

Nonperturbative spectrum of two-dimensional (1,1) super-Yang-Mills theory at finite and large N

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We consider the dimensional reduction of $\mathcal{N}=1$ SYM $_{2+1}$ to 1+1 dimensions, which has (1,1) supersymmetry. The gauge groups we consider are $U(N)$ and $SU(N)$, where N is a finite variable. We implement discrete light-cone quantization to determine nonperturbatively the bound states in this theory. A careful analysis of the spectrum is performed at various values of N , including the case where N is large (but finite), allowing a precise measurement of the $1/N$ effects in the quantum theory. The low energy sector of the theory is shown to be dominated by string-like states. The techniques developed here may be applied to any two dimensional field theory with or without supersymmetry. [S0556-2821(98)02418-7]

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I. INTRODUCTION

Solving for the nonperturbative properties of quantum field theories—such as QCD—is typically an intractable problem. In order to gain some insight, however, a number of lower dimensional models have been proposed as useful laboratories in which to study QCD related phenomena (for a review see [1]).

In recent times, the role of low dimensional quantum field theories has shifted rather dramatically following the remarkable developments in string or M theory. The present literature on this subject is immense, but a common theme appears to be emerging: there is more interesting physics in Yang-Mills theory than was once thought reasonably possible. In addition to the M(atrix) model conjecture, which formulates M theory in terms of supersymmetric quantum mechanics [2], there is also a proposal by Maldacena [3] that large N super-Yang-Mills theories in various dimensions are related to certain supergravity solutions.

All of these developments suggest that it would be desirable to have a better understanding of the non-perturbative properties of super-Yang-Mills theory at large (but finite) N , and in any dimension. Towards this end, we choose to study in detail the bound state structure and spectrum of a two dimensional field theory, which may be obtained by dimensionally reducing 2+1 dimensional $\mathcal{N}=1$ super Yang Mills. Such a theory has already been investigated in the $N=\infty$ (or planar) approximation [4], and is believed to exhibit the property of screening [5,6]. In this work we will allow the number of gauge colors, N , to be a finite variable. This means we will be able to monitor the behavior of the spectrum as N is varied and made arbitrarily large. Special attention is given to measuring the precise effects on the spectrum due to $1/N$ contributions in the quantum theory.

Although we focus on one particular model in this paper, the techniques we develop here are applicable to *any* two dimensional field theory, with or without supersymmetry.

The organization of the paper may be summarized as follows; in Sec. II, we discuss the relevant features of a (1,1) super-Yang-Mills theory in 1+1 dimensions, giving explicit expressions for the (quantized) light-cone supercharges for-

mulated in the light-cone gauge. Formulation of the discrete light-cone quantization (DLCQ) bound-state problem of this theory is the subject of Sec. III, followed by a detailed analysis of the corresponding numerical bound-state solutions in Sec. IV. In Sec. V, we conclude with a perspective on future applications of non-perturbative finite N calculations for arbitrary (super) Yang-Mills theories.

II. (1,1) SUPER-YANG-MILLS THEORY IN 1+1 DIMENSIONS

The theory we wish to study is readily obtained by dimensionally reducing $\mathcal{N}=1$ $D=3$ super-Yang-Mills to 1+1 dimensions. The resulting theory has (1,1) supersymmetry, and can be formulated in the light-cone frame. The details of this light-cone formulation appear in [4], to which we refer the reader for explicit derivations. We simply note here that the light-cone Hamiltonian P^- is given in terms of the supercharge Q^- via the supersymmetry relation $\{Q^-, Q^-\} = 2\sqrt{2}P^-$, where

$$Q^- = 2^{3/4}g \int dx^- \text{tr} \left\{ (i[\phi, \partial_- \phi] + 2\psi\psi) \frac{1}{\partial_-} \psi \right\}. \quad (1)$$

In the above, $\phi_{ij} = \phi_{ij}(x^+, x^-)$ and $\psi_{ij} = \psi_{ij}(x^+, x^-)$ are $N \times N$ Hermitian matrix fields representing the physical boson and fermion degrees of freedom (respectively) of the theory, and are remnants of the physical transverse degrees of freedom of the original 2+1 dimensional theory. This is a special feature of light-cone quantization in light-cone gauge: all unphysical degrees of freedom present in the original Lagrangian may be explicitly eliminated. There are no ghosts.

