

## Quantum geometry with intrinsic local causality

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The space of states and operators for a large class of background independent theories of quantum spacetime dynamics is defined. The SU(2) spin networks of quantum general relativity are replaced by labelled compact two-dimensional surfaces. The space of states of the theory is the direct sum of the spaces of invariant tensors of a quantum group  $G_q$  over all compact (finite genus) oriented 2-surfaces. The dynamics is background independent and locally causal. The dynamics constructs histories with discrete features of spacetime geometry such as causal structure and multifingered time. For SU(2) the theory satisfies the Bekenstein bound and the holographic hypothesis is recast in this formalism.

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### I. INTRODUCTION

In this article we describe a new class of quantum geometries that is to be used in developing a theory of quantum gravity. These are natural extensions of the spin network states that have been shown to comprise the non-perturbative state space of quantum general relativity [1,2].<sup>1</sup> In this formulation the labeled graphs on which spin networks are based are replaced by 2-manifolds and the invariant tensors of a quantum group  $G_q$  associated with them. The motivations for this generalization comes partly from results of non-perturbative quantum gravity and string theory.

This formulation has a kinematical part and a dynamical part. The kinematical part, which is described in the next four sections, generalizes the spin network states in two ways. The first is that the SU(2) group of the spin network states of quantum general relativity is replaced by an arbitrary quantum group  $G_q$ . Within the framework of non-perturbative, diffeomorphism invariant quantum field theories, this is the natural way to extend the degrees of the freedom of the theory to include gauge fields [6] and supersymmetry [7]. The quantum deformation is motivated from physics by three considerations. First, in quantum general relativity the introduction of a cosmological constant  $\Lambda$  requires a quantum deformation of SU(2) with  $q = e^{2\pi/k+2}$  defined by [8,9,10]

$$k = \frac{6\pi}{G^2\Lambda}. \quad (1)$$

Second, the truncation in the number of representations with  $q$  at a root of unity improves the formulation of the dynamics by making the sums involved in the path integral less divergent. It also introduces new symmetries in the theory which

are not present in the classical case when  $q \rightarrow 1$ . These are analogous to the duality symmetries of perturbative string theory. As we argue below, this may play a role in the interpretation of the theory.

The second sense in which our proposal extends the spin network states of quantum general relativity is that the states are defined intrinsically, without the use of a background manifold. In quantum general relativity the spin network states are diffeomorphism classes of embeddings of graph in a fixed three-manifold  $\Sigma$  [1,2]. We go beyond this to a purely algebraic definition of the state space which depends on no prior specification of a manifold.

One result of non-perturbative quantum gravity has been the discovery that geometrical quantities, including area [11,2], volume [11,2,12] and length [13] have discrete spectra. This is true before the introduction of dynamics or matter couplings and signals that the combination of diffeomorphism invariance and quantum theory requires that quantum geometry be essentially discrete. At the same time, the application of these techniques to the Hamiltonian constraint of general relativity [14,15,16] leads to a theory without a good continuum limit [17,18]. Given this, it seems more natural to construct the theory purely from algebra and combinatorics and let continuum notions arise in the classical limit of the theory.

We may note that the dualities of string theory suggest that one and the same physical situation may sometimes be described in two different ways, which differ in the topology and manifold structures of the underlying manifolds [19]. Other results show that in string theory there are continuous phase transitions whose semiclassical description involves abrupt changes of the topology of the underlying manifold [20]. These suggest that the fundamental, non-perturbative, description should not be based on fixed topological manifolds.

But without background manifolds the theory cannot be formulated in terms of the embeddings of surfaces or membranes. The alternative is to construct the states and operators that are to represent quantum geometry algebraically, using only combinatorics and representation theory. This is the main goal of this paper. In [21] and [22] results are presented

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<sup>1</sup>For recent reviews see [3,4]. Spin networks were originally introduced by Penrose [5] as a model of quantum geometry.

consistent with the hypothesis [4] that the resulting extension of the spin networks formalism may serve as framework for non-perturbative string theory.<sup>2</sup>

A theory formulated without reference to any background manifold still requires dynamics and that dynamics should have built into it some notion of local causality. Below, in Sec. VII, we show that this can be achieved by an extension of a formulation of spin network dynamics proposed earlier by one of us [23]. The dynamics is based on discrete histories  $\mathcal{M}$ , which are combinatorial structure which have two properties shared by classical spacetime:

(1) Each history  $\mathcal{M}$  contains a finite set  $\mathcal{E}$  of elements that may be called “events.” This set of events is a partially ordered set. We thus have the finite element analogue of the points of a Lorentzian spacetime.

(2) Each history  $\mathcal{M}$  contains a large number of connected sets of causally unrelated events, which may be called “quantum spacelike surfaces.” Each spacelike surface is also a quantum state. Thus, the theory has a discrete analogue of the many-fingered time of general relativity, which means that a discrete analogue of spacetime diffeomorphism invariance is built in.

Each history is then given an amplitude which is a product of factors each associated to a local transition in the quantum geometry. Causality and locality impose restrictions on the choice of these amplitudes which are discussed below. The issue of the choice of dynamics and the related question of the continuum limit is discussed in [24,22].

In the next three sections we introduce the space of states that we propose extends spin networks and describe useful decompositions of them which are based on 3- and 4-punctured spheres. Section V introduces an algebra of operators that act on the states and the interpretations of some of them, which yields a picture of quantum geometry, is the subject of Sec. VI. The dynamics of the theory is described in Sec. VII, while Sec. VIII discusses coarse-grained observables and entropy and their relationship to the holographic hypothesis and Bekenstein bound. The conclusion is largely devoted to describing ongoing work that will be reported in other papers.

## II. THE SPACE OF STATES

The space of states that we investigate here is both the extension of  $SU(2)$  spin networks to a quantum group  $G_q$  and of the spin network states of canonical quantum gravity to the non-embedded case.

Given a quantum group  $G_q$  and a compact oriented 2-

surface  $\mathcal{S}$ , let  $\mathcal{V}_{G_q}^{\mathcal{S}}$  be the space of  $G_q$  invariant tensors on  $\mathcal{S}$ .<sup>3</sup> We then define the space of states  $\mathcal{H}_{G_q}$  of  $G_q$  quantum gravity to be

$$\mathcal{H}_{G_q} = \bigoplus_{\mathcal{S}} \mathcal{V}_{G_q}^{\mathcal{S}} \quad (2)$$

where the sum is over all compact 2-surfaces of finite genus. Each  $\mathcal{V}_{G_q}^{\mathcal{S}}$  is finite dimensional when  $q$  is at a root of unity.  $\mathcal{H}_{G_q}$  is equipped with the natural inner product [see Eq. (5) below] and is a Hilbert space.

The sense in which these states may be considered to constitute an extension of the spin network states of quantum general relativity will be discussed shortly, but we note that this is not a new notion. It is known that the quantum deformation of spin networks requires that their edges be enlarged to ribbons or tubes [28,26,27]. This is to allow dependence of the states on twistings of the edges, necessary for the  $q$ -deformed case [28,26,30,29]. In the next sections we investigate properties of these states that are important for their physical interpretation.

## III. TRINION DECOMPOSITION: BASIS STATES

We begin by reviewing some of the properties of the state spaces  $\mathcal{V}_{G_q}^{\mathcal{S}}$  that we will need to discuss their role in representing the states of quantum gravity.<sup>4</sup> For the purposes of describing the states and operators on  $\mathcal{H}_{G_q}$  it will be very useful to understand the behavior of the states in  $\mathcal{V}_{G_q}^{\mathcal{S}}$  under decompositions of the surface  $\mathcal{S}$  into a union of punctured spheres. We begin by discussing the decomposition of a genus  $g$  surface  $\mathcal{S}$  into 3-punctured spheres, or trinions. Given a surface  $\mathcal{S}$  we may choose a maximal set of non-intersecting elements of  $\pi^1[\mathcal{S}]$ , which we shall call circles,  $c_\alpha$ . Cutting  $\mathcal{S}$  along the circles  $c_\alpha$  decomposes it into a set of  $N$  trinions,  $B_I^3$ ,  $I=1, \dots, N$ . The trinions are joined on their punctures so that each circle  $c_\alpha$  corresponds to the punctures on two trinions. [See Fig. 1(a).] This may be done in several different ways. [See Fig. 1(b).]

A trinion decomposition will be called *non-degenerate* if no two trinions meet at more than two circles (see Fig. 2).

