(z)U(z')\rangle=|z-z'|^{-(1/4\pi\beta)}. \qquad (3.40)$$

On the other hand, if $2\pi\beta<4$ (the high-temperature or strong coupling phase), an instability occurs and the creation of well-separated vortices is favored and disorder increases. In the high-temperature phase, one has a plasma of free charges. The vortex expectation decays exponentially yielding exponential decay of $\langle U(z)U(z')\rangle$,

$$\langle U(z)U(z')\rangle=|z-z'|^{-(1/4\pi\beta)}e^{-m(\beta)|z-z'|}. \quad (3.41)$$

Hence a naive estimate of the critical temperature is obtained:

$$\beta_c=\frac{2}{\pi}, \quad g_c^2=\pi^2. \quad (3.42)$$

This is the phase transition without the appearance of a spontaneous magnetization. The phase transition can be interpreted as a dipole condensation. The critical point separates the dissociated dipole phase from the condensed phase.

In order to calculate the Wilson loop, we use the equivalence of the Coulomb gas to the sine-Gordon model (see Appendix A for a proof). The sine-Gordon model is defined by the action and the partition function

$$S_{\rm sG}(\phi):=\alpha\int d^dx\left[\frac{1}{2}(\partial_\mu\phi(x))^2-h\cos\phi(x)\right], \quad (3.43)$$

$$Z_{\rm sG}(h):=\int[d\phi]\exp[-S_{sG}(\phi)]. \quad (3.44)$$

This is equivalent to the partition function of a globally neutral gas of particles of charges $Q_i=\pm1$ through a Coulomb potential in d dimensions:

$$Z_{C,\pm1}(h)=\sum_{n=0}^{\infty}\frac{z^{2n}}{(n!)^2}\prod_{i=1}^{n}\int d^dx_i d^dy_j \times\exp\left\{-\frac{1}{T}\left[\sum_{i<j}[V(|x_i-x_j|)+V(|y_i-y_j|)] -\sum_{i,j}V(|x_i-y_j|)\right]\right\} \quad (3.45)$$

at temperature T with the fugacity z,

$$\alpha=T=\frac{1}{4\pi^2\beta}=\frac{g^2}{8\pi^3}, \quad \alpha h=2z=2\zeta. \quad (3.46)$$

Note that $\alpha\beta=1/(4\pi^2)$. It is known that the transition point of the sine-Gordon model is

$$\alpha_c=\frac{1}{8\pi}, \quad (3.47)$$

which is in agreement with Eq. (3.42). The relation of φ and ϕ is given by

$$\phi(x,t)=\int_x^\infty dy\,\dot{\varphi}(y,t) \quad (3.48)$$

or

$$\partial_\mu\varphi=\epsilon_{\mu\nu}\partial^\nu\phi. \quad (3.49)$$

This implies that the fields φ and ϕ are dual variables. Therefore, O(2) NLSM is equivalent to the Coulomb gas and moreover it is equivalent to the sine-Gordon model when the charge of the Coulomb gas [or vorticity of O(2) NLSM] is restricted to $Q_i=\pm1$. Taking into account $Q_i>1$ will lead to the $\cos(Q\phi)$ term. The above consideration can be transferred into the lattice formulation, see Ref. [23].

C. Wilson loop and area decay

The Wilson loop expectation is calculated using the equivalent sine-Gordon model. The generating functional for the charge density ρ

$$\rho(x):=\sum_i Q_i\delta^{(2)}(x-x_i) \quad (3.50)$$

is obtained as

$$\frac{Z_{\rm sG}[\eta]}{Z_{\rm sG}(0)}=\langle e^{i\int d^2x\rho(x)\eta(x)}\rangle_{\rm sG}=\left\langle e^{i\sum_i Q_i\eta(x_i)}\right\rangle_{\rm sG}, \quad (3.51)$$

$$Z_{\rm sG}[\eta]=\int[d\phi]\exp\left\{-\alpha\int d^2x\left[\frac{1}{2}[\partial_\mu\phi(x)]^2 -h\cos[\phi(x)+\eta(x)]\right]\right\}. \quad (3.52)$$