For completeness, we write the additional relation $\{Q^+, Q^+\} = 2\sqrt{2}P^+$ for the light-cone momentum P^+ , where

$$Q^+ = 2^{1/4} \int dx^- \text{tr} [\phi \partial_- \psi - \psi \partial_- \phi]. \quad (2)$$

The (1,1) supersymmetry of the model follows from the fact $\{Q^+, Q^-\} = 0$. In order to quantize ϕ and ψ on the light-cone, we first introduce the following expansions at fixed light-cone time $x^+ = 0$:

$$\begin{aligned} \phi_{ij}(x^-, 0) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} (a_{ij}(k^+) e^{-ik^+x^-} \\ &\quad + a_{ji}^\dagger(k^+) e^{ik^+x^-}); \end{aligned} \quad (3)$$

$$\begin{aligned} \psi_{ij}(x^-, 0) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty dk^+ (b_{ij}(k^+) e^{-ik^+x^-} \\ &\quad + b_{ji}^\dagger(k^+) e^{ik^+x^-}). \end{aligned} \quad (4)$$

We then specify the commutation relations

$$[a_{ij}(p^+), a_{lk}^\dagger(q^+)] = \{b_{ij}(p^+), b_{lk}^\dagger(q^+)\} = \delta(p^+ - q^+) \delta_{il} \delta_{jk} \quad (5)$$

for the gauge group $U(N)$, or

$$\begin{aligned} [a_{ij}(p^+), a_{lk}^\dagger(q^+)] &= \{b_{ij}(p^+), b_{lk}^\dagger(q^+)\} \\ &= \delta(p^+ - q^+) \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right) \end{aligned} \quad (6)$$

for the gauge group $SU(N)$.¹

For the bound state eigenproblem $2P^+P^-|\Psi\rangle = M^2|\Psi\rangle$, we may restrict to the subspace of states with fixed light-cone momentum P^+ , on which P^+ is diagonal, and so the bound state problem is reduced to the diagonalization of the light-cone Hamiltonian P^- . Since P^- is proportional to the square of the supercharge Q^- , any eigenstate $|\Psi\rangle$ of P^- with mass squared M^2 gives rise to a natural degeneracy in the spectrum because of the supersymmetry algebra—all four states below have the same mass:

$$|\Psi\rangle, \quad Q^+|\Psi\rangle, \quad Q^-|\Psi\rangle, \quad Q^+Q^-|\Psi\rangle. \quad (7)$$

Although this degeneracy is realized in the continuum formulation of the theory, this property will not necessarily survive if we choose to discretize the theory in an arbitrary manner. However, a nice feature of DLCQ is that it does preserve the supersymmetry (and hence the *exact* four-fold degeneracy) for any resolution. In the context of numerical calculations, this reduces (by a factor of four) the size of the DLCQ matrix that needs to be diagonalized.

The explicit expression for Q^- , in the momentum representation is now obtained by substituting the quantized field expressions (3) and (4) directly into the definition of the supercharge (1). The result is

$$\begin{aligned} Q^- &= \frac{i2^{-1/4}g}{\sqrt{\pi}} \int_0^\infty dk_1 dk_2 dk_3 \delta(k_1 + k_2 - k_3) \left\{ \frac{1}{2\sqrt{k_1 k_2}} \frac{k_2 - k_1}{k_3} [a_{ik}^\dagger(k_1) a_{kj}^\dagger(k_2) b_{ij}(k_3) - b_{ij}^\dagger(k_3) a_{ik}(k_1) a_{kj}(k_2)] \right. \\ &\quad + \frac{1}{2\sqrt{k_1 k_3}} \frac{k_1 + k_3}{k_2} [a_{ik}^\dagger(k_3) a_{kj}(k_1) b_{ij}(k_2) - a_{ik}^\dagger(k_1) b_{kj}^\dagger(k_2) a_{ij}(k_3)] \\ &\quad + \frac{1}{2\sqrt{k_2 k_3}} \frac{k_2 + k_3}{k_1} [b_{ik}^\dagger(k_1) a_{kj}^\dagger(k_2) a_{ij}(k_3) - a_{ij}^\dagger(k_3) b_{ik}(k_1) a_{kj}(k_2)] \\ &\quad \left. + \left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} \right) [b_{ik}^\dagger(k_1) b_{kj}^\dagger(k_2) b_{ij}(k_3) + b_{ij}^\dagger(k_3) b_{ik}(k_1) b_{kj}(k_2)] \right\}. \end{aligned} \quad (8)$$

In ordinary DLCQ calculations, one chooses to discretize the light-cone Hamiltonian P^- . However it was pointed out in [4] that supersymmetric theories admit a natural DLCQ formulation in terms of discretized supercharges. This ensures that *supersymmetry is preserved even in the discretized theory*. Before proceeding with the DLCQ formulation of the bound state problem, we note that for the gauge group $U(N)$, massless states appear automatically because of the decou-

pling of the $U(1)$ and $SU(N)$ degrees of freedom that constitute $U(N)$. More explicitly, we may introduce the $U(1)$ operators

$$\alpha(k^+) = \frac{1}{N} \text{tr}[a(k^+)], \quad \beta(k^+) = \frac{1}{N} \text{tr}[b(k^+)], \quad (9)$$

which allow us to decompose any $U(N)$ operator into a sum of $U(1)$ and $SU(N)$ operators:

$$\begin{aligned} a(k^+) &= \alpha(k^+) \cdot \mathbf{1}_{N \times N} + \tilde{a}(k^+), \\ b(k^+) &= \beta(k^+) \cdot \mathbf{1}_{N \times N} + \tilde{b}(k^+), \end{aligned} \quad (10)$$