Associated to each trinion decomposition of  $\mathcal{S}$  is a class of bases of  $\mathcal{V}_{G_q}^{\mathcal{S}}$ , which is constructed as follows.  $G_q$  has a list of irreducible representations, which we will label by  $j_\alpha$ . (For the  $q$  taken at a root of unity, which we will assume, this is a finite list.) Each of the three circles of a trinion  $B_I^3$  may be labeled by a representation  $j_\alpha$ ,  $\alpha=1,2,3$ . For each choice of the representations  $j_\alpha$  there is a linear space  $\mathcal{V}_{j_1 j_2 j_3}^I$  of intertwiners  $\mu_I$ . The intertwiners are the maps

<sup>2</sup>In fact the basic idea of the present formulation is rather like the idea behind the transition from quantum field theory to perturbative string theory. Just as Feynman diagrams are replaced by string worldsheets, the present generalization of quantum general relativity extends spin network states to 2-dimensional surfaces and the states of field theories defined on the surfaces.

<sup>3</sup>Equivalently this is the space of conformal blocks of the Wess-Zumino-Witten (WZW) theory corresponding to level  $k$  on  $\mathcal{S}$  [25,26,27] or the space of states of  $G_q$  Chern-Simons theory on  $\mathcal{S}$ , seen as a spatial slice of some 3-manifold [28].

<sup>4</sup>Complete characterizations of  $\mathcal{V}_{G_q}^{\mathcal{S}}$  may be found in [25,26].

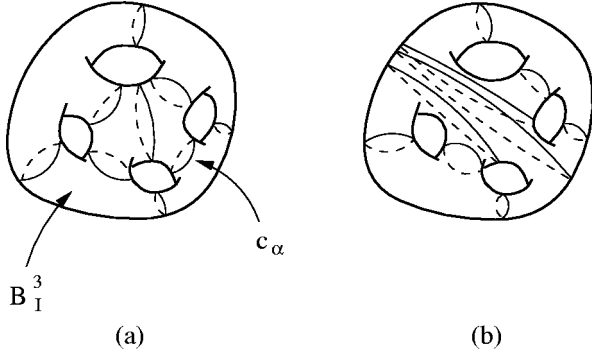


FIG. 1. (a) A genus 4 surface cut into six trinions  $B_I^3$  by circles  $c_\alpha$ . (b) The same surface in a different trinion decomposition.

$$\mu_I : j_1 \otimes j_2 \otimes j_3 \rightarrow \mathbf{1}. \quad (3)$$

A choice of a set of  $j_\alpha$  on the punctures of a trinion is called consistent if the corresponding  $\mathcal{V}_{j_1 j_2 j_3}^I$  has strictly positive dimension.

The space of states  $\mathcal{V}_{G_q}^S$  (subset of  $\mathcal{H}_{G_q}$ ) associated with the surface  $S$  is constructed by taking direct products of all the constituent spaces  $\mathcal{V}_{j_1 j_2 j_3}^I$  and summing over the representations,

$$\mathcal{V}_{G_q}^S = \sum_{j_\alpha} \otimes_I \mathcal{V}_{\{j\}_I}^I \quad (4)$$

where  $I$  labels an arbitrary trinion  $B_I^3$  in  $S$  with labels  $\{j\}_I$ .

A generic state in  $\mathcal{V}_{G_q}^S$  will be denoted  $|\mathcal{S}, \Psi\rangle$ . A basis in  $\mathcal{V}_{G_q}^S$  is then constructed as follows. We choose an orthogonal basis of intertwiners in the space  $\mathcal{V}_{\{j\}_I}^I$  of each of the trinions, denoted  $\mu_I^p$ . A basis of states in  $\mathcal{V}_{G_q}^S$  is then given by a choice of  $j_\alpha$  on each circle  $c_\alpha$  in  $S$  and a choice of a basis element  $\mu_I^p$  on each trinion. These basis states are denoted  $|\mathcal{S}, j_\alpha, \mu_I^p\rangle$ .

Given a trinion decomposition of every finite genus 2-surface  $S$ , the states  $|\mathcal{S}, j_\alpha, \mu_I^p\rangle$  provide an orthonormal basis for the state space  $\mathcal{H}_{G_q}$ . The inner product on  $\mathcal{H}_{G_q}$  is given by

$$\langle \mathcal{S}, j_\alpha, \mu_I^p | \mathcal{S}', j'_\alpha, \nu_I^{\bar{p}} \rangle = \delta_{\mathcal{S}\mathcal{S}'} \prod_\alpha \delta_{j_\alpha j'_\alpha} \prod_I \langle \mu_I^p | \nu_I^{\bar{p}} \rangle_I \quad (5)$$

where the same trinion decomposition is assumed for the two states when  $\mathcal{S} \cong \mathcal{S}'$  and  $\langle \mu_I^p | \nu_I^{\bar{p}} \rangle_I$  is the inner product in the space of intertwiners  $\mathcal{V}^I$  on the  $I$ -th trinion.

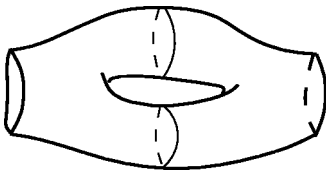


FIG. 2. This trinion decomposition is degenerate because the two trinions have two circles in common.

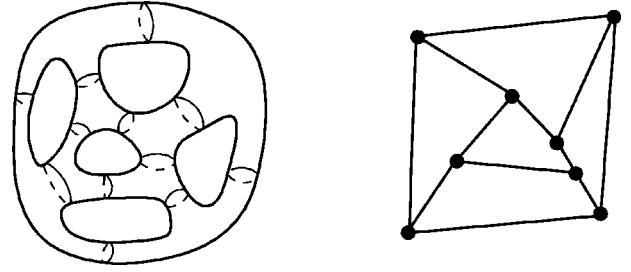


FIG. 3. A trinion decomposition of a genus 5 surface reduced to a spin network graph.

Note that, given a particular trinion decomposition of  $S$ , the states in the basis  $|\mathcal{S}, j_\alpha, \mu_I^p\rangle$  may be thought of as generalized combinatorial trivalent spin networks. (See Fig. 3 but note that for  $q \neq 1$  these are quantum spin networks [30].) The edges  $e_\alpha$  of the corresponding graph  $\Gamma$  are labeled with the same representations  $j_\alpha$  as the corresponding circles  $c_\alpha$ , while the trivalent nodes  $v_I$  associated to the trinions are labeled by the intertwiners  $\mu_I$ . Because of this association we sometimes call the basis states  $|\mathcal{S}, j_\alpha, \mu_I^p\rangle$  *tubular spin networks*.

The assignment of a graph  $\Gamma$  to the surface  $S$  depends on the choice of the trinion decomposition and the same is thus true of the basis  $|\mathcal{S}, j_\alpha, \mu_I^p\rangle$ . If we choose a different trinion decomposition of  $S$ , based on a different maximal set of non-intersecting elements of  $\pi^1[S]$ , we have a different basis for  $\mathcal{V}_{G_q}^S$ . The recoupling identities of the representation theory of  $G_q$  [30] then provide the change of basis formulas. Alternatively, they may be computed using the modular transformations of the corresponding rational conformal field theory as in [25,26].

We may note that when  $q \rightarrow 1$  the spaces  $\mathcal{V}_{G_q}^S$  become infinite dimensional as there are an infinite number of representations  $j_\alpha$ . Then, the Moore-Seiberg operators are no longer well defined unitary operators. Thus, in the limit  $q \rightarrow 1$  the states in  $\mathcal{H}_{G_q}$  are the usual combinatorial spin network states of  $SU(2)$ .

#### IV. DECOMPOSITION IN 4-PUNCTURED SPHERES

Just as the trinion decomposition is related to an extension of trivalent spin networks, we can associate an extension of 4-valent spin networks to the bases of states in  $\mathcal{V}^S$  that come from decomposing  $S$  into 4-punctured spheres (from now on we drop the suffix  $G_q$  of  $\mathcal{V}_{G_q}^S$ ). To accomplish this we pick a (non-maximal) set of non-intersecting circles  $c_\alpha$  on  $S$  that decompose it into 4-punctured spheres  $B_I^4$ . As before we can label these circles with representations  $j_\alpha$ .

