Finite size corrections in the massive Thirring model

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We calculate for the first time the finite size corrections in the massive Thirring model. This is done by numerically solving the equations of periodic boundary conditions of the Bethe ansatz solution. It is found that the corresponding central charge extracted from the $1/L$ term is around 0.4 for the coupling constant of $g_0 = -\pi/4$ and decreases down to zero when $g_0 = -\pi/3$. This is quite different from the predicted central charge of the sine-Gordon model as well as the light cone six vertex model. [S0556-2821(98)01420-9]

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I. INTRODUCTION

In two dimensional field theory, there is a remarkable correspondence between the fermionic and bosonic field theories. This was first recognized by Coleman [1], and he proved that the sine-Gordon field theory and the massive Thirring model are equivalent to each other in that the arbitrary order of the correlation functions turn out to be the same.

Recently, however, Klassen and Melzer [2] argue that the equivalence between the sine-Gordon and the massive Thirring models may be violated at the finite size correction. They proved by using the perturbed conformal field theory that these two models are different in finite-volume energy levels, for example.

In this paper, we calculate the finite size corrections to the ground state energy. We solve numerically the equations of the periodic boundary condition in the Bethe ansatz solutions of the massive Thirring model [3–5]. The ground state energy can be expressed as

$$E_v = E_0 L - \frac{\pi\tilde{c}}{6L} + \cdots , \tag{1.1}$$

where L denotes the box size. $\tilde{c}$ corresponds to a central charge at the massless limit [6,7].

The present calculation shows that the corresponding central charge $\tilde{c}$ in the negative coupling constant regions (no bound states) is around 0.4 for $g_0 = -\pi/4$ and that it becomes zero when $g_0 = -\pi/3$. These values can be compared with those calculated for the sine-Gordon field theory [8,9]. The central charge for the sine-Gordon field theory with the massless limit can be expressed as

$$c = 1 - \frac{6}{p(p+1)}, \tag{1.2}$$

where p is an integer and is related to the coupling constant g_0 as

$$g_0 = -\frac{\pi}{2}\left(1 - \frac{1}{p}\right). \tag{1.3}$$

In Fig. 1, we summarize the calculated central charge as the function of the coupling constant for the sine-Gordon model by Itoyama and Moxhay, and for the massive Thirring model by the present calculations. One can see that the values of the central charge predicted for the two models are very different from each other.

Furthermore, Destri and de Vega [10] show that the central charge of the light cone six vertex model turns out to be unity for the massless limit. Since this light cone six vertex model is shown to be equivalent to the massive Thirring model at the continuum limit [11], it also predicts a different central charge from the sine-Gordon and the massive Thirring models.

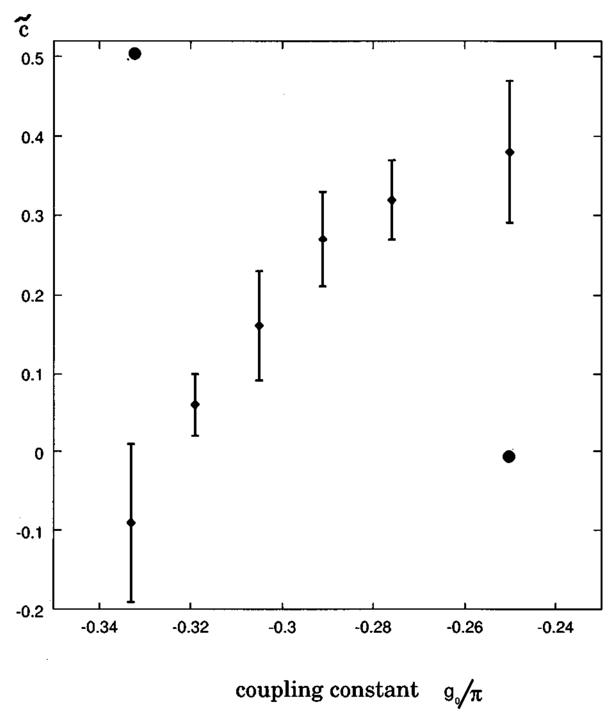

FIG. 1. The calculated values of the $\tilde{c}(g_0)$ (the black diamonds with error bars) are plotted as the function of the coupling constant g_0/π. The black circle shows the predictions by Itoyama-Moxhay.

