Instantons, three dimensional gauge theories, and monopole moduli spaces

Christophe Fraser*

*Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
and Department of Physics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, United Kingdom*

David Tong[†]

Department of Physics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, United Kingdom

We calculate instanton corrections to three dimensional gauge theories with $N=4$ and $N=8$ supersymmetry and $SU(n)$ gauge groups. The $N=4$ results give new information about the moduli space of n BPS $SU(2)$ monopoles, including the leading order non-pairwise interaction terms. A few comments are made on the relationship of the $N=8$ results to membrane scattering in matrix theory. [S0556-2821(98)03716-3]

I. INTRODUCTION

In the past year a remarkable relationship between three dimensional gauge theories and monopole moduli spaces has been uncovered. Following the work of Seiberg and Witten [1], Chalmers and Hanany [2] were the first to conjecture that the moduli space of n Bogomol'nyi-Prasad-Sommerfield (BPS) $SU(2)$ monopoles is equivalent to the vacuum moduli space of $SU(n)$ gauge theory in three dimensions with $N=4$ supersymmetry. This proposal found its natural setting in the work of Hanany and Witten [3], where configurations of 5 branes and 3 branes in type IIB string theory lead directly to the result.

The $SU(2)$ theory has subsequently been subjected to a first principles instanton calculation [4]. In this case the vacuum moduli space is severely restricted by the (super)-symmetries and perturbative sector of the theory, allowing for just a one parameter family of metrics. A one instanton calculation is sufficient to fix this parameter and the resulting metric is indeed that of the two monopole moduli space, known as the Atiyah-Hitchin metric [5].

In the following section we consider $N=4$ $SU(n)$ gauge theory in three dimensions. The corresponding n monopole $SU(2)$ moduli space is known only for well-separated monopoles [6]. We calculate instanton corrections in the three dimensional theory which correspond to the first exponential corrections to this metric. In three dimensions the relevant instantons are BPS monopole configurations.[1] We review such configurations in higher rank gauge groups with a Higgs field transforming under a global R symmetry. In the presence of extra Higgs fields, the zero modes of instantons are fewer than the single Higgs results [7,8] in a manner crucial for the interpretation of n-particle scattering. The non-zero modes around the background of the instanton are treated in the Gaussian approximation and, as in the $SU(2)$

case [4], there is a non-cancellation of bosonic and fermionic modes. However, unlike the situation for $SU(2)$ instantons, there exist *curves of marginal stability* (CMS) within the weak coupling regime of the moduli space of vacua upon which certain non-zero modes become zero-modes; i.e., the instanton moduli space is enlarged. For these modes the Gaussian approximation is not sufficient and we treat them exactly using the method of constrained instantons [9]. A potential is introduced on the enlarged instanton moduli space, reflecting the fact that these configurations are not in general solutions to the full equations of motion. We find that the potential is generated by the norm of the $U(1)$ Killing vectors of the instanton moduli space.

In Sec. III, we translate these results into the language of the moduli space of n monopoles of $SU(2)$ and find the leading order exponentially suppressed corrections to the metric of Gibbons and Manton [6]. The non-cancellation of the instanton background fluctuations leads to a structure for the metric corrections corresponding to non-pairwise interactions between monopoles. These corrections become singular in the limit of co-linear monopoles due to the extra zero modes appearing on the CMS. These singularities are resolved by the constrained instanton approach and we find the expected behavior in the limit of co-linearity.

Further applications of three dimensional instantons have arisen in the context of matrix theory [10]. Polchinski and Pouliot [11] related the dynamics of two membranes scattering with momentum transfer in the longitudinal direction to instantons in three dimensional $SU(2)$ gauge theory, this time with $N=8$ supersymmetries. The k-instanton corresponds to k units of transferred momentum. A one instanton calculation performed in [11] was found to be in agreement with the equivalent eleven dimensional supergravity calculation. Dorey, Khoze and Mattis [12] later performed the all-instanton calculation, retaining agreement with supergravity. The k-instanton contribution is proportional to the Euler character of the k-monopole moduli space (up to certain boundary terms which are proposed to vanish). In Sec. IV, we generalize this result to $SU(n)$ gauge groups. The extra supersymmetry means that the background fluctuations now cancel between bosons and fermions, ensuring that the corresponding membrane scattering acts in a pairwise manner.

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*Electronic address: pycf@swan.ac.uk

[†]Electronic address: pydt@swan.ac.uk

[1]To avoid confusion we will refer to these configurations as ''instantons'' with the term ''monopole'' reserved for the vacuum moduli space.

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II. THREE DIMENSIONAL INSTANTONS

$N=4$ supersymmetric gauge theory in three dimensions is best viewed as the dimensional reduction of the six dimensional $\mathcal{N}=1$ theory. The bosonic sector contains the three dimensional gauge field, A_μ, with field strength, $F_{\mu\nu}$, and three scalars, ϕ^i, $i=1,2,3$. The scalars transform as a vector under a global $SO(3)$, the remanent of the six dimensional Lorentz group. Following [1] we denote the double cover of this group as $SU(2)_N$.

The Weyl fermion of six dimensions decomposes as four two-component Majorana fermions in three dimensions, χ_α^m, $m=1,...,4$, $\alpha=1,2$. There exists a second R-symmetry, denoted $SU(2)_R$, under which the scalars are singlets. The fermions transform under both global symmetry groups, as the **4** of $Spin(4)\simeq SU(2)_N\times SU(2)_R$. All fields transform in the adjoint of the gauge group.

As is usual in theories with extended supersymmetry, the scalar potential, $V(\phi)=\frac{1}{2}\Sigma_{i,j}[\phi^i,\phi^j]^2$, has flat directions. The vacuum expectation value (VEV) of the scalars is taken to reside in **H**, the $(n-1)$-dimensional Cartan subalgebra (CSA) of $SU(n)$:[2]

$$\langle\phi^i\rangle=\mathbf{v}^i\cdot\mathbf{H}; \quad i=1,2,3. \tag{1}$$

For maximal symmetry breaking, $SU(n)\to U(1)^{n-1}$, we require $\|\mathbf{v}^i\cdot\boldsymbol{\alpha}\|\neq 0$, for all roots $\boldsymbol{\alpha}$ where $\|$ denotes the norm of the $SO(3)_N$ vector. This is assumed for the remainder of the paper.

Unlike the situation with a single Higgs field, for a generic vacua, the VEVs (1) do not pick out a unique set of simple roots, an observation at the heart of the zero mode structure for instantons in these theories. Although there is no unique choice, positive roots $\boldsymbol{\alpha}^A$, $A=1,...,\frac{1}{2}n(n-1)$, may always be defined by choosing a suitable constant 3-vector, ρ^i, and requiring $\rho^i\mathbf{v}^i\cdot\boldsymbol{\alpha}^A\geq 0$. We normalize the roots as $\boldsymbol{\alpha}^A\cdot\boldsymbol{\alpha}^A=1$ (no sum over A). Decomposing the fields into the Cartan-Weyl basis, those residing along the step operators $E_{\pm A}$ pick up masses $M_A=\|\mathbf{v}^i\cdot\boldsymbol{\alpha}^A\|$ by the adjoint Higgs mechanism. The fields residing in the Cartan subalgebra remain massless. The choice of positive roots defines a set of simple roots, $\boldsymbol{\beta}^a$, $a=1,...,n-1$, which we choose to define a (non-orthogonal) basis for the massless gauge fields:

$$A_\mu^a=\mathrm{Tr}(A_\mu\boldsymbol{\beta}^a\cdot\mathbf{H}), \quad a=1,\dots,n-1, \tag{2}$$

with similar definitions for the supersymmetric partners.

Concerning ourselves just with the massless fields the classical approximation to the Euclidean low-energy Lagrangian is a free Abelian theory, with bosonic sector

$$S_B=\frac{2\pi}{e^2}\int d^3x(K^{-1})_{ab}\left(\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^b+\frac{1}{2}\partial_\mu\phi^{ia}\partial_\mu\phi^{ib}\right) \tag{3}$$

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[2] We use bold type to denote vectors in the root space and a superscript i for 3-vectors transforming under $SO(3)_N$.

where the inverse Cartan matrix, K^{-1}, makes an appearance as the metric of the classical sigma model.

In the maximally broken Abelian theory, a surface term is included to count the winding of the gauge field at infinity. Defining $n-1$ winding numbers,

$$n_a=\frac{1}{8\pi}(K^{-1})_{ab}\int d^3x\epsilon_{\mu\nu\rho}\partial_\mu F_{\nu\rho}^b\in Z, \tag{4}$$

the surface term is given by $S_S=in_a\sigma^a$. The parameters, σ^a, can be thought of as Lagrange multipliers for the $U(1)$ Bianchi identities and, as is clear from Eq. (4), they range from 0 to 2π. Promoting each σ^a to a dynamical field, we integrate out the field strengths in favor of these periodic scalars to obtain the dual description of the classical low energy effective action with $4(n-1)$ massless scalars and $4(n-1)$ massless Majorana fermions,

$$S=\frac{2\pi}{e^2}\int d^3x(K^{-1})_{ab}\left(\frac{1}{2}\partial_\mu\phi^{ia}\partial_\mu\phi^{ib}\right.$$
$$\left.+\frac{e^4}{\pi^2(8\pi)^2}\frac{1}{2}\partial_\mu\sigma^a\partial_\mu\sigma^b+\frac{i}{2}\chi^{am}\gamma_\mu\partial_\mu\chi^{bm}\right) \tag{5}$$

where we take the three dimensional gamma matrices, γ_μ, to be the Pauli matrices $(\sigma^3,-\sigma^1,\sigma^2)$.