It will also be useful to work with general  $n$ -punctured spheres. In general, a 2-sphere with  $n$  punctures, denoted  $B_I^n$ , is labelled by representations  $j_1, \dots, j_n$  of the group  $G_q$ . Given  $B_I^n$  and the labels  $j_1, \dots, j_n$ , there is a linear space of intertwiners,  $\mathcal{V}_{j_1, \dots, j_n}^I$ , consisting of the invariant maps

$$\mu_I : j_1 \otimes \dots \otimes j_n \rightarrow \mathbf{1}. \quad (6)$$

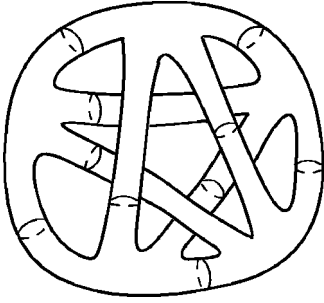


FIG. 4. The tubular 4-simplex  $\mathcal{P}$ , a genus 6 surface decomposed to 5 4-punctured spheres.

As in the 3-punctured case, the dimension of  $\mathcal{V}_{j_1, \dots, j_n}^l$  is required to be non-zero otherwise the choice of  $j_1, \dots, j_n$  is inconsistent and not allowed. Now, given any decomposition of  $\mathcal{S}$  into  $n$ -punctured spheres along a set of circles  $c_\alpha$ , we have representation of the states in  $\mathcal{V}^{\mathcal{S}}$  in terms of triples  $|\mathcal{S}, j_\alpha, \mu_I\rangle$ . The formulas (4) and (5) still hold.

Returning to the decompositions in terms of 4-punctured spheres, we may note that such decompositions may be made, at least locally, in a surface  $\mathcal{S}$  by grouping the trinions in some trinion decomposition into pairs. Finally, as in the case of trinions, we call a decomposition of a surface  $\mathcal{S}$  into 4-punctured spheres *non-degenerate* if no two 4-punctured spheres share more than one puncture.

### A. The tubular 4-simplex

In fact, a genus  $g$  surface always has a non-degenerate decomposition into 4-punctured spheres for  $g \geq 6$ . It is easy to see that the smallest number of 4-punctured spheres that can fit together non-degenerately is 5. These make up a genus 6 surface which may be thought of as a tubular generalization of the 4-simplex (see [23]), as every 4-punctured sphere  $B_I^4$ ,  $I = 1, \dots, 5$  is connected to every other one once. (See Fig. 4.) This surface plays a special role in the dynamics. We shall call it  $\mathcal{P}$  and refer to as the *generating surface*. Together with ten fixed circles  $c_{IJ}$  connecting  $B_I^4$  and  $B_J^4$  that decompose it into five such 4-punctured spheres  $\mathcal{P}$  will be called the *tubular 4-simplex*. Its  $q \rightarrow 1$  limit is a 4-valent graph with 4-valent nodes  $v_I$  for each 4-punctured sphere  $B_I^4$  of  $\mathcal{S}$  and an edge  $e_{IJ}$  for each circle  $c_{IJ}$ . Labeling the circles  $c_{IJ}$  by representations  $j_\alpha$  and the  $B_I^4$  by a basis  $\mu_I^\rho$  in the corresponding spaces of intertwiners  $\mathcal{V}_{\{j_\alpha\}}^{B_I^4}$  we have a basis of states  $|\mathcal{P}, j_\alpha, \mu_I^\rho\rangle$ . Each of these is a coloring of the 4-simplex.

### B. Tubular evolution moves

Consider a non-degenerate decomposition of a surface  $\mathcal{S}$  into  $n$  4-punctured spheres,

$$\mathcal{S} = \bigodot_{I=1}^n B_I^4, \quad (7)$$

where  $\bigodot$  denotes the gluing of a pair of punctures with the same labels. Given  $\mathcal{S}$ , there is a set of local moves each of

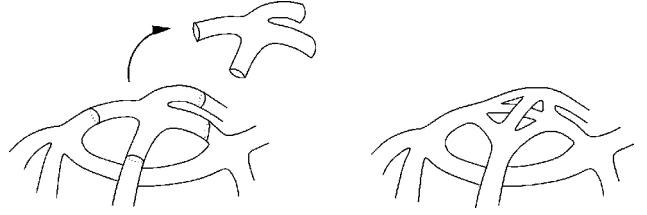


FIG. 5. An elementary substitution move.

which yields another surface  $\mathcal{S}'$  expressed as a non-degenerate composition of 4-punctured spheres  $\mathcal{S}' = \bigodot_{I=1}^m B_I^4$  where in general  $m \neq n$ .<sup>5</sup>

To define these moves let us now put forward some notation. An *elementary local region*,  $L$ , is a set of  $n \leq 4$  4-punctured 2-spheres  $B_I^4$ ,

$$L = \bigodot_{I=1}^n B_I^4 \quad n \leq 4, \quad (8)$$

each pair of which is connected by exactly one tube.  $L$ , therefore, is a 2-surface with 4 or 6 punctures, their number given by

$$\text{number of free punctures} = 4n - n(n-1). \quad (9)$$

$n$  has to be at most 4 for  $L$  to have any free punctures. For  $n=1,2$  the genus (not counting punctures) of  $L$  is 0, for  $n=3$  it is 1 and for  $n=4$  it is 3.

Given these definitions, a local move is the following. Given a decomposition  $\mathcal{S} = \bigodot_{I=1}^n B_I^4$ , remove an elementary local region  $L$  in it and replace it with a new one  $L'$  that has the same number of punctures and same labels on the punctures. (See Fig. 5.)

The topology of the new local region is determined by requiring that  $L$  and  $L'$  can be composed along their common punctures to form the generating surface,  $\mathcal{P}$  (see Fig. 6),

$$L' \bigodot L = \mathcal{P}_{L' \bigodot L}. \quad (10)$$

Namely,  $L'$  is the complement of  $L$  in  $\mathcal{P}_{L' \bigodot L}$ .<sup>6</sup> We call such a substitution a *tubular evolution move*. (See Figs. 5, 6 and 7). The result is a new 2-manifold  $\mathcal{S}'$  which has a decomposition into 4-punctured spheres that fall into two sets, those in  $\mathcal{S}-L$  and those in  $L'$ .

It is clear that if the original decomposition of  $\mathcal{S}$  into 4-punctured spheres is non-degenerate then so is the new one. No two 4-punctured spheres in  $\mathcal{S}-L$  share more than one connection because the original decomposition is non-degenerate. The same is true for the 4-punctured spheres in  $L'$  because every elementary local region is non-degenerate.

<sup>5</sup>These moves are a generalization of the Pachner moves from combinatorial topology [31] that played an important role in the evolution of spin networks in [23].

<sup>6</sup>In terms of the vector space representations of  $L$  and  $L'$ , Eq. (10) is the tensor product of the vector space of  $L$  with the dual vector space of  $L'$ ,  $\mathcal{V}^L \otimes (\mathcal{V}^{L'})^D$ .

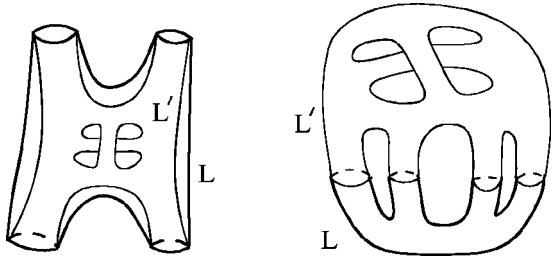


FIG. 6. Left: The substitution move seen as a three manifold that defines a cobordism from  $L$  to  $L'$ . Right: Joining the  $L$  and  $L'$  together makes a generating surface  $\mathcal{P}$ .

The non-degeneracy of  $L'$  implies that there can be at most one connection between any sphere in  $L'$  and one in  $\mathcal{S}-L$ .<sup>7</sup>

There are four kinds of tubular evolution moves, depending on the number of 4-punctured spheres in the old and new elementary regions  $L$  and  $L'$ . As in the case of the Pachner moves used in [23], these are denoted the  $1 \rightarrow 4$ ,  $4 \rightarrow 1$ ,  $2 \rightarrow 3$  and  $3 \rightarrow 2$  moves. In terms of the corresponding surfaces one can see from Fig. 7 that these result in a change of genus by  $+3$ ,  $-3$ ,  $+1$  and  $-1$  respectively. This means that starting with the tubular 4-simplex  $\mathcal{P}$  which has genus 6, one can make  $r$  successive  $2 \rightarrow 3$  moves to reach a surface of any genus  $g = 6 + r$ . Therefore, each surface with genus  $g \geq 6$  has a non-degenerate decomposition into 4-punctured spheres.

