Bright solitons as black holes

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2D Jackiw-Teitelboim gravity is represented as a completely integrable nonlinear reaction-diffusion system, whose Euclidean version leads to the nonlinear Schrödinger equation. The solitonlike solutions, called dissipatons, to such systems characterize completely the black holes of the considered gravity model (the black hole horizon, the Hawking temperature, and the causal structure). The collision of black holes is described in terms of elastic scattering of dissipatons, which shows a novel transmissionless character, creating a metastable state with a specific lifetime. Finally, alternative descriptions of the model in terms of other completely integrable systems are overlooked. [S0556-2821(98)02716-7]

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I. INTRODUCTION

It is well known that a 2D gravity theory in general can be always locally represented in the conformal gauge. This naturally leads to the so-called Liouville model of gravity. For example, this is true in the Jackiw-Teitelboim (JT) model [1,2], obtained as a dimensional reduction of the $2+1$ Einstein-Hilbert action, or also from a spherical reduction of a 4D dilaton-gravity Einstein-Hilbert-Maxwell action [3]. Now, the classical Liouville equation is a completely integrable system and its general solution is provided in terms of free fields [4,5]. However, its quantum version is a challenging and not completely solved problem (for a recent review, see, for instance, Refs. [6,7]). On the other hand, the study of this low-dimensional gravity model received a great deal of attention recently since black hole (BH) solutions were discovered [8,9]. This allowed one to have available a lower-dimensional analogue of realistic 4D black holes, for which the key features could be exhibited without unnecessary complications. Moreover, the existence of a black hole implies a nontrivial causal structure, which is related to the Hawking radiation phenomenon [10] and to interesting thermodynamical properties [3,11–13]. However, the nontrivial causal structure associated with a BH is hard to describe in the Liouville gauge. Hence, one is encouraged to look for alternative formulations. For instance, in Ref. [14] it is studied the connection between JT gravity and the sine-Gordon equation. In a more direct approach, we have recently investigated certain noncovariant gauge choices in the context of the gauge formulation of the JT model [15]. In particular, we proposed a nonrelativistic gauge choice related to the SO(2,1) Heisenberg model [15]. This choice leads to a completely integrable reaction-diffusion (RD) system for the Zweibein fields. This multicomponent system admits the dissipative analogue of the nonlinear Schrödinger (NLS) equation solitons, which we will call "dissipatons." The main goal of the present work is to show that the dissipatons correspond to the BH of the JT theory, in the sense that we can describe the horizon position and the causal structure of the solution. This is intimately related to the dissipaton boundary conditions. In its turn, different types of boundary conditions are admissible in correspondence of different signature of the nonlinear coupling constant, proportional to the cosmological constant Λ introduced in the JT model. In particular, in accordance with Refs. [8,9], for $\Lambda<0$ from one side the BH's correspond to the anti–de Sitter space time and, from another side, to the attractive NLS case, as shown in the present paper. Then, BH's and bright solitons are correlated. Secondly, since the dissipaton amplitude scales as Λ^{-1}, strong nonlinear effects are not longer negligible at sufficiently large scales, also when $\Lambda\rightarrow 0$. Furthermore, only dissipatons moving at velocities less than a critical value lead to BH solutions. In this sense, the relativistic bound on the allowed velocities is recovered in the present nonrelativistic picture. Finally, from these preliminaries we will outline the properties of the RD system, which are nontrivial, although it can be seen as an analytical continuation of the NLS equation.

In Sec. II we review the connection among the RD and NLS systems and the JT model, giving its explicit reformulation in this new formalism. In Sec. III we provide the link between the one-dissipaton solution and the black hole of the JT gravity. This is made both for the static and for the moving dissipaton, characterizing the set of soliton parameters meaningful in the present context. Section IV is devoted to the analysis of a type of collision of two black holes in terms of elastic scattering of dissipatons. This scattering has a transmissionless character (in contrast with the reflectionless solitons) and creates a metastable state of BH's with a specific life time. Section V contains the Euclidean version of the previous treatment, which enables us to compute the Hawking temperature in terms of the NLS soliton parameters. In Sec. VI we review the connection among the gravity model and certain relevant completely integrable systems: the self-dual σ model, the nonlinear σ model, the Korteweg–de Vries (KdV) and the modified KdV equations (MKdV). All of them are possible alternative representations of the JT model, enjoing several nice properties at the classical level. Of course, the equivalence of two classical theories does not imply the equivalence of the corresponding quantum versions. In the particular case of the JT model,

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there are some indications that classically equivalent formulations are nonequivalent at the quantum level [16]. But the quantization of a classically completely integrable model can be performed also by the quantum spectral transform [17], which in the present case may provide a different quantum integrable model of the gravity. Some interesting properties could arise from the Poisson structure of the RD system, being the same of the NLS equation, which is simpler than for the Liouville equation.

II. GAUGE FORMULATION OF THE JT GRAVITY

In order to fix some concepts and the notation, let us overview the gauge formulation of the JT model. The main idea of a gauge field theory of gravity is that the general coordinate transformations are implemented by the gauge ones. These do not act on the original metric tensor, but on the Vielbein and spin connection. Thus, the local gauge group induces on a base space the space-time metric tensor [18]. This approach has been used for the JT model [19–22], given by

$$S=\int_{\mathcal{M}}\sqrt{-g}V_0(R-\Lambda)dx^1dx^2, \tag{2.1}$$

where R is the scalar curvature, Λ the cosmological constant, and V_0 is a world scalar Lagrange multiplier, or dilaton field. Then, one introduces the ''rotated'' Zweibein fields $q_a^{\pm}$ $(a=1,2)$, defined by

$$g_{ab}=-\frac{4}{\Lambda}(q_a^+q_b^-+q_b^+q_a^-) \tag{2.2}$$

and a spin connection V_a, taken as independent variables and combined into the connection one-form

$$J=J_adx^a, \tag{2.3}$$

$$J_a=\frac{i}{2}(q_a^++q_a^-)\tau_1-\frac{i}{2}(q_a^+-q_a^-)\tau_2+\frac{i}{4}V_a\tau_0 \tag{2.4}$$

$$=\frac{i}{4}\tau_0V_a+\begin{pmatrix}0 & q_a^-\\ q_a^+ & 0\end{pmatrix}\quad (a=1,2), \tag{2.5}$$

where τ_i $(i=0,1,2)$ are the basis elements of the sl(2,$\mathbf{R}$) algebra, satisfying the set of relations

$$\tau_i\tau_j=h_{ij}+ic_{ijk}\tau_k, \tag{2.6}$$

with $(h_{ij})=\mathrm{diag}(-1,-1,1)$ and $c_{ijk}=-\epsilon_{ijl}h_{lk}$. This parametrization realizes a $\mathbf{Z}_2$ graduation of the connection algebra with isotropy group $O(1,1)$. In order to keep contact with a more usual notation, we write

$$P_0=\frac{i}{2}\tau_0,\quad P_a=(-1)^{a+1}\frac{i}{2}\sqrt{-\frac{\Lambda}{2}}\tau_a\quad (a=1,2), \tag{2.7}$$

$$[P_1,P_2]=-\frac{\Lambda}{2}P_0,\quad [P_a,P_0]=\epsilon_a{}^bP_b, \tag{2.8}$$

where P_a $(a=1,2)$ and P_0 are the generators of translations and of the Lorentz transformations, respectively, generating the de Sitter group SO(1,2). Our convention for the locally flat Minkowski metric is $\eta_{ab}=\mathrm{diag}(-1,1)$. Thus, the metric tensor (2.2) takes the usual Zweibein form

$$g_{ab}=e_a^ce_b^d\eta_{cd},\quad \text{where}\quad q_a^{\pm}=\frac{1}{2}\sqrt{-\frac{\Lambda}{2}}(e_a^0\pm e_a^1). \tag{2.9}$$

