2D induced gravity as an effective WZNW system

M. Blagojević* and B. Sazdović†
Institute of Physics, 11001 Belgrade, P.O. Box 57, Yugoslavia
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We introduce a dynamical system given by a difference of two simple $SL(2,R)$ Wess-Zumino-Novikov-Witten actions in 2D, and define the related gauge theory in a consistent way. It is shown that gauge symmetry can be fixed in such a way that, after integrating out some dynamical variables in the functional integral, one obtains the induced gravity action. [S0556-2821(98)03218-4]

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I. INTRODUCTION

Two-dimensional (2D) gravity naturally appears in the string functional integral in subcritical dimensions, where it represents an effective theory of quantum fluctuations of matter fields coupled to the metric of the string world sheet [1]. The induced, effective action is closely related to the Weyl anomaly of the original string theory, and represents a gravitational analogue of the usual Wess-Zumino action in gauge theories. The dynamical structure of 2D gravity is, therefore, an important aspect of string theory, but it also represents a useful model for the theory of gravitational phenomena in four dimensions.

Polyakov and his collaborators [2] demonstrated that in the light-cone gauge the *n*-point functions of the effective 2D gravity can be explicitly found. Although the gauge is fixed, these solutions display a hidden chiral $SL(2,R)$ symmetry, which turned out to be very important for the analysis of quantum dynamics. These results motivated the investigation of the structure of 2D gravity in the conformal gauge, where it becomes the standard Liouville theory [3]. Although the $SL(2,R)$ symmetry is naturally connected to the light-cone gauge, there exists a canonical formulation of the theory in terms of gauge independent variables, the $SL(2,R)$ currents, which demonstrates the importance of this symmetry for the general structure of the theory [4].

Dynamical significance of the $SL(2,R)$ symmetry, and strong analogy between the induced gravity and the usual Wess-Zumino action, inspired detailed investigations of the relation between the $SL(2,R)$ Wess-Zumino-Novikov-Witten (WZNW) theory and the induced gravity. Polyakov found the connection between the $SL(2,R)$ WZNW theory and the induced gravity in the *light-cone gauge* [5]. Similar results in the light-cone gauge have been also obtained in Refs. [6,7]. The same problem was discussed in the *conformal* gauge in Ref. [8], where it was shown that Liouville theory may be regarded as the WZNW theory, reduced by certain conformally invariant constraints. These constraints can be automatically produced if one considers gauge extension of the $SL(2,R)$ WZNW model, based on two gauge fields [9].

In a recent letter [10] we used a general method of constructing canonical gauge invariant actions to establish the connection between 2D induced gravity and a WZNW system, defined by a difference of two simple WZNW actions for $SL(2,R)$ group:

$$I(g_1,g_2)=I(g_1)-I(g_2), \quad g_1,g_2\in SL(2,R). \tag{1.1}$$

In this paper we set up the Lagrangian framework for this connection, starting from a gauge invariant extension of the WZNW system (1.1). The connection is established in a *covariant* way, fully respecting the diffeomorphism invariance of both theories. The approach will be very useful for constructing and studying properties of general solutions of the induced gravity, in terms of the related simpler solutions of the WZNW system.

After recalling some basic properties of the WZNW theory in Sec. II, we introduce in Sec. III a consistent formulation of the gauge invariant extension of our basic object, the WZNW system (1.1). By taking the difference of *two* WZNW actions we are able to overcome the usual difficulties which one encounters in the process of gauging a *single* WZNW theory [11,12]. In Sec. IV we explicitly choose the gauge group, a four-parameter subgroup of $SL(2,R)\times SL(2,R)$ leading to four gauge fields, which is sufficient for generating an effective transition to the induced gravity. In Sec. V we show that new gauge invariance can be fixed in such a way that, after integrating out some variables, one arrives at the induced gravity action:

$$I(\phi,g_{\mu\nu})=\int d^2\xi\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}\alpha\phi R-M(e^{2\phi/\alpha}-1)\right]. \tag{1.2}$$

In this process, the original symmetry of the action (1.1) under conformal rescalings is also fixed. Appendixes A, B, and C are devoted to some details concerning geometrical properties of spacetime Σ and the group manifold $SL(2,R)$, and gauge properties of the WZNW model.