In two dimensions, we can introduce the dual vector field

$$H_\mu=\epsilon_{\mu\nu}\omega_\nu. \quad (3.53)$$

Then the dual field is connected with the charge density ρ as follows:

$$\oint_C\omega_\mu(z)dz^\mu=\int_S\epsilon_{\mu\nu}\partial_\mu\omega_\nu(z)d^2z=\int\partial_\mu H_\mu(z)d^2z =\frac{2\pi}{g}\int\rho(z)d^2z. \quad (3.54)$$

Note that the rotation of ω_μ or the divergence of H_μ,

$$\epsilon_{\mu\nu}\partial_\mu\omega_\nu=\partial_\mu H_\mu=\frac{2\pi}{g}\rho, \quad (3.55)$$

measures the density of the topological charge. If we identify the right-hand side with the magnetic charge, this implies Dirac quantization condition

$$g_m=\frac{2\pi}{g}Q \quad (Q:\ \text{integer}). \quad (3.56)$$

The Wilson loop is calculated from the η given by

$$\eta(x)=\frac{q}{g}\oint_C dz^\mu\epsilon_{\mu\nu}\frac{(z-x)_\nu}{(z-x)^2}, \quad (3.57)$$

since

$$q\oint_C \omega_\mu dz^\mu = \int d^dx \rho(x)\eta(x) = \sum_i Q_i \eta(x_i). \quad (3.58)$$

Note that $\eta(x)=0$ if the argument x of $\eta(x)$ is outside the loop, while $\eta(x)=2\pi q/g$ if x is inside the loop.

In the high-temperature phase, the photon is massive, whereas the photon is massless in the low-temperature phase. This is because in the high-temperature phase, the random distribution of free vortices with long-range interaction spoils the correlation. Therefore, in the high-temperature phase, the Wilson loop expectation is estimated by the steepest descent as

$$\langle W_C[\omega]\rangle_{\rm sG} \cong \exp\left\{-\alpha\int d^2x\left[\frac{1}{2}\{\partial_\mu[\phi_{cl}(x)-\eta(x)]\}^2 - h\cos\phi_{cl}(x)\right]\right\}, \quad (3.59)$$

where ϕ_{cl} is determined by the Debye equation

$$\nabla^2[\phi_{cl}(x)-\eta(x)] = h\sin\phi_{cl}(x). \quad (3.60)$$

Corrections to this field due to fluctuation are exponentially small in the high-temperature phase. The loop C is placed in two-dimensional plane. We can perform the calculation in the same way as done by Polyakov [3].

Instead of repeating a similar calculation to Ref. [3], we use the Villain form [27]

$$e^{J\cos\phi} \to e^J \sum_{m\in Z} e^{-(J/2)(\phi-2\pi m)^2} \quad (3.61)$$

to estimate the Wilson loop expectation. Then the partition function is replaced with (apart from field-independent constants)

$$\begin{aligned} Z_{\rm sG}[\eta] &= \int [d\phi] e^{-\alpha\int d^dx\{1/2[\partial_\mu\phi(x)]^2\}} \prod_{x\in R^d} e^{\alpha h\cos[\phi(x)+\eta(x)]} \\ &= \int [d\phi] e^{-\alpha\int d^dx\{(1/2)[\partial_\mu\phi(x)]^2\}} \times \prod_{x\in R^d}\sum_{m(x)\in Z} e^{-(\alpha h/2)[\phi(x)+\eta(x)-2\pi m(x)]^2} \\ &= \sum_{\{m(x)\in Z; x\in R^d\}} \int [d\phi] e^{-\alpha\int d^dx(1/2)[-\phi(x)\partial^2\phi(x)]} \times e^{-(\alpha h/2)\int d^dx[\phi(x)+\eta(x)-2\pi m(x)]^2} \\ &= \sum_{\{m(x)\in Z; x\in R^d\}} e^{-(\alpha h/2)\int d^dx[\eta(x)-2\pi m(x)]^2} \int [d\phi] \times e^{-\int d^dx\{(\alpha/2)\phi(x)(-\partial^2+h)\phi(x)+\alpha h\phi(x)[\eta(x)-2\pi m(x)]\}} \\ &= \sum_{\{m(x)\in Z; x\in R^d\}} \exp\left[-\frac{\alpha h}{2}\int d^dx\{[\eta(x)-2\pi m(x)]^2 - h[\eta(x)-2\pi m(x)](-\partial^2+h)^{-1} \times [\eta(x)-2\pi m(x)]\}\right]. \end{aligned} \quad (3.62)$$