Now, from the vanishing of the curvature two-form $F=dJ+J\wedge J$ one gets the equation

$$\begin{aligned}F_{12}&=\partial_1J_2-\partial_2J_1+[J_1,J_2]\\&=\frac{i}{4}\epsilon_{ab}(\partial_aV_b-4q_a^+q_b^-)\tau_0\\&\quad+\frac{i}{2}\epsilon_{ab}(D_a^-q_b^++D_a^+q_b^-)\tau_1\\&\quad-\frac{i}{2}\epsilon_{ab}(D_a^-q_b^+-D_a^+q_b^-)\tau_2=0,\end{aligned} \tag{2.10}$$

where $D_a^{\pm}\equiv\partial_a\pm(1/2)V_a$ represents the covariant derivative and the antisymmetric tensor ϵ is given by $\epsilon_{12}=1$. The first component of Eq. (2.10) provides the curvature of the connection (2.5), while the second and third components dictate the torsionless condition. Solving them with respect to the spin connection V_a and substituting into the curvature equation we obtain that the scalar curvature R is equal to Λ, as prescribed by Eq. (2.1). However, Eq. (2.10) can be derived from the action functional

$$\begin{aligned}S&=\int_{\mathcal{M}}\epsilon_{ab}\Big[q_0^+D_a^+q_b^-+q_0^-D_a^-q_b^+\\&\quad+\frac{1}{8}V_0(\partial_aV_b-4q_a^+q_b^-)\Big]dx^1dx^2\\&=\int_{\mathcal{M}}\mathrm{Tr}(J_0F_{12})dx^1dx^2,\end{aligned} \tag{2.11}$$

where $q_0^{\pm}$ are new Lagrangian multipliers and J_0 is defined accordingly to Eq. (2.5). Actions of the form (2.11) are known as BF theories, whose classical and quantum issues have been well studied (see Ref. [23] for a review on the topic). In particular, the action (2.11) is invariant with respect to infinitesimal SL$(2,\mathbf{R})$ gauge transformations, which are equivalent to general coordinate invariance if the equation of motion are used [24,25]. Moreover, the equation for the multiplier J_0 is given by $\mathcal{D}_aJ_0=0$, where $\mathcal{D}_a=\partial_a-[J_a,]$. The total set of the $O(1,1)$ gauge invariant equations arising from the variation of the action (2.11) can be written in the form

$$D_\mu^\mp q_\nu^\pm = D_\nu^\mp q_\mu^\pm,$$
$$\partial_\mu V_\nu - \partial_\nu V_\mu = 4(q_\mu^+ q_\nu^- - q_\nu^+ q_\mu^-), \tag{2.12}$$

with $\mu,\nu=0,1,2$ and under the assumption that all the derivatives of $q_a^\pm$ and V_a ($a=1,2$) with respect to an auxiliary variable x^0 vanish. Then, Eq. (2.12) is again the zero-curvature condition in 3D for the connection J. Actually, Eq. (2.12) represents the Euler-Lagrange equation for the Chern-Simons (CS) action on the SL(2,**R**) group [24,25], which can be seen as a subgroup of the corresponding Poincaré group. In 2D the local symmetry group identifies with the de Sitter group. Thus, the obtained structure is a relic of the 3D CS theory under the effects of the dimensional reduction. Of course, the space of all solutions of the classical field equations, modulo SL(2,**R**) gauge transformations, is finite dimensional. This is specific of the topological character of the gauge field theory (2.11). In fact, the general solution of the system (2.12) can be given in the form of the right-invariant chiral current $J_\mu = G^{-1}\partial_\mu G$, where G is a differentiable mapping on SL(2,**R**). But, if we introduce the so-called moving trihedrals frame $\{\mathbf{n}_i\}$ [26,27] by the local adjoint representation of the algebra expressed by

$$G\tau_i G^{-1} = n_i^k \tau_k, \quad \mathbf{n}_i = (h_{jj} n_j^i), \tag{2.13}$$

one sees that it would satisfy the orthonormal conditions

$$(\mathbf{n}_i \mathbf{n}_j) = h_{ij}, \quad \mathbf{n}_i \wedge \mathbf{n}_j = c_{ijk}\mathbf{n}_k, \tag{2.14}$$

induced by the relations (2.6). Moreover, the moving frame changes accordingly to the adjoint representation of J_μ, that is,

$$\partial_\mu \mathbf{n}_i = (J_\mu)_{ik}^{(ad)} \mathbf{n}_k, \tag{2.15}$$

where $J_\mu^{(ad)}$ are matrices in the adjoint representation. Equation (2.15) can be seen as a linear system for $\{\mathbf{n}_i\}$. Its integrability is assured by the zero curvature condition, satisfied by the chiral currents J_μ, namely, by Eq. (2.12). Now, we assign to $\mathbf{n}_0 = \mathbf{s}$ the special role of (pseudo)spin variable. It has the one-sheeted hyperboloid $(\mathbf{s},\mathbf{s}) = -1$ as phase space. The vector fields $\mathbf{n}_1$, $\mathbf{n}_2$ describe the tangent plane of such a hyperboloid and can be locally rotated, corresponding to a local Lorentz transformation. Indeed, introducing the new basis $\mathbf{n}_\pm = \mathbf{n}_1 \pm \mathbf{n}_2$ one can perform a local SO(1,1) gauge transformation generated by an arbitrary real function α:

$$\mathbf{s} \to \mathbf{s}, \mathbf{n}_+ \to e^{+\alpha}\mathbf{n}_+, \mathbf{n}_- \to e^{-\alpha}\mathbf{n}_-. \tag{2.16}$$

Furthermore, from Eqs. (2.15) and (2.14) one can see that

$$V_\mu = 2(\mathbf{n}_2, \partial_\mu \mathbf{n}_1), \quad q_\mu^\pm = \pm\frac{1}{2}(\mathbf{s}, \partial_\mu \mathbf{n}_\pm). \tag{2.17}$$

The transformation (2.16) for these quantities reads

$$V'_\mu = V_\mu + 2\partial_\mu \alpha, \quad q_\mu^{+\prime} = e^\alpha q_\mu^+, \quad q_\mu^{-\prime} = e^{-\alpha} q_\mu^-. \tag{2.18}$$

The linear system (2.15), whose integrability condition is Eq. (2.12), takes the form

$$D_\mu^\mp \mathbf{n}_\pm = \mp 2 q_\mu^\pm \mathbf{s},$$
$$\partial_\mu \mathbf{s} = q_\mu^+ \mathbf{n}_- - q_\mu^- \mathbf{n}_+. \tag{2.19}$$

Finally, in establishing the relation between the local trihedrals and the JT metric, we easily find, by using the definition (2.2) and the relations (2.17), the expression

$$g_{ab} = 2(\partial_a \mathbf{s}, \partial_b \mathbf{s}). \tag{2.20}$$

This formula enables us to give a gravitational interpretation of the σ model we are going to discuss.

The idea we follow is to add to Eq. (2.19) a differential constraint in the (x^1,x^2) space for **s**, such that a completely integrable dynamics is introduced in order to fix partially the gauge freedom in a controlled fashion and allowing a residual local Lorentz covariance. Moreover, for the moment we forget all that concerns the variable x^0 and the current J_0. Precisely, we consider as a constraint the classical continuous Heisenberg model realized on the SL(2,**R**)/SO(1,1) coset space

$$\partial_2 \mathbf{s} = \mathbf{s} \wedge \boldsymbol{\partial}_1^2 \mathbf{s}. \tag{2.21}$$

Substitution from Eq. (2.21) into Eq. (2.19) yields

$$q_2^+ = D_1^- q_1^+, \quad q_2^- = -D_1^+ q_1^-. \tag{2.22}$$

Taking account of these relations, the field equations (2.12) can be written as

$$D_2^\mp q_1^\pm \mp (D_1^\mp)^2 q_1^\pm = 0,$$
$$\partial_2 V_1 - \partial_1 V_2 = 4\partial_1(q_1^+ q_1^-). \tag{2.23}$$

Defining the flat connection

$$A_2 = V_2 + 4(q_1^+ q_1^- - a), \quad A_1 = V_1, \tag{2.24}$$

where a is an arbitrary real constant, and gauging out it by a local SO(1,1) transformation $A_j = 2\partial_j \lambda$, $q_1^\pm = q^\pm e^{\pm\lambda}$ for a regular real function λ, we get the nonlinear reaction-diffusion (RD) system

$$\partial_2 q^\pm \mp \partial_1^2 q^\pm \pm 2(q^+ q^- - a)q^\pm = 0. \tag{2.25}$$

Here only the global SO(1,1) invariance $q^\pm \to e^{\pm\alpha} q^\pm$ survives. Equation (2.25) represents a particular form of a two-component reactive-diffusive system, playing an important role in synergetics [28–30]. However, the unusual negative value for the second diffusion coefficient is crucial for the existence of Hamiltonian structure and the integrability of the model. In fact, performing the ''Wick rotation'' $x^2 \to ix^2$ and assuming that $q^\pm$ are complex functions, the system (2.25) becomes the NLS equation and its complex conjugate. However, the appearance of thermodynamical properties of the black holes may be related to some ''dissipative'' features of Eq. (2.25). Moreover, this system is very similar to the ''fictitious'' or ''mirror-image'' systems with *negative* friction, which appear into the thermofield approach to the damped harmonic oscillator treated in Ref. [31] (see also Ref. [32] for connections with the CS theory). The energy is drained from the ''real'' oscillator to its ''image,'' which mimics inaccessible states hidden in a thermostate. In this way the total energy is conserved and the Lagrangian description is allowed. This analogy works effectively when one considers homogeneous configurations of the system (2.25). Then, it reduces to a system of two coupled harmonic oscillators with damping and in the massless limit.