Let us re-examine the symmetries of the low energy theory. The VEVs generically spontaneously break the $SU(2)_N$ symmetry completely [for the $SU(2)$ gauge group, there remains an unbroken $U(1)_N$]. The low energy action, Eq. (5), has $n-1$ new Abelian symmetries, $\sigma^a\to\sigma^a+c^a$ for any constants c^a. Because of the additive nature of this transformation, these too are spontaneously broken. At the classical level, the vacuum moduli space is $(\mathbf{R}^3\times\mathbf{S}^1)^{n-1}/S_{n-1}$ where S_{n-1} is the Weyl group of $SU(n)$. This moduli space inherits the metric from the low-energy sigma model, classically, the inverse Cartan matrix acting on the $n-1$ copies of $\mathbf{R}^3\times\mathbf{S}^1$.

The $4(n-1)$ massless scalars (and fermions) remain massless in the full quantum theory [1]. The Wilsonian low-energy effective action, obtained by integrating out all massive modes, replaces $\delta_{ij}\times K_{ab}^{-1}$ with the quantum corrected metric g_{aibj}, now depending on the vacuum expectation values, (VEVs) of ϕ's and σ's, with $i,j=1,2,3,\sigma$. Four supersymmetries force g_{aibj} to be hyper-Kähler, while a non-anomalous $SO(3)_N$ global symmetry requires g_{aibj} to admit an $SO(3)$ isometry. It is proposed that g_{aibj} is the metric of the moduli space of n BPS monopoles with $SU(2)$ gauge group [1–3]. This metric is known to be complete, implying that the singularities of the classical vacuum moduli space arising as $M_A\to 0$ are resolved by strong coupling quantum effects.

Perturbatively, the $U(1)$ symmetries shifting the σ's are respected and corrections to the metric must contain $n-1$ Abelian isometries. Chalmers and Hanany [2] have confirmed that the perturbative corrections to g_{aibj} do indeed reproduce the asymptotic form of the n monopole moduli space discovered by Gibbons and Manton [6]. In the monopole picture, the $U(1)$ isometries correspond to the conser-

vation of electric charge of each individual dyon. Nonperturbatively, these $U(1)$ symmetries of the field theory are violated by instantons, which in the monopole picture leads to charge exchange between dyons as their cores overlap. We now examine these instantons in more detail.

A. Instanton zero modes

For $SU(n)$ we have different species of three dimensional instantons labelled by their winding number (4), which we take to define a charge vector in the root lattice, $\mathbf{g} = n_a \boldsymbol{\beta}^a$. The instantons of interest satisfy the Bogomol'nyi equation [7,14]

$$\mathcal{D}_\mu \phi^i = \lambda_{\mathbf{g}}^i B_\mu, \quad [\phi^i, \phi^j] = 0 \tag{6}$$

where $B_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\mu\nu}$ and $\lambda_{\mathbf{g}}^i$ is given by

$$\lambda_{\mathbf{g}}^i = \frac{\mathbf{v}^i \cdot \mathbf{g}}{\|\mathbf{v}^i \cdot \mathbf{g}\|}. \tag{7}$$

Solutions of Eqs. (6) have the property that they are annihilated by half the supersymmetries. The action of such a solution saturates the Bogomol'nyi bound and is given by

$$S_{\mathbf{g}} = \frac{8\pi^2}{e^2} \lambda_{\mathbf{g}}^i \mathbf{v}^i \cdot \mathbf{g} + i n_a \sigma^a. \tag{8}$$

The $i n_a \sigma^a$ term was first introduced by Polyakov [13] to incorporate the long range effects of instantons in the dilute gas approximation. In the present context it appears through the surface term of the action, S_S.

A class of explicit solutions can be constructed by embedding charge k $SU(2)$ instantons in the $SU(2)$ subgroup associated with $\boldsymbol{\alpha}^A$:

$$t^1 = \frac{1}{\sqrt{2}} (E_A + E_{-A})$$

$$t^2 = \frac{1}{\sqrt{2}i} (E_A - E_{-A})$$

$$t^3 = \boldsymbol{\alpha}^A \cdot \mathbf{H} \tag{9}$$

with a $\mathbf{g} = k \boldsymbol{\alpha}^A$ instanton solution obtained by

$$\phi^i = \lambda^i \phi^m(v) t^m + (\mathbf{v}^i - (\mathbf{v}^i \cdot \boldsymbol{\alpha}^A) \boldsymbol{\alpha}^A) \cdot \mathbf{H}$$

$$A_\mu = A_\mu^m(v) t^m \tag{10}$$

where $\phi^m(v)$ and $A_\mu^m(v)$ are the solution for BPS monopoles in $SU(2)$ with a single Higgs field of expectation value $v = \|\mathbf{v}^i \cdot \boldsymbol{\alpha}^A\|$.[3] In fact, we will see that the extra Higgs fields ensure that for generic values of the expectation values all

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[3]For a detailed review of BPS monopoles as 3D instantons see Appendix C of [4].

instantons are of this form. Below we apply the Callias index theorem and will infer that the only solutions of Eqs. (6) have charge vector $\mathbf{g} \propto \boldsymbol{\alpha}^A$ for some root $\boldsymbol{\alpha}^A$. This is in contrast to the situation with a single Higgs field where instantons exist for all charge vectors, $\mathbf{g} = \Sigma_a m_a \boldsymbol{\beta}^a$, with $4 \Sigma_a m_a$ zero modes [8]. That this is no longer the situation here is known from an analysis of monopoles in four dimensions [7,14], and can be anticipated from Eq. (1); the three VEVs do not pick out a unique set of simple roots, $\boldsymbol{\beta}^a$.

Rather than counting directly the number of bosonic zero modes, we determine the number of fermionic zero modes in the background of each instanton. The unbroken supersymmetry then pairs fermionic and bosonic zero modes. The Dirac equation reads

$$\Delta_{mn} \chi^n = (i \gamma_\mu \mathcal{D}_\mu \delta_{mn} - \eta_{mn}^i \phi^i) \chi^n \tag{11}$$

where η^i are the self-dual 't Hooft matrices and the covariant derivative is referred to the background field of the instanton.

Introducing the projection operators $P_\pm = \frac{1}{2}(1 \pm i \lambda_{\mathbf{g}}^i \eta^i)$ acting on the $Spin(4)$ vector space, we take products with the adjoint operator:

$$\Delta \Delta^\dagger = -\mathcal{D}_\mu \mathcal{D}_\mu - 2\gamma_\mu B_\mu P_+ + \phi^i \phi^i$$

$$\Delta^\dagger \Delta = -\mathcal{D}_\mu \mathcal{D}_\mu - 2\gamma_\mu B_\mu P_- + \phi^i \phi^i. \tag{12}$$

Observing that $\Delta^\dagger \Delta P_+$ is positive definite, all zero modes of Δ must lie in the eigenspace of P_- where $\Delta \Delta^\dagger$ is itself positive definite. Let Tr_- be the trace function restricted to this space, and following Weinberg [8] define

$$\mathcal{I}(\mu^2) = \mathrm{Tr}_- \left(\frac{\mu^2}{\Delta^\dagger \Delta + \mu^2} \right) - \mathrm{Tr}_- \left(\frac{\mu^2}{\Delta \Delta^\dagger + \mu^2} \right). \tag{13}$$

The number of fermionic zero modes is given by the limit $\mu^2 \to 0$ of $2\mathcal{I}(\mu^2)$. We rewrite $\phi^i \phi^i = \Phi \Phi + \hat{\phi}^i \hat{\phi}^i$ where

$$\Phi = \lambda_{\mathbf{g}}^i \phi^i, \quad \hat{\phi}^i = (\delta^{ij} - \lambda_{\mathbf{g}}^i \lambda_{\mathbf{g}}^j) \phi^j. \tag{14}$$

Note that the three $\hat{\phi}^i$ have only two independent degrees of freedom. With the exception of the extra Higgs fields, $\hat{\phi}^i \hat{\phi}^i$, Eq. (13) is the same as Weinberg's function [8]. In Appendix A we repeat Weinberg's calculation with this term and find

$$\mathcal{I}(\mu^2) = 2 \sum_A \frac{\mu^2 \lambda_{\mathbf{g}}^i (\mathbf{v}^i \cdot \boldsymbol{\alpha}^A)(\mathbf{g} \cdot \boldsymbol{\alpha}^A)}{(\|\hat{\mathbf{v}}^i \cdot \boldsymbol{\alpha}^A\|^2 + \mu^2)(\|\mathbf{v}^i \cdot \boldsymbol{\alpha}^A\|^2 + \mu^2)^{1/2}}. \tag{15}$$

We now see the consequences of the extra Higgs fields for the zero mode structure. For most charges, $\mathbf{g}$, the factor of μ^2 in the numerator of Eq. (15) means that there are no zero modes at all. For a non-zero $\mathcal{I}(M^2 \to 0)$, we require

$$\|\hat{\mathbf{v}}^i \cdot \boldsymbol{\alpha}^A\|^2 = (\mathbf{v}^i \cdot \boldsymbol{\alpha}^A)(\mathbf{v}^i \cdot \boldsymbol{\alpha}^A) - \lambda_{\mathbf{g}}^i \lambda_{\mathbf{g}}^j (\mathbf{v}^i \cdot \boldsymbol{\alpha}^A)(\mathbf{v}^j \cdot \boldsymbol{\alpha}^A) = 0 \tag{16}$$

for some root $\boldsymbol{\alpha}^A$ (no sum over A). For the case of generic VEV, the only solution is $\mathbf{g}$ aligned with $\boldsymbol{\alpha}^A$, say $\mathbf{g} = k \boldsymbol{\alpha}^A$. In this case we have

$$\lim_{\mu^2 \to 0} \mathcal{I}(\mu^2) = 2k \frac{\lambda_{\mathbf{g}}^i \mathbf{v}^i \cdot \boldsymbol{\alpha}^A}{\|\mathbf{v}^i \cdot \boldsymbol{\alpha}^A\|} = 2k. \tag{17}$$

For $\mathbf{g}$ not proportional to a root, Eqs. (6) have no zero modes and so at most only isolated solutions. Yet a soliton must have translational zero modes and we infer that, for generic expectation values, the Bogomol'nyi equation has no solutions in these sectors.