When $\eta=0$, the denominator is obtained,

$$Z_{\rm sG}[0] = \sum_{\{m(x)\in Z; x\in R^d\}} \exp\left[-(2\pi)^2\frac{\alpha h}{2}\int d^dx \times\{m(x)^2 - hm(x)(-\partial^2+h)^{-1}m(x)\}\right]. \quad (3.63)$$

Note that the field $\eta(x)$ has its support on $S(\partial S=C)$ and has the value $2\pi(q/g)$. If q is an integral multiple of g (the elementary charge), we have $\eta\in 2\pi Z$. This is absorbed by the shift of m. Therefore, in this case, charge confinement does not occur. This is interpreted as the charge screening.

A naive estimate of the ratio $Z_{\rm sG}[\eta]/Z_{\rm sG}[0]$ is given when $\eta\notin 2\pi Z$ in Appendix B. Finally we obtain the area decay of the Wilson loop expectation

$$\langle W_C[\omega]\rangle_{\rm sG} \cong e^{-\sigma A(C)}, \quad (3.64)$$

$$\sigma = \left(2\pi\frac{q}{g}\right)^2\frac{\alpha h}{2} = \left(2\pi\frac{q}{g}\right)^2\zeta \sim e^{-S^{(1)}}. \quad (3.65)$$

This implies the linear static potential

$$V(R) = \sigma R \quad (3.66)$$

between two fixed electric charges and an electric string with uniform energy density σ which is called the string tension. Therefore condensation of topological nontrivial configuration leads to quark confinement. From Eqs. (3.17), (3.19), and (3.66), the total static potential is given by

$$V(R) = \sigma R - \frac{g^2}{4\pi}\frac{1}{R} + \text{const.} \quad (3.67)$$

Even the continuum Abelian U(1) gauge theory has a confinement phase, exhibiting a rich phase structure similar to lattice compact U(1) gauge theory.

IV. DISCUSSION

In four-dimensional pure U(1) gauge theory, we have proved the existence of a strong coupling phase where the fractional electric charge is confined by the linear static po-

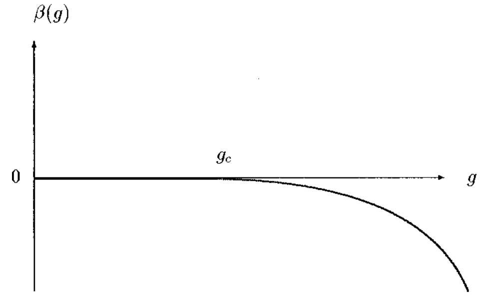

FIG. 1. Renormalization group beta function of U(1) gauge theory.

tential due to vortex condensation. In the following we discuss a few points of perspective.

A. Renormalization group and non-Gaussian fixed point

The O(N) NLSM has the renormalization group beta function

$$\beta(g) := \mu \frac{dg(\mu)}{d\mu} = -\frac{N-2}{8\pi^2} g^3 + O(g^5). \tag{4.1}$$

At low temperature $(0 < T < T_c)$, therefore, the O(2) NLSM has vanishing beta function $\beta \equiv 0$ $(T \ll 1)$ and $0 < T < T_c$ is the line of fixed points. This is consistent with the fact that at low temperature, the inverse correlation length or mass $m = \xi^{-1}$ of the O(2) NLSM or XY model vanishes, i.e., $m(T) \equiv 0$ for all $T < T_c$. The theory is conformal invariant. For high temperature $(T > T_c)$, the renormalization group study of the XY model shows that the mass behaves as

$$m(T) \sim \exp\left(-\frac{C}{\sqrt{T - T_c}} \right), \quad T \downarrow T_c, \tag{4.2}$$

with a constant C.