The main issues concerning the integrability structures associated with the system (2.25) [33,34,15] can be obtained by a proper treatment of those for the NLS equation. However, although most of the algebraic forms are sligthly generalized with respect to the NLS case, the analytical aspects are less trivially extended, because of the reality of the fields $q^{\pm}$ and of their boundary conditions. The system (2.25) admits the Lax pair

$$L_1=\partial_1+\begin{pmatrix}\zeta & q^- \\ q^+ & -\zeta\end{pmatrix},$$

$$L_2=\partial_2+\begin{pmatrix}2\zeta^2-(q^+q^--a) & -(\partial_1-2\zeta)q^- \\ (\partial_1+2\zeta)q^+ & -2\zeta^2+(q^+q^--a)\end{pmatrix}, \tag{2.26}$$

the Bäcklund transformations

$$\partial_1(q^{\pm}-\tilde{q}^{\pm})=\sqrt{(q^+-\tilde{q}^+)(q^--\tilde{q}^-)-\mu}(q^{\pm}+\tilde{q}^{\pm}), \tag{2.27}$$

$$\partial_2(q^{\pm}-\tilde{q}^{\pm})=\pm\sqrt{(q^+-\tilde{q}^+)(q^--\tilde{q}^-)-\mu}\,\partial_1(q^{\pm}+\tilde{q}^{\pm}) \mp(q^+q^-+\tilde{q}^+\tilde{q}^-)(q^{\pm}-\tilde{q}^{\pm}), \tag{2.28}$$

and it belongs to the bi-Hamiltonian hierarchy of commuting flows

$$\partial_2\begin{pmatrix}q^+ \\ q^-\end{pmatrix}=(LJ^{-1})^n\begin{pmatrix}q^+ \\ -q^-\end{pmatrix}, \tag{2.29}$$

where L and J are symplectic operators with respect to the usual bilinear form in $\mathcal{L}^2_{\mathbf{R}^2}$ defined by

$$J=-i\sigma_2,\quad L=\begin{pmatrix}2q^+\int^{x^1}q^+ & \partial_1-2q^+\int^{x^1}q^- \\ \partial_1-2q^-\int^{x^1}q^+ & 2q^-\int^{x^1}q^-\end{pmatrix}, \tag{2.30}$$

with $\int^x f\equiv\frac{1}{2}(\int_{-\infty}^x f dx-\int_x^{+\infty}f dx)$. Equation (2.25) is obtained for $n=2$ in Eq. (2.29). The Lax pair (2.26) is of the Zakharov-Shabat type [35], where a rotation of $\pi/2$ in the complex plane of the spectral parameter is required. However, we notice that in general the Galilei transformations allowed by Eq. (2.25) $x^1\to x^1+2Vx^2$, $x^2\to x^2$, $q^{\pm}\to e^{\mp\alpha(x^{1\prime},x^{2\prime})}q^{\pm}(x^{1\prime},x^{2\prime})$, where $\alpha(x^1,x^2)=V^2x^2+Vx^1$ do not preserve the boundary conditions. Then, solutions with nice asymptotics $q^{\pm}\to 0$ for $x^1\to\pm\infty$ may become unbounded in reference frames moving at sufficiently high velocity.

In the gauge fixed by Eq. (2.21) the metric tensor (2.2) takes the form

$$g_{00}=\frac{8}{\Lambda}\partial_1q^+\partial_1q^-,$$

$$g_{11}=-\frac{8}{\Lambda}q^+q^-,$$

$$g_{01}=g_{10}=-\frac{4}{\Lambda}(q^-\partial_1q^+-q^+\partial_1q^-). \tag{2.31}$$

We observe that the components g_{11} and g_{01} are densities of the simplest conserved quantities, i.e., the ''mass'' and the ''momentum,'' respectively.

By using the Bäcklund transformations, one can find several types of solutions [15]. Here we will consider only few of them. In particular, we will take under consideration the analogous of the bright soliton solution for the NLS equation $(a=0)$. In a moving frame coordinate $\xi=x^1-vx^2+\xi_0$, such a solution is given by

$$q^{\pm}(x^1,x^2)=\pm k\ \exp\pm[(k^2-\tfrac{1}{4}v^2)x^2 -\tfrac{1}{2}v(\xi-\xi_0)]\mathrm{sech}k\xi, \tag{2.32}$$

depending on the two real parameters k and v. But in contrast with the bright soliton of the NLS equation, the above solution does not preserve the amplitude for the components $q^{\pm}$ independently. During the time evolution one of the fields is exponentially growing, while the other one is decaying. At the same time, the product q^+q^- has the usual solitonic shape. Since this is similar to the pattern formation in the context of the dissipative structures, for the solution (2.32) the name of ''dissipaton'' is suggested. For the N-dissipaton solution see Ref. [15]. Notice that the parameter v corresponds to the real part of the spectrum of the NLS solitons. Furthermore, in the space of the parameters (v,k) there exists the critical value $v_{\rm crit}=2k$. For the solution (2.32) obtained with $v<v_{\rm crit}$, at the infinity one has $q^{\pm}\to 0$. At the critical value the solution is a steady state in the moving frame $q^{\pm}=\pm ke^{\pm k\xi_0}(1\mp\tanh k\xi)$, with constant asymptotics $q^{\pm}\to\pm 2ke^{\pm k\xi_0}$ for $x^1\to\mp\infty$ and $q^{\pm}\to\pm 0$ for $x^1\to\pm\infty$. In the over-critical case $v>v_{\rm crit}$, we are led to $q^{\pm}\to\pm\infty$ for $x^1\to\mp\infty$ and $q^{\pm}\to\pm 0$ for $x^1\to\pm\infty$.

III. DISSIPATON AS BLACK HOLE

Static case

We first analyze the dissipaton (2.32) at rest, which describes the stationary metric of JT gravity (we will use the simplest notation $x^1\to x$ and $x^2\to t$)

$$ds^2=-\frac{8k^2}{\Lambda}\cosh^{-2}kx[k^2\tanh^2kx(dt)^2-(dx)^2]. \tag{3.1}$$

This metric is regular everywhere, except for a causal singularity at $x=0$ and it is vanishing for $x\to\infty$. However, one can promptly check that

$$\Gamma^0_{00}=\Gamma^0_{11}=\Gamma^1_{01}=\Gamma^1_{10}=0, \tag{3.2}$$

$$\Gamma^0_{01}=k\frac{1-\sinh^2 kx}{\sinh kx\cosh kx},$$

$$\Gamma^1_{00}=k^3\frac{(1-\sinh^2 kx)\sinh kx}{\cosh^3 kx},\quad \Gamma^1_{11}=-k\tanh kx, \tag{3.3}$$

$$R=g^{00}R_{00}+g^{11}R_{11}=\frac{1}{g_{00}}R_{00}+\frac{1}{g_{11}}R_{11}=\Lambda, \tag{3.4}$$

satisfying everywhere the equation provided by the action (2.1). Hence, the dissipaton maximum position $x=0$ has to be interpreted as the event horizon. If we introduce a new spacelike coordinate, similar to the ''tortoise'' coordinate for the Schwarzschild solution, defined by

$$x'=\frac{1}{k}x+\frac{1}{k^2}\ln|1-e^{-2kx}|, \tag{3.5}$$

the metric (3.1) takes the conformal flat form

$$ds^2=-\frac{8k^4}{\Lambda}\cosh^{-2}kx\tanh^2 kx[(dt)^2-(dx')^2]. \tag{3.6}$$

Now, introducing a system of coordinates of the Kruskal-Szekeres (KS) type, namely,

$$2v=e^{k^2(x'+t)},\quad 2u=-e^{k^2(x'-t)}, \tag{3.7}$$

we get a metric in the form

$$ds^2=-\frac{8}{\Lambda}\frac{du\,dv}{(1-uv)^2}. \tag{3.8}$$

The metric (3.8) has exactly the form given in Refs. [36,37]. It suggests to represent the one dissipaton metric (3.1) in the de Sitter form, with event horizon at $x=0$. In the next section we will show that indeed it can be done also for a moving dissipaton solution. In the coordinate system

$$v=U+V,\quad u=U-V, \tag{3.9}$$

where

$$uv=U^2-V^2, \tag{3.10}$$

we have the diagram reported in Fig. 1. The original constant curved (x,t) space-time corresponds to region I. The soliton maximum position $x=0$ corresponds to the diagonal coordinates lines $u=0$ $(U=V)$ and $v=0$ $(U=-V)$. It is clear that the maximally extended KS coordinates admit the physical singularity at $uv=1$, represented by the past and future hyperbola $U^2-V^2=1$. In the original variables this singularity corresponds to the analytical extension to the complex x plane. Since $uv=-\sinh^2 kx$, one easily finds the singularity for pure imaginary $x=\pm i(\pi/2k)$. This suggests that singular solutions of the NLS type equation have a physical interpretation as singularities in gravitational theories.