¹We assume the normalization $\text{tr}[T^a T^b] = \delta^{ab}$, where the T^a 's are the generators of the Lie algebra of $SU(N)$.

where $\tilde{a}(k^+)$ and $\tilde{b}(k^+)$ are traceless $N \times N$ matrices. If we now substitute the operators above into the expression for the supercharge (8), we find that all terms involving the U(1) factors $\alpha(k^+), \beta(k^+)$ vanish—only the SU(N) operators $\tilde{a}(k^+), \tilde{b}(k^+)$ survive. i.e. starting with the definition of the U(N) supercharge, we end up with the definition of the SU(N) supercharge. In addition, the (anti)commutation relations $[\tilde{a}_{ij}(k_1), \alpha^\dagger(k_2)] = 0$ and $\{\tilde{b}_{ij}(k_1), \beta^\dagger(k_2)\} = 0$ imply that this supercharge acts only on the SU(N) creation operators of a Fock state—the U(1) creation operators only introduce degeneracies in the SU(N) spectrum. Clearly, since Q^- has no U(1) contribution, any Fock state made up of only U(1) creation operators must have zero mass. The non-trivial problem is therefore solving for SU(N) bound states.

III. DISCRETIZED LIGHT-CONE QUANTIZATION AT FINITE N

In order to implement the DLCQ formulation [7] of the theory, we simply restrict the momenta k_1, k_2 and k_3 appearing in Eq. (8) to the following set of allowed momenta: $\{P^+/K, 2P^+/K, 3P^+/K, \dots\}$. Note that we omit the zero momentum modes [8,9], which are not expected to affect the massive spectrum. Here, K is some arbitrary positive integer, and must be sent to infinity if we wish to recover the continuum formulation of the theory. The integer K is called the *harmonic resolution*, and $1/K$ measures the coarseness of our discretization.² Physically, $1/K$ represents the smallest unit of longitudinal momentum fraction allowed for each parton. As soon as we implement the DLCQ procedure, which is specified unambiguously by the harmonic resolution K , the integrals appearing in the definition of Q^- are replaced by finite sums, and the eigen-equation $(Q^-)^2|\Psi\rangle = \lambda|\Psi\rangle$ is reduced to a finite matrix problem. For sufficiently small values of K (in this case for $K \leq 4$) this eigen-problem may be solved analytically. For values $K \geq 5$, we may still compute the DLCQ supercharge analytically as a function of N , but the diagonalization procedure must be performed numerically.

The details of how to construct the DLCQ light-cone supercharges in the model studied here appear in reference [4]. A similar model was also studied using this approach in [11]. The only modification we make here is that we allow the number of gauge colors, N , to be a finite (algebraic) variable. This complicates things considerably. The reason is rather simple. In the $N = \infty$ formulation, all fockstates may be written as a *single* trace of creation operators,

$$|\Psi\rangle \sim \text{tr}[c^\dagger(k_1^+) \cdots c^\dagger(k_n^+)]|0\rangle \quad (11)$$

$[c^\dagger(k^+)]$ represents either a boson or fermion carrying longitudinal momentum k^+ , since individual Fockstates that in-

volve a product of two or more traces couple to these states like $1/N$, and are therefore completely decoupled in the limit $N = \infty$. This gives rise to decoupled sectors that are characterized by the number of traces appearing in each Fock state. In addition, color interactions in the light-cone Hamiltonian (or supercharge) simplify when $N = \infty$, since splitting or joining interactions occur between *adjacent* color-contracted partons in a Fock state. This dramatically simplifies the representation of any light-cone operator on the Hilbert space of single trace Fock states. This property also tremendously simplifies the evaluation of inner products. It is sometimes helpful to think of a single trace state as a closed string made up of “string bits” [12]. Multiple-trace states are therefore multi-string states, and the string coupling is given by $1/N$. For $N = \infty$, these multi-trace states are just free non-interacting closed “strings.” Splitting and joining of these strings is only possible when N is finite.

Of course, as soon as we allow N to be finite, we have to give up all of these wonderful simplifications. In computational terms, this usually means that the most time consuming part of a DLCQ calculation is the evaluation of inner products for many parton Fock states, which is relatively trivial in the $N = \infty$ case. Of course, the processing time involved in calculating the representation of the light-cone Hamiltonian relative to the discretized Fock basis is augmented considerably due to these complications.