These results established in the two-dimensional model would be translated into the four-dimensional Abelian gauge theory, provided that the two theories have the same renormalization-group β function. We assume that in these two theories the mass $m(g)$ is generated by the dimensional transmutation in such a way that the beta function $\beta(g)$ is related to the mass $m(g)$ through the well-known relation

$$m(g) = \mu f(g) = \mu \exp\left(-\int^g \frac{dg}{\beta(g)} \right), \tag{4.3}$$

where $\beta(g)$ is defined by

$$\beta(g) := \mu \frac{dg(\mu)}{d\mu} = -\frac{f(g)}{f'(g)}. \tag{4.4}$$

In the weak coupling phase $(g < g_c)$, the gauge field is massless, $m(g) \equiv 0$, and the beta function of the renormalization group [22] is identically zero,

$$\beta(g) \equiv 0 \quad (0 < g < g_c). \tag{4.5}$$

Therefore, $0 < g < g_c$ is the line of fixed points. In strong coupling phase $(g > g_c)$, the beta function behaves as (see Fig. 1)

$$\beta(g) = -\frac{1}{B g_c}(g^2 - g_c^2)^{3/2} < 0, \quad (g > g_c, g \cong g_c), \tag{4.6}$$

which is compatible with the BKT mass (4.2),

$$m(g) = A \exp\left(-\frac{B}{\sqrt{g^2 - g_c^2}} \right), \quad g \downarrow g_c, \tag{4.7}$$

with constants $A, B > 0$. Note that $\beta(g)$ is independent of A.

It is worth remarking that a recent lattice computer simulation [28,29] indicates the existence of a non-Gaussian fixed point in four-dimensional pure compact U(1) gauge theory. This should be compared with the old results [30,31]. In the simulation [28,29] the continuous phase transition was found and analyzed according to the power law scaling, although our investigation suggests a scaling behavior of essential singularity type and the data do not exclude the essential singularity. It will be rather difficult to specify the essential singularity in computer simulations, since the lattice size available is not yet large enough to confirm this issue. Anyway, it will be interesting to find any relationship to fill the gap between two approaches.

If the mass scale is generated by dimensional transmutation, the above results are quite analogous to the situation found for the dynamical fermion mass and the β function in quenched massless QED [32,33]. The relationship of these results with quenched QED is more suggestive using the method of bosonization or fermionization. The two-dimensional sine-Gordon model is equivalent to the massive Thirring model [34,35] with an action

$$S[\psi, \bar{\psi}] = \int d^2x \left[\bar{\psi}(i\gamma_\mu \partial_\mu + m)\psi - \frac{G}{2}(\bar{\psi}\gamma_\mu \psi)^2 \right]. \tag{4.8}$$

The correspondence between two theories is given by

$$1 + \frac{G}{\pi} = 4\pi\alpha, \tag{4.9}$$

$$\bar{\psi}\gamma_\mu \psi = -\frac{1}{2\pi}\epsilon_{\mu\nu}\partial_\nu \phi, \tag{4.10}$$

$$m\bar{\psi}\psi \rightarrow -\alpha h \cos \phi. \tag{4.11}$$

Our result shows that the transition point $\alpha_c = 1/(8\pi)$ corresponds to $G_c = -\pi/2$. By making use of this equivalence, the Wilson loop (3.64) in the topological nontrivial sector of four-dimensional U(1) gauge theory can be calculated in the two-dimensional massive Thirring model. The details will be given elsewhere.

B. Lower- and higher-dimensional cases

Using the equivalence between the TQFT part of $U(1)_D$ gauge theory and $O(2)_{D-2}$ NLSM, we can study other dimensional cases. For $D=3$, the equivalent O(2) NLSM is one dimensional, $O(2)_1$. This is not a field theory model, but a quantum-mechanical model of the plane rotor. There is no phase transition in this model. This implies that three-dimensional U(1) gauge theory only has a confinement phase. This can be understood as the tunneling effect among classical vacua in the sense that the double-well anharmonic oscillator is related to the one-dimensional Ising model. It will be interesting to see the agreement (or disagreement) of the confinement mechanism between our approach and the Polyakov approach [2,3]. For $D=5$, the O(2) NLSM is three dimensional, $O(2)_3$. The three-dimensional O(2) NLSM has two phases on the lattice [36]. The phase transition is first order. This result is consistent with the mean field study of five-dimensional lattice U(1) gauge theory [24]. This theory has a finite nonzero critical coupling g_c. In the strong coupling phase, quark confinement is expected to occur. However, the phase transition is first order. Therefore, on the lattice, it is impossible to take the continuum limit at this point. In view of this, the construction of continuum U(1) gauge theory from lattice regularized theory will be problematic in five and higher dimensions, unless the action is modified.