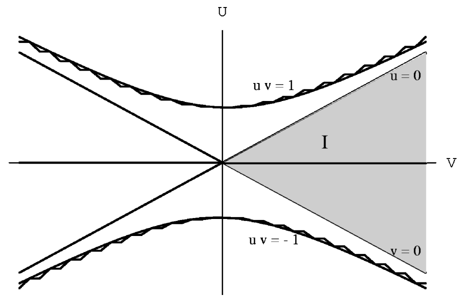

FIG. 1. Standard KS diagram for a static BH. The physical region is I.

Moving case

When we choose $v\neq 0$ in the solution (2.32), the corresponding components of the metric tensor are

$$g_{00}=\frac{8}{\Lambda}q^+q^-\left[k^2\tanh^2 k(x-vt)-\frac{1}{4}v^2\right],$$

$$g_{11}=-\frac{8}{\Lambda}q^+q^-,\quad g_{01}=\frac{8}{\Lambda}q^+q^-\frac{v}{2}, \tag{3.11}$$

which lead to the line element

$$ds^2=-\frac{8}{\Lambda}k^2\cosh^{-2}\rho \times\left[\left(k^2\tanh^2\rho-\frac{1}{4}v^2\right)(dt)^2-\frac{1}{k^2}(d\rho)^2-\frac{v}{k}d\rho dt\right], \tag{3.12}$$

where we have introduced the moving frame coordinate of the soliton $\rho=k(x-vt)$. In this system of coordinates the time t cannot be defined globally because of the cross term $d\rho dt$. Nevertheless, we can find a ''synchronized'' system of coordinates, which describe a static (i.e., time translation and reflection symmetric) spacetime, as in the case of static dissipaton illustrated above. Indeed, if we consider a new coordinate system (ρ,T) defined by

$$dT=dt-\frac{v}{2k(k^2\tanh^2\rho-v^2/4)}d\rho, \tag{3.13}$$

we obtain the metric element

$$ds^2=-\frac{8}{\Lambda}k^2\cosh^{-2}\rho\left[\left(k^2\tanh^2\rho-\frac{1}{4}v^2\right)(dT)^2 -\frac{\tanh^2\rho}{(k^2\tanh^2\rho-v^2/4)}(d\rho)^2\right]. \tag{3.14}$$

This metric shows a horizon singularity at

$$\tanh\rho = \pm\frac{v}{2k} \quad \text{only if} \quad |v| \leq 2|k| \equiv |v_{\max}| \tag{3.15}$$

for the dissipaton velocity. Consequently, a BH cannot move faster than the maximal value of the velocity $|v_{\max}| = 2|k|$. We emphasize this phenomenon, which arises in a purely nonrelativistic treatment, provided by the RD system (2.25) and the Heisenberg constraint (2.21). The key point is given by the the boundary values for the dissipaton. In fact, the above limiting value $v_{\max}$ coincides with the critical value v_{crit} treated at the end of Sec. II. In other words, horizon singularities exist only when both the components of the Zweibein fields $q^{\pm}$ go to zero at infinity, or they are bounded as discussed above for the critical case. If we increase the dissipaton velocity to a value $v > v_{\max} \equiv v_{\text{crit}}$, the synchronization of clocks is still possible, but the metric (3.14) becomes regular everywere except at $|\rho| \to \infty$.

In order to represent the metric (3.14) in a form closer to a Schwarzschild type form, let us define the spacelike variable

$$r = |k|\cosh^{-1}\rho, \quad 0 < r < |k|. \tag{3.16}$$

Then, assuming that Eq. (3.15) is not satisfied, i.e., $k^2 < v^2/4$, we obtain the line element

$$ds^2 = -\frac{8}{\Lambda}\left[-r^2(r^2+r_0^2)(dT)^2 + \frac{(dr)^2}{(r^2+r_0^2)}\right], \tag{3.17}$$

where $r_0 = (v^2/4 - k^2)^{1/2}$. Thus, the only singular point is $r = 0$, which corresponds to the asymptotics $\rho \to \pm\infty$. Anyhow, these points do not represent physical singularity, since the relation $R = \Lambda$ still holds. Furthermore, by performing the transformation

$$z = \frac{2r^2 + r_0^2}{r_0^4}, \tag{3.18}$$

one finds that the line element takes the Schwarzschild form

$$ds^2 = -\frac{2r_0^4}{\Lambda}\left[-(r_0^4 z^2 - 1)(dT)^2 + \frac{(dz)^2}{(r_0^4 z^2 - 1)}\right], \tag{3.19}$$

and the spacelike variable runs over the finite interval $1/r_0^2 = z_{\min} < z \leq z_{\max} = (1/r_0^2)[(v^2+4k^2)/(v^2-4k^2)]$, noticing that now $z_{\min}$ corresponds to $\rho \to \pm\infty$. We interpret this result saying that this metric describes the region outside the BH event horizon, while the inside part never can be reached for $v^2 > 4k^2$.

A much more interesting situation occurs when $k^2 > v^2/4$ [see Eq. (3.15)]. In fact, in terms of the variable r defined in Eq. (3.16) the metric takes the form

$$ds^2 = -\frac{8}{\Lambda}\left[-r^2(r^2-r_H^2)(dT)^2 + \frac{(dr)^2}{(r^2-r_H^2)}\right], \tag{3.20}$$

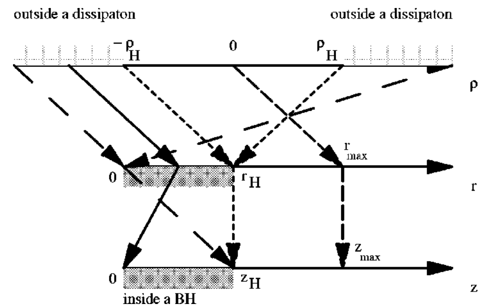

FIG. 2. Mapping a moving dissipaton into a black hole.

which is singular at $r = r_H = (k^2 - v^2/4)^{1/2}$. In the moving frame, this point corresponds to

$$\rho_H = \pm \operatorname{arccosh}[1 - v^2/4k^2]^{-1/2} \tag{3.21}$$

or

$$x_H = \frac{\rho_H}{k} + vt. \tag{3.22}$$

These expressions say that for $v = 0$ the singularity is located at $r_H = r_{\max} = |k|$, namely, at $\rho = x = 0$ as in the static case discussed before. Increasing the velocity the horizon position r_H decreases regularly up to the minimal value $r_H = 0$, when the critical value $|v_{\max}| = 2|k|$ is reached. Thus, in the variable ρ for $v \to v_{\max}$ the horizon goes to infinity, where it remains as seen in the discussion above.

For the slowly moving dissipaton satisfying Eq. (3.15), we can introduce the analogue of the transformation (3.18)

$$z = \frac{|2r^2 - r_H^2|}{r_H^4}. \tag{3.23}$$

This new variable runs over the interval

$$0 \leq z \leq z_{\max} = \frac{1}{r_H^2}\frac{4k^2+v^2}{4k^2-v^2}, \tag{3.24}$$

and the singular point r_H corresponds to $z_H = 1/r_H^2 \leq z_{\max}$. The metric of the Schwarzschild type reads

$$ds^2 = -\frac{2r_H^4}{\Lambda}\left[-(r_H^4 z^2 - 1)(dT)^2 + \frac{(dz)^2}{(r_H^4 z^2 - 1)}\right], \tag{3.25}$$

which is singular at z_H but, differently from the case (3.19), it can be continued for smaller z. We could interpret this result by saying that for $z \in [0, z_H)$ the metric (3.25) describes the region inside the event horizon, while for $z \in (z_H, z_{\max}]$ it describes the outside region. In order to make more clear our argument we refer to Fig. 2. From this diagram one can see that the central part of the dissipaton given by $|\rho| < \rho_H$ corresponds to the outside of the BH. Vice versa, the outside of the dissipaton, i.e., $|\rho| > \rho_H$, gives the inside of the BH.