Nevertheless, we feel justified in dealing with these complications, since a number of interesting physical properties associated with the dynamics of super-Yang-Mills theory are expected to arise as “ $1/N$ effects” in the quantum theory.³

In practical terms, the complexities cited above simply restrict how large the harmonic resolution, K , is allowed to be in numerical computations. In the present study, we could manage only $K \leq 8$ (about 2000 states altogether for $K = 8$), and we expect that higher values of K could be probed if more powerful machines and more efficient code were available.⁴

Before proceeding to discuss our numerical results, we point out that one may significantly reduce the computational complexity of setting up the DLCQ supercharge by taking advantage of the simple fact that the U(N) and SU(N) supercharges are equivalent. From a computational point of view, the commutation relations for U(N) matrices [Eq. (5)] are simpler than the SU(N) relations [Eq. (6)], and so it would be desirable to work with the U(N) basis even when we are interested in solving for SU(N) bound states. It turns out that if one constructs a basis of U(N) Fock states, and then discards all states that contain a trace of a single parton, then the corresponding spectrum of the U(N) theory on this modified basis yields the same spectrum as the SU(N) theory. Of course, constructing the U(N) supercharge re-

²Recently, Susskind has proposed a connection between the harmonic resolution arising from the DLCQ of M theory, and the integer N appearing in the U(N) gauge group for M(atr)ix theory (namely, they are the same) [10].

³Maldacena has recently argued that the $1/N$ effects for a particular class of super-Yang-Mills theories account for Hawking radiation in a corresponding class of space-time geometries [3].

⁴Numerical calculations were performed using a desk-top PC, and the computer code was written for Mathematica version 3.0.

quires much less computational effort, and we therefore employ this strategy when solving for $SU(N)$ bound states when K is large. A more thorough discussion of this technique will appear elsewhere [13]. Of course, the DLCQ program we use can do both $SU(N)$ and $U(N)$ independently, and the above procedure can be checked explicitly for $K \leq 6$ (it works). This method is expected to play a crucial role when solving for $SU(N)$ bound states in more complicated two dimensional theories.

IV. NUMERICAL BOUND STATE SOLUTIONS

There are two parameters in the DLCQ formulation of the theory; the harmonic resolution K , and the number of gauge colors N . This dependence on K is of course an artifact of the light-cone compactification scheme, $x^- = x^- + 2\pi R$, and in practice it is eliminated by extrapolating the results at finite K to the continuum limit $K = \infty$. Reliable extrapolations require careful analysis of the theory as K is steadily increased. From Eq. (1), one sees that the two dimensional Yang-Mills coupling constant g factors out in the definition of the light-cone supercharge, and so the only adjustable coupling constant in the theory is the parameter $1/N$. This quantity measures the strength of interactions between different trace sectors in the Hilbert space, where each sector is characterized by the number of traces in each Fock state. Of course, in the limit $N = \infty$, these sectors are completely decoupled. It is therefore an interesting physical problem to investigate the behavior of the theory when N is allowed to be finite and large. Bound states will be a superposition of Fock states containing any number of traces, but interactions between different sectors will be weak.

Our numerical work involved solving the DLCQ $SU(N)$ bound state equations for $2 \leq K \leq 8$, and then extrapolating the results to obtain estimates for continuum bound state masses. Supersymmetry in the DLCQ formulation gives rise to an obvious exact two-fold mass degeneracy between bosons and fermions, but there is an additional two-fold degeneracy for each massive boson and fermion bound state. We therefore have an exact four-fold degeneracy in the spectrum of massive bound states. Figure 1 is a summary of the low mass spectrum (i.e. only eigenvalues less than 30 are plotted) that were obtained for $3 \leq K \leq 8$, and for $N = 10$. The vertical axis measures the bound state mass squared $M^2 \equiv 2P^+P^-$, in units of g^2N/π .

At resolution $K=2$ there are precisely two massless $SU(N)$ bound states (one boson and one fermion), each consisting of two partons. At $K=3$, massive bound states begin to appear in the spectrum, and are four-fold degenerate (two bosons and two fermions). If these solutions signify the presence of true bound states in the continuum, one expects that their structure will persist as K is increased. In order to check this, one must look at the Fock state content of a bound state at different resolutions, and see whether the same approximate structure is preserved as we increase K ; whether the wave functions begin to converge or not is an indicator of whether the continuum bound state might be normalizable or not. Of course, as we continue to increase K , new states will appear in the (discretized) Fock space that are not related to

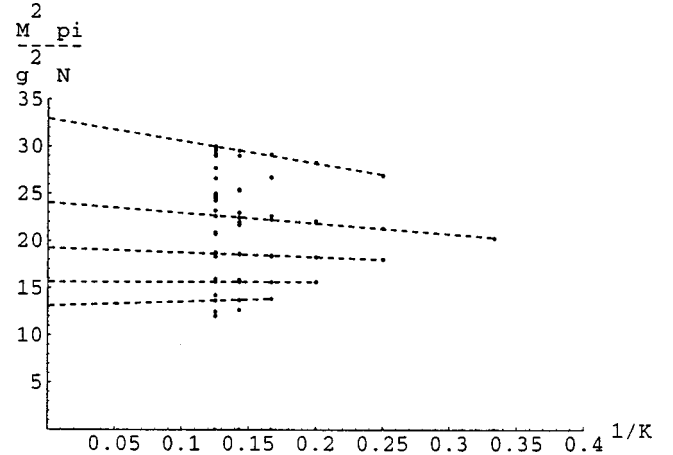


FIG. 1. Bound state masses M^2 (in units g^2N/π) versus $1/K$ for $N=10$.