Note added in proof. A more recent result of simulations of compact U(1) lattice gauge theory is found in [37] where the phase transition is claimed to be first order in contrast with the results of Refs. [28,29]. In view of these results, it seems rather difficult to decide the order in the foreseeable future numerically beyond any doubt.

ACKNOWLEDGMENTS

The author would like to thank Jiri Jersak and Volodya Miransky for informing him of the recent results of their computer simulation [28,29]. This work was supported in part by a Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

APPENDIX A: EQUIVALENCE BETWEEN COULOMB GAS AND SINE-GORDON MODEL

The partition function can be rewritten as

$$Z_{\rm sG}(h)=\int [d\phi]e^{-\alpha\int d^dx(1/2)[\partial_\mu\phi(x)]^2}\sum_{n=0}^{\infty}\frac{(\alpha h)^n}{n!}\times\left[\int d^dx\ \cos\ \phi(x)\right]^n$$

$$=\int [d\phi]e^{-\alpha\int d^dx(1/2)[\partial_\mu\phi(x)]^2}\sum_{n=0}^{\infty}\frac{(\alpha h)^{2n}}{(2n)!}\frac{1}{2^{2n}}\times\left[\int d^dx(e^{i\phi(x)}+e^{-i\phi(x)})\right]^{2n}$$

$$=\int [d\phi]e^{-\alpha\int d^dx(1/2)[\partial_\mu\phi(x)]^2}\sum_{n=0}^{\infty}\frac{(\alpha h)^{2n}}{(2n)!}\frac{1}{2^{2n}}\binom{2n}{n}\times\prod_{i=1}^{n}\int d^dx_id^dy_ie^{i\phi(x_i)-i\phi(y_i)}$$

$$=\int [d\phi]e^{-\alpha\int d^dx(1/2)[\partial_\mu\phi(x)]^2}\sum_{n=0}^{\infty}\frac{(\alpha h/2)^{2n}}{(n!)^2}\times\prod_{i=1}^{n}\int d^dx_id^dy_ie^{i\phi(x_i)-i\phi(y_i)}. \quad \text{(A1)}$$

Note that

$$[[e^{\int d^dxJ(x)\phi(x)}]] := \int [d\phi]\exp\left[-\frac{\alpha}{2}\int d^dx[\partial_\mu\phi(x)]^2+\int d^dxJ(x)\phi(x)\right] =\exp\left[\frac{1}{2\alpha}\int d^dx\int d^dyJ(x)\Delta(x,y)J(y)\right], \quad \text{(A2)}$$

where $\Delta(x,y)$ is the massless scalar field propagator

$$\Delta(x,y) := \int\frac{d^dp}{(2\pi)^d}\frac{e^{ip(x-y)}}{p^2}. \quad \text{(A3)}$$

In particular, for $J(x)=i\Sigma_jq_j\delta(x-x_j)$,

$$\left[\left[\prod_{j=1}^{n}e^{iq_j\phi(x_j)}\right]\right]=\begin{cases}\exp\left[-\frac{1}{2\alpha}\sum_{j,k}q_jq_k\Delta(x_j,x_k)\right] & \text{for } \sum_j q_j=0,\\ 0 & \text{for } \sum_j q_j\neq 0,\end{cases} \quad \text{(A4)}$$

where the latter case is a result of invariance under constant translation of the field $\varphi(z)\to\varphi(x)+c$. Using this result for $q_j=\pm 1$,

$$Z_{\rm sG}(h)=\sum_{n=0}^{\infty}\frac{(\alpha h/2)^{2n}}{(n!)^2}\prod_{i=1}^{n}\int d^dx_id^dy_i\times\int [d\phi]e^{-\alpha\int d^dx(1/2)[\partial_\mu\phi(x)]^2}\prod_{i=1}^{n}e^{i\phi(x_i)-i\phi(y_i)}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha h/2)^{2n}}{(n!)^2}\prod_{i=1}^{n}\int d^dx_id^dy_i\times\exp\left\{-\frac{1}{2\alpha}\left[\sum_{i<j}[\Delta(x_i-x_j)+\Delta(y_i-y_j)]-\sum_{i,j}\Delta(x_i-y_j)\right]\right\}. \quad \text{(A5)}$$