In certain vacua known as CMS,[4] where $\lambda_A^i = \lambda_B^i$ for some $A \neq B$, Eq. (16) has more solutions and the moduli space of the corresponding instanton enlarges. In the special case of $\lambda_A^i = \lambda_B^i$ for all roots $\boldsymbol{\alpha}^A$ and $\boldsymbol{\alpha}^B$, Eq. (15) reduces to Weinberg's expression [8].

Thus the extra Higgs fields have two effects: enforcing a democracy among roots and removing any off-root solution to the Bogomol'nyi equations. The former is already well known from studies of BPS spectra in four-dimensional theories with both $\mathcal{N}=2$ [7] and $\mathcal{N}=4$ [14] supersymmetries. The $\hat{\phi}^i \hat{\phi}^i$ terms punish any deviation from the $SU(2)$ subgroup (9), restricting the solution to be of the form (10). In [14] this is used to explain how, for a generic VEV, S-duality of the BPS spectrum of $\mathcal{N}=4$ theories with arbitrary simple gauge group is reduced to the equivalent problem in $SU(2)$, at least for charges proportional to roots. That no BPS solitons exist for charges not aligned with a root completes this argument.[5]

For the three dimensional case in hand, this null result for the zero modes of instantons with charge not aligned with a root means the action of any instanton in these sectors is raised above the Bogomol'nyi bound and thus the configurations break all of the supersymmetries. Acting with these broken supersymmetries gives rise to too many fermionic zero modes to contribute to the low energy Wilsonian effective action with two derivatives and four fermions. Similarly, on the CMS, instantons in these sectors must have at least eight fermionic zero modes, again too many to contribute. Thus we restrict our attention to charges aligned with roots, $\mathbf{g} = k\boldsymbol{\alpha}^A$, and denote $\lambda_{\mathbf{g}}^i = \lambda_A^i$.

Every such instanton has at least four bosonic modes, three corresponding to translations in space and time and one global $U(1)$ gauge transformation. The contribution of these modes to the bosonic measure is

$$\int d\mu_B = \int \frac{d^3 x}{(2\pi)^{3/2}} (\mathcal{J}_X)^{3/2} \int_0^{2\pi/k} \frac{d\theta}{(2\pi)^{1/2}} (\mathcal{J}_\theta)^{1/2} \tag{18}$$

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[4] The name derives from studies of spectra in four dimensional $\mathcal{N}=2$ theories where certain solitonic states are at threshold for decay. For gauge groups of rank $r \geq 3$ the CMS extend into the weak coupling regime.

[5] The above argument does not forbid "middle multiplets" with electric charge not parallel to magnetic charge. The zero modes for solutions with electric charge require a more delicate handling of the index theorem. We thank T. Hollowood for explaining this.

where $\mathcal{J}_X$ and $\mathcal{J}_\theta$ are the Jacobians resulting from the change of variables from fields to zero modes. These are calculated in Appendix C of [4] (see also [12]) for gauge group $SU(2)$. In the present case the instanton is restricted to an $SU(2)$ subgroup and the calculation of [4] generalizes trivially. For the instanton with $\mathbf{g} = \boldsymbol{\alpha}^A$, $\mathcal{J}_X = 8\pi^2 M_A/e^2$ and $\mathcal{J}_\theta = 8\pi^2/M_A e^2$.

Similarly, each configuration has at least two fermionic collective coordinates corresponding to broken supersymmetry generators. Supersymmetry transformations on the fermions with parameters ξ_α^m yield

$$\chi^m = -i\gamma_\mu B_\mu (P_-)_{mn} \xi^n. \tag{19}$$

For instantons, the required broken supersymmetries have ξ^m residing in the eigenspace of the projection operator P_-. If ξ^1 and ξ^2 are the two eigenvectors of P_-, the contribution of these modes to the fermionic measure is

$$\int d\mu_F = \int d^2 \xi^1 d^2 \xi^2 (\mathcal{J}_\xi)^{-2}. \tag{20}$$

The fermionic Jacobians, $\mathcal{J}_\xi = 16\pi^2 M_A/e^2$, are also calculated in Appendix C of [4].

The Grassmann integrations of Eq. (20) are saturated by the insertion of four Fermi fields in the path integral. If the instanton solution is to contribute to the low energy effective action at the two derivative and four Fermi field level, any further fermionic zero modes must be lifted. In Sec. IV, we will discuss the case of an adjoint massless matter multiplet ($N=8$ supersymmetry) where such lifting does indeed occur [12]. In the $N=4$ theory with no matter multiplets there is no mechanism for lifting extra fermionic zero modes and the only instantons that contribute must have the zero modes of equations Eqs. (18) and (20) and no others. Thus we restrict our attention yet again to $\mathbf{g} = \boldsymbol{\alpha}^A$ for each root $\boldsymbol{\alpha}^A$. Further contributions come from perturbative (two-loop) corrections about the background of these solutions and various numbers of instanton–anti-instanton pairs.

B. Instanton non-zero modes

Before integrating over zero modes, we must first deal with the non-zero fluctuations around the background of the instanton. Expanding about the configurations to quadratic order, the Gaussian integrations yield determinants of the quadratic fluctuation operators. In [4] these were found to be non-trivial for the case of $SU(2)$ and for higher rank gauge groups we find even more structure. Choosing the background gauge, $\mathcal{D}_\mu \delta A_\mu - i[\phi^i, \delta\phi^i] = 0$, we find that the contribution from the ghost fields cancels the fluctuations around $\hat{\phi}^i$. Supersymmetry ensures that the remaining bosonic and fermionic fluctuations are related and we find

$$R = \left(\frac{\det(-\mathcal{D}_\mu \mathcal{D}_\mu + \phi^i \phi^i)}{\det'(-\mathcal{D}_\mu \mathcal{D}_\mu - 2\gamma_\mu B_\mu + \phi^i \phi^i)} \right)^{1/2}$$

$$= \left(\frac{\det(\Delta\Delta^\dagger P_- + \Delta^\dagger \Delta P_+)}{\det'(\Delta\Delta^\dagger P_+ + \Delta^\dagger \Delta P_-)} \right)^{1/2} \tag{21}$$

where the operator in the denominator has zero eigenvalues and det$'$ denotes the removal of these from the determinant. Although supersymmetry insists the non-zero eigenvalues of the two operators in Eq. (21) are equal, the densities of these values are not. This was first noticed by Kaul [15] in the context of mass renormalization of monopoles. As explained in [4], the calculation of this ratio is essentially equivalent to the index calculation of Appendix A. More precisely, for the instanton $\mathbf{g} = \boldsymbol{\alpha}^A$, we have

$$R = \lim_{\mu \to 0} \left[\mu^2 \exp\left(\int_\mu^\infty \frac{d\nu}{\nu} \mathcal{I}(\nu) \right) \right]^{1/2}$$

$$= 2 M_A \prod_{B \neq A} \left[\frac{1 + \lambda_A^i \lambda_B^i}{1 - \lambda_A^i \lambda_B^i} \right]^{\boldsymbol{\alpha}^A \cdot \boldsymbol{\alpha}^B}. \qquad (22)$$

The expression for the determinants clearly diverges as the VEVs approach the CMS i.e. $\lambda_A^i = \lambda_B^i$. This is not unexpected. On the CMS the instanton moduli space enlarges and the quadratic fluctuation operators gain extra zero eigenvalues. The singularities are the result of treating these modes in the Gaussian approximation. To make progress we must treat these modes exactly. We then expect the instanton calculation to yield zero when the VEVs lie on the CMS, for the instanton will have fermionic zero modes which cannot be saturated in the path integral.

C. Instanton soft modes

Close to the CMS, the modes that become zero modes are soft; that is the eigenvalues of the quadratic fluctuation operators are small. The existence of these modes is reminiscent of the more familiar situation of four dimensional instantons where a self-dual field strength ceases to satisfy the equations of motion when an adjoint scalar has a non-zero VEV and the one-instanton moduli space is lifted, leaving just the singular point at the origin. However, the self-dual configurations retain their importance in the semi-classical expansion. The correct technique for dealing with such modes is known as the constrained instanton [9] (for a detailed account applied to four dimensional $\mathcal{N} = 2$ theories see also [16]). At short distances the equations of motion are solved perturbatively in $g^2 \rho^2 v^2$ where ρ is the scale size of the instanton. This allows all self-dual configurations to be treated exactly in the semi-classical expansion. The action of these configurations gains a ρ dependence, ensuring that the contribution of the larger instantons to the path integral are suitably suppressed.

The three dimensional situation is analogous. The VEVs of $\hat{\phi}^i$ lift certain solutions to the equations of motion which still remain important in the semi-classical expansion near the CMS.[6] However, the details of the lifting of the instanton

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[6]This correspondence is emphasized further if we trace the three dimensional theory back to its $\mathcal{N} = 2$ four dimensional roots, combining A_μ and Φ to construct a self-dual gauge field. The two independent degrees of freedom in $\hat{\phi}^i$ create the complex scalar field and the generic self-dual field strength no longer satisfies the equations of motion when this scalar has a VEV.

moduli spaces are more complicated. A zero VEV for all $\hat{\phi}^i$ corresponds to the intersection of all the CMS and the moduli space of solutions is given by the single Higgs results of Weinberg [8]. Turning on VEVs for $\hat{\phi}^i$ generically means a departure from the CMS and the moduli space is lifted, although a non-trivial submanifold may remain. Moreover, by varying the VEVs in special directions along the CMS intermediate situations are possible with submanifolds of exact solutions of varying dimensions.