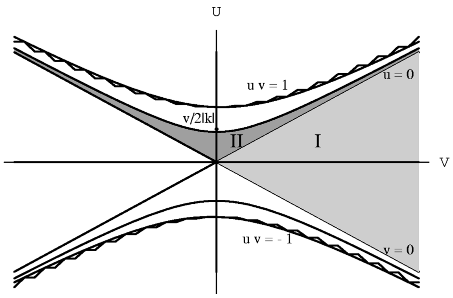

FIG. 3. Standard KS diagram for a moving BH. The physical regions are I and II.

Now we follow closely the treatment of the static dissipaton, first giving to the BH metric (3.14) the conformal form

$$ds^2=\frac{2}{\Lambda}r_H^4[(dT)^2-(dR)^2]\begin{cases}\operatorname{cosech}^2(r_H^2R) & \text{for } r>r_H,\\ -\operatorname{sech}^2(r_H^2R) & \text{for } r<r_H,\end{cases} \tag{3.26}$$

where $R=(1/2r_H^2)\ln|1-r_H^2/r^2|$ with $-\infty<R<\infty$. Then, the metric of the KS type (3.7) is obtained by the change of variables

$$v=e^{r_H^2(R+T)}, \quad u=\operatorname{sgn}(r-r_H)e^{r_H^2(R-T)}. \tag{3.27}$$

Finally, we get an analogous KS diagram as obtained for the static case (see Fig. 3), by using again the variables (3.9).

However, some comments are in order. In fact, the diagonal lines $u=0$ and $v=0$ correspond now to the BH horizon $r=r_H<|k|$. The region I corresponds to the inner part of the BH, i.e., for $0<r<r_H$ or equivalently $\rho>\rho_H$. The part of the regions II (IV) below (above) the hyperbola $U^2-V^2=v^2/4k^2$ describes the outer region to the BH, with $r>r_H$ or $\rho<\rho_H$. It approaches the singularity of the KS metric given by the hyperbola $uv=U^2-V^2=1$, when the dissipaton velocity reaches the critical value $|v|_{\max}$.

IV. METASTABLE STATES OF TWO BLACK HOLES

The integrability of the RD model (2.25) implies the existence of N-dissipaton solutions, the superposition formula of which is shown in Ref. [15]. It is well known [35] that the N soliton is asymptotically decomposed in the individual solitons when $t\to\pm\infty$. Furthermore, the collision of two individual solitons is elastic and, asymptotically, the only effect of the interaction is a shift of phase and position. We could expect a similar behavior also for the dissipatons. But this suggests that the horizon of the individual black hole, which is related to the dissipaton, will shift as well. A detailed analysis of this problem for general initial data is still under investigation. Actually, the main difficulty is due to the nonstationary character of the metric corresponding to the colliding dissipatons. This fact makes the synchronization problem highly nontrivial. However, in this section we describe a particular, but interesting situation.

We consider the following two-dissipaton solution to Eq. (2.25):

$$q^{\pm}(x,t)=\pm\frac{2}{\Delta}\left(\frac{k_1+k_2}{k_1-k_2}\right)(k_1\cosh\theta_2 e^{\pm k_1^2t}+k_2\cosh\theta_1 e^{\pm k_2^2t}), \tag{4.1}$$

where

$$\Delta=\cosh(\theta_1+\theta_2)+\left(\frac{k_1+k_2}{k_1-k_2}\right)^2\cosh(\theta_1-\theta_2)+\frac{4k_1k_2}{(k_1-k_2)^2}\cosh(k_1^2-k_2^2)t, \tag{4.2}$$

$$\theta_i=k_i(x-x_{0i}), \quad i=1,2. \tag{4.3}$$

At any fixed time t this solution is exponentially decaying at space infinity. If one of the parameters vanishes, say, $k_2=0$, the solution (4.1) reduces to the ''static'' dissipaton with amplitude k_1, located at the point x_{01} for $t=0$.

Now, let us put the dissipaton labeled by 1 at the origin of the coordinates [i.e., $x_{01}=0$ in Eq. (4.3)] and the second one displaced by a certain amount, say, $x_{02}=d$ with $d>0$. Then, we look at the limit $d\to\infty$ in a fixed bounded domain of (x,t). Since the solution is exponentially decaying, we expect to obtain a well separated dissipaton for sufficiently large values of d. Indeed, from Eq. (4.1) we get

$$q^{\pm}=\pm k_1\frac{e^{\pm k_1^2t}}{\cosh k_1(x-x_0)}, \quad \text{where} \quad x_0=-\frac{1}{k_1}\ln\left|\frac{k_1+k_2}{k_1-k_2}\right|. \tag{4.4}$$

In other words, the dissipaton labeled by 2 goes far apart on the right, while the first one suffers a negative shift of the position. As was shown in Sec. III, this type of solution corresponds to a static BH with the horizon at the dissipaton location. Since the second dissipaton is far apart, it is located inside the event horizon. However, in this case it cannot escape outside, that is overcome the horizon position. Consequently, the interaction is repulsive. Moreover, it is a long-range interaction, which induces a shift of the horizon of the first one. Because of the left-right symmetry, the horizon shift of the same amount will occur if the second dissipaton is sent to $x\to-\infty$, but in the right direction. This situation is quite unusual for the solitons. For instance [35], the fastest soliton of a N-soliton solution of the NLS equation overcomes the slower ones and, asymptotically, it recovers the original shape (i.e., that at $t\to-\infty$), suffering only a phase and a position shift. In the present case, similarly to the bound states of two off-phase solitons with equal amplitudes [38,39], two dissipatons start to move from $\pm\infty$ with opposite velocities $v_1=-v_2=k_1-k_2$ and after a repulsive interaction go back to $\pm\infty$. This is evident from the analysis of the mass density (see Figs. 4–6)

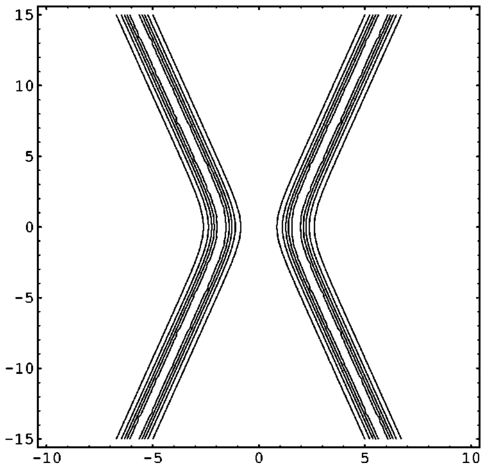

FIG. 4. Two-dissipaton collision for $d=0$.

$$q^+q^-=-4\frac{k_+^2k_-^2}{\tilde{\Delta}^2}[k_1^2\cosh^2\theta_2+k_2^2\cosh^2\theta_1 +2k_1k_2\cosh\theta_1\cosh\theta_2\cosh(k_1^2-k_2^2)t], \tag{4.5}$$

where

$$\tilde{\Delta}=k_-^2\cosh\theta_++k_+^2\cosh\theta_-+4k_1k_2\cosh(k_1^2-k_2^2)t,$$

$$k_\pm=k_1\pm k_2,\quad \theta_\pm=\theta_1\pm\theta_2. \tag{4.6}$$

The expression (4.5) is symmetric under the time reflection $t\to-t$ and the space reflection $x\to-x, x_{0i}\to-x_{0i}$ $(i=1,2)$. At large times and distances, say for $x\to+\infty$ and $t\to+\infty$ in a uniformly moving frame with velocity $v_1=k_1-k_2$, we have

$$q^+q^-\sim-\frac{k_+^2}{4}\frac{1}{\cosh^2(k_+/2)(x-x_{0+}-\overline{x}_0-k_-t)}, \tag{4.7}$$

where the asymptotic position shift is

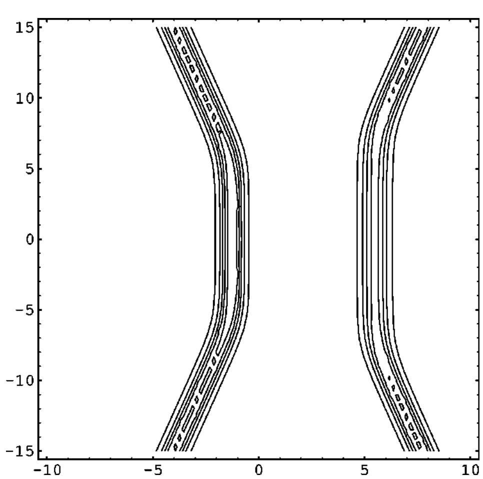

FIG. 5. Two-dissipaton collision for $d=4$.