Note also that if  $B^4$  is a 4-punctured sphere with labels  $j_1, j_2, j_3, j_4$  it may be decomposed along a circle  $c_1$  into two trinions  $B_1^3$  and  $B_2^3$ . If we call the label on  $c_1$  by  $l$  we have

$$\mathcal{V}_{j_1 j_2 j_3 j_4}^{B^4} = \sum_l \mathcal{V}_{j_1 j_2 l}^{B_1^3} \otimes \mathcal{V}_{l j_3 j_4}^{B_2^3}. \quad (11)$$

Thus, we see that there are many trinion decompositions of a surface  $\mathcal{S}$  that are subdivisions of a decomposition of  $\mathcal{S}$  into 4-punctured spheres. In terms of the analogy to spin networks this corresponds to what has been called decomposing a 4-valent node of a spin network in terms of two trivalent nodes and an internal, or ‘‘virtual,’’ edge. In the present context all of these are connected by elements of the modular group [25,26].

Clearly, a given surface  $\mathcal{S}$  has more than one inequivalent decompositions into 4-punctures spheres. As an example, consider the tubular 4-simplex of Fig. 4. The relationship between these different compositions correspond to transformations between two bases in which the roles of the representations and the intertwiners are exchanged. This has interesting consequences for quantum geometry that we will discuss below, when we describe how the geometrical interpretation of the theory is constructed.

We will use the tubular evolution rules to define the dynamics of the theory. But first we have to define operators on  $\mathcal{H}_{G_q}$  that implement them.

<sup>7</sup>This is an extension of the basic fact that the Pachner moves applied on a  $PL$  triangulation preserves non-degeneracy of triangulations, i.e. when no two tetrahedra share more than one face.

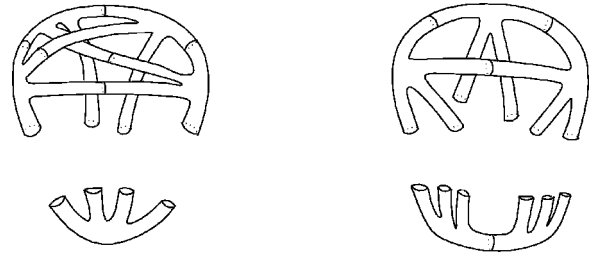


FIG. 7. The four elementary substitution moves,  $1 \leftrightarrow 4$  and  $2 \leftrightarrow 3$ .

## V. TUBE OPERATORS

We now turn to the operators on the space of states  $\mathcal{H}_{G_q}$ . The Moore-Seiberg [25] operators are a set of unitary operators that act inside each  $\mathcal{V}^{\mathcal{S}}$ . However, if our theory is to be a generalization of spin networks there must be operators that take us from states in one  $\mathcal{V}^{\mathcal{S}}$  to states in another  $\mathcal{V}^{\mathcal{S}'}$  on a different surface  $\mathcal{S}'$ . We will see here that several useful sets of operators can be constructed, which will play a role in the interpretation and dynamics of the theory. They are analogous to the loop operators whose algebra defines the loop representation of general relativity [32]. Here, because the states are defined without any reference to a background manifold, the operators are defined relationally, in terms of decompositions of the surfaces  $\mathcal{S}$  into pieces.

Let  $Y$  denote a genus  $g$  compact oriented 2-surface with  $n \geq 1$  punctures  $j_k$  ( $k = 1, \dots, n$ ). Given a compact  $\mathcal{S}$  let  $r_I$  denote the maps

$$r_I: Y \rightarrow \mathcal{S} \quad (12)$$

taking  $Y$  homomorphically to a component of  $\mathcal{S}$ . In general there will be a set of such maps; they are distinguished by the index  $I$ .

For each  $I$  the map picks out a set of  $n$  non-intersecting circles  $c_k^I$ ,  $k = 1, \dots, n$  in  $\mathcal{S}$ . Cutting  $\mathcal{S}$  on these circles decomposes it into the two pieces  $r_I(Y)$  and  $(\mathcal{S} - r_I(Y))$ . The space of intertwiners  $\mathcal{V}^{\mathcal{S}}$  decomposes as

$$\mathcal{V}^{\mathcal{S}} = \sum_k \mathcal{V}_{j_1 \dots j_n}^Y \otimes \mathcal{V}_{j_1 \dots j_n}^{(\mathcal{S} - r_I(Y))}. \quad (13)$$

A state  $|\mathcal{S}, \Psi\rangle \in \mathcal{V}^{\mathcal{S}}$  then decomposes to a sum over the representations  $j_1, \dots, j_n$  of the product of a state in  $\mathcal{V}_{j_k}^Y$  and a state in  $\mathcal{V}_{j_k}^{(\mathcal{S} - r_I(Y))}$ ,

$$|\mathcal{S}, \Psi\rangle = \sum_k |Y, j_k, \Psi^1\rangle \otimes |(\mathcal{S} - r_I(Y)), j_k, \Psi^2\rangle. \quad (14)$$

Using this decomposition we then define three classes of operators. The first two are block diagonal in the decomposition (2), while the third changes the topology of the surface  $\mathcal{S}$ .

### A. Surface operators

Let  $F(\{j_k\})$  be a symmetric function of  $n$  representation labels  $\{j_k\}$ . Given such a function and a 2-surface  $Y$  with  $n$  punctures, there is an Hermitian operator  $\hat{\mathcal{F}}_Y$  that acts in  $\mathcal{H}_{G_q}$  as follows. On the spaces  $\mathcal{V}_{j_1 \dots j_n}^Y$ ,  $\hat{\mathcal{F}}_Y$  is the diagonal operator equal to  $F(\{j_k\})$ . Then on a general state

$$\hat{\mathcal{F}}_Y |\mathcal{S}, \Psi\rangle = \sum_I \sum_k F(\{j_k\}) |Y, j_k, \Psi^1\rangle \otimes |(\mathcal{S} - r_I(Y)), j_k, \Psi^2\rangle. \quad (15)$$

This operator looks for the instances of the submanifold  $Y$  in each surface  $\mathcal{S}$  and, in each state, measures a property of the boundary separating  $Y$  from the remainder  $\mathcal{S} - r_I(Y)$  given by the function  $F(\{j_k\})$ . The punctured surface  $Y$  can be thought of as the algebraic representation of a 3-dimensional region  $\mathcal{R}$  in the quantum geometry that a state in  $\mathcal{V}^{\mathcal{S}}$  represents. The surface operators thus measure properties of boundaries of regions in space. This interpretation will be developed in the next section, where we will see that an example in the case of  $SU(2)$  is given by the area operator obtained in quantum general relativity [11,2].

### B. Bulk operators

Once a region  $\mathcal{R}$  of an abstract quantum geometry has been identified by a map  $r_I: Y \rightarrow \mathcal{S}$  we can also try to measure bulk properties of that region. These will be eigenvalues of operators that act on the space  $\mathcal{V}_{j_1 \dots j_n}^Y$  and depend on the topology of  $Y$  and hence on  $\dim \mathcal{V}_{j_1 \dots j_n}^Y$ . To define such an operator let us choose, for every  $j_1 \dots j_n$  an operator  $\hat{B}_{j_1 \dots j_n}$  on  $\mathcal{V}_{j_1 \dots j_n}^Y$ . The corresponding bulk operator  $\hat{\mathcal{B}}_{j_1 \dots j_n}^Y$  is defined on the state space  $\mathcal{H}_{G_q}$  as

$$\hat{\mathcal{B}}_{j_1 \dots j_n}^Y |\mathcal{S}, \Psi\rangle = \sum_I \sum_k \hat{B} |Y, j_k, \Psi^1\rangle \otimes |(\mathcal{S} - r_I(Y)), j_k, \Psi^2\rangle. \quad (16)$$

Examples of bulk operators are the volume operators which we will describe in the next section.

### C. Substitution operators

In the last section we defined the tubular evolution moves. These are examples of a large class of substitution operations that take us from one manifold  $\mathcal{S}$  to a different manifold  $\mathcal{S}'$  by cutting out a piece,  $Y^1$  of  $\mathcal{S}$  and replacing it with a different manifold  $Y^2$  with the same boundary. The tubular evolution moves are examples of these. For such substitutions we can define linear operators that act on  $\mathcal{H}_{G_q}$  and take states from  $\mathcal{V}^{\mathcal{S}}$  to those in  $\mathcal{V}^{\mathcal{S}'}$ .