any states at smaller resolution. These will signify the onset of additional bound states that may also be followed as K is increased. The first appearance of a state at a given resolution may be thought of as a “trail head” [14] of the corresponding continuum state. At large N , this procedure of finding “trails of bound states” is rather straightforward to carry out (although tedious). For intermediate values—say $N=10$ —the Fock state content of any bound state is complicated considerably due to the non-trivial mixing between states with differing numbers of traces, and the procedure of following the trail of a bound state at different resolutions must be administered with care.

In Fig. 1, we illustrate this procedure for the case $N = 10$, where the dashed lines represent trails of particular bound states, and we have extrapolated these curves to estimate continuum ($K = \infty$) bound state masses (we move from right to left on these curves as K is increased). In general, as we increase the resolution, states pick up additional Fock state contributions with a larger number of partons, but the approximate structure seen at lower resolutions is still clearly visible provided N is large. In order to determine the trails of these states for intermediate values of N , say $N=10$, we first consider the trail of a state for large N (say $N = 100$ or 1000), and then tune the value of N down to the desired smaller value; this effects a smooth change in wave function amplitudes, but the type of Fock states in the Fock state expansion remains unchanged. One can therefore be confident that one is tracking the correct state.

There are a number of striking features in the DLCQ spectrum of the theory. First, the low energy spectrum appears to be dominated by string-like states; each time we increase the resolution, a new massive state appears in the spectrum which is lighter than any of the massive states that appeared at a lower resolution. This can be seen in Fig. 1. In addition, the average number of partons in these states increases commensurately with the resolution K . So in the continuum $K \rightarrow \infty$, one can expect the existence of very light bound states that have an arbitrarily large number of partons. Since the spectrum is bounded from below (by supersymmetry), one deduces the existence of an accumulation point in the mass spectrum, which we denote by M_c . Bound states

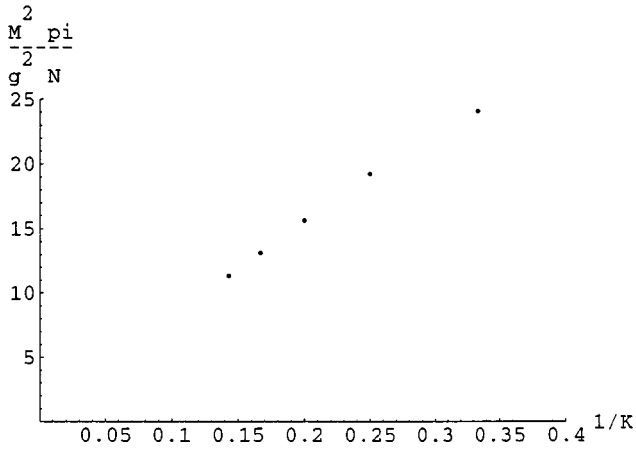


FIG. 2. Extrapolated continuum masses of lightest massive bound states for different resolutions K , and for $N=10$. The vertical axis is the extrapolated continuum mass of the lightest non-zero mass eigenstates that first appear at resolution K . The extrapolation of these points to $K=\infty$ gives an estimate for the accumulation point M_c^2 in the spectrum.

with masses at (or at least sufficiently near) this point are expected to behave like strings made out of an essentially infinite number of string “bits.” Evidently, pair creation of partons seems to be energetically favored. This behavior directly contrasts what is observed in many other models studied in the same framework [1]; namely, the mass of a state generally increases with the average number of partons in its Fock state expansion.

One interesting question that we are unable to answer is whether M_c is zero or not. We know that massless states do exist at any resolution (there are in fact $2^{(K-1)}$ of them at the resolution K), and it might seem reasonable that these light massive states approach the already existing massless states in the $K\rightarrow\infty$ limit. We see from Fig. 2 that the prediction for M_c , which is the extrapolation of the points to $K=\infty$, appears to be very close to zero, if not exactly zero. The horizontal axis is specified by $1/K$, where K is the resolution at which the lightest non-zero mass eigenstate first appears (i.e. has a “trail head”), and the vertical axis is its extrapolated continuum mass (i.e. where the extrapolation curves in Fig. 1 intersect the vertical axis). Due to extrapolation errors, and the low resolutions that were attainable, the uncertainties in Fig. 2 are expected to be quite large.