II. WZNW MODEL ON CURVED MANIFOLDS

Basic properties of two-dimensional WZNW model are defined by the action

*Email address: mb@phy.bg.ac.yu
†Email address: sazdovic@phy.bg.ac.yu

$$I(g)=I_0(v)+n\Gamma(v)=\frac{1}{2}\kappa\int_\Sigma(^*v,v)+\frac{1}{3}\kappa\int_M(v,v^2),$$
$$v\equiv g^{-1}dg, \tag{2.1}$$

where n is an integer, $\kappa=n\kappa_0$, and κ_0 is a normalization constant. The first term is a σ-model action which provides dynamics for a group-valued field g, defined over a two-dimensional, Riemann manifold Σ, and taking values in a semisimple Lie group G, while the second term is the topological Wess-Zumino term, defined on a three-manifold M whose boundary is $\partial M=\Sigma$. Here, v is the Maurer-Cartan (Lie algebra valued) 1-form, $^*v$ is the dual of v, and $(X,Y)=1/2\,\mathrm{Tr}(XY)$ is the Cartan-Killing bilinear form on the Lie algebra of G (Tr denotes the ordinary matrix trace operation in the adjoint representation of G). The normalization factor κ_0 is chosen in such a way that the Wess-Zumino term is well defined modulo a multiple of 2π, which is irrelevant in the functional integral $Z=\int Dg\,\exp[iI(g)]$.

Let us now parametrize the group elements by some local coordinates q^α, $g=g(q^\alpha)$, so that

$$v=E^a t_a\equiv dq^\alpha E^a{}_\alpha t_a,$$

where t_a are the generators of G, satisfying the Lie algebra $[t_a,t_b]=f_{ab}{}^c t_c$. Then,

$$(^*v,v)={}^*dq^\alpha dq^\beta\gamma_{\alpha\beta},\qquad \gamma_{\alpha\beta}(q)\equiv E^a{}_\alpha E^b{}_\beta\gamma_{ab},$$
$$(v,v^2)=\frac{1}{2}E^aE^bE^cf_{abc}=-6d\tau,$$

where $\gamma_{ab}=(t_a,t_b)$ is the Cartan metric on G, and $f_{abc}=f_{ab}{}^e\gamma_{ec}$. The last equation is based on the theorem that any closed form is locally exact. Therefore, the WZNW action on the group manifold takes the form

$$I(q)=\kappa\int\left(\frac{1}{2}{}^*dq^\alpha dq^\beta\gamma_{\alpha\beta}-dq^\alpha dq^\beta\tau_{\alpha\beta}\right), \tag{2.2a}$$

where we used $\tau=\frac{1}{2}dq^\alpha dq^\beta\tau_{\alpha\beta}$.

Next, we introduce local coordinates ξ^μ ($\mu=0,1$) on Σ, and rewrite the action as

$$I(q)=\kappa\int_\Sigma d^2\xi\left(\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu q^\alpha\partial_\nu q^\beta\gamma_{\alpha\beta}-\varepsilon^{\mu\nu}\partial_\mu q^\alpha\partial_\nu q^\beta\tau_{\alpha\beta}\right),$$

where $g^{\mu\nu}$ is the inverse metric on Σ. It is convenient to define an orthonormal basis of tangent vectors $\partial_i=e_i{}^\mu\partial_\mu$ ($i=+,-$), in which the metric $\eta_{ij}=e_i{}^\mu e_j{}^\nu g_{\mu\nu}$ takes the light-cone form: $\eta_{-+}=\eta_{+-}=1$. In this basis,

$$I(q)=\kappa\int_\Sigma d^2\xi\sqrt{-g}\left(\frac{1}{2}\eta^{ij}\partial_i q^\alpha\partial_j q^\beta\gamma_{\alpha\beta}-\varepsilon^{ij}\partial_i q^\alpha\partial_j q^\beta\tau_{\alpha\beta}\right). \tag{2.2b}$$

Note that the action $I(q)$ is invariant under conformal transformations $g_{\mu\nu}\to g_{\mu\nu}e^{2F}$ (which implies $\partial_i\to e^{-F}\partial_i$).