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II. MASSIVE THIRRING MODEL AND BETHE ANSATZ SOLUTIONS

Here, we briefly review the massive Thirring model whose Lagrangian density can be written as [12]

$$\mathcal{L}=\bar{\psi}(i\gamma_\mu\partial^\mu-m_0)\psi-\frac{1}{2}g_0 j^\mu j_\mu \tag{2.1}$$

with the fermion current $j_\mu=:\bar{\psi}\gamma_\mu\psi:$. Choosing a basis where γ_5 is diagonal, we write the Hamiltonian as

$$H=\int dx\left[-i\left(\psi_1^\dagger\frac{\partial}{\partial x}\psi_1-\psi_2^\dagger\frac{\partial}{\partial x}\psi_2\right)+m_0(\psi_1^\dagger\psi_2+\psi_2^\dagger\psi_1)+2g_0\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1\right]. \tag{2.2}$$

The Hamiltonian Eq. (2.2) can be diagonalized by the Bethe ansatz wave functions $\Psi(x_1,\dots,x_N)$ with N particles

$$\Psi(x_1,\dots,x_N)=\exp\left(im_0\sum x_i\sinh\beta_i\right)\times\prod_{1\leqslant i<j\leqslant N}[1+i\lambda(\beta_i,\beta_j)\times\epsilon(x_i-x_j)], \tag{2.3}$$

where β_i is related to the momentum k_i and the energy E_i of ith particle as

$$k_i=m_0\sinh\beta_i, \tag{2.4a}$$

$$E_i=m_0\cosh\beta_i, \tag{2.4b}$$

where β_i's are complex variables.

$\epsilon(x)$ is a step function and is defined as

$$\epsilon(x)=\begin{cases}-1, & x<0,\\ 1, & x>0.\end{cases} \tag{2.5}$$

$\lambda(\beta_i,\beta_j)$ is related to the phase shift function $\phi(\beta_i-\beta_j)$ as

$$\frac{1+i\lambda(\beta_i,\beta_j)}{1-i\lambda(\beta_i,\beta_j)}=\exp[i\phi(\beta_i-\beta_j)]. \tag{2.6}$$

The phase shift function $\phi(\beta_i-\beta_j)$ can be explicitly written as

$$\phi(\beta_i-\beta_j)=-2\tan^{-1}\left[\frac{1}{2}g_0\tanh\frac{1}{2}(\beta_i-\beta_j)\right]. \tag{2.7}$$

From the definition of the rapidity variable β_i's, one sees that for positive energy particles, β_i's are real while for negative energy particles, β_i takes the form $i\pi-\alpha_i$ where α_i's are real.

Since the Bethe ansatz wave functions diagonalize the Hamiltonian, we demand that they satisfy the periodic boundary conditions (PBC) with the box length L [3],

$$\Psi(x_i=0)=\Psi(x_i=L). \tag{2.8}$$

This leads to the following PBC equations:

$$m_0L\sinh\beta_i=2\pi n_i-\sum_j\phi(\beta_i-\beta_j), \tag{2.9}$$

where n_i's are integer. Here, we note that we cannot take the antiperiodic boundary condition since it does not reproduce the boson spectrum in the positive coupling constant regions [5].

III. NUMERICAL SOLUTIONS

The parameters we have here are the box length L and the particle number N. In this case, the density of the system ρ becomes

$$\rho=\frac{N}{L}. \tag{3.1}$$

Here, the system is fully characterized by the density ρ.