Employing the constrained instanton, we relax the conditions on the configurations about which we perform the semi-classical expansion. Rather than insisting that configurations be a minimum of the action, in the short distance regime, $x \ll 1/M_A$, we solve the equations of motion perturbatively in an appropriate parameter, generically $e^2 r^2 \|\hat{\mathbf{v}}^i\|^2 / \|\mathbf{v}^i\|^2$ where r denotes all radial parameters on the largest instanton moduli space. In the language of Weinberg [8], for large r, they are the collective coordinates obtained by separating two "fundamental" instantons.

The approach of solving the equations perturbatively also applies to the auxiliary fields which we have so far neglected. There exist three auxiliary fields, F^i, one for each $N = 1$ scalar multiplet. There are no auxiliary fields from gauge multiplets in three dimensions. In the $N = 4$ theory F^i satisfy the equation of motion

$$F^i = -i \epsilon^{ijk} \phi^j \phi^k. \qquad (23)$$

Writing $\hat{F}^i = F^i - \lambda_A^i \lambda_A^j F^j$, the defining equations for the constrained instanton at short distance are given by

$$\mathcal{D}_\mu \Phi = B_\mu \qquad (24)$$

$$\gamma_\mu \mathcal{D}_\mu \chi^m - [\Phi, \chi^m] = 0 \qquad (25)$$

$$(P_-)_{mn} \chi^n = \chi^m \qquad (26)$$

$$\mathcal{D}_\mu \mathcal{D}_\mu \hat{\phi}^i - [\Phi, [\Phi, \hat{\phi}^i]] = -\eta_{mn}^i \chi^m \chi^n \qquad (27)$$

$$\hat{F}^i = i(\lambda_A^i \lambda_A^j \epsilon^{jkl} - \epsilon^{ikl}) \phi^k \phi^l \qquad (28)$$

$$\lambda_A^i F^i = 0. \qquad (29)$$

The bosonic moduli space of such solutions is determined solely by solutions to Eq. (24). In the topological sector $\mathbf{g} = \boldsymbol{\alpha}^A$, the VEV $\lambda_A^i \mathbf{v}^i$ picks auxiliary simple roots, $\boldsymbol{\gamma}^a$ such that $\boldsymbol{\alpha}^A = \Sigma_a m_a \boldsymbol{\gamma}^a$. The moduli space of solutions is of dimension $4 \Sigma_a m_a$ [8]. Notice that the roots $\boldsymbol{\gamma}^a$ differ from sector to sector and need not coincide with the $\boldsymbol{\beta}^a$ defined earlier. This means the relative dimensions of the moduli spaces of Eq. (24) in different sectors do not follow the simple pattern of the single Higgs boson model. Moreover, in varying the $\mathbf{v}^i$ it is possible for $\lambda_A^i \mathbf{v}^i$ to cross the wall of a Weyl chamber without the associated non-maximal symmetry breaking of the single Higgs boson model and thus the moduli space may change discontinuously. However, after integration over these manifolds, the final instanton calculation will be smooth.

The general moduli space decomposes into the form

$$\mathcal{M} = \mathbf{R}^3 \times \frac{\mathbf{R} \times \tilde{\mathcal{M}}_d}{Z}. \quad (30)$$

$\tilde{\mathcal{M}}_d$ are complete hyper-Kähler manifolds with coordinates X^a and metric $\tilde{g}_{ab}$. For $\mathbf{g} = \boldsymbol{\alpha}^A$, $\tilde{\mathcal{M}}_d$ are the Lee-Weinberg-Yi spaces [17]. $\tilde{\mathcal{M}}_d$ has dimension $4(d-1)$ where d is the height of the root $\boldsymbol{\alpha}^A$ as measured by $\boldsymbol{\gamma}^a$ i.e. $d = \Sigma_a m_a$. In the standard notation, $\tilde{\mathcal{M}}_d = \tilde{\mathcal{M}}_{(1,1,\ldots,1)}$ where there are d 1's in the string. For $\boldsymbol{\alpha}^A$ simple with respect to $\lambda_A^i \mathbf{v}^i$ ($d=1$), $\tilde{\mathcal{M}}_1$ is taken to be a single point.

The $\mathbf{R}^3$ factor in Eq. (30) corresponds to space-time translations of the instanton while the $\mathbf{R}$ factor is generated by global $U(1)$ gauge transformations

$$\mathbf{Q} \cdot \mathbf{H} = \frac{\Sigma_a(\lambda_A^i \mathbf{v}^i \cdot \boldsymbol{\gamma}^a) \boldsymbol{\omega}^a \cdot \mathbf{H}}{\Sigma_b \lambda_A^i \mathbf{v}^i \cdot \boldsymbol{\gamma}^b} \quad (31)$$

where $\boldsymbol{\omega}^a$ are the fundamental weights defined by $\boldsymbol{\gamma}^a \cdot \boldsymbol{\omega}^b = \frac{1}{2}\delta^{ab}$. When the ratios of the $\lambda_A^i \mathbf{v}^i \cdot \boldsymbol{\gamma}^a$ are rational the $\mathbf{R}$ factor collapses to S^1 and the Z to the cyclic subgroup Z_d.

The remaining $n-2$ $U(1)$ gauge transformations generated by elements of the CSA orthogonal to $\boldsymbol{\alpha}^A$ result in up to $n-2$ $U(1)$ isometries of $\tilde{\mathcal{M}}$. We denote as $\mathbf{K}^a$ the Killing vector of $\mathcal{M}$ generated by $\mathbf{H}$. If a particular element of the CSA acts trivially on the configuration, the corresponding Killing vector is taken to be zero.

Configurations satisfying Eqs. (24)–(29) raise the action above the Bogomol'nyi bound (8). Moreover, this action will have a dependence on the collective coordinates of $\tilde{\mathcal{M}}_d$; we can consider the action as defining a potential on $\tilde{\mathcal{M}}_d$. The bosonic contribution to this potential is

$$\Delta S_{\text{boson}} = \frac{2\pi}{e^2} \int d^3x \frac{1}{2} \mathcal{D}_\mu \hat{\phi}^i \mathcal{D}_\mu \hat{\phi}^i + \frac{1}{2} F^i F^i$$

$$= \frac{2\pi}{e^2} \int d^3x \frac{1}{2} \mathcal{D}_\mu \hat{\phi}^i \mathcal{D}_\mu \hat{\phi}^i - \frac{1}{2}[\Phi, \hat{\phi}^i]^2. \quad (32)$$

We will now show that this potential is related to the $U(1)$ Killing vectors on the moduli space $\mathcal{M}$. First, consider a configuration of A_μ and Φ satisfying Eq. (24) and thus corresponding to a point $\mathcal{M}$. Act on these fields with a gauge transformation parametrized by $\hat{\phi}^i$:

$$\delta^i A_\mu = \mathcal{D}_\mu \hat{\phi}^i, \quad \delta^i \Phi = -i[\Phi, \hat{\phi}^i]. \quad (33)$$

Such transformations clearly satisfy the linearized version of the Bogomol'nyi equation (24). A global (large) gauge transformation is generated by $\hat{\mathbf{v}}^i \cdot \mathbf{H}$, the VEV of $\hat{\phi}^i$, while the spatially dependent part is a small gauge transformation. However, in order to be true zero modes of the configuration, the transformations must also satisfy a gauge condition, which for the purely bosonic theory is Gauss' law:

$$\mathcal{D}_\mu \delta A_\mu - i[\Phi, \delta\Phi] = 0. \quad (34)$$

We see that, up to Fermi bilinears, this equation is indeed satisfied by virtue of Eq. (27). Thus the gauge transformations above are indeed true zero modes. Moreover, gauge transformations move the configuration infinitesimally along the isometries of $\mathcal{M}$. By construction $\boldsymbol{\alpha}^A \cdot \hat{\mathbf{v}}^i = 0$ and gauge transformations (33) correspond to elements of the CSA responsible for generating the Killing vectors on $\tilde{\mathcal{M}}_d$ together with a VEV dependent shift along the $\mathbf{R}$ factor. Thus, the bosonic part of the action can be rewritten as

$$\Delta S_{\text{boson}} = \frac{4\pi^2}{M_A e^2} \|\mathbf{Q} \cdot \hat{\mathbf{v}}^i\| + \frac{1}{2} \tilde{g}_{ab}(\hat{\mathbf{v}}^i \cdot \mathbf{K}^a)(\hat{\mathbf{v}}^i \cdot \mathbf{K}^b). \quad (35)$$

Equations (24)–(29) are covariant under supersymmetry transformations parametrized by $(P_+)_{mn}\xi^n$, ensuring that the total action of these configurations inherits a supersymmetry acting on the collective coordinates. Indeed, we may always replace the sum over Killing vectors in the second term of Eq. (35) by a single Killing vector, $\tilde{g}_{ab}(\hat{\mathbf{v}}^i \cdot \mathbf{K}^a)(\hat{\mathbf{v}}^i \cdot \mathbf{K}^b) = \tilde{g}_{ab}(\mathbf{V} \cdot \mathbf{K}^a)(\mathbf{V} \cdot \mathbf{K}^b)$, which is the form dictated by supersymmetry [18]. The fermionic part of the action is simply the supersymmetric completion of Eq. (35):

$$\Delta S_{\text{fermion}} = \frac{i}{2} \nabla_a(\mathbf{V} \cdot \mathbf{K}_b) \psi^a \psi^b \quad (36)$$

where ∇_a is the covariant derivative on $\mathcal{M}$ with respect to the Levi-Civita connection and ψ^a are fermionic collective coordinates.