We now turn our attention to $G=SL(2,R)$. Using the fact that any element g of $SL(2,R)$ in a neighborhood of identity admits the Gauss decomposition $g=g_+g_0g_-$, where g_+, g_0 and g_- are defined in terms of group coordinates $q^\alpha=(x,\varphi,y)$ as in Eqs. (B3a) and (B3b), one can find explicit expressions for $\gamma_{\alpha\beta}$ and τ, Eqs. (B5) and (B6), and obtain

$$I(q)=\kappa\int_\Sigma d^2\xi\sqrt{-g}\left[\frac{1}{2}\eta^{ij}\partial_i\varphi\partial_j\varphi+2(\eta^{ij}-\varepsilon^{ij})\partial_i x\partial_j y e^{-\varphi}\right]$$
$$=\kappa\int_\Sigma d^2\xi\sqrt{-g}\left(\partial_+\varphi\partial_-\varphi+4\partial_+x\partial_-ye^{-\varphi}\right). \tag{2.3}$$

III. GAUGE EXTENSION OF THE WZNW ACTION

We shall now discuss how one can gauge the WZNW theory starting from the existence of global symmetries, as usual. The action $I(g)$, where g belongs to $SL(2,R)$, is invariant under the *global* transformations on the $SL(2,R)$ manifold:

$$g\to g'=\Omega g\bar{\Omega}^{-1},\qquad dg\to(dg)'=\Omega(dg)\bar{\Omega}^{-1}.$$

where $(\Omega,\bar{\Omega})$ is an element of $SL(2,R)\times SL(2,R)$. We want to introduce the corresponding gauge theory, having the following properties.

(a) It should be invariant under the *local* transformations

$$g'=\Omega g\bar{\Omega}^{-1},\quad \Omega=\Omega(\xi^-,\xi^+),\quad \bar{\Omega}=\bar{\Omega}(\xi^+,\xi^-), \tag{3.1}$$

where $(\Omega,\bar{\Omega})$ belongs to a subgroup H of $SL(2,R)\times SL(2,R)$ (which may be equal to the whole group).

(b) It should be defined as a field theory on Σ. It is well known that the second requirement can not be fulfilled for every gauge group H [11,12], since the WZ term $n\Gamma$, originally defined on M, does not have a gauge invariant extension that can be reduced to an integral over $\Sigma=\partial M$. Possible solutions of this problem will be discussed after clarifying the meaning of the first requirement (a).

The transformation law of dg under gauge transformation is changed, but the change can be compensated by introducing the *covariant derivative*:

$$Dg\equiv dg+Ag-gB,\quad (Dg)'=\Omega(Dg)\bar{\Omega}^{-1}, \tag{3.2}$$

where (A,B) are *gauge fields* (Lie algebra valued 1-forms). The covariant derivative Dg transforms homogeneously under local transformations, provided the gauge fields transform according to

$$A'=\Omega(A+d)\Omega^{-1},\quad B'=\bar{\Omega}(B+d)\bar{\Omega}^{-1}. \tag{3.3}$$

Having defined Dg, one can try to gauge the WZNW action by replacing $dg\to Dg$, i.e., by replacing 1-form $v=g^{-1}dg$ with the corresponding covariant 1-form V:

$$V\equiv g^{-1}Dg=v+g^{-1}Ag-B, \quad V'=\bar{\Omega}V\bar{\Omega}^{-1}. \tag{3.4}$$

It is also useful to define the field strengths, $F_A=dA+A^2$ and $F_B=dB+B^2$, which transform as follows: $F'_A=\Omega F_A\Omega^{-1}$, $F'_B=\bar{\Omega}F_B\bar{\Omega}^{-1}$.

Now, we apply this procedure to *formally* define a gauge invariant extension of the WZNW action (2.1):

$$I(g,A,B)=I_0(V)+n\Gamma(V)$$
$$=\frac{1}{2}\kappa\int_\Sigma(^*V,V)+\frac{1}{3}\kappa\int_M(V,V^2). \tag{3.5}$$

The first term $I_0(V)$ is both (a) gauge invariant, and (b) defined over Σ, so that it represents an acceptable gauge invariant action. It can be written as

$$I_0(V)=I_0(v)+\Delta_0,$$
$$\Delta_0=\kappa\int_\Sigma\frac{1}{2}\,\mathrm{Tr}\left[-{}^*\bar{v}A-{}^*vB-{}^*(g^{-1}Ag)B+\frac{1}{2}(^*AA+{}^*BB)\right], \tag{3.6a}$$

where $\bar{v}=gdg^{-1}=-gvg^{-1}$.

The second term $n\Gamma(V)$ is defined as an integral of a three-form on M, which is gauge invariant. However, this form is in general not exact, so that $n\Gamma(V)$ cannot be expressed as an integral over Σ; therefore, it can not be used as part of the σ-model action on Σ.