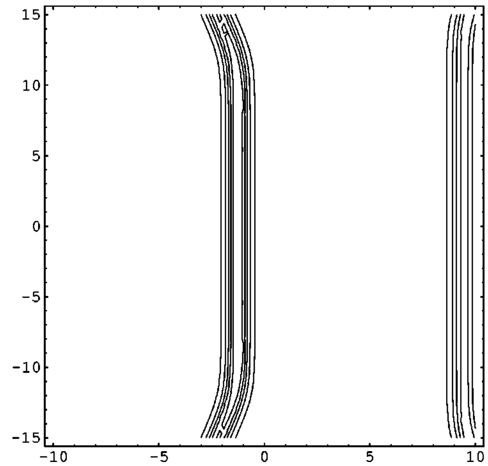

FIG. 6. Two-dissipaton collision for $d=8$.

$$\overline{x}_0=\frac{1}{k_+}\ln\frac{4k_1k_2}{k_-^2}. \tag{4.8}$$

Performing an analogous computation for $x\to-\infty$, $t\to+\infty$ yields

$$q^+q^-\sim-\frac{k_+^2}{4}\frac{1}{\cosh^2(k_+/2)(x-x_{0+}+\overline{x}_0+k_-t)}, \tag{4.9}$$

but in a reference frame uniformly moving with velocity $v_2=-v_1$. Notice that the amplitude of the individual dissipatons are the same and equal to $k_+/2$.

The momentum density of solution (4.1) is

$$q^+q_x^- - q^-q_x^+ = -\frac{4k_+^2k_-^2}{\tilde{\Delta}^2}k_1k_2 \times(k_+\sinh\theta_-+k_-\sinh\theta_+)\sinh(k_1^2-k_2^2)t. \tag{4.10}$$

By integrating this expression, we obtain the vanishing of the total momentum $P=\int_{-\infty}^{\infty}(q^+q_x^- - q^-q_x^+)dx=0$. In Figs. 4–6 we consider a dissipaton-dissipaton collision for a particular choice of the amplitude parameters $k_1=2$ and $k_2=1$. In this case we always have $v=k_-<2k=k_+$. Then, we can apply the discussion at the end of the previous section. In particular, we can compute the position of the horizons x_H by Eq. (3.22), at least in the asymptotics $t\to\pm\infty$. Our parameters are chosen in an interval in which $x_H\ll 1/k$, where the latter quantity estimates the coherence length of the dissipaton. This means that the horizon is located very close to the dissipaton central position $x_{\rm diss}=x_{0+}\pm(\overline{x}_0+k_-t)$. Moreover, since we defined the BH interior by $x>|x_H-x_{\rm diss}|$, the horizon surface is now given by two disconnected parts, which never can overcome each other. This is a sort of hard core interaction, which drastically changes the dissipatons interaction. Furthermore, in Fig. 4–6 we recognize the behavior of the position shift described by Eq. (4.8).

Finally, another interesting and completely new phenomenon appears, as is evident from the same pictures. Indeed, after reaching a minimal distance $\sim d$ the dissipatons form a metastable bound state, with the lifetime depending on d. Then, after this period the metastable state decays on the original two dissipatons. This is similar to what happens for a resonance in the elementary particle physics. However, in the present case the involved ''particles'' are well separated and preserve their individual structure during the interaction, as a result of the long-range character of the interaction. To estimate this lifetime we compute the solution (4.1) on the position x_0 given by Eq. (4.4), with $x_{01}=0$, $x_{02}=d$, and for $t>0$. Then the density mass (4.5) becomes a quite simple rational function of $\cosh(k_1^2-k_2^2)t$. Comparing the values of this function with the involved coefficients and in the limit of large d, one obtains the duration of the metastable state

$$\Delta T\approx\frac{2}{(k_1^2-k_2^2)}\left(k_2 d+\ln\left[\frac{k_1(k_1^2-k_2^2)}{2k_2}\right]\right). \qquad (4.11)$$

This formula is in a good agreement with computer calculations represented in Figs. 4–6. When $d\to\infty$, the metastable state approaches the stable one, describing isolated dissipatons.

V. EUCLIDEAN GRAVITY AND BLACK HOLE TEMPERATURE

The analytical continuation of the previous black hole space times is important to understand the quantum and thermodynamical aspects of the proposed 2D gravity theory. This is now more relevant because of the strict relation with some well-known completely integrable systems. Indeed, in order to build up the Euclidean version of the JT gravity, we need to replace the de Sitter group SL$(2,\mathbf{R})$ into the orthogonal SO(3). The corresponding isotropy subgroup $O(1,1)$ on the tangent plane is replaced by the real rotations group $O(2)$. Consequently, the construction developed in Sec. II can be repeated with small changes. Indeed, instead of Eq. (2.5) the $\mathbf{Z}_2$ graduation of the connection one-form J is given by

$$J_a=i/4\sigma_3 V_a+\begin{pmatrix}0 & -\kappa\bar{\psi}_a\\ \psi_a & 0\end{pmatrix}, \qquad (5.1)$$

where $\kappa=\pm 1$ for su(2) or su(1,1) and ψ_a is a complex function. Then, one can introduce the moving trihedral frame $\{\mathbf{n}_i\}$ as in Eqs. (2.13)–(2.15). Furthermore, the (pseudo)spin variable $\mathbf{s}\equiv\mathbf{n}_0$ satisfies the constraint $(\mathbf{s},\mathbf{s})=1$, which means that it belongs to the two-dimensional sphere $\mathcal{S}^2$, or to the pseudosphere $\mathcal{S}^{1,1}$, depending on the chosen isotropy group. The two vector fields $(\mathbf{n}_1,\mathbf{n}_2)$ form a basis in the tangent plane to $\mathbf{s}(x_1,x_2)$, enjoying the local U(1) symmetry, in contrast to the SO(1,1) invariance shown in Eq. (2.16). Again the quantities V_a and ψ_a can be formally expressed as in Eq. (2.17), which transform as a U(1) gauge field and a complex matter field, respectively. The analogue of Eq. (2.19), that is, the system describing the admissible transformations of the moving frame $(\mathbf{s},\mathbf{n}_+,\mathbf{n}_-)$, is now

$$D_\mu\mathbf{n}_+=-2\kappa\psi_\mu\mathbf{s},$$
$$\partial_\mu\mathbf{s}=\psi_\mu\mathbf{n}_-+\bar{\psi}_\mu\mathbf{n}_+, \qquad (5.2)$$

where $D_\mu\equiv\partial_\mu-i/2V_\mu$ is the U(1) covariant derivative. Its integrability condition reads

$$D_a\psi_b=D_b\psi_a,$$
$$[D_2,D_1]=2\kappa(\bar{\psi}\psi_{12}-\bar{\psi}_2\psi_1). \qquad (5.3)$$

The gauge fixing constraint (2.21) reads formally the same, recalling that now $\mathbf{s}$ belongs to $\mathcal{S}^2$ or to $\mathcal{S}^{1,1}$. Hence, in the tangent plane formalism one has

$$\psi_2=iD_1\psi, \qquad (5.4)$$

where for simplicity we have put $\psi_1=\psi$.

Using these relations into the system (5.3) and eliminating the residual local U(1) symmetry, one obtains the NLS equation

$$i\partial_\tau\psi+\partial_x^2\psi+2\kappa|\psi|^2=0, \qquad (5.5)$$

where we set $x_1\to x$ and $x_2\to\tau$. As is well known [35], for $\kappa=1$ this equation admits the ''bright'' soliton solution

$$\psi=ik\frac{e^{-i(V^2/4-k^2)\tau+iVx/2}}{\cosh k(x-V\tau)}, \qquad (5.6)$$

where k and V are real quantities, such that $\lambda=-V/2+ik$ plays the role of the spectral parameter in the inverse spectral transform method. But, by comparison with the dissipaton solution (2.32), we see that its analytical continuation is provided by Eq. (5.6) through

$$t=i\tau,\quad v=-iV,\quad \psi=iq^+,\quad \bar{\psi}=iq^-. \qquad (5.7)$$

In this sense the bright soliton can be considered as the Euclidean version of the dissipaton. As we saw in the previous sections, the dissipaton admits an interpretation in terms of gravitational BH in the pseudo-Euclidean metric. Therefore, the bright soliton can be considered as the BH in the corresponding Euclidean gravity, usually interpreted as the gravitational instanton. This is a complete nonsingular positive definite metric solution of the vacuum Einstein equation. Moreover, those metrics which are asymptotically flat in spatial directions and periodic in the imaginary time direction, contribute to the thermal canonical ensemble and are relevant for the thermodynamical properties of the BH [40,41]. Below, we show that the bright soliton (5.6) defines an asymptotically constant curvature gravitational instanton of the JT gravity.