Start with two punctured surfaces,  $Y_1$  and  $Y_2$ , each with an ordered set of  $n$  punctures, with labels  $j_1, \dots, j_n$ . They can be represented by vector spaces  $\mathcal{V}_{j_1 \dots j_n}^{Y_1}$  and  $\mathcal{V}_{j_1 \dots j_n}^{Y_2}$ . Note that in general  $\dim \mathcal{V}_{j_1 \dots j_n}^{Y_1} \neq \dim \mathcal{V}_{j_1 \dots j_n}^{Y_2}$ . Given two

vector spaces, we have the space of linear maps from the first to the second, denoted  $\text{hom}(\mathcal{V}^{Y_1}, \mathcal{V}^{Y_2})$ . A particular  $\hat{c} \in \text{hom}(\mathcal{V}^{Y_1}, \mathcal{V}^{Y_2})$  acts on a state,  $|Y_1, j_k, \Psi^1\rangle \in \mathcal{V}_{j_1 \dots j_n}^{Y_1}$  as

$$\hat{c} |Y_1, j_k, \Psi^1\rangle = |Y_2, j_k, \Psi^2\rangle \in \mathcal{V}_{j_1 \dots j_n}^{Y_2}, \quad (17)$$

giving a state in  $\mathcal{V}_{j_1 \dots j_n}^{Y_2}$ . Given any such  $\hat{c}$  we construct a substitution operator  $\hat{\mathcal{C}}_{Y_1, Y_2, \hat{c}}$ , defined by

$$\hat{\mathcal{C}}_{Y_1, Y_2, \hat{c}} |\mathcal{S}, \Psi\rangle = \sum_I \sum_k |(\mathcal{S} - r_I(Y_1)), j_k, \Psi^1\rangle \otimes [\hat{c} |Y^1, j_k, \Psi^1\rangle]. \quad (18)$$

The action of  $\hat{\mathcal{C}}_{Y_1, Y_2, \hat{c}}$  is pictured in Fig. 5.

Note that we may also glue  $Y_1$  and  $Y_2$  along their identical boundaries as in the right hand figure of Fig. 6.

## VI. GEOMETRICAL INTERPRETATIONS

So far, we have defined states in  $\mathcal{H}_{G_q}$  in terms of labelled 2-dimensional manifolds. We shall now interpret them in terms of observables related to 3-dimensional space. These arise as natural extensions of the observables of quantum general relativity: the area and volume operators.

The subtlety is that here there is no background manifold. All of the properties of space, including its topological and metric properties, must be coded into the states. In the absence of any background manifold to provide surfaces and regions, geometrical observables are constructed relationally, from information coded into the states.

Let us begin with the space of states  $\mathcal{V}^{\mathcal{S}}$  associated to a given 2-surface  $\mathcal{S}$ . A *microscopical geometrical interpretation* of these states exists for every decomposition of  $\mathcal{S}$  into a set of  $n$ -punctured 2-spheres,  $B_I^n$ , with  $n \geq 3$ , joined on a set of circles,  $c_\alpha$ . Let us consider a basis of states which is (partially) determined by definite values for the representations  $j_\alpha$  for these circles. This state is of the form,  $|\mathcal{S}, j_\alpha, \mu_I\rangle$  with intertwiners  $\mu_I \in \mathcal{V}_{j_1 \dots j_n}^{B_I^n}$  for each of the punctured spheres.

The geometrical interpretation is constructed as follows. Associated to each  $B_I^n$  is a region  $\mathcal{R}_I$ . These regions have three kinds of properties:

**Surface properties:** A surface property of a region  $\mathcal{R}_I$  is a function  $F(j_1, \dots, j_n)$  of the labels on the punctures of the corresponding  $B_I^n$ . Surface properties are measured by surface operators (15).

**Bulk properties:** A bulk property of a region  $\mathcal{R}_I$  is measured by a Hermitian operator  $\hat{B}$  (16) in the space of intertwiners  $\mathcal{V}_{j_1 \dots j_n}^{B_I^n}$ .

**Shared properties:** Two regions  $\mathcal{R}_I$  and  $\mathcal{R}_J$  may have shared properties if they have a set of common punctures with labels, say,  $j_1, \dots, j_k$ . If this set is non-empty,

then  $j_1, \dots, j_k$  is the common boundary of  $\mathcal{R}^I$  and  $\mathcal{R}^J$ . A shared property of  $\mathcal{R}^I$  and  $\mathcal{R}^J$  is then a function  $G(j_1, \dots, j_k)$ .

In the  $SU(2)$  case we may import the kinematical structure from quantum general relativity found in [11,2,1] to give us examples of each kind of observable:

The area of  $\mathcal{R}_I$  is a surface property. It is given by  $F(j_1, \dots, j_n) = l_{Pl}^2 \sum_{\alpha=1}^n \sqrt{j_\alpha(j_\alpha+1)}$ .

The volume of the interior of  $B_I^n$  is an example of a bulk property. As we know from [2,12] the volume operator is a Hermitian operator  $\hat{V}[j_1, \dots, j_n]$  that acts in the space or intertwiners  $\mathcal{V}_{j_1, \dots, j_n}^{B_I^n}$ <sup>8</sup>.

The area also gives an example of a shared property. If  $B_I^n$  and  $B_J^n$  share a set of  $k$  spins  $j_1, \dots, j_k$  then the area of the common boundary of  $\mathcal{R}_I$  and  $\mathcal{R}_J$  is given by  $l_{Pl}^2 \sum_{\alpha=1}^k \sqrt{j_\alpha(j_\alpha+1)}$  summed over the common punctures of the two regions.

Note that, given a division of  $\mathcal{S}$  into punctured spheres, we may simultaneously diagonalize all of the area and volume operators on the corresponding regions  $\mathcal{R}_I$ . Thus, a common eigenstate  $|\mathcal{S}, j_\alpha, \mu_I\rangle$  may be called a *microscopic quantum geometry*. It is a set of regions together with (i) an area for the boundary of each one, (ii) an area for each common boundary, such that the area of each is the sum of its common boundaries with the others and (iii) a volume for each region.

For a general  $G_q$ , we expect a *generalized microscopic quantum geometry* to be the maximal set of simultaneous eigenvalues of surface and bulk observables for a decomposition of  $\mathcal{S}$  into punctured spheres. There is also a notion of a coarse-grained quantum geometry. We will discuss this in Sec. VIII.

### A. Duality between edges and intertwiners

The reader may have noticed that the geometrical interpretations available to the states in  $\mathcal{V}^{\mathcal{S}}$  are not determined by  $\mathcal{S}$ . There is a geometrical interpretation for every way of dividing  $\mathcal{S}$  into punctured spheres. We regard this freedom as an intrinsic and attractive feature of the generalization from spin network states to the space of states  $\mathcal{H}_{G_q}$ . For example, in the 4-valent spin networks two kinds of edges appear: real edges connecting the nodes and ‘‘virtual’’ edges that may be used to label the intertwiners of the 4-valent nodes. Thus, in the usual spin network formalism they play different roles.

For example, consider the tubular 4-simplex  $\mathcal{P}$  and the two different decompositions into 4-punctured spheres illustrated in Fig. 4. These may be described in terms of two sets of circles  $c_\alpha$  and  $c_{\alpha'}$ , as shown in Fig. 8. [The full set  $(c_\alpha, c_{\alpha'})$  make up a maximal set of non-intersecting circles

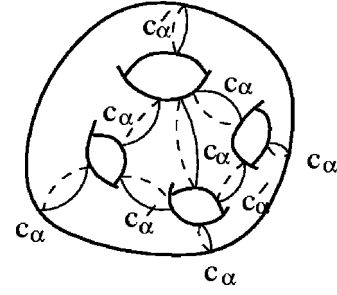


FIG. 8. A choice of decomposition into 4-punctured spheres where  $c_\alpha$  are representations and  $c'_\alpha$  intertwiners.

on  $\mathcal{P}$  and define a trinion decomposition of  $\mathcal{P}$ .] This decomposition represents states  $|\mathcal{P}, j(c_\alpha), \mu_I\rangle$ . If we now read the decomposition with a different set of circles, including  $c'_\alpha$ , separating the five 4-punctured spheres, we obtain a different set of basis states  $|\mathcal{P}, j(c'_\alpha), \mu_I\rangle$ . This shows that the distinction between spins and intertwiners in this formalism is dependent on the choice of  $n$ -punctured spheres. Therefore, so is the geometrical interpretation.

## VII. CAUSAL EVOLUTION

We now discuss the evolution of the states in  $\mathcal{H}_{G_q}$ . The dynamics of the theory will be based on the evolution moves defined in Sec. IV B. By composing the moves we produce sequences of states that we call histories. These discrete histories share three characteristics of Lorentzian spacetimes. (i) There is a set of events which is a discrete partially ordered set with no closed causal loops. This is a discrete analogue of a Lorentzian spacetime. (ii) There are connected sets of causally unrelated events, the combinatorial analogues of space-like surfaces. (iii) A history can be decomposed in many ways into sequences of spacelike surfaces, leading to a discrete analogue of many fingered time.