We turn finally to the measure. As neither the metric nor the integrand depend upon the coordinates associated with the $U(1)$ isometries, the bosonic and fermionic measures for the $\mathbf{R}^3 \times \mathbf{R}$ factor of the moduli space are given (after taking into account the discrete group Z) by Eqs. (18) and (20) respectively. This leaves us with the integrations over the multi-cover of the Lee-Weinberg-Yi space, $\tilde{\mathcal{M}}_d$. Although expressions for the corresponding zero modes are not known, the Jacobians depend on the metric, $\tilde{g}_{ab}$ only,

$$\int d\tilde{\mu}_B = \int \prod_{a=1}^{4(d-1)} dX^a \frac{\sqrt{\det \tilde{g}}}{(2\pi)^{2(d-1)}}$$

$$\int d\tilde{\mu}_F = \int \prod_{a=1}^{4(d-1)} d\psi^a (\det \tilde{g})^{-1/2}. \quad (37)$$

Note that the metric dependences in the bosonic and fermionic measures cancel.

D. Instanton calculation

Having analyzed the various fluctuations around the background of the instanton, it is now possible to put all the pieces together.

In each topological sector, defined by $\boldsymbol{\alpha}^A$, we must calculate the height, d, of $\boldsymbol{\alpha}^A$ with respect to $\boldsymbol{\gamma}^\mu$. If $\lambda_A^i \mathbf{v}^i \cdot \boldsymbol{\alpha}^B = 0$ for some root $\boldsymbol{\alpha}^B$, the VEV lies on the wall of a Weyl chamber; it does not define a unique set of $\boldsymbol{\gamma}^\mu$ and the height of $\boldsymbol{\alpha}^A$ and hence the moduli space $\tilde{\mathcal{M}}_d$ is ambiguous. We will comment on this case at the end of this section. For now we assume that the VEV $\lambda_A^i \mathbf{v}^i$ lies strictly within a Weyl chamber. The integration over soft modes is then given by

$$L_d(\mathbf{v}^i) = (2\pi)^{2(1-d)} \int_{\tilde{\mathcal{M}}_d} dX^a d\psi^a$$

$$\times \exp(-\Delta S_{\text{boson}} - \Delta S_{\text{fermion}}). \qquad (38)$$

For $d=1$ we set $L_1=1$.

To avoid overcounting, we must divide by the Gaussian approximation for these modes which we have already taken into account when integrating over non-zero modes (22). The recipe for this is to transform to polar coordinates for $\tilde{\mathcal{M}}_d$ such that in the vicinity of $r=0$, the metric is of the form $\tilde{g}_{ab} = \tilde{g}_{ab}^{\text{flat}}[1+O(r^2)]$ where r denotes all $d-1$ radial coordinates on the space. The Gaussian approximation requires a truncation of this metric to $\tilde{g}_{ab}^{\text{flat}}$ with the corresponding truncation to the bosonic potential and fermionic potential. We will denote this integral as G_d, again with $G_1=1$. Note that the first term in Eq. (35) is independent of collective coordinates and will be cancelled after division by the Gaussian approximation.

For $d=2$ the relevant manifold is Taub-NUT (Newman-Unti-Tamburino) space [19,20]. In Appendix B we calculate L_2 and its Gaussian approximation.

With all zero mode fields now confined to an $SU(2)$ subgroup the remainder of the instanton calculation for the four fermion vertex now proceeds as in [4], with the resulting correlator related to the Riemann tensor of the monopole metric. We choose instead to calculate instanton contributions to the scalar propagator which will provide direct information about the inverse metric. To saturate the fermionic zero modes of the instanton, the scalars must themselves pick up Fermi bilinears. Acting on ϕ^i with a finite supersymmetry transformation, $\exp(-\xi^m Q^m)$, yields

$$\phi^i \rightarrow \phi^i - B_\mu \eta_{mn}^i \xi^m \gamma_\mu \xi^n. \qquad (39)$$

For ξ^m, an eigenvector of P_-, only $\hat{\phi}^i$ pick up these bilinears. Φ remains unchanged by the supersymmetry transformations. The contributions from the $\boldsymbol{\alpha}^A$ instanton are thus

$$\langle \Phi^a \Phi^b \rangle_A = 0$$

$$\langle \Phi^a \hat{\phi}^{ib} \rangle_A = 0 \qquad (40)$$

and

$$\langle \hat{\phi}^{ia} \hat{\phi}^{jb} \rangle_A = (\boldsymbol{\beta}^a \cdot \boldsymbol{\alpha}^A)(\boldsymbol{\beta}^b \cdot \boldsymbol{\alpha}^A) \frac{L_d(\mathbf{v}^i)}{G_d(\mathbf{v}^i)}$$

$$\times \prod_{B \neq A} \left[\frac{1+\lambda_A^i \lambda_B^i}{1-\lambda_A^i \lambda_B^i} \right]^{\boldsymbol{\alpha}^A \cdot \boldsymbol{\alpha}^B} \langle \hat{\phi}^i \hat{\phi}^j \rangle_{\lambda^i \mathbf{v}^i \cdot \boldsymbol{\alpha}^A} \quad (41)$$

where $\langle \hat{\phi}^i \hat{\phi}^j \rangle_{\lambda^i \mathbf{v}^i \cdot \boldsymbol{\alpha}^A}$ is the scalar propagator in an $SU(2)$ gauge theory with VEV $\lambda \mathbf{v}^i \cdot \boldsymbol{\alpha}^A$. It was shown explicitly in [4] that this scalar propagator reproduces the leading order exponential corrections to the inverse Atiyah-Hitchin metric.

Close to the CMS, L_d/G_d is small and cancels the singularities in the product factor. Far from the CMS, L_d/G_d is exponentially close to unity. Furthermore, in this regime these exponential deviations from unity are of the same magnitude as other effects that we have neglected such as two-loop perturbation theory around the background of the instanton and instanton–anti-instanton pairs.

Finally we turn the the situation where $\lambda_A^i \mathbf{v}^i \cdot \boldsymbol{\alpha}^B = 0$ for some root $\boldsymbol{\alpha}^B$, i.e., $\lambda_A^i \mathbf{v}^i$ lies on the wall of a Weyl chamber. In this case the moduli space $\mathcal{M}_d$ is not well defined. However, this occurs in a regime far from the CMS and thus corrections from the constrained instanton are not important here. Nevertheless, it is interesting to note that L_d/G_d does indeed remain smooth as $\lambda_A^i \mathbf{v}^i$ crosses the wall of the Weyl chamber.

III. MONOPOLE MODULI SPACES

We now translate the results of the previous section into the metric on the moduli space of n BPS $SU(2)$ monopoles. This $4n$ dimensional space has the form

$$\mathcal{M}_n = R^3 \times \frac{S^1 \times \tilde{\mathcal{M}}_n}{Z_n}. \qquad (42)$$

R^3 corresponds to (Euclidean) space-time translations of the center of mass and S^1 to global $U(1)$ gauge transformations. $\tilde{\mathcal{M}}_n$ is the relative n-monopole moduli space; it has dimension $4(n-1)$, is complete and hyper-Kähler.

The perturbative sector of the three dimensional $SU(n)$ gauge theory reproduces the asymptotic metric on $\tilde{\mathcal{M}}_n$ [2]. In this regime monopoles interact pairwise via velocity dependent $U(1)$ electric, magnetic and scalar forces and the metric takes a simple form discovered by Gibbons and Manton [6]:

$$ds^2 = M_{ij} d\vec{x}_i \cdot d\vec{x}_j + M_{ij}^{-1} \left(d\theta_i + \sum_k \vec{W}_{ik} \cdot d\vec{x}_k \right)$$

$$\times \left(d\theta_j + \sum_l \vec{W}_{jl} \cdot d\vec{x}_l \right) \qquad (43)$$

where

$$M_{ii} = 1 - \sum_{j \neq i} \frac{1}{r_{ij}}, \quad M_{ij} = \frac{1}{r_{ij}} \quad (i \neq j)$$

$$\vec{W}_{ii} = -\sum_{j \neq i} \vec{w}_{ij}, \quad \vec{W}_{ij} = \vec{w}_{ij} \quad (i \neq j) \tag{44}$$

with the Dirac potential, $\vec{w}_{ij}$, defined by $\nabla_i \times \vec{w}_{ij} = \nabla_i(1/r_{ij})$. The $\vec{x}_i$ are the positions of the well-separated monopoles. This is the metric on $\mathcal{M}_n$; that is it includes the motion of the center of mass and center of charge of the monopole configuration. This corresponds to the three dimensional $U(n)$ gauge theory. In order to compare with the $SU(n)$ results above we must freeze the centers of mass and charge from this metric. For well separated monopoles, the $4 \times (n-1)$ coordinates on this space are a basis chosen from the $4 \times \frac{1}{2}n(n-1)$ relative separations and relative charges:

$$\vec{r}_{ij} = \vec{x}_i - \vec{x}_j, \quad \psi_{ij} = \theta_i - \theta_j. \tag{45}$$