We write the PBC equations for the vacuum which is filled with negative energy particles ($\beta_i=i\pi-\alpha_i$):

$$\sinh\alpha_i=\frac{2\pi n_i}{L_0}-\frac{2}{L_0}\sum_{j\neq i}\tan^{-1}\left[\frac{1}{2}g_0\tanh\frac{1}{2}(\alpha_i-\alpha_j)\right], \tag{3.2}$$

where $n_i=0,\pm1,\pm2,\dots,\pm N_0$ with $N_0=\frac{1}{2}(N-1)$ and $L_0=m_0L$.

In this case, the vacuum energy E_v can be written as

$$E_v=-\sum_{i=-N_0}^{N_0}m_0\cosh\alpha_i. \tag{3.3}$$

In this paper, we have carried out the numerical calculations of the PBC equations. The numerical method to solve the PBC equations is explained in detail in Ref. [5].

Now, the calculated vacuum energy can be parametrized as

$$E_v=E_0L-\frac{\pi\tilde{c}(g_0)}{6L}+\cdots, \tag{3.4}$$

where $\tilde{c}(g_0)$ corresponds to the central charge at the massless limit. In what follows, we call this $\tilde{c}(g_0)$ as the central charge even though we are solving the massive field theory. Here, we take the massless limit ($m_0\to0$) as we discuss later. It should be noted that the first term in Eq. (3.4) can be evaluated analytically by taking the thermodynamic limit [3].

Since we can vary the values of L and N, we obtain the corresponding central charge $\tilde{c}(g_0)$. Although we have still rather small particle number ($N\sim10000$), we believe that the values extracted for the central charge must be reasonably reliable.

Now, we want to obtain the central charge $\tilde{c}(g_0)$ at the field theory limit $\rho\to\infty$. In Fig. 2, we show the calculated central charge $\tilde{c}(g_0)$ as the function of the effective density $\rho_0=N_0/L_0$. It is quite interesting to observe that the calculated central charge can be well parametrized by the following simple formula [13]:

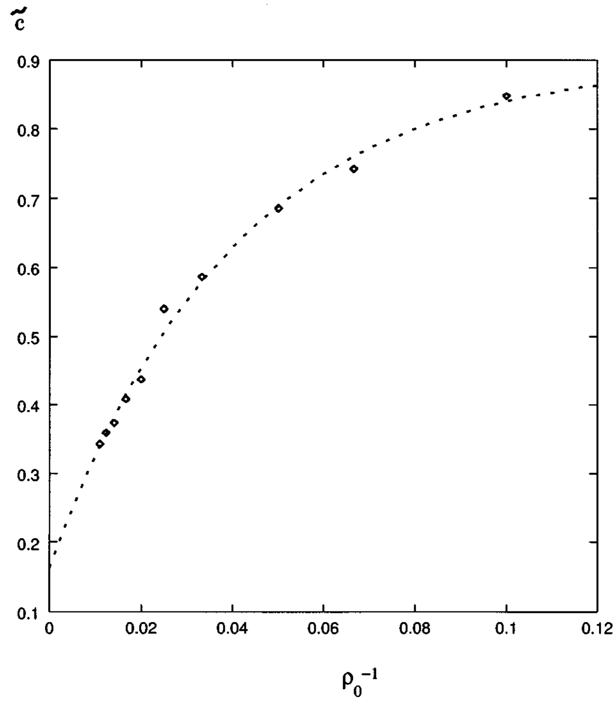

FIG. 2. We show the calculated $\tilde{c}(g_0)$ (white diamonds) as the function of ρ_0^{-1}. The broken line denotes the fit by Eq. (3.5). Here, the coupling constant is at $g_0/\pi = -0.305$.

$$\tilde{c}(g_0) = A + B \exp\left(-\frac{\kappa}{\rho_0}\right), \tag{3.5}$$

where A, B, and κ are constants. Therefore, the field theory limit can be easily taken since we can let ρ_0 infinity. It should be noted that this corresponds to the limit of $\rho \to \infty$ with the mass $m_0 \to 0$ [5].