Another interesting feature of the DLCQ spectrum is that the extrapolation curves (dashed lines in Fig. 1) are relatively flat. This means that by performing a relatively trivial calculation at $K=3$ or 4, one is able to estimate the continuum bound state mass perhaps within ten percent of the actual continuum value (assuming no pathologies in the DLCQ spectrum for extremely large K). Of course, for the massless states, the curve is perfectly flat, and so we obtain exact information about the continuum spectrum (i.e. that there are massless states). There is an additional curious property about these massless states that appear in the DLCQ spectrum. It was shown recently [15] that any normalizable massless bound state in this theory is a superposition of an infinite number of Fock states. Of course, when one works in the

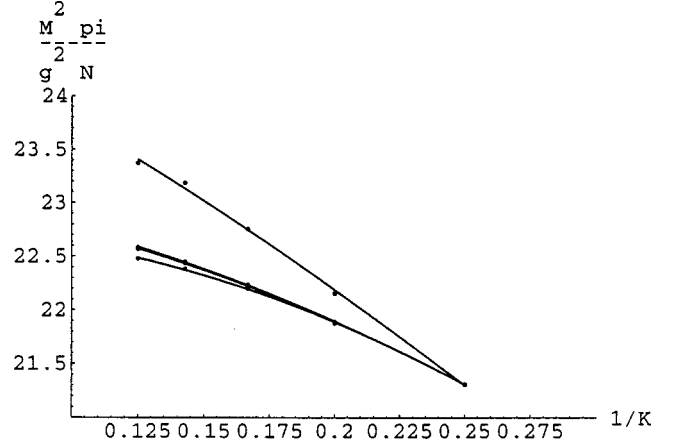


FIG. 3. Bound state masses versus $1/K$ for different N ; (a) $N=3$ (top curve), (b) $N=5$ (bottom curve), (c) $N=10$ (third from top) and (d) $N=100$ (second from top). The $N=1000$ curve is indistinguishable from the $N=100$ curve.

DLCQ formulation, the number of Fock states is finite. In our numerical analysis, however, we observe that the states are exactly massless at *any* resolution; increasing the resolution increases the complexity of the Fock state expansions of these massless states, but the masses are always precisely zero. Some very special cancellations are evidently responsible for protecting these massless states from receiving corrections due to the change in resolution. Note that this is suggestive of some kind of “duality;” namely, for K small, the problem is relatively easy to solve, and has a simple description in terms of a small number of degrees of freedom, while for $K\rightarrow\infty$, the complexity of the DLCQ problem increases dramatically, and the precise description of corresponding bound states is in terms of many more degrees of freedom. Nevertheless, the masses of certain states are preserved. It would be interesting to understand this from another point of view.

Since N is an algebraic variable in our calculations, we are able to investigate the changes in the masses of states as N is varied. As an illustration, we consider the mass of a state which has a “trail head” mass of $M^2=20.25g^2N/\pi$. See Fig. 3. The values of N are 3, 5, 10, 100 and 1000, and we consider the range $3\leq K\leq 8$ as usual.

Evidently, for $N=3$, the coupling $1/N$ is no longer negligible, and there is an apparent shift in the estimated continuum mass of the bound state. For $N>5$, convergence to the large N limit appears to be rather rapid. The general behavior at large N would be consistent with the interpretation $M^2\pi/g^2N\sim a-b/N^2$, where a and b are positive constants. We do not have a term linear in $1/N$ since one can show directly that $M^2\pi/g^2N$ is even under the interchange⁵ $N\rightarrow -N$. For $N=3$, it is clear from Fig. 3 that this picture is no longer valid, and one would expect relevant contributions at higher order in the $1/N$ expansion for M^2 .

⁵Recall that the DLCQ Hamiltonian is an algebraic function of N , and so we may analytically continue N to non-integer or negative integer values by direct substitution.

The multi-particle spectrum is a feature of the DLCQ spectrum that has only recently become of interest. It was pointed out in [14] that there may be bound states in the DLCQ spectrum at resolution K which may be thought of as two non-interacting bound states; this was verified by determining the masses $M^2(K-n)$ and $M^2(n)$ of bound states at resolutions $K-n$ and n respectively (n is positive integral), and showing that the light-cone energy relation for two free particles,

$$\frac{M^2(K)}{K} = \frac{M^2(K-n)}{K-n} + \frac{M^2(n)}{n}, \quad (12)$$

was obeyed. It is perhaps surprising that such a spectrum was found in [14], since the calculation was performed for $N = \infty$, where the Hilbert space consists of Fock states that are only *single* traces of parton creation operators. There is no obvious way of identifying single or multi-particle states in such a basis. In the case of finite but very large N , it is easy to see how the spectrum approaches a many body continuum; the basis now consists of multi-trace states, but interactions between bound states consisting of predominantly single trace Fock states are suppressed by $1/N$. Two body continuum in the spectrum are therefore obvious in our finite analysis, and we will discuss this in more detail shortly. Nevertheless, it is tempting to speculate on a possible explanation for the presence of these “multi-particle” states in the $N = \infty$ analysis; namely, following the work [15], one expects even at very large (but finite) N that any predominantly two trace bound state has a contribution from single trace Fock states. One sees this directly in the DLCQ analysis, of course. This suggests that it might be possible to “see” multi-trace bound states in the $N = \infty$ spectrum (where there are no multi-trace states in the Hilbert space) by virtue of the surviving single trace contributions. A more thorough numerical investigation will need to be carried out before this question can be properly resolved, since the above argument rests heavily on the dynamical properties of the theory.