We shall now analyze some additional restrictions under which an acceptable gauge extension of $I(g)$ *can* be defined. First we note that, after some algebra, the second term can be rewritten as

$$n\Gamma(V)=n\Gamma(v)+\Gamma_1+\Gamma_2+\Gamma_3,$$
$$\Gamma_1=\kappa\int_\Sigma\frac{1}{2}\,\mathrm{Tr}[-\bar{v}A+vB+g^{-1}AgB],$$
$$\Gamma_2=\kappa\int_M\frac{1}{2}\,[\omega_3(B,F_B)-\omega_3(A,F_A)],$$
$$\Gamma_3=\kappa\int_M\frac{1}{2}\,\mathrm{Tr}[F_A(Dg)g^{-1}+F_Bg^{-1}(Dg)], \tag{3.6b}$$

where $\omega_3(A,F_A)=\mathrm{Tr}(AF_A-1/3A^3)$ is the Chern-Simons three-form. Formal extension of $I(g)$, obtained in Eqs. (3.5) and (3.6a), and (3.6b) can be written as

$$I(g,A,B)=I^r(g,A,B)+\Gamma_2(A,B)+\Gamma_3(g,A,B),$$
$$I^r(g,A,B)\equiv I(g)+\Delta_0(g,A,B)+\Gamma_1(g,A,B), \tag{3.7}$$

where Γ_2 and Γ_3 are defined not on Σ but on M, violating thereby the basic requirement (b). Now, one should observe that the term Γ_3 is gauge invariant, therefore it can be removed from $I(g,A,B)$, leaving us with the gauge invariant combination $I^r(g,A,B)+\Gamma_2(A,B)$. Since Γ_2 is a three-form on M, only $I^r(g,A,B)$ can be included as part of the action for the σ-model on Σ, but it is not gauge invariant:

$$\delta I^r(g,A,B)=-\delta\Gamma_2(A,B)$$
$$=-\kappa\int_M\frac{1}{2}\,\delta[\omega_3(B,F_B)-\omega_3(A,F_A)]\neq 0.$$

The action $I^r(g,A,B)$ is very close to what we want: it is defined on Σ, and its variation under gauge transformations gives an expression which depends on gauge fields (A,B), but not on g. Can one find a mechanism that compensates this noninvariance, and yields an acceptable gauge invariant extension of the WZNW action (2.1)?

In the analogous four-dimensional model Witten [13] solved the problem by requiring the constraint $\omega_3(B,F_B)-\omega_3(A,F_A)=0$ on the gauge group H, sufficient for gauge invariance. In string models one can simply remove Γ_2 without assuming any constraint on H, while the gauge invariance of the theory is ensured by the presence of some additional field in the action, with ''anomalous'' transformation law [12]. In this paper we shall solve the Γ_2 problem by considering a gauge extension of the action (1.1), describing a system of *two* simple WZNW models, in which the problematic Γ_2 term in the first sector will cancel the corresponding term in the second sector, leading to the theory which is both (a) gauge invariant and (b) defined on Σ.

The construction goes as follows. We start with the formal extension of $I(g,A,B)$, as obtained in Eq. (3.7). Next, using gauge invariance of $\Gamma_3(g,A,B)$ we define a simpler gauge invariant action:

$$I'(g,A,B)=I^r(g,A,B)+\Gamma_2(g,A,B).$$

It is now easy to see that an acceptable gauge extension of the action (1.1) for the WZNW system can be defined by

$$I(1,2)=I'(g_1,A,B)-I'(g_2,A,B), \tag{3.8}$$

where $g_1=g(x_1,\varphi_1,y_1)$ and $g_2=g(x_2,\varphi_2,y_2)$ are different fields, belonging to the same representation of $SL(2,R)$. Indeed, since $\Gamma_2(A,B)$ does not depend on g, the contribution of two Γ_2 terms to $I(1,2)$ vanishes. Thus, the gauge invariant action (3.8) can be written in the simpler form

$$I(1,2)=I^r(g_1,A,B)-I^r(g_2,A,B), \tag{3.9a}$$

where we clearly see that it is an action defined on Σ.