APPENDIX B: ESTIMATE OF SINE-GORDON PARTITION FUNCTION

Note that the quantity

$$\varrho(x)^2 - h\varrho(x)(-\partial^2+h)^{-1}\varrho(x) = \varrho(x)\frac{-\partial^2}{-\partial^2+h}\varrho(x) \tag{B1}$$

is positive, since $(-\partial^2)/(-\partial^2+h)$ is a positive operator. Therefore,

$$e^{-(\alpha h/2)(2\pi)^2\int_S d^dx\{\varrho(x)^2 - h\varrho(x)(-\partial^2+h)^{-1}\varrho(x)\}},$$

$$\varrho(x) = \left|\frac{\eta(x)}{2\pi} - m(x)\right| \tag{B2}$$

is monotonically (rapidly) decreasing in $\{|\rho(x)|;\ x\in S\}$. Therefore, in the partition function,

$$Z_{\rm sG}[\eta] = \sum_{\{m(x)\in Z; x\in R^d\}} \exp\left[-\frac{\alpha h}{2}(2\pi)^2 \times \int d^dx\{\varrho(x)^2 - h\varrho(x)(-\partial^2+h)^{-1}\varrho(x)\}\right], \tag{B3}$$

the most dominant contribution comes from a set of configurations $\{|\varrho(x)|;\ x\in S\}$ which gives the smallest value for $\int d^dx\{\varrho(x)^2 - h\varrho(x)(-\partial^2+h)^{-1}\varrho(x)\}$.

If the argument x of $\eta(x)$ is outside the loop $C(C=\partial S)$, $\eta(x)=0$, while $\eta(x)=2\pi q/g$ if x is inside the loop. For $Z_{\rm sG}[0]$, $\{m(x)\equiv 0\}$ gives the most dominant contribution. Hence, we see

$$Z_{\rm sG}[0] = 1 + \cdots. \tag{B4}$$

For $Z_{\rm sG}[\eta]$, the most dominant contribution comes from a set of integers $\{m(x)\}$ whose value is the nearest to q/g where $\varrho(x)=2\pi|q/g-m(x)|$. Since the integral part of q/g is absorbed in the shift of $m(x)$, it is sufficient to consider the case $0<q/g<1$ without loss of generality. For the half integer q/g, i.e., $q/g=\pm 1/2$, we see that $\{m(x)\equiv 0,\pm 1\}$ gives the smallest value of $2\pi|q/g-m(x)|$ for $x\in S$. For $0<q/g<1/2$ $(1/2<q/g<1)$, the most dominant contribution is given by $\{m(x)\equiv 0\}$ $(\{m(x)\equiv 1\})$. Thus, the rough estimate, for example, in the case of $0<q/g<1/2$ leads to

$$Z_{\rm sG}[\eta] = e^{-(\alpha h/2)\int_S d^dx\{\eta(x)^2 - h\eta(x)(-\partial^2+h)^{-1}\eta(x)\}} + \cdots. \tag{B5}$$

A naive estimate of the ratio $Z_{\rm sG}[\eta]/Z_{\rm sG}[0]$ is given for $\eta\notin 2\pi Z$ by

$$\langle W_C[\omega]\rangle_{\rm sG} = \frac{Z_{\rm sG}[\eta]}{Z_{\rm sG}[0]} = e^{-(\alpha h/2)\int_S d^dx\,\eta(x)[1-h(-\partial^2+h)^{-1}]\eta(x)} + \cdots. \tag{B6}$$

Here $(-\partial^2+h)^{-1}(x,y)$ is the massive scalar propagator with mass $m\sim\sqrt{h}$ and hence has an exponential damping factor $e^{-m|x-y|}$. Therefore, the integral $\int_S d^dx\,\eta(x)(-\partial^2+h)^{-1}\eta(x)$ converges to a finite value even for large S. The term $\int_S d^2x\,\eta(x)^2$ is proportional to the area of S. This term leads to the area decay of the Wilson loop expectation.