In Table I, we show the values of A, B, and κ for some values of the coupling constant g_0. The central charge becomes $A+B$ at the field theory limit. The calculated values of the central charge are shown as the function of the coupling constant g_0 in Fig. 1. We also plot the central charge calculated for the sine-Gordon theory by Itoyama and Moxhay [9]. As can be seen from Fig. 1, the two values of the central charge are quite different from each other.

At the same time, the calculated central charge of the light cone six vertex model is found to be unity at the massless limit for any value of the coupling constant [10]. This is also quite different from the massive Thirring as well as the sine-Gordon models.

TABLE I. We show the values of A, B, and κ for some coupling constant g_0 together with the $\tilde{c}(g_0)$ at the field theory limit.

g_0/π	A	B	κ	$\tilde{c}(g_0)$
$-1/4$	0.941	-0.562	25	0.38 ± 0.09
-0.276	1.06	-0.745	16.5	0.32 ± 0.05
-0.291	1.06	-0.795	16	0.27 ± 0.06
-0.305	0.901	-0.739	25	0.16 ± 0.07
-0.319	0.854	-0.790	30	0.06 ± 0.04
$-1/3$	0.793	-0.879	40	-0.09 ± 0.10

IV. DISCUSSIONS

How can we interpret these differences? The first possibility is that the three theories (sine-Gordon, massive Thirring and light cone six vertex models) are different from each other at the finite volume. We do not know whether this difference can show up as the central charge or not. However, a simple-minded physical intuition suggests that the central charge which should correspond to the heat capacity cannot be different if all the correlation functions of the models are the same with each other. In particular, concerning the equivalence between the sine-Gordon and the massive Thirring models, we should rather check the convergence of the perturbation expansions in Coleman's proof since it crucially depends on the convergence of the expansions. For the negative values of the coupling constant, we do not know whether this convergence is already verified or not.

The second possibility is that neither of the calculations are accurate enough to argue the difference between them. To this, we should comment on the accuracy of the present calculations. Since we have only the limited number of particles, we always face the criticism that the real nature (even though $1+1$ dimension) must be with the infinite number of particles. We have varied the number of particles from 1000 to 10000. It seems to us that the extracted central charge may well be reliable to within a few tens of percents. At least, we believe that the calculation must be rather reliable for the coupling constant around $g_0 = -\pi/4$ where the extracted central charge is not very small. On the other hand, the present calculation may involve somewhat large errors for the coupling constant around or smaller than $g_0 = -\pi/3$ since the extracted central charge is rather small. This is in contrast to the bound state problems [5,14,15] where there is some possibility of controlling the accuracy of the numerical calculations. However, the evaluation of the central charge involves rather complicated processes of extracting it since we have to obtain it from the term proportional to $1/L$ in the vacuum energy. Therefore, the error bars of the calculations we have shown in Fig. 1 may well be still optimistic numbers.

Concerning the central charge of the sine-Gordon model, we do not know whether the central charge predicted by Itoyama and Moxhay can be taken to be exact or not. Here, we only make a comment on the *string* hypothesis in the massive Thirring model when they employ the thermodynamic Bethe ansatz [16]. As discussed in Refs. [5, 13], the *string* picture in the massive Thirring model in the positive values of the coupling constant turns out to be invalid in the sense that they do not satisfy the PBC equations. However, in the negative values of the coupling constant, we do not know whether there is a *string*-like solution that satisfies the PBC equations [17].

For the central charge obtained for the light cone six vertex model, it looks reasonable that the central charge of the light cone six vertex model turns out to be unity at the massless limit. This is because the central charge of the massless Thirring model is known to be unity. However, it is also known that the massless limit in the massive Thirring model corresponds to a singularity, and thus it may not be so simply related to the massless limit of the Lagrangian level.

Finally, we comment on the zero central charge. We believe that our procedure of $\rho_0 \to \infty$ should correspond to the massless limit. Concerning the zero central charge, we do not know which operators may correspond to it in the massive Thirring model [18].

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