The reduced action (3.7) can be written as

$$I^r(g,A,B)=I(g)+\kappa\int_\Sigma\frac{1}{2}\,\mathrm{Tr}[-(^*\bar{v}+\bar{v})A-(^*v-v)B-(^*B+B)(g^{-1}Ag)], \tag{3.9b}$$

where the g-independent term $1/2(^*AA+{}^*BB)$ in Δ_0 is ignored, as its contribution to sectors 1 and 2 in Eq. (3.9a) is canceled. The absence of this term implies that $I(1,2)$ does not depend on $(^*A+A)\sim A_-$ and $(^*B-B)\sim B_+$ [see Eq. (A6)], i.e., self-dual and anti-self-dual parts of the gauge fields A and B, respectively. As we shall see, the absence of these parts will greatly influence the dynamical structure of the gauged WZNW system.

IV. $H_+\times H_-$ GAUGE THEORY

In this section we shall specify the gauge group, and derive an explicit expression for the gauge action (3.9) in terms of the group coordinates (x_1,φ_1,y_1) and (x_2,φ_2,y_2).

Using now the matrix $R(g)$, that defines the adjoint representation of the gauge group, $g^{-1}t_b g=-t_c R^c{}_b$, where $R^c{}_b\equiv E^c{}_\alpha \bar{E}^\alpha{}_b$ (Appendix B), the reduced action takes the form

$$I^r(g,A,B)=I(g)+2\kappa\int_\Sigma d^2\xi\sqrt{-g}[-\overline{v}^a_-A^b_+ -v^a_+B^b_-+B^a_-R^b{}_cA^c_+]\gamma_{ab}. \tag{4.1}$$

As we mentioned, the action $I(1,2)$ does not contain variables (A_-,B_+), which implies the existence of an *extra* gauge symmetry, allowing an arbitrary change of the absent components. This is a specific feature of the action for the WZNW system. To simplify further considerations we shall fix this symmetry by imposing the following gauge conditions:

$$A_-=0,\quad B_+=0. \tag{4.2}$$

Up to now we did not specify the gauge group H. We could take H to be the whole $SL(2,R)\times SL(2,R)$, but for our purposes this is not necessary. We assume that H is a subgroup of $SL(2,R)\times SL(2,R)$, defined by

$$H=H_+\times H_-, \tag{4.3}$$

where H_+ and H_- are subgroups of $SL(2,R)$ defined by the generators (t_+,t_0) and (t_0,t_-), respectively. When compared to $SL(2,R)\times SL(2,R)$, our choice means that the gauge fields should be restricted as follows:

$$A^{(-)}=0,\quad B^{(+)}=0. \tag{4.4}$$

The gauge symmetry $H_+\times H_-$ is defined in terms of the following gauge fields and gauge parameters:

$$(A_+^{(+)},A_+^{(0)},B_-^{(-)},B_-^{(0)}),\quad(\varepsilon^{(+)},\varepsilon^{(0)},\overline{\varepsilon}^{(-)},\overline{\varepsilon}^{(0)}).$$

Gauge transformations of dynamical variables are given in Eqs. (C4) and (C5).

Using the general relations

$$v_i=g^{-1}\partial_i g=t_aE^a{}_\alpha\partial_i q^\alpha,$$

$$\overline{v}_i=g\partial_i g^{-1}=t_\alpha\bar{E}^a{}_\alpha\partial_i q^\alpha,$$

and assuming the restrictions (4.2) and (4.4), one obtains

$$\overline{v}^a_-A^b_+\gamma_{ab}=2\overline{v}^{(-)}_-A^{(+)}_+ +\overline{v}^{(0)}_-A^{(0)}_+$$
$$=-2e^{-\varphi}\partial_-y[A^{(+)}_+ +xA^{(0)}_+]-\partial_-\varphi A^{(0)}_+,$$

$$v^a_+B^b_-\gamma_{ab}=2v^{(+)}_+B^{(-)}_- +v^{(0)}_+B^{(0)}_-$$
$$=2e^{-\varphi}\partial_+x[B^{(-)}_- +yB^{(0)}_-]+\partial_+\varphi B^{(0)}_-.$$

Next, with the help of the expression (B8) for R_{ab} we find

$$B^a_-R_{ab}A^b_+=-2e^{-\varphi}[yB^{(0)}_-A^{(+)}_+ +xyB^{(0)}_-A^{(0)}_+ +B^{(-)}_-A^{(+)}_+ +xB^{(-)}_-A^{(0)}_+]-B^{(0)}_-A^{(0)}_+.$$