### A. The evolution operator

The evolution of states is generated by an operator that implements the evolution moves described in Sec. IV B. This will be a substitution operator of the form defined in Sec. V C. To do this let  $\rho = 1, 2, 3, 4$  correspond to the four kinds of Pachner moves  $1 \rightarrow 4, 2 \rightarrow 3, 3 \rightarrow 2, 4 \rightarrow 1$ . Then take  $L_\rho$  to be the elementary local region consisting of  $\rho$  4-punctured spheres, so that  $L'_\rho$ , the complement of  $L_\rho$  in  $\mathcal{P}$  consists of  $5 - \rho$  4-punctured spheres. We will call  $L_\rho$  the past set and  $L'_\rho$  the future set of the  $\mathcal{P}$  associated with the  $\rho \rightarrow 5 - \rho$  move (see Fig. 7).

For  $\rho = 1$  and 4 the  $L_\rho$  and  $L'_\rho$  each have 4 punctures, which are labeled by representations  $j_\gamma, \gamma = 1, \dots, 4$ . For  $\rho = 2, 3$  there are six punctures and  $\gamma = 1, \dots, 6$ . For each  $\rho$  and sets of 4 or 6 representations  $j_\gamma$  we may choose operators  $\hat{c}_{\rho, j_\gamma} \in \text{hom}(\mathcal{V}_{j_\gamma}^{L_\rho}, \mathcal{V}_{j_\gamma}^{L'_\rho})$ . The  $\rho$ th move is then implemented by the substitution operator

<sup>8</sup>Here we take the definition given in [2] that does not require any assumptions about structure not present in our case such as linear relations among tangent vectors at the nodes.

$$\begin{aligned}
\hat{H}^\rho |S, \Psi\rangle &= \hat{C}_{L_\rho, L'_\rho, \hat{c}_\rho} |S, \Psi\rangle \\
&= \sum_I \sum_{k=1}^{n_\rho} |(\mathcal{S} - r_I(L_\rho)), j_\gamma, \Psi\rangle \\
&\quad \otimes [\hat{c}_\rho |L_\rho, j_\gamma, \Psi\rangle]. \tag{19}
\end{aligned}$$

The total evolution operator is then given by

$$\hat{H} = \sum_\rho \hat{H}^\rho. \tag{20}$$

To see how these act, let us start with an initial state  $|S, \Psi\rangle$  and act on it with one of the  $H^\rho$ . If  $\mathcal{S}$  is large enough, there will be numerous regions in it homeomorphic to  $L_\rho$ . To each of them there is a map  $r_I: L_\rho \rightarrow \mathcal{S}$ . For each  $I$  we then cut from  $\mathcal{S}$  the region  $r_I(L_\rho)$  and replace it by  $L'_\rho$ . This results each time in a new 2-surface which we call  $\mathcal{S}_{\rho, I}$ . The result of the application of  $\hat{H}^\rho$  is then a superposition of the states given by the action (19). The exact map from the old states to the new states is given by the linear maps  $\hat{c}_\rho$ . (We suppress the dependence of  $\hat{c}_\rho$  on the representations  $j_\gamma$ .)

The operator  $H$  is Hermitian when each of the  $\hat{c}_\rho$  are appropriately chosen. In this case a formal unitary evolution operator may be written down as

$$\hat{U} = e^{i\hat{H}t} \tag{21}$$

where  $t$  is a parameter having nothing to do with the physical time (it just scales the operators  $\hat{c}_\rho$ .) The amplitude for an initial state  $|initial\rangle = |\mathcal{S}_{initial}, \Psi_{initial}\rangle$  to evolve to a final state  $|final\rangle = |\mathcal{S}_{final}, \Psi_{final}\rangle$  is formally given by

$$\mathcal{A}[|initial\rangle \rightarrow |final\rangle] = \langle final | \hat{U} | initial \rangle. \tag{22}$$

### B. Amplitudes for causal evolution and a discrete path integral

By decomposing the action of  $\hat{U}$  at each order  $n$  of the action of  $(\hat{H})^n$  in terms of 4-punctured spheres produced by the evolution moves, the amplitude (22) can be given in terms of a sum over a set of histories,  $\mathcal{M} = \{|1\rangle, |2\rangle, |3\rangle, \dots\}$  in which each  $|I+1\rangle$  results from the previous  $|I\rangle$  by the application of one of the four moves. The theory gives an amplitude to each transition from an initial state to one of its successor states. The amplitude is given by

$$\mathcal{A}_{L \rightarrow L'} = \langle L', j_{k'}, \mu'_I | \hat{c}^\rho | L, j_k, \mu_I \rangle, \tag{23}$$

where  $|L, j_k, \mu_I\rangle$  is a trinion basis state for the initial elementary local region to be cut out and  $|L', j_{k'}, \mu'_I\rangle$  is a basis state on the elementary local region that replaces it.

Consider now an  $(N-1)$ -step history  $\mathcal{M} = \{|1\rangle, |2\rangle, \dots, |N\rangle\}$ . Each transition is a generalized evolution move which has an amplitude  $\mathcal{A}^I$  given by (23) for the transition from  $|I\rangle$  to  $|I+1\rangle$ ,  $I = \{1, \dots, N-1\}$ . The amplitude of the history  $\mathcal{M}$  is then given by

$$\mathcal{A}[\mathcal{M}] = \prod_I \mathcal{A}^I. \tag{24}$$

Let us then have two states,  $|initial\rangle$  and  $|final\rangle$ . There is an infinite number of histories  $\mathcal{M}$  such that the first state is equal to  $|initial\rangle$  and the last state is equal to  $|final\rangle$ . By analogy to the simplicial case we may denote this as  $\partial\mathcal{M} = |initial\rangle \cup |final\rangle$ . The transition amplitude to evolve to  $|final\rangle$  given  $|initial\rangle$  is then,

$$\mathcal{A}[|initial\rangle \rightarrow |final\rangle] = \sum_{\mathcal{M} | \partial\mathcal{M} = |initial\rangle \cup |final\rangle} \mathcal{A}[\mathcal{M}]. \tag{25}$$

As this is an infinite sum one may first compute the amplitude for  $|initial\rangle$  to evolve to  $|final\rangle$  in  $N$  steps. This is given by

$$\mathcal{A}^N[|initial\rangle \rightarrow |final\rangle] = \sum_{\mathcal{M} | |1\rangle = |initial\rangle, |N\rangle = |final\rangle} \mathcal{A}[\mathcal{M}], \tag{26}$$

i.e., the sum over  $(N-1)$ -step histories that take the initial to the final state. However, note that while the full amplitude (22) is formally unitary by construction the same is not the case for the  $N$  step amplitude (26).

### C. The causal structure

We now show that each history  $\mathcal{M}$  has defined on it a discrete causal structure as a result of its construction from the evolution moves. Each history consists of  $N$  states  $|I\rangle$  which are elements of  $\mathcal{H}_{G_j}$ . Furthermore, the states  $|I\rangle$  come as labeled spin-tubes. Each one has a set of descriptions in terms of generalized areas and volumes because of its decompositions into  $n$ -punctured spheres. Each history may be thought of as consisting of a succession of quantum 3-geometries. Besides the representations and intertwiners, there is another structure defined on the histories: each history  $\mathcal{M}$  is a causal set, whose structure is determined as follows.

Each history  $\mathcal{M}$  is also a set of genus-6 elementary spin-tubes  $\mathcal{P}_i$ . Each  $\mathcal{P}_i$  is divided into two parts  $L_i$  and  $L'_i$  corresponding to the elementary local regions that were removed and inserted. The 4-punctured spheres in  $L_i$  are the *past set* of  $\mathcal{P}_i$ . The remaining 4-punctured spheres, which are in the complement  $L'_i$  are the *future set* of  $\mathcal{P}_i$ . Now, consider a particular 4-punctured sphere  $s$  in the future set  $L'_i$  in some  $\mathcal{P}_i$ . Let us assume that  $s$  has been acted on by at least one generalized evolution move  $\mathcal{P}_j$  for  $j > i$ . Then  $s$  also belongs to the past subset  $L_j$  of  $\mathcal{P}_j$ . If now  $s'$  is a 4-punctured sphere in the future subset  $L'_j$  of  $\mathcal{P}_j$ , we will say that  $s'$  is to the *immediate causal future* of  $s$ .