An explicit hyper-Kähler quotient of the Gibbons-Manton metric yields a messy result, essentially because the metric is symmetric in all $\frac{1}{2}n(n-1)$ of the relative coordinates but is expressed in only an $(n-1)$ dimensional subset of these. To retain manifest permutation symmetry of the relative-monopole metric we choose to write it as a metric on a larger $4 \times \frac{1}{2}n(n-1)$ dimensional manifold, the pullback of which yields the required quotient of the Gibbons-Manton metric. We take the relative distances, r^A, relative Euler angles, θ^A and ϕ^A, and relative charges, ψ^A, $A = 1,...,\frac{1}{2}n(n-1)$ and write the metric in the form

$$ds^2 = \frac{1}{2} \sum_A f^2(r^A) dr^A dr^A + a^2(r^A)(\sigma_1^A)^2$$
$$+ b^2(r^A)(\sigma_2^A)^2 + c^2(r^A; r^B)(\sigma_3^A)^2 \tag{46}$$

where all summations have been kept explicit. f, a, b and c take the form

$$f(r^A) = -\left(\frac{2}{n} - 2M_A\right)^{1/2}$$

$$a(r^A) = b(r^A) = r^A \left(\frac{2}{n} - 2M_A\right)^{1/2}$$

$$c(r^A; r^B) = \left(\frac{1}{2n} - \frac{1}{2}(M^{-1})_A\right)^{-1/2} \tag{47}$$

where $M_A = M_{ij}$ for A labelling the separation (ij). The one-forms σ_i^A are defined as

$$\sigma_1^A = -\sin \psi^A d\theta^A + \cos \psi^A \sin \theta^A d\phi^A$$

$$\sigma_2^A = \cos \psi^A d\theta^A + \sin \psi^A \sin \theta^A d\phi^A$$

$$\sigma_3^A = d\psi^A + \frac{2}{n-1} \sum_B \Omega^{AB} \cos \theta^B d\phi^B \tag{48}$$

where Ω^{AB} is non-zero only if the separations A and B have a monopole in common. More precisely, if the relative Cartesian separation vector $\vec{r}^A$ goes from the ith to the jth monopole and $\vec{r}^B$ from the kth to the lth,

$$\Omega^{AB} = \frac{1}{2}(\delta^{ik} + \delta^{jl} - \delta^{il} - \delta^{jk}). \tag{49}$$

To recover the relative-monopole Gibbons-Manton metric, we must first pick a linearly independent set of $n-1$ separations (3-vector and charge). Labelling the monopoles in some arbitrary manner, we choose the separations between the ith monopole and the $(i+1)^{\text{th}}$. Labelling this set of linearly independent coordinates with a subscript a, $(\vec{r}^a, \psi^a)$, the metric on $\tilde{\mathcal{M}}_n$ is obtained by pulling back with the map, f_a^A, relating $\vec{r}^A$ and $\vec{r}^a$:[7]

$$\vec{r}^A = f_a^A \vec{r}^a. \tag{50}$$

Pulling back the flat metric gives the inverse Cartan matrix that was the classical metric of Eq. (3),

$$\sum_A f_a^A f_b^A = \frac{n}{2}(K^{-1})_{ab}. \tag{51}$$

At this stage we need to introduce a dictionary between objects in three dimensions and the above coordinates on the monopole moduli space. First, we note that the choice of separations between n monopoles discussed above is mimicked by the root structure of $su(n)$. The map f_a^A is the map between simple roots, $\boldsymbol{\beta}^a$, and roots, $\boldsymbol{\alpha}^A$. Defining the roots as $\boldsymbol{\alpha}^A = (\mathbf{e}^i - \mathbf{e}^j)/\sqrt{2}$ and $\boldsymbol{\alpha}^B = (\mathbf{e}^k - \mathbf{e}^l)/\sqrt{2}$, where $\mathbf{e}^i$ are n orthonormal vectors, then $\Omega^{AB} = \boldsymbol{\alpha}^A \cdot \boldsymbol{\alpha}^B$. We must identify the three scalar VEVs along the $\boldsymbol{\alpha}^A$ direction of root space with the vector distance between the ith and j^{th} monopole:[8]

$$\vec{r}^a = r^{ai} = \frac{8\pi^2}{e^2} \mathbf{v}^i \cdot \boldsymbol{\beta}^a$$

$$\psi^a = \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\beta}^a. \tag{52}$$

Other expressions from the instanton calculation also have a simple geometrical meaning. In the topological sector, $\mathbf{g} = \boldsymbol{\alpha}^A$, we have $\lambda_A^i = r^{Ai}/r^A$ and

$$\mathbf{v}^i \cdot \mathbf{v}^i = \left(\frac{e^2}{8\pi^2}\right)^2 \frac{2}{n} \sum_B (r^B)^2$$

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[7]Pulling back in polar coordinates is extremely messy. The simplest way to see this result is to first add terms corresponding to the center of mass and charge, change to Cartesian coordinates and pull back to recover the original Gibbons-Manton metric.

[8]To retain agreement between Eqs. (3) and (46), the metric should be premultiplied by $e^2/16n\pi^3$. This will not be important in what follows.

$$\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i = \left(\frac{e^2}{8\pi^2}\right)^2 \frac{2}{n} \sum_B (r_\perp^B)^2 \qquad (53)$$

where $r_\perp^{Bi}$ is the component of r^{Bi} perpendicular to r^{Ai}.

The semi-classical approximation translates to the requirement that the monopoles be well separated; we will find the leading order corrections to the Gibbons-Manton metric. Geometrically, the curves of marginal stability correspond to three or more monopoles becoming colinear. In this regime the corrections from the constrained instanton (38) will have the most impact. The other point of interest is when the VEV $\lambda_A^i \mathbf{v}^i$ lies on the wall of a Weyl chamber. This corresponds to $r^{Ai} r^{Bi} = 0$ for some separation B. The form of the corrections to the metric will be different on each side of this situation but will meet smoothly on the wall itself.

We are now in a position to recast the instanton contribution to the scalar propagator (41) as the exponential corrections to the n-monopole metric. Corrections to the scalar propagator equate to corrections to the inverse metric

$$\delta g^{aibj} = \sum_A (\boldsymbol{\alpha}^A \cdot \boldsymbol{\beta}^a)(\boldsymbol{\alpha}^A \cdot \boldsymbol{\beta}^b)\langle \phi^i \phi^j \rangle_A. \qquad (54)$$

To leading order in $1/r$, the inverse of this is

$$\delta g_{aibj} = -(K^{-1})_{ac}(K^{-1})_{db} \sum_A (\boldsymbol{\alpha}^A \cdot \boldsymbol{\beta}^c)(\boldsymbol{\alpha}^A \cdot \boldsymbol{\beta}^d)\langle \phi^i \phi^j \rangle_A^{-1} \qquad (55)$$

which can be written simply as the pullback of the diagonal metric with entries $\langle \phi^i \phi^j \rangle_A^{-1}$. The first exponential corrections to the functions f, a, b and c are thus given by

$$f(r^A; \vec{r}^B, \psi^B) = -\left(\frac{2}{n} - 2M_A\right)^{1/2}$$

$$a(r^A; \vec{r}^B, \psi^B) = r^A \left(\frac{2}{n} - 2M_A\right)^{1/2}$$
$$- 4(r^A)^2 e^{-r^A} \frac{L_d}{G_d} \prod_{B \neq A} \left(\frac{1 + \cos\Theta_{AB}}{1 - \cos\Theta_{AB}}\right)^{\Omega^{AB}}$$

$$b(r^A; \vec{r}^B, \psi^B) = r^A \left(\frac{2}{n} - 2M_A\right)^{1/2}$$
$$+ 4(r^A)^2 e^{-r^A} \frac{L_d}{G_d} \prod_{B \neq A} \left(\frac{1 + \cos\Theta_{AB}}{1 - \cos\Theta_{AB}}\right)^{\Omega^{AB}}$$

$$c(r^A; \vec{r}^B, \psi^B) = \left(\frac{1}{2n} - \frac{1}{2}(M^{-1})_A\right)^{-1/2} \qquad (56)$$

where $\Theta_{AB} = \hat{r}^{iA} \hat{r}^{iB}$.

Some comments are probably in order. First, the product factor in Eqs. (56) corresponds to non-pairwise scattering of monopoles. The interaction between a pair of monopoles depends on the distances between the pair and all other monopoles. As one monopole is taken to infinity, the corresponding part of the product tends towards unity and the monopole
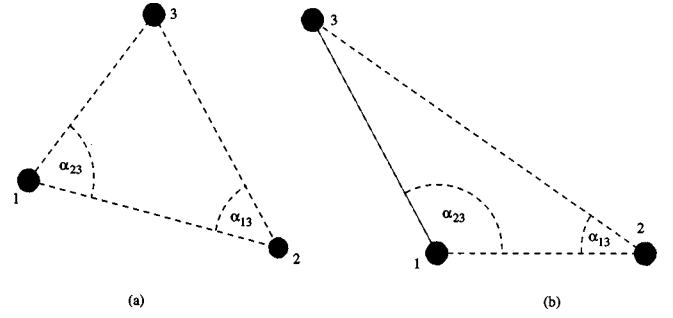

FIG. 1. For triangle (a), the height of the root corresponding to the interaction between the first and second monopoles is 2. For triangle (b), the root is simple.

decouples from the interaction as expected. As monopoles become colinear, the product factor in Eqs. (56) becomes singular, but this singularity is cancelled by L_d/G_d and in each case the overall correction to the metric is zero.

The situation of three monopoles is depicted in Fig. 1. For the interaction between the first and second monopoles, the moduli space for the constrained instanton jumps discontinuously when $\lambda_{(12)}^i \mathbf{v}^i \cdot \boldsymbol{\alpha}^{(12)} = |\lambda^i \mathbf{v}^i| \cos(\pi/3)$. In the monopole picture. this corresponds to $\alpha_{23} = \pi/2$ or $\alpha_{13} = \pi/2$. Thus, if both α_{23} and α_{13} are acute, as in triangle (a), the correction from the constrained instanton is L_2/G_2. If either is obtuse as in triangle (b), the correction is $L_1/G_1 = 1$.