The final result for the reduced action takes the form

$$I^r(g,A,B)=\kappa\int d^2\xi\sqrt{-g}[\partial_-\varphi\partial_+\varphi+2A^{(0)}_+\partial_-\varphi -2B^{(0)}_-\partial_+\varphi+4D_+xD_-ye^{-\varphi}-2B^{(0)}_-A^{(0)}_+], \tag{4.5}$$

where

$$D_+x=[\partial_++A^{(0)}_+]x+A^{(+)}_+,$$

$$D_-y=[\partial_--B^{(0)}_-]y-B^{(-)}_-,$$

are covariant derivatives on the group manifold, Eqs. (C2a) and (C2b). The last, g-independent term in Eq. (4.5) will be canceled in the complete action $I(1,2)$, Eq. (3.9a).

V. INDUCED GRAVITY FROM GAUGED WZNW SYSTEM

In this section we shall consider the functional integral of the theory defined by the action (3.9a), and (3.9b) and show, by performing a suitable gauge fixing and integrating out some dynamical variables, that this theory leads to the induced gravity action (1.2).

A. Effective theory for gauged WZNW system

It is useful to introduce auxiliary fields $f_{1\pm},f_{2\pm}$, and rewrite the part of the action $I(1,2)$ given by

$$\Upsilon\equiv4D_+x_1D_-y_1e^{-\varphi_1}-4D_+x_2D_-y_2e^{-\varphi_2}, \tag{5.1}$$

in the form

$$\Upsilon=f_{1-}D_+x_1+f_{1+}D_-y_1-\frac{1}{4}f_{1-}f_{1+}e^{\varphi_1}-f_{2-}D_+x_2 -f_{2+}D_-y_2+\frac{1}{4}f_{2-}f_{2+}e^{\varphi_2},$$

or, more explicitly,

$$\Upsilon=-B^{(-)}_-(f_{1+}-f_{2+})+A^{(+)}_+(f_{1-}-f_{2-}) +f_{1-}[\partial_++A^{(0)}_+]x_1+f_{1+}[\partial_--B^{(0)}_-]y_1$$

$$-\frac{1}{4}f_{1-}f_{1+}e^{\varphi_1}-f_{2-}[\partial_+ + A_+^{(0)}]x_2 - f_{2+}[\partial_- - B_{(-)}^{(0)}]y_2 + \frac{1}{4}f_{2-}f_{2+}e^{\varphi_2}.$$

Gauge transformations determined by $\varepsilon^{(+)}$ and $\overline{\varepsilon^{(-)}}$ have the form

$$\delta x_1=\varepsilon^{(+)},\quad \delta x_2=\varepsilon^{(+)},$$

$$\delta A_+^{(+)}=-[\partial_+ + A_+^{(0)}]\varepsilon^{(+)},$$

$$\delta y_1=-\overline{\varepsilon^{(-)}},\quad \delta y_2=-\overline{\varepsilon^{(-)}},$$

$$\delta B_-^{(-)}=-[\partial_- - B_-^{(0)}]\overline{\varepsilon^{(-)}}.$$

This part of the complete gauge symmetry can be fixed by imposing the following gauge conditions:

$$x_2=0,\quad y_2=0.$$

Integrations $\int dA_+^{(+)}dB_-^{(-)}$ and $\int df_{2+}df_{2-}$ in the functional integral lead to the elimination of $f_{2\pm}$ from the action: $f_{2\pm}=f_{1\pm}$. Writing $f_\pm$ instead of $f_{1\pm}$ for simplicity, one obtains

$$\Upsilon = f_-[\partial_+ + A_+^{(0)}]x_1 - f_+[\partial_- - B_-^{(0)}]y_1 - \frac{1}{4}f_-f_+(e^{\varphi_1}-e^{\varphi_2}).$$

After that the integration $\int dx_1dy_1$ produces

$$\delta[(\nabla_+ - A_+^{(0)})f_-]\cdot\delta[(\nabla_- + B_-^{(0)})f_+],$$

where $\nabla_\pm$ is the covariant derivative on Σ (Appendix A). Then, $\int dA_+^{(0)}dB_-^{(0)}$ yields

$$A_+^{(0)}=-\omega_+ + \partial_+\ \ln|f_-|,\quad B_-^{(0)}=-\omega_- - \partial_-\ \ln|f_+|, \tag{5.2}$$

where $\omega_\pm$ is the connection on Σ (Appendix A), and an additional factor $[\det(f_-f_+)]^{-1}$ appears in the functional measure. Consequently, we find

$$\Upsilon = -\frac{1}{4}f_-f_+(e^{\varphi_1}-e^{\varphi_2}), \tag{5.3}$$

and the reduced action becomes

$$I^r(\varphi,f_-,f_+) = \kappa\int d^2\xi\sqrt{-g}\bigg[\partial_-\varphi\partial_+\varphi - 2(\omega_+ - \partial_+\ \ln|f_-|)\partial_-\varphi + 2(\omega_- + \partial_-\ \ln|f_+|)\partial_+\varphi - \frac{1}{4}f_-f_+e^{\varphi}\bigg]. \tag{5.4}$$