Now, consider a sequence of  $r$  4-punctured spheres  $s_i$ ,  $i = 1, \dots, r$ , such that for each  $s_i$ ,  $i < r$  either (i)  $s_{i+1}$  is to the immediate causal future of  $s_i$ , or (ii) there is some  $|I\rangle \equiv |\mathcal{S}_I, \Psi_I\rangle \in \mathcal{M}$  such that  $s_i$  and  $s_{i+1}$  are both in the surface  $\mathcal{S}_I$  and  $s_i \cap s_{i+1} \neq \emptyset$ . (This, is either each 4-punctured sphere in the sequence is to the immediate causal future of its pre-



decessor, or it and its predecessor overlap in a single surface associated with a state  $|I\rangle$  in the history.) When this is the case we will say that  $s_r$  is to the causal future of  $s_1$ ,  $s_r > s_1$ .

It is clear that the relation  $>$  is transitive and that given two 4-punctured spheres  $s_1$  and  $s_2$ ,  $s_1 > s_2 > s_1$  is never the case. Thus, the 4-punctured spheres in each history  $\mathcal{M}$  constitute a causal set, which is defined in [33] to be a partially ordered set with no closed causal loops which is locally finite. The latter means that given any  $s_1$  and  $s_2$  the set contained in the causal past of  $s_2$  and the future of  $s_1$  is finite. As argued in [33,34] a discrete set that has on it a causal structure is a candidate for a discrete model of spacetime.

The 4-punctured spheres of a history  $\mathcal{M}$ , defined by the evolution moves that construct it, are then the *events* of  $\mathcal{M}$ . We will call the set of events  $\mathcal{E}$ . By construction,  $\mathcal{E}$  is a causal set. It differs from the causal set of Sorkin and collaborators [33] in that there is additional structure, associated to a notion of space.

Each history  $\mathcal{M}$  may be foliated by a number of sets of causally unrelated events of  $\mathcal{M}$  that we will call the spacelike slices  $\Gamma$ . A spacelike slice of  $\mathcal{M}$  is a subset  $\{s_a\}$  of  $\mathcal{E}$  glued together according to the following rules:

- (1) No two  $s_a$  in  $\Gamma$  may be causally related.
- (2) Two events  $s_a$  and  $s_b$  in  $\Gamma$  may be glued together if there is a state  $|I\rangle \in \mathcal{M}$  in which they are glued along some circle.
- (3) The set  $\Gamma$  is maximal in that no  $s_a$  may be added to it without violating these conditions.

Associated with  $\Gamma$  is a state  $|\Gamma\rangle \in \mathcal{H}_{G_q}$  given by  $|\mathcal{S}_\Gamma, j, \mu_a\rangle$ . Here the intertwiners  $\mu_a$  are fixed because  $s_a \in \Gamma$  are given. Similarly, each circle  $c_{ab}$  along which two adjacent 4-punctured spheres  $s_a$  and  $s_b$  are glued is in fact a circle labeled by a fixed representation  $j$ . Hence the labels on the state  $|\Gamma\rangle = |\mathcal{S}_\Gamma, j, \mu_a\rangle$  are uniquely determined by the history  $\mathcal{M}$ .

The  $N$  original states  $\{|1\rangle, |2\rangle, \dots, |I\rangle, \dots, |N\rangle\}$  are spacelike slices according to this definition. But there are many more sequences which may be constructed given the history  $\mathcal{M} = \{|1\rangle, |2\rangle, \dots, |I\rangle, \dots, |N\rangle\}$  that have  $|1\rangle$  as the initial state and  $|N\rangle$  as final. We call the set of such states  $W_{\mathcal{M}}$ . One may in general select other sequences of elements of  $W_{\mathcal{M}}$ , e.g.  $\mathcal{M}' = \{|1\rangle, |2'\rangle, \dots, |I'\rangle, \dots, |N\rangle\}$ , that have the property that every event in  $\mathcal{E}$  is a 4-punctured sphere in a decomposition of at least one  $|I'\rangle$ . As far as the local geometry and causal structure are concerned these are equivalent descriptions of the history  $\mathcal{M}$ . Thus, this quantum theory has a discrete analogue of multi-fingered time.

Thus, a discrete history  $\mathcal{M}$  combines discrete analogues of both the canonical picture of quantum gravity and the spacetime causal structure. It is the marriage of both kinds of structure within a completely discrete approach to quantum gravity that we believe gives this approach its particular power.

#### D. Connection with spin foam and membranes

In a number of recent papers [35–39] a concept Baez calls “spinfoam” has been introduced. These are networks of colored 2-surfaces embedded in a four-dimensional spacetime whose slices by three-manifolds are spin networks. Gupta [40] has shown that the spin foam can be given a Lorentzian formulation by the addition of a causal structure and that formulation is in a particular sense dual to the formulation of [23]. There is an analogous spacetime foam structure associated with the histories  $\mathcal{M}$ , although it has not been so far investigated. It can be constructed by noting that each of the evolution moves may be seen as three-dimensional cobordisms between the two surfaces  $L_\rho$  and  $L'_\rho$  (see Fig. 6). The resulting three-manifolds may be joined together to construct a three-dimensional timelike combinatorial manifold associated to each history  $\mathcal{M}$ . This is a non-perturbative, background independent membrane.

### VIII. COARSE GRAINING, ENTROPY AND THE HOLOGRAPHIC HYPOTHESIS

Before closing we make some comments about coarse graining and entropy that will enable us to comment also on the relationship of our proposal to the holographic hypothesis [41,42] and the Bekenstein bound [43].

The basic idea is that in addition to the fine grained observables discussed previously there are coarse grained observables that describe statistical information about the states defined in Sec. II. There are two kinds of course grainings which are relevant. In the first we retain information about the topology of the surface  $\mathcal{S}$  while in the second we retain only information that can be measured by observers at the boundaries of the regions.

Before describing these we may note that the existence of coarse grained observables in itself means that the theory genuinely has local observables that are not determined by the values of the coarse grained observables.

#### A. Coarse graining by topology

We can coarse grain the information in a state  $|\mathcal{S}, \Psi\rangle$  by forgetting the information about the state  $\Psi \in \mathcal{V}^{\mathcal{S}}$  and retaining only statistical information about the surface  $\mathcal{S}$ . This results in a density matrix which is constructed by tracing over the representations  $j_a$  and intertwiners  $\mu_i^j$ . To each surface  $\mathcal{S}$  is then associated a density matrix which is  $\rho_{\mathcal{S}} = P_{\mathcal{S}}$ , the projection operator onto  $\mathcal{V}^{\mathcal{S}}$ . There is an entropy associated with this coarse graining. Associated to each surface  $\mathcal{S}$  is an entropy  $S[\mathcal{S}] = \ln(\dim \mathcal{V}^{\mathcal{S}})$ .

As the dynamics changes the topology an entropy change can be associated with the evolution operators defined in the last section. This makes possible a thermodynamic treatment of the evolution, which will be described elsewhere.

**B. Coarse graining by regions**

Rather than coarse graining by the topology of  $\mathcal{S}$  we can coarse grain by splitting space into regions and measuring statistical information about each region. To do this we must take into account what we learned from our discussion of geometrical interpretations, which is that as the topology and geometry of space are defined from the states, the splitting of space into regions must be defined intrinsically in terms of the states. We then define a coarse grained quantum geometry as a coarse grained interpretation of a quantum state  $|\mathcal{S}, \{j\}, \{\mu\}\rangle$ . Let us then consider a decomposition of  $\mathcal{S}$  into a set of regions  $R_i$  along  $m_i$  circles  $c_\gamma$ . Each piece consists of a component of  $\mathcal{S}$  we will call  $W_i$ . Each  $W_i$  is a punctured surface, punctured by the  $m_i$  labels  $j_\gamma$  on the circles  $c_\gamma$ .

To each region we will also associate a punctured  $S^2$ , with  $m_i$  punctures with the same labels as the  $S_i$ . Coarse graining will mean that for each region  $R_i$  we forget the details of the topology of the component  $W_i$ . This means that all observables concerning the region must be representable as operators in the space of intertwiners on the associated punctured  $S^2$ . There are then two spaces of intertwiners which are relevant,  $\mathcal{V}_{j_\gamma}^{W_i}$  and  $\mathcal{V}_{j_\gamma}^{S^2}$ . Coarse graining consists of replacing a microscopic state, which is a vector in  $\mathcal{V}_{j_\gamma}^{W_i}$  with a density matrix in  $\mathcal{V}_{j_\gamma}^{S^2}$ .