The product factor of Eq. (56) is

$$\left(\frac{1 + \cos\alpha_{23}}{1 - \cos\alpha_{23}}\right)^{1/2} \left(\frac{1 - \cos\alpha_{13}}{1 + \cos\alpha_{13}}\right)^{-1/2}. \qquad (57)$$

For triangle (a), L_2/G_2 can be read off from Eq. (B9) of Appendix B,

$$1 - \left(1 + \frac{1}{18} r_{12} \tan\alpha_{13} \tan\alpha_{23}\right)$$

$$\times \exp\left(-\frac{1}{18} r_{12} \tan\alpha_{13} \tan\alpha_{23}\right). \qquad (58)$$

As the third monopole is brought between the other two, the singularity in Eq. (56) is cancelled by Eq. (58) and the interaction between the first and second vanishes. As the third monopole is taken to infinity, $\alpha_{23} = \pi - \alpha_{13}$, the two factors in Eq. (57) cancel. The corrections from the constrained instanton (58) retain some information about the position of the third monopole. However, the window in which such corrections are to be applied becomes vanishingly small and the third monopole decouples from the interaction as expected.

Note that for a right angle triangle, Eq. (58) is unity and thus we have a smooth transition between the acute and obtuse triangles. However, we once again emphasize that in this regime the exponential corrections in Eq. (58) are subleading with respect to other corrections that we have not dealt with.

The function $f(r)$ receives no instanton corrections. In the expansion of the functions f, a, b and c given in [22], f has

exponential corrections of the same order as a and b. However, the function f has no invariant meaning and the coordinates employed by the authors of [22] differ by exponential corrections to the coordinates defined by Eqs. (52). Such issues are common in dealing with exact results in supersymmetry (SUSY) gauge theories, namely that the vacuum moduli space is parametrized by coordinates of the low energy theory which have a complicated dependence on the coordinates defined in the original Lagrangian. For instance, this behavior is seen to arise in finite $\mathcal{N}=2$ theories in four dimensions [21].

It would be gratifying to compare Eqs. (56) with known metrics on subspaces of the full moduli space. The first situation where such metrics are known is the case with monopoles co-linear and equidistant. For co-linear monopoles, there are only interactions between adjacent monopoles which are independent of the positions of the others and the corrections are just of the form $r \exp(-r)$. If we further impose equidistance, we are left with the leading order corrections of the Atiyah-Hithin metric in agreement with [23].

The one other case in which exact metrics are known is for four monopoles. The metric on the one dimensional submanifold of tetrahedrally symmetric manifolds has been computed using Nahm data and is found to have the leading exponential corrections occurring at $\exp(-2r)$ [24]. From the three dimensional perspective, instanton contributions to this metric are of the form $\langle \Phi\Phi \rangle$ and so vanish. Naively, it appears that the two pictures are in agreement, the $\exp(-2r)$ term corresponding to an instanton–anti-instanton pair. However, the same coordinate problems arise as with the $f(r)$ term in the Atiyah-Hitchin metric and the conclusion is that, while consistent, agreement between the two remains ambiguous.

IV. $N=8$, $SU(n)$

In this final section, we turn our attention to the $N=8$ theory. Dimensionally reducing the $\mathcal{N}=1$ ten dimensional theory to 3 dimensions, the field content of the $N=4$ theory is augmented by the addition of 4 scalars and 4 Majorana fermions. While the algebra has a $Spin(8)$ automorphism group, only a $Spin(7)$ R-symmetry is manifest in the Lagrangian description with the vector transforming in a singlet, the scalars in **7** and the fermions in **8** [25].

In four dimensions, a non-renormalization theorem for the $\mathcal{N}=4$ theories prevents instanton corrections to eight Fermi vertices [26,27]. This is no longer the case in three dimensions and eight Fermi (or four derivative) vertices [26,11,12]. The counting of zero modes proceeds as in Sec. II; $\mathbf{g} = k\boldsymbol{\alpha}^A$ generically has $2k$ Fermi zero modes while $\mathbf{g}$ not aligned with a root has none. Instantons in the latter sector again break all the supersymmetries and so fail to contribute. However, the addition of adjoint massless matter multiplets enhancing $N=4$ supersymmetry to $N=8$ allows for the lifting of zero modes not protected by supersymmetry and sectors labelled by $\mathbf{g} = k\boldsymbol{\alpha}^A$ contribute for all k and all $\boldsymbol{\alpha}^A$.

The instanton calculation now proceeds identically to the $SU(2)$ case. The reader is referred to [12] for details. The lifting of the zero modes is such that integration over them

yields the volume contribution to the Euler character of the relative instanton moduli space, which generically for $\mathbf{g} = k\boldsymbol{\alpha}^A$ is the $\tilde{\mathcal{M}}_k$ of Eq. (42). This can be identified with the Euler character of $\tilde{\mathcal{M}}_k$ only up to boundary terms. For the case $k=2$, the metric is known explicitly and the boundary terms vanish [28]. For higher charges, the Gibbons-Manton metric corresponding to well-separated monopoles has the required asymptotic flatness for the boundary term to vanish, but there may be contributions from ''clustering regions'' at the boundary where at least one pair of monopoles remains close. As in [12], we assume that this is not the case and that the integral over zero modes does indeed yield the Euler character.

The Euler characters of the k-monopole $SU(2)$ relative monopole moduli spaces are determined to be [29]

$$\chi(\tilde{\mathcal{M}}_k) = k. \tag{59}$$

Importantly, just as in the $SU(2)$ case, the integrations over non-zero fluctuations about the background of the instanton, which were ultimately responsible for the non-pairwise interaction of monopoles in the $N=4$ case, now give unity. The singularities which occurred on the CMS are no longer there. It is plausible that the extra supersymmetry takes care of the soft modes, circumventing the need for the constrained instanton approach of Sec. II, although a proof of this would require knowledge of the Euler characters and boundary terms of moduli spaces for higher rank gauge groups. Let us see how this might occur.

Suppose we do not use the constrained instanton. Then on the CMS the instanton moduli space jumps discontinuously from $\tilde{\mathcal{M}}_k$ to $\tilde{\mathcal{M}}_{k(1,1,\ldots,1)}$, that is from an $SU(2)$ to an $SU(d)$, $d \leq n$, moduli space. Unlike the $N=4$ case, the extra fermionic zero modes are lifted and the resulting integration becomes the Euler character of the enlarged moduli space (up to the complication of boundary terms). The instanton contribution to the correlation function is continuous over the CMS provided

$$\chi(\tilde{\mathcal{M}}_{k(1,1,\ldots,1)}) = \chi(\tilde{\mathcal{M}}_k) = k. \tag{60}$$

This scenario remains more or less the same if we do use the machinery of the constrained instanton. The integration over soft modes and zero modes now yields the volume contribution to the G-index generalization of the Gauss-Bonnet integral [18] and again, up to boundary terms, continuity of the instanton calculation requires the Euler characters to be given by Eq. (60).

There are also other instanton contributions which occur only on the CMS, namely those with charges not proportional to a root. Recall, that for the generic situation the actions of such instantons are lifted above the Bogomol'nyi bound. On the CMS, the contributions from such solutions are again proportional to the Euler character of the moduli space. For continuity we require this to vanish. It is interesting to note that these spaces are expected to contain no square-integrable harmonic forms. While not a direct prediction of S-duality [30], were these forms to exist, states in the four dimensional $\mathcal{N}=4$ theory would appear at no point in

the moduli space except on the curves of marginal stability, where they are at their most vulnerable for decay.

Finally we turn to a rather different application of three dimensional gauge theories. Polchinski and Pouliot [11] relate the dynamics of $N=8$ $SU(2)$ theory to the scattering of two membranes in matrix theory. Instanton processes of charge k correspond to scattering with momentum exchange in the eleventh direction, giving an important test of the eleven dimensional Lorentz invariance of matrix theory. The $SU(n)$ theory considered here is of course related to the scattering of n membranes. The transverse distances and interactions between branes follow the same pattern as the monopoles in the $N=4$ theory. To model moving membranes, the VEVs are allowed time dependence and the four time derivative vertex of the low-energy action becomes a quartic velocity term in the low velocity effective action for interacting membranes. Usually such actions only make sense up to terms quadratic in the velocity, but for purely gravitational systems and systems with constant charge to mass ratio, the backreaction from the fields enters at the 5th order in the velocities and the expansion may continue to quartic terms (see pp. 165 and 337 of [31]). Indeed, in the present context the moduli space of membrane solutions in eleven dimensional supergravity is flat and the quadratic terms vanish.

The cancellation of the non-zero mode fluctuations ensures that the four derivative term is just the sum over pairs of membranes. Unlike the monopole case, the longitudinal scattering of membranes in matrix theory occurs pairwise. It would be interesting to see if this behavior is reproduced in supergravity.