The complete action $I(1,2)$ is invariant under the remaining piece of gauge transformations:

$$\delta f_- = -\varepsilon^{(0)}f_-,\quad \delta f_+ = \overline{\varepsilon^{(0)}}f_+,$$

$$\delta\varphi_1=\varepsilon^{(0)}-\overline{\varepsilon^{(0)}},\quad \delta\varphi_2=\varepsilon^{(0)}-\overline{\varepsilon^{(0)}}.$$

Now, we introduce gauge invariant variables

$$\phi_1=\varphi_1+\ln|f_+f_-|,\quad \phi_2=\varphi_2+\ln|f_+f_-|,$$

in terms of which, after some cancellation of ϕ-independent pieces, we obtain

$$I(1,2)=I^r(\phi_1)-I^r(\phi_2),$$

$$I^r(\phi)=\kappa\int_\Sigma d^2\xi\sqrt{-g}\bigg[\partial_-\phi\partial_+\phi + 2\omega_-\partial_+\phi - 2\omega_+\partial_-\phi - \frac{1}{4}e^{\phi}\bigg]. \tag{5.5}$$

Note that the part of the functional integral depending on f_-,f_+ is decoupled from the rest, and can be absorbed into the normalization factor. Thus, using gauge invariant variables effectively restricts the space of dynamical variables; it is, essentially, equivalent to a gauge-fixing corresponding to $(\varepsilon^{(0)},\overline{\varepsilon^{(0)}})$ transformations, e.g., $f_-=\mu_-$, $f_+=\mu_+$, followed by the integration $\int df_-df_+$.

Thus, the final form of the effective action for the gauged WZNW system is given by Eq. (5.5).

B. Transition to the induced gravity

To show that the effective theory (5.5) is equivalent to the induced gravity (1.2), let us observe that (5.5) is invariant under the following conformal rescalings:

$$g_{\mu\nu}\to e^{2F}g_{\mu\nu},\quad \phi_1\to\phi_1-2F,\quad \phi_2\to\phi_2-2F, \tag{5.6}$$

which imply $\partial_\pm\to e^{-F}\partial_\pm$, and $\omega_\pm\to e^{-F}(\omega_\pm\mp\partial_\pm F)$. This symmetry is directly connected to the invariance of the original WZNW theory under conformal rescalings. It can be gauge-fixed by demanding

$$\phi_2=\ln\ \mu,$$

whereafter the effective action becomes

$$I(\phi,g_{\mu\nu}) = \kappa\int_\Sigma d^2\xi\sqrt{-g}\bigg[\partial_-\phi\partial_+\phi + 2\omega_-\partial_+\phi - 2\omega_+\partial_-\phi - \frac{1}{4}\mu(e^{\phi}-1)\bigg], \tag{5.7}$$

where we introduced $\phi=\phi_1-\ln\ \mu$. Now, partial integrations together with Eq. (A3), and the replacement $\phi\to\phi/\sqrt{\kappa}$, lead to the induced gravity action (1.2), where $\alpha=2\sqrt{\kappa}$, and $M=\kappa\mu/4$.

VI. CONCLUDING REMARKS

We presented here the connection between the gauge extension of the WZNW system (1.1) and the induced gravity action (1.2), fully respecting the diffeomorphism invariance of both theories.

It is well known that an acceptable gauge extension of the simple WZNW model does not exist unless one requires specific constraints on the gauge group [11]. The reason for this unusual behavior steams from the fact that gauged WZ term $n\Gamma$ does not represent, in general, a field theory on a 2D manifold Σ. If one tries to select an action defined on Σ, one looses gauge invariance, and vice versa. In string models one can overcome these problems with the help of an additional field [12]. Following the ideas developed in Ref. [10], we introduced in this paper an acceptable action, which is