First let us consider the static bright soliton [$V=0$ in Eq. (5.6)]. One easily sees that it is a periodic function in the time, with period $T=2\pi/k^2$. Moreover, the corresponding metric

$$ds_E^2 = \xi_E^2 d\tau^2 + \frac{dx^2}{P}$$
$$= -\frac{8k^2}{\Lambda}\cosh^{-2}kx[k^2\tanh^2 kx(d\tau)^2 + (dx)^2] \quad (5.8)$$

defines the Euclidean BH spacetime with the horizon fixed at $x=0$. As in Ref. [40] we introduce the proper distance from the Euclidean horizon l, which is defined by the equation $dl = dx/P^{1/2}$ and in our case takes the explicit form

$$l = 4\sqrt{\frac{2}{|\Lambda|}}\left[\arctan e^{kx} - \frac{\pi}{4}\right]. \quad (5.9)$$

Then, near the horizon the metric is approximately given by

$$ds_E^2 \approx \kappa^2 l^2 d\tau^2 + dl^2, \quad (5.10)$$

where the so-called surface gravity

$$\kappa = \lim_{l\to 0} P^{1/2}\frac{d\xi}{dx} = k^2 \quad (5.11)$$

has been computed for the bright soliton metric. Furthermore, the Euclidean time is ranging over $\tau \in (0, 2\pi/\kappa)$, and the Hawking temperature of the black hole is

$$T_H = \frac{k^2}{2\pi}. \quad (5.12)$$

Thus, the static soliton amplitude square takes the meaning of BH temperature.

Now we consider the moving dissipaton with $v \neq 0$. In this case the solution (5.6) is not periodic in time and the off-diagonal term for the metric is nonvanishing. Then, in order to get the Hawking temperature we have to use the BH metric (3.25). For imaginary time $T_E = -iT$ and $z > 1/r_H^2$ this metric is positive definite. By using the new coordinate $y = (r_H^4 z^2 - 1)^{-1/2}$ it becomes of the de Sitter form

$$ds_E^2 = -\frac{2}{\Lambda r_H^2}(\sinh^2 y\, d\tau^2 + dy^2), \quad (5.13)$$

where $\tau = r_H^2 T_E$. Following Hawking [42], we can see that the apparent singularity in the (y, τ) plane at $y=0$ is similar to the singularity of the plane metric in polar coordinates. Indeed, for small $y \ll 1$ one has

$$ds_E^2 \approx -\frac{2}{\Lambda r_H^2}(y^2 d\tau^2 + dy^2), \quad (5.14)$$

where we are interpreting y as a radius and τ as an angle. This means that T_E is a defined modulo of period $2\pi/r_H^2$, then the Hawking temperature is

$$T_H = \frac{r_H^2}{2\pi} = \frac{k^2 - V^2/4}{2\pi}. \quad (5.15)$$

In the particular case $V=0$ we recover the previous result (5.12). Finally, the features of the dissipaton collisions outlined above should imply a sort of temperature conservation in the scattering of two BHs.

Now let us keep a more strict contact with the JT action (2.1), in order to express it in terms of the ψ variable. This can be achieved repeating the procedure sketched above, but considering as gauge condition $V_1 = 0$ and putting $\kappa = -\Lambda/8$. Then, by recalling [1,2] that the scalar curvature in the Zweibein formalism is given by $R = \epsilon^{ab}\partial_a V_b/\det(e_{ab})$, where e_{ab} is defined in Eq. (2.9) in terms of the q_a, one easily sees that

$$R = 2i\frac{\partial_2 V_1 - \partial_1 V_2}{\bar{\psi}\psi_2 - \bar{\psi}_2\psi} = \Lambda. \quad (5.16)$$

From the above consideration we conclude that the cosmological constant plays the role of the nonlinear coupling. When $\Lambda = 0$ we have the linear Schrödinger equation with the wave function as coordinate in the tangent plane, while the time variable is the Euclidean time of the gravity model. The nonvanishing cosmological term leads to the nonlinear modification of the Schrödinger equation. The de Sitter space with positive cosmological constant $\Lambda > 0$ corresponds to the repulsive (defocusing) NLS equation, while $\Lambda < 0$ for the anti–de Sitter space and we have the attractive (focusing) NLS model. Moreover, since the amplitude of the soliton solutions of the NLS equation depends on the coupling constant κ as $|\kappa|^{-1/2}$, in our case it scales as $2/|\Lambda|^{1/2}$. Thus, in the considered gauges also small values of the cosmological constant provide nontrivial solutions.

VI. FORMULATION OF THE JT GRAVITY IN TERMS OF OTHER INTEGRABLE SYSTEMS

KdV and MKdV hierarchies

In the previous sections we studied the deep relation between the 2D JT gravity and the RD system (2.25). Equations (2.29)–(2.30) tell us that there exist infinitely many completely integrable PDE's, which correspond to different higher order gauge conditions of the type (2.21). These equations have in common the family of the integrals of motion in involution, being themselves fluxes in commutation [33,34]. In particular, we are going to show that for special reductions the MKdV and KdV equations, and their hierarchies, naturally appear. It becomes clear that all these structures are relevant in lineal gravity, which also occur in the context of the matrix model formulation of 2D quantum gravity [43]. For instance, it turns out that the partition function of topological gravity is a tau-function for the KdV equation [44–47].

By using Eqs. (2.29) and (2.30) for $n=3$ one easily finds the system

$$\partial_2 q^{\pm} = -\partial_1^3 q^{\pm} + 6q^+ q^- \partial_1 q^{\pm}. \quad (6.1)$$

Because of the symmetry of these equations, we can consider the reduction

$$q^+ = q^- \equiv u. \tag{6.2}$$

Then we get the MKdV equation

$$\partial_2 u = -\partial_1^3 u + 6u^2 \partial_1 u. \tag{6.3}$$

On the other hand, the nonsymmetric reduction

$$q^+ \equiv u, \quad q^- = 1, \tag{6.4}$$

leads to the KdV equation

$$\partial_2 u = -\partial_1^3 u + 6u\partial_1 u. \tag{6.5}$$

These two reductions imply that the properly reduced RD hierarchy should contain the MKdV and KdV hierarchies. Indeed, taking only the odd members of the RD hierarchy (2.29) with $n=2k+1$ one sees that the reduction (6.2) is allowed for any k. Then we can write a unique scalar equation of the form [33,34]

$$\partial_2 u = R^k_{\mathrm{MKdV}}(\partial_1 u) = \left(\partial_1^2 - 4u^2 - 4(\partial_1 u) \int^{x_1} u(x)dx \right)^k (\partial_1 u). \tag{6.6}$$

Following the same procedure as in the previous case we can get the KdV hierarchy by observing that for all odd members of the RD hierarchy (2.29) the reduction (6.4) is allowed. Then, one arrives at the set of scalar differential equations [33,34]

$$\partial_2 u = R^k_{\mathrm{KdV}}(\partial_1 u) = \left(\partial_1^2 - 4u - 2(\partial_1 u) \int^{x_1} dx \right)^k (\partial_1 u). \tag{6.7}$$

The self-dual σ model

In the previous sections we analyzed in detail the use of a special gauge for the JT gravity in the BF formulation (2.11). However, since the key point resides in Eq. (2.21), one could consider different integrable nonlinear σ models as the gauge fixing conditions. Thus, besides the usually conformal flat metric, leading to the Liouville equation, we will study other types of metrics related to completely integrable hierarchies of equations.

The self-duality equation for the unitary spin vector $\mathbf{s}$ ($\mathbf{s}^2 = -1$) is

$$\partial_2 \mathbf{s} - \mathbf{s} \wedge \partial_1 \mathbf{s} = 0. \tag{6.8}$$

We remind the reader that a similar equation, but with compact $O(3)$ phase space, was obtained in Ref. [48] for the description of the 1D antiferromagnets in the long wave approximation. In light cone coordinates this equation becomes

$$\partial_+ \mathbf{s} - \mathbf{s} \wedge \partial_+ \mathbf{s} = 0, \quad \partial_- \mathbf{s} + \mathbf{s} \wedge \partial_- \mathbf{s} = 0, \tag{6.9}$$

where $\partial_\pm = \partial_2 \pm \partial_1$. By introducing the stereographic projection

$$s_\pm = \frac{2\xi_\pm}{1+\xi_+\xi_-}, \quad s_3 = \frac{1-\xi_+\xi_-}{1+\xi_+\xi_-}, \tag{6.10}$$

of the one-sheet hyperboloid to the real plane $\mathcal{R}^2$, the evolution equation (6.8) takes the form of the hyperbolic Cauchy-Riemann relations

$$\partial_2 \xi_+ = -\partial_1 \xi_-, \quad \partial_1 \xi_+ = -\partial_2 \xi_-. \tag{6.11}$$

The general solution to this system can be written as

$$\xi_+ = \frac{1}{2}[F(x_2-x_1) + G(x_2+x_1)],$$

$$\xi_- = \frac{1}{2}[F(x_2-x_1) - G(x_2+x_1)], \tag{6.12}$$

for arbitrary F and G.