In correspondence with the different notions of properties we may define a *coarse grained surface property* of the region  $W_i$  to be a function of the labels  $j_1, \dots, j_{i_m}$  and a *coarse grained bulk property* to be an operator in  $\mathcal{V}_{j_\gamma}^{S^2}$ . Finally, two regions may share properties when the corresponding  $W_i$ 's are glued along punctures. Moreover, given a full set of labelings on the punctured surface  $W_i$  we have a state in  $\mathcal{V}_{j_\gamma}^{S^2}$  by considering the  $W^i$  as a framed spin network embedded in the interior of the surface  $S_i$  in  $R^3$ .

A coarse grained description of the quantum geometry is then given by a density matrix in the spaces  $\mathcal{V}_{j_\gamma}^{S^2}$  that corresponds to each of the regions  $R_i$ . It corresponds to what observers may measure about the world, assuming they can only measure on boundaries.

**C. Connection with the holographic hypothesis and Bekenstein bound**

The possibility of describing coarse grained properties in this way also suggests a formulation of the holographic [42,41] hypothesis that is entirely non-perturbative and background independent. This arises in the case that we split the universe into two regions, and assume that we can only make measurements in one of them.

Let us introduce a splitting of a surface  $\mathcal{S}$  along a set of  $p$  non-intersecting elements of  $\pi^1[\mathcal{S}]$ , which we will call the  $c_\gamma$ ,  $\gamma=1, \dots, p$ . The two halves may be called  $\mathcal{S}^+$  and  $\mathcal{S}^-$ ; the  $c_\alpha$  are in each case their ends. Let us further consider a basis of states in which there are definite representations  $j_\gamma$  defined on the surfaces.

In the absence of a background manifold we will simply represent the splitting by a  $p$ -punctured  $S^2$ , labeled by the  $j_\gamma$ . Each half  $\mathcal{S}^\pm$  then has on it a space of intertwiners  $\mathcal{V}_{j_\gamma}^{S^\pm}$ . An element  $\mathcal{V}_{j_\gamma}^{S^\pm}$  defines what we will call a *quantum geometry with boundary*. Given a quantum geometry, i.e., a state in a  $\mathcal{V}_S$ , there are many ways to split it into two halves, giving two quantum geometries with boundaries. The splitting of the world into two parts constitutes a simple coarse graining of it.

Now consider an observer who lives in one half,  $\mathcal{S}^+$ , who is for some reason unable to measure any information about the topology or state of  $\mathcal{S}$  in the other half  $\mathcal{S}^-$ . This might, for example, arise if the causal structure (which we have shown makes sense at this, non-perturbative background independent level) does not enable him or her to receive any information from the other half. In this case the observer effectively lives in a quantum geometry with boundary defined by the half  $\mathcal{V}_{j_\gamma}^{S^+}$ .

What information can the observer have about the physics of the other half  $\mathcal{V}_{j_\gamma}^{S^-}$ ? All they can measure is correlations between measurements they may make at the  $p$  ends. This means that the possible states they may distinguish by their measurements are given exactly by the space of conformal blocks on the  $p$ -punctured  $S^2$  associated with their boundary. This is the space  $\mathcal{V}_{j_\gamma}^{S^2}$  which we described before.

To summarize, the following may be considered a *non-perturbative formulation of the holographic hypothesis*: When an observer is unable to measure information corresponding to the interior of a region of a quantum geometry, because of the presence of a causal horizon, or for any other reason, the information accessible to them by measuring observables at the boundary of that region is represented by a finite dimensional space of states  $\mathcal{V}_{j_\gamma}^{S^2}$  for some  $p$ -punctured  $S^2$ .

This has several further consequences. First, in the SU(2) case it is known that [8]

$$\ln(\dim[\mathcal{V}_{j_\gamma}^{S^2}]) \leq \frac{c}{4} \frac{A[j_\gamma]}{l_{Planck}^2} \tag{27}$$

for large numbers of punctures, where  $c=8 \ln(2)/\sqrt{3}$ . Here  $A[j_\gamma]$  is the area operator of quantum general relativity [11,2] with eigenvalues  $\sum_\gamma J_{Planck}^2 \sqrt{j_\gamma(j_\gamma+1)}$ . Thus, *the Bekenstein bound [43] is automatically satisfied.*<sup>9</sup>

In the case of a general  $G_q$  we do not know which ob-

<sup>9</sup>We may note that the constant  $c$  is not equal to one. This is not surprising given that the quantity  $l_{Planck}$  in the area formula is given by the bare Newton's constant. Unless the theory has a continuum limit the macroscopic, renormalized Newton's constant which plays a role in black hole thermodynamics cannot be defined. This result suggests then predicts that in those cases  $G_{ren} = cG_{bare}$ .

servable corresponds to the area. It may be any surface property, which means it must be an additive function  $A$  of the casimers of  $G_q$ . The Bekenstein bound gives us a constraint on that definition, which is that

$$A[j_\gamma] < 4l_{Plank}^2 \ln(\dim[\mathcal{V}_{j_\gamma}^{S^2}]). \quad (28)$$

We may note that the Bekenstein bound (28), together with certain other assumptions is, as Jacobson has shown [44], equivalent to the Einstein equations. Jacobson's argument in [44] can be interpreted to imply that any finite theory of quantum gravity that has a classical limit such that (a) the relationship (28) is satisfied on every horizon which exists by virtue of an observer being accelerated and (b) quantum fields behave as conventional free fields in the limit of low curvatures, then the field equations of general relativity are true to leading order in curvatures as a consequence of the ordinary laws of thermodynamics [44]. This suggests that statistical assumptions about the dynamics, together with Eq. (28) may be sufficient to derive the classical limit of the theory.

### IX. CONCLUSION

The general framework introduced here becomes a theory with two inputs: a group or algebra  $G_q$ , and a choice of the dynamical operators  $\hat{H}^\rho$  that define the evolution. The main question that must be investigated is how these operators are to be chosen. Good choices should lead to a theory with a good continuum limit which reproduces classical general relativity with matter fields. This is currently being investigated in several directions.

- (1) The algebra of the tube operators introduced here should be worked out. It will be interesting to see if there is a set of local operators that generate the algebra and if they are related to the loop algebra of quantum gravity [32] in the  $q \rightarrow 1$  limit.
- (2) It appears possible to choose the evolution operator  $\hat{c}^\rho$  to agree with the dynamics generated by the Lorentzian Hamiltonian constraint of Thiemann [16]. A path integral representation of Thiemann's Lorentzian constraint, along the lines of [36], may then be possible. It seems that the evolution generated by Thiemann's constraints is ultralocal [17,18]. However we may note that the evolution generated by the  $1 \rightarrow 4$  and  $4 \rightarrow 1$  moves are ultralocal in the sense that they do not lead to long-range propagation. As suggested already in the Euclidean context in [36] it follows that the other moves are necessary in order to have long-range propagation.

- (3) More generally, the relation of the causal theory to the Euclidean path integral approaches [35–39] should be investigated. In this direction, Gupta in [40] has formulated a causal spin foam.
- (4) All of the above involve so far only the SU(2) spin networks. The extension to other groups is important. The SO(8) case is of special interest because of its connection to supersymmetry and triality. It is currently under investigation with Asok. The general case of a supergroup should be investigated.
- (5) Two connections with string theory have been investigated. In [21], we take  $G_q$  to be the projective group of the circle. Its representations are parametrized by relatively prime pairs of integers  $(p, q)$ . The states in this case turn out to be combinatorial  $(p, q)$  string networks [45] whose dynamics is a simple case of Sec. VII. Second, in [22] perturbations of the SU(2) theory have been studied which are given by a  $(1+1)$ -dimensional system with couplings determined by  $\hat{c}^\rho$ . When the full theory has a good continuum limit, the action for the  $1+1$  system is given to leading order by the Nambu action of bosonic string theory. An argument may be given that if, for some choice of  $G_q$  and  $\hat{c}^\rho$  the induced  $1+1$  dimensional theory is a consistent perturbative string theory then the continuum limit of the non-perturbative theory exists.
- (6) In [24] we argued that the existence of a continuum limit can be seen as a critical phenomenon which is analogous to directed percolation. To investigate this we have invented a set of simple models that have dynamical causal structure of the type described here [46]. Further, these models are discrete dynamical systems since evolution proceeds by discrete local steps. This leads to proposals for the evaluation of the path integrals proposed here which are discussed in [47]. Finally, given the remarks in the previous section, one may use statistical mechanics to make general statements about the evolution of the states based on the entropy  $S[S]$ .

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