[1] A.M. Polyakov, *Gauge Fields and Strings* (Harwood Academic, London, 1987).
[2] A.M. Polyakov, Phys. Lett. **59B**, 82 (1975).
[3] A.M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[4] K.-I. Kondo, Phys. Rev. D (to be published), hep-th/9801024.
[5] K. Wilson, Phys. Rev. D **10**, 2445 (1974).
[6] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
[7] N.D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); N.D. Mermin, J. Math. Phys. **8**, 1061 (1967); P.C. Hohenberg, Phys. Rev. **158**, 383 (1967); S. Coleman, Commun. Math. Phys. **31**, 259 (1973).
[8] V.L. Berezinskii, Sov. Phys. JETP **32**, 493 (1971); J.M. Kosterlitz, J. Phys. C **7**, 1046 (1974); J.M. Kosterlitz and D.V. Thouless, *ibid.*, **6**, 1181 (1973).
[9] J. Fröhlich, Commun. Math. Phys. **47**, 233 (1976).
[10] J.V. Jose, L.P. Kadanoff, S. Kirkpatrick, and D.R. Nelson, Phys. Rev. B **16**, 1217 (1977).
[11] D.J. Amit, Y.Y. Goldschmidt, and G. Grinstein, J. Phys. A **13**, 585 (1980).
[12] J. Fröhlich and T. Spencer, Commun. Math. Phys. **81**, 527 (1981).
[13] T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977).
[14] J. Glimm and A. Jaffe, Commun. Math. Phys. **56**, 195 (1977).
[15] M.E. Peskin, Ann. Phys. (N.Y.) **113**, 122 (1978).
[16] K.-I. Kondo, Phys. Rev. D **57**, 7467 (1998)
[17] K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) **110**, 440 (1978).
[18] S. Elitzur, R.B. Pearson, and J. Shigemitsu, Phys. Rev. D **19**, 3698 (1979).
[19] A. Ukawa, P. Windey, and A. Guth, Phys. Rev. D **21**, 1013 (1980).
[20] A. Guth, Phys. Rev. D **21**, 2291 (1980).
[21] J. Fröhlich and T. Spencer, Commun. Math. Phys. **83**, 411 (1982).

[22] K.G. Wilson and J. Kogut, Phys. Rep., Phys. Lett. **12C**, 75 (1974).
[23] J.B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[24] J.-M. Drouffe and J.-B. Zuber, Phys. Rep. **102**, 1 (1983).
[25] E. Seiler, *Gauge Theories as a Problem of Constructive Field Theory and Statistical Mechanics*, Vol. 159 of Lecture Note in Physics (Springer, Berlin, 1982).
[26] K.-I. Kondo, Phys. Rev. D (to be published), hep-th/9805153.
[27] J. Villain, J. Phys. (Paris) **36**, 581 (1975).
[28] J. Jersak, C.B. Lang, and T. Neuhaus, Phys. Rev. Lett. **77**, 1933 (1996).
[29] J. Cox, W. Franzki, J. Jersak, C.B. Lang, T. Neuhaus, and P.W. Stephenson, Nucl. Phys. **B499**, 371 (1997).
[30] H.G. Evertz, J. Jersak, T. Neuhaus, and P.M. Zerwas, Nucl. Phys. **B251 [FS13]**, 279 (1985).
[31] J. Jersak, T. Neuhaus, and P.M. Zerwas, Nucl. Phys. **B251 [FS13]**, 299 (1985).
[32] V.A. Miransky, Nuovo Cimento A **90**, 149 (1985).
[33] K.-I. Kondo, T. Mino, and K. Yamawaki, Phys. Rev. D **39**, 2430 (1989).
[34] S. Coleman, Phys. Rev. D **11**, 2088 (1975).
[35] K.-I. Kondo, Prog. Theor. Phys. **94**, 899 (1995).
[36] P.R. Thomas and M. Stone, Nucl. Phys. **B144**, 513 (1978).
[37] I. Campos, A. Cruz, and A. Tarancon, Nucl. Phys. **B528**, 325 (1998).