In the tangent space representation one again introduces the V_i and $q_i^\pm$ variables as defined in Eq. (2.17) which, properly combined in the form $q^+_\pm = q^+_2 \pm q^+_1$,$q^-_\pm = q^-_2 \pm q^-_1$, provide $q^+_- = 0, q^-_+ = 0$ [the analogue of Eq. (2.22)] and the hyperbolic self-dual Chern-Simons model

$$D^\mp_\mp q^\pm_\pm = 0,$$

$$\partial_- V_+ - \partial_+ V_- = -4q^+_+ q^-_-, \tag{6.13}$$

the analogue of system (2.23). Expressing $V_\pm$ from the first equation in terms of q's,

$$V_- = 2\partial_- \ln q^+_+, \quad V_+ = -2\partial_+ \ln q^-_-, \tag{6.14}$$

and substituting into the last one, we obtain the hyperbolic Liouville equation

$$\partial_+ \partial_- \phi = 2e^\phi, \tag{6.15}$$

where $q^+_+ q^-_- = e^\phi$.

However, Eq. (6.13) can be left in a two component system of first order equations, by introducing the new irrotational gauge potential

$$A_2 = V_2 - 2\int^x q^+_+ q^-_-, \quad A_1 = V_1. \tag{6.16}$$

Performing a suitable SO(1,1) gauge transformation, we find

$$\partial_2 q^\pm_\pm = \pm \partial_1 q^\pm_\pm \pm q^\pm_\pm \int^x q^+_+ q^-_-. \tag{6.17}$$

This nonlinear evolution equation can be considered as a sort of ''square root'' of the Liouville equation. In fact, it is easy to see that the combination $q^+_+ q^-_- = e^\phi$ fulfills the Liouville equation (6.15), with conformally flat metric tensor

$$g_{++} = g_{--} = 0, \quad g_{+-} = g_{-+} = -\frac{4}{\Lambda} e^\phi. \tag{6.18}$$

Nonlinear σ model

The same procedure can be applied to the nonlinear σ model

$$\partial_+\partial_-\mathbf{s}-(\partial_+\mathbf{s},\partial_-\mathbf{s})\mathbf{s}=0, \tag{6.19}$$

which contains the model (6.13) above as a Bogomolniy limit. In contrast with the compact case, the model (6.19) admits two types of the nonlinear spin wave solutions. The hyperbolic waves are $s_3=\text{const}$ and $s_\pm=s_1\pm s_2=\pm\sqrt{s_3^2-1}\exp[\pm(kx_+-s_3k^2x_-)]$. The elliptic type waves are expressed by $s_2=\text{const}$ and $s_+=s_1+is_3=\sqrt{s_2^2+1}\exp[i(kx_++s_2k^2x_-)]$.

The zero curvature field equations read

$$D_+^-q_-^+=D_-^-q_+^+,\quad D_-^+q_+^-=D_+^+q_-^-, \tag{6.20}$$

$$\partial_-V_+-\partial_+V_-=4(q_-^+q_+^--q_+^+q_-^-), \tag{6.21}$$

and must be supplied with the additional constraints

$$D_-^-q_+^+=0,D_-^+q_+^-=0. \tag{6.22}$$

The resulting system

$$\partial_2 q_\pm^\pm=\pm\partial_1 q_+^+\pm q_\pm^\pm\int^{x_+}\left(q_+^+q_-^--\frac{U_+U_-}{q_+^+q_-^-}\right)dx'_+, \tag{6.23}$$

where $U_-=q_-^+q_-^-,U_+=q_+^+q_+^-$ satisfies the equations $\partial_+U_-=0,\partial_-U_+=0$, is gauge equivalent to the σ model (6.19). The quantity $q_+^+q_-^-=\exp\phi$ obeys the conformal hyperbolic sinh-Gordon equation

$$\partial_+\partial_-\phi=2(e^{\phi}-U_+U_-e^{-\phi}), \tag{6.24}$$

which reduces to the usual sinh-Gordon equation when $U_+U_-=1$.

In order to give a gravitational interpretation of the above equations, we resort to Eq. (2.20). Without any restriction on the σ model, the stereographic projections (6.10) in the light cone variables yield

$$g_{++}=\frac{8}{\Lambda}\frac{\partial_+\xi_+\partial_+\xi_-}{(1+\xi_+\xi_-)^2},\quad g_{--}=\frac{8}{\Lambda}\frac{\partial_-\xi_-\partial_-\xi_+}{(1+\xi_+\xi_-)^2}, \tag{6.25}$$

$$g_{+-}=g_{-+}=\frac{4}{\Lambda}\frac{\partial_+\xi_+\partial_-\xi_-+\partial_-\xi_+\partial_+\xi_-}{(1+\xi_+\xi_-)^2}. \tag{6.26}$$

We notice that the components g_{+-} and g_{-+} are equal to the Lagrangian density for the σ model $g_{+-}=g_{-+}=-(2/\Lambda)\partial_+\mathbf{s}\partial_-\mathbf{s}=-(4/\Lambda)(e^{\phi}+U_+U_-e^{-\phi})$.

In terms of the conformal sinh-Gordon model (6.24) the metric tensor components are

$$g_{00}=\left(-\frac{2}{\Lambda}\right)(U_++U_-+e^{\phi}+U_+U_-e^{-\phi}), \tag{6.27}$$

$$g_{11}=\left(-\frac{2}{\Lambda}\right)(U_++U_--e^{\phi}-U_+U_-e^{-\phi}), \tag{6.28}$$

$$g_{01}=\left(-\frac{2}{\Lambda}\right)(U_+-U_-). \tag{6.29}$$

Two particular special cases arise for (i) $U_+=1,U_-=1$, leading to the sinh-Gordon metric

$$g_{00}=\left(-\frac{8}{\Lambda}\right)\cosh^2\frac{\phi}{2},\quad g_{11}=-\left(-\frac{8}{\Lambda}\right)\sinh^2\frac{\phi}{2},\quad g_{01}=0 \tag{6.30}$$

and (ii) $U_+=1,U_-=-1$, providing

$$g_{aa}=\left(-\frac{4}{\Lambda}\right)\sinh\phi\,\eta_{aa},\quad g_{01}=\left(-\frac{4}{\Lambda}\right), \tag{6.31}$$

related to the cosh-Gordon equation

$$\partial_+\partial_-\phi=4\cosh\phi. \tag{6.32}$$

The last two models are no longer conformal invariant, but still completely integrable systems. We note that the BH solution in the sine-Gordon context was studied recently in Ref. [14].

VII. CONCLUSIONS

We have investigated the JT model of gravity in the context of a gauge field formulation. The JT model, despite its simplicity, is remarkably similar and is the most appropriate two-dimensional analogue of Einstein theory in higher dimensions. Although the gravity theory is local in itself, there are nontrivial global effects, such as the existence of the event horizon of a BH. This implies a nontrivial causal structure, which in turn generates interesting nontrivial thermodynamical behaviors. This makes gravity in 2D a potentially useful model for obtaining new insight and understanding into higher-dimensional gravity.

At first sight the conditions of gauge fixing (2.22) we chose may not appear very natural, however, they surely lead to wide classes (hierarchies of countable many equations) of completely integrable systems. In particular, we have studied a sort of NLS equation, whose solitonlike solutions, dissipatons, can be interpreted as BH's. Among several cases, the most interesting is given by the moving soliton with a limited velocity. In such a case the metric associated with the dissipaton describes the inner and the outer part of a BH in compact space. The position of the event horizon is determined in terms of the two dissipaton parameters. Furthermore, the interaction of two dissipatons has been reinterpreted in terms of scattering of the event horizons of the two BH's. Finally, the surface gravity and the Hawking temperature within the Euclidean version of the theory have been related to the soliton parameters. Thus, the novelty of our RD reformulation is the possibility to describe analytically not only the one BH solution, which was known before, but to derive explicitly the N-BH solutions and to study collision of two BH's of equal mass. As a result, we get two new phenomena: (a) the shift of the BH horizon and (b) the BH metastable state formation. The analogy with a fourdimensional BH may be used as a simple model in order to study the processes involving real BH's. As is well known, the collisions of two BH's is considered to be one of the most promising sources of gravitational waves [49]. However, it has been shown that the standard numerical calculations encounter great difficulties, due to the coordinate singularities and the numerical instabilities [50]. This is why any exact analytical results are important.

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