We now return to the issue of multi-particle bound states in the context of our finite N calculations. For very large values of N , it is straightforward to identify in the bound state spectrum those states that are essentially two loosely bound particles; namely, any bound state that is predominantly a superposition of two-trace Fock states are obvious candidates. Of course, one needs to verify relation (12) before concluding that the bound state admits such a “two free-particle” interpretation. A representation of such calculations is given in Fig. 4.

After solving the DLCQ bound state equations for different resolutions, we are able to identify a predominantly single trace bound state with three partons in the trace. The extrapolated continuum mass is estimated by the solid curve in Fig. 4. One also discerns many massless states consisting of two partons. At resolution $K=5$, one finds a bound state consisting of two trace Fock states that is readily identified as the two bound states mentioned above that are essentially non-interacting. Its mass is predicted exactly by Eq. (12). At resolution $K=6$, there are two ways to form the state and at resolutions $K=7$ and $K=8$ there are three and four ways

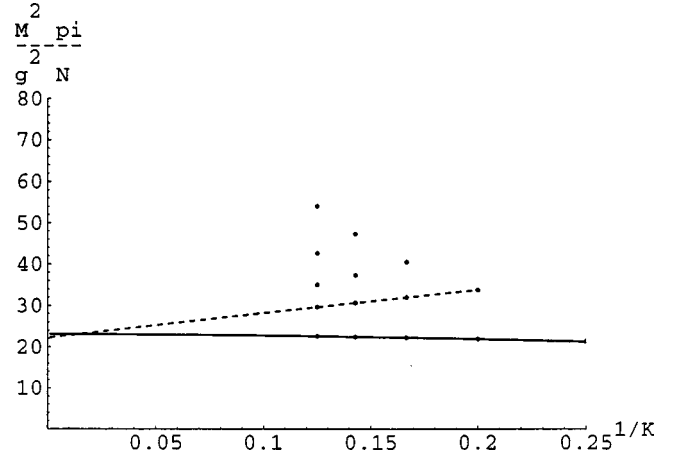


FIG. 4. Single and multi-particle bound state masses for $N = 1000$. The solid line represents the trail line of a predominantly single trace bound state. The remaining points represent this state bound with a massless bound state. Equation (12) predicts these masses to high precision, and we therefore expect two body continua in the spectrum in the limit $K \rightarrow \infty$.

respectively to form the state (see Fig. 4). We find all these states have a mass that is predicted by (12) to very high precision. What we are seeing is therefore the discrete realization of the two body continuum spectrum. We have made a best fit to the lowest two-body mass at each resolution, and we see that the extrapolated value coincides (within error) with the mass of the three parton massive bound state. This is of course expected, since the mass at threshold of a massive and massless state is precisely the mass of the massive state. In the DLCQ calculation, one finds that the masses of these two body states are highly degenerate. The degeneracy of the massive state is 4, while the degeneracy of the massless states is $2^{(\tilde{K}-1)}$, where \tilde{K} is the resolution of the two parton massless states appearing in the two-trace states. The total degeneracy is therefore expected to be $2^{(\tilde{K}+1)}$, which is indeed observed in the spectrum. As we increase the resolution the density of points will increase and effectively fill the continuum as a dense subset. What we have presented is an illustrative example, and we in fact find the same pattern for other combinations of states as well, including three body spectra, which all occur at the expected mass.

Of course, the above observations are expected as trivial realizations of the $1/N$ expansion. What is of interest is the modifications in the spectrum due to small but measurable contributions if we allow $1/N$ interactions to become important. As we mentioned earlier, studying the trail lines (i.e. tracking a particular state at different resolutions as in Fig. 1) becomes increasingly difficult if N is not very large. To assist one in establishing the correct trail lines, it is helpful to first consider identifying states at large N (say $N=1000$), and then following the state as N is lowered to the desired value. In Fig. 5 we perform this procedure, starting with the two body spectrum represented in Fig. 4 (where $N=1000$), and then eventually arriving at the spectrum for $N=10$. The obvious difference between Fig. 4 and Fig. 5 is that at $N = 10$, the mass splittings in the spectrum become discernible.