APPENDIX A: INSTANTON ZERO MODES

In this appendix we calculate the number of zero modes of the Dirac equation in the background of an instanton with several Higgs fields. We follow Weinberg [8] closely, concentrating on points that differ from the original calculation. First, we define a set of 16×16 6-dimensional gamma matrices (this unconventional representation arises from using 't Hooft matrices, η^i, rather than Pauli matrices, σ_i)

$$\Gamma_{\mu=1,2,3}=-\gamma_\mu\otimes1\otimes\sigma_2, \quad \Gamma_{i=4,5,6}=-i\otimes\eta^i\otimes\sigma_1 \tag{A1}$$

and $\Gamma_7=1_2\otimes1\otimes\sigma_3$. Writing Δ_M as the 6-dimensional covariant derivative

$$\Gamma_M\Delta_M=\begin{pmatrix}0 & \Delta \\ -\Delta^\dagger & 0\end{pmatrix} \tag{A2}$$

where Δ is defined in Eq. (11) and the explicit matrix in Eq. (A2) refers to the last of the three direct products. We now rewrite Eq. (13) as

$$I(\mu^2)=-\mathrm{Tr}_-\left(\Gamma_7\frac{\mu^2}{-(\Gamma\cdot\Delta)^2+\mu^2}\right)$$
$$=-\mathrm{Tr}\left(P_-\Gamma_7\frac{\mu^2}{-(\Gamma\cdot\Delta)^2+\mu^2}\right). \tag{A3}$$

Using

$$P_-\lambda_\mathbf{g}^i\eta^i=\lambda^i\eta^iP_-, \quad P_-(\eta^i-\lambda_\mathbf{g}^i\lambda_\mathbf{g}^j\eta^j)=(\eta^i-\lambda_\mathbf{g}^i\lambda_\mathbf{g}^j\eta^j)P_+ \tag{A4}$$

and tracing liberally over η and γ matrices, it may rewritten as

$$I(\mu^2)=\int d^3x\ \mathrm{tr}\ P_-\Gamma_7\Gamma_\mu\mathcal{D}_\mu\langle x|(\Gamma\cdot\Delta+\mu)^{-1}|x\rangle \tag{A5}$$

$$+\int d^3x\ \mathrm{tr}\ \frac{i}{2}P_-\Gamma_7\Gamma_i\langle x|\hat{\phi}^i(\Gamma\cdot\Delta+\mu)^{-1}|x\rangle \tag{A6}$$

where tr denotes the trace over group and (6-dimensional) spinor indices only. The second term differs from Weinberg's calculation and is due to the extra Higgs fields. We multiply on top and bottom by $-\Gamma\cdot\Delta+\mu$, take spinor traces and move this term to the left hand side of the equation, to arrive at

$$-\mathrm{Tr}\left(P_-\Gamma_7\frac{\mu^2+\hat{\phi}^i\hat{\phi}^i}{-(\Gamma\cdot\Delta)^2+\mu^2}\right)$$
$$=-\int d^3x\ \mathrm{tr}[P_-\Gamma_7\Gamma_\mu\partial_\mu\langle x|(\Gamma\cdot\Delta+\mu)^{-1}|x\rangle]. \tag{A7}$$

At this point it is important to note that although we have traced over spinor indices extensively, the derivation of Eq. (A7) does not rely on the trace over group or spatial indices, allowing us to divide by the numerator of the left hand side. Integration by parts then yields

$$I(\mu^2)=\int_{\Sigma_\infty} d^2S^\mu\ \mathrm{tr}\left(P_-\Gamma_7\frac{\mu^2}{\hat{\phi}^i\hat{\phi}^i+\mu^2}\Gamma_\mu\frac{1}{\Gamma\cdot\Delta+\mu}\right) \tag{A8}$$

$$-\int d^3x\ \mathrm{tr}\left[P_-\Gamma_7\Gamma_\mu\mathcal{D}_\mu\left(\frac{\mu^2}{\hat{\phi}^i\hat{\phi}^i+\mu^2}\right)\frac{1}{\Gamma\cdot\Delta+\mu}\right]. \tag{A9}$$

The second term vanishes using $\mathcal{D}_\mu\hat{\phi}^i=0$. The first term is very similar to the corresponding expression in Weinberg. Again multiply top and bottom by $-\Gamma\cdot\Delta+\mu$ and perform the trace over the spinor indices. The presence of P_- in the numerator destroys all but the Φ term in Δ. Expanding,

$$\frac{1}{-(\Gamma \cdot \Delta)^2 + \mu^2} = \frac{1}{-\mathcal{D}^2 + \phi^i \phi^i + \mu^2} + \frac{1}{-\mathcal{D}^2 + \phi^i \phi^i + \mu^2}$$

$$\times 2 \gamma_\mu B_\mu \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} \frac{1}{-\mathcal{D}^2 + \phi^i \phi^i + \mu^2}$$

$$+ \cdots \tag{A10}$$

and keeping terms of $O(1/x^2)$, evaluation of this first term now proceeds as in Weinberg. Tracing over group indices indeed yields Eq. (15) as claimed.

APPENDIX B: INTEGRATION OVER TAUB-NUT SPACE

If $\boldsymbol{\alpha}^A$ is of height 2, say $\boldsymbol{\alpha}^A = \boldsymbol{\gamma}^1 + \boldsymbol{\gamma}^2$, the relevant moduli space $\tilde{\mathcal{M}}_{(1,1)}$ is Taub-NUT space [19,20]. Defining the reduced mass, $\mu = (\lambda_A^i \mathbf{v}^i \cdot \boldsymbol{\gamma}^1)(\lambda_A^j \mathbf{v}^j \cdot \boldsymbol{\gamma}^2)/M_A$, the metric is given by

$$ds^2 = \frac{8\pi^2}{e^2} \left(\mu V(r) d\mathbf{r}^2 + \frac{1}{4} \mu^{-1} V(r)^{-1} (d\psi + \cos \theta d\phi)^2 \right) \tag{B1}$$

where $0 \le \psi \le 4\pi$ and

$$V(r) = 1 + \frac{1}{2\mu r}. \tag{B2}$$

The Killing vector ∂_ψ is generated by $(\boldsymbol{\gamma}^1 - \boldsymbol{\gamma}^2) \cdot \mathbf{H}/3$. Thus the action of the constrained instanton configurations parametrized by this space is raised by

$$\frac{1}{2} \tilde{g}_{ab} (\hat{\mathbf{v}}^i \cdot \mathbf{K}^a)(\hat{\mathbf{v}}^i \cdot \mathbf{K}^b) = \frac{2\pi^2}{3e^2} \hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i \frac{r}{1 + 2\mu r} \tag{B3}$$

together with the $\|\mathbf{Q} \cdot \hat{\mathbf{v}}^i\|$ term of Eq. (35). Note that this potential flattens out as $r \to \infty$. This is a generic feature of all potentials on Lee-Weinberg-Yi spaces generated by the $U(1)$ Killing vectors. Because the Lee-Weinberg-Yi spaces are non-compact, the integral over this potential alone will diverge. The integral is rendered finite by the corresponding integration over fermionic coordinates. In the present case there is only one Killing vector on the moduli space and the potential is already in the form $(\mathbf{V} \cdot \mathbf{K}^a)(\mathbf{V} \cdot \mathbf{K}_b)$ with $\mathbf{V} \cdot \mathbf{V} = \hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i$. The fermionic potential is given by

$$\sum_i \frac{i}{2} \nabla_a (\mathbf{V} \cdot \mathbf{K}_a) \psi^a \psi^b = \frac{2i\pi^2}{\sqrt{3}e^2} \frac{(\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i)^{1/2}}{(1 + 2\mu r)^2}$$

$$\times [\cos \theta \psi^r \psi^\phi + \psi^r \psi^\psi$$

$$- r \sin \theta (1 + 2\mu r) \psi^\theta \psi^\phi]. \tag{B4}$$

Using the measures of integration Eqs. (37), the integration over fermionic coordinates brings down two factors

of the fermionic potential, isolating the $\psi^r \psi^\psi \psi^\theta \psi^\phi$ term and leaving us with the following expression for $\tilde{L}_2 = L_2 \exp(+4\pi^2 \|\mathbf{Q} \cdot \hat{\mathbf{v}}^i\|/M_A e^2)$:

$$\tilde{L}_2 = \frac{\pi^2}{3e^2} \hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i \int dr d\phi d\theta d\psi r (1 + 2\mu r)^{-3}$$

$$\times \sin \theta \exp\left[-\frac{2\pi^2}{3e^2} \hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i \frac{r}{1 + 2\mu r} \right]$$

$$= \frac{12}{\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i} \left[1 - \left(1 + \frac{2\pi^2}{3e^2} \frac{\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i}{2\mu} \right) \exp-\left(\frac{2\pi^2}{3e^2} \frac{\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i}{2\mu} \right) \right]. \tag{B5}$$

Notice that for small $\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i$, $L_2 \sim (\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i)^2$.

In order not to integrate over these modes twice, we must divide by the Gaussian approximation to L_2. The coordinate system used in Eq. (B1) has a coordinate singularity at the origin, $r = 0$. In order to present a metric that is smooth at the origin, we transform to the coordinate, R, where $r = \frac{1}{2} \mu R^2$. In this basis, the metric is

$$ds^2 = \frac{8\pi^2 \mu}{e^2} \left[(1 + \mu^2 R^2) dR^2 \right.$$

$$+ \frac{1}{4} R^2 (1 + \mu^2 R^2)(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\left. + \frac{1}{4} R^2 (1 + \mu^2 R^2)^{-1} (d\psi^2 + \cos \theta d\phi^2) \right]. \tag{B6}$$

Note that to leading order in R, this is the flat metric on $\mathbf{R}^4$ in Euler angle coordinates. The recipe for the Gaussian approximation is to truncate the Taub-NUT metric (B6) to the flat metric and repeat the calculation above using this. The Gaussian approximation to the bosonic potential is thus

$$\frac{\pi^2 \mu}{3e^2} \hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i R^2 \exp-\left(\frac{4\pi^2 \|\mathbf{Q} \cdot \hat{\mathbf{v}}^i\|}{M_A e^2} \right) \tag{B7}$$

and the fermionic counterpart

$$\frac{2i\pi^2 \mu}{\sqrt{3}e^2} (\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i)^{1/2} \left(R \cos \theta \psi^R \psi^\phi + R \psi^R \psi^\psi \right.$$

$$\left. - \frac{1}{2} R^2 \sin \theta \psi^\theta \psi^\phi \right). \tag{B8}$$

Once more performing the integrations, we deduce $G_2 = 12/(\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i) \exp(-4\pi^2 \|\mathbf{Q} \cdot \mathbf{H}\|/M_A e^2)$ and, thus,

$$\frac{L_2}{G_2} = 1 - \left(1 + \frac{\pi^2 (\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i)}{3e^2 \mu} \right) \exp\left[-\frac{\pi^2 (\hat{\mathbf{v}}^i \cdot \hat{\mathbf{v}}^i)}{3e^2 \mu} \right]. \tag{B9}$$

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