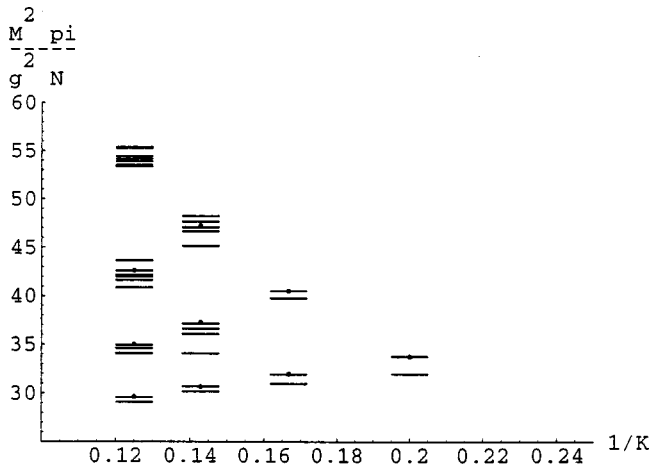


FIG. 5. Mass splittings for multi-particle bound states for $N = 10$. The horizontal lines are bound state masses, and the points are masses predicted by the two free particle formula (12). One sees the formation of bound states, suggesting that for very large (but finite) N , the asymptotic degeneracy of the spectrum could be quite complicated.

Note that there is a discontinuous change in the number of degrees of freedom at $N = \infty$, since at this point, the bound states at large but finite N will dissociate into their constituent particles at $N = \infty$. The presence of multi-particle bound states for finite N evidently provides scope for an exponential growth in the density of states.

The points in Fig. 5 are the values predicted by Eq. (12) at $N_c = 10$. Most of the states are below the threshold indicating that at $N_c = 10$ the interaction that mixes the various trace sectors is attractive and we consider these states to be bound states. Some of the states are above the threshold implying that they are candidate continuum states. The mass splittings introduced by $1/N$ interactions may push states above and/or below threshold, depending on the details of the interactions, and so determining the number of bound states in a theory at finite N is a highly non-trivial dynamical question.

At this point, we remark that the additional interactions we introduce as a result of working with finite N are suggestive of a system of weakly interacting hadrons; the case $N = \infty$ is analogous to a system of non-interacting colorless bound states, while the $1/N$ effects introduce the many subtle interactions that arise between colorless hadrons.

V. DISCUSSION

To summarize, we find that the low energy spectrum of (1,1) $SU(N)$ super-Yang-Mills theory in 1+1 dimensions is dominated by string-like states. This followed from the observation that increasing the DLCQ resolution introduces new lighter states that have on average more partons in their Fock state expansion than states at smaller resolutions. There is also strong numerical evidence that these states are normalizable, since one can keep track of these solutions as the resolution is increased, and we find that the Fock state amplitudes converge rapidly (see Fig. 1). It is therefore clear that pair creation of partons in this theory is not energetically

suppressed. String-like states were also found in a theory involving complex adjoint fermions, although their Fock state content was much simpler [16,17].

This immediately raises a question about the detailed structure of the spectrum. From supersymmetry, masses are bounded from below, and we therefore infer the existence of an accumulation point in the spectrum. Near this point, bound states consist of an arbitrarily large number of partons. Whether this accumulation point occurs at zero or positive mass was partly addressed in Fig. 2, and this still remains an open question. Nevertheless, these results suggest that the fundamental degrees of freedom in the theory (i.e. the normalizable bound states) may give rise to a continuous spectrum starting at (or close to) zero mass. Whether this is the signature of an additional hidden dimension, as was discussed in [11] in the context of the non-critical superstring in 2+1 dimensions, or the manifestation of screening [5,6], is still unclear. Nevertheless, it is clear that the model exhibits remarkably complicated low energy dynamics.

One of the main goals of this work was to go beyond the $N = \infty$ (or planar) approximation of gauge theories in order to study $1/N$ effects (e.g see Fig. 3). In the present context, the quantity $1/N$ plays the role of a coupling constant, and measures the strength of interactions between sectors in the Hilbert space that are characterized by the number of colorless traces in each Fock state. For N large (but finite), it is easy to identify two ‘‘loosely bound’’ particles in the spectrum, since it will be made up of predominantly two trace Fock states. We showed that the same strategy adopted in [14] to calculate the mass of two freely interacting bound states in the DLCQ spectrum applies equally well in the present context. Figure 4 illustrates the manifestation of such ‘‘two body’’ continua in the DLCQ spectrum.

For intermediate values of N , it is possible to measure the effects of $1/N$ interactions, and we have presented an illustration of the mass splittings that occur in Fig. 5. It is a dynamical question whether an attractive force will develop between particles that freely interact in the $N = \infty$ limit. Evidently, the formation of bound states is favored in the present model, and we are therefore faced with the interesting problem of counting the asymptotic degeneracies in the spectrum if N is made arbitrarily large (but finite). We were unable to address this question here. Note that the presence of very light string-like states suggests that the quantity $1/N$ plays the role of a string coupling constant [12]. It would be interesting to pursue these ideas further in the context of a two dimensional super-Yang-Mills realization of the ten dimensional critical string [18].

Finally, it has become evident recently that the properties of low dimensional super-Yang-Mills theory may provide a nonperturbative formulation of quantum theories with gravity. It would be interesting to explore this connection further by performing the sort of nonperturbative analyses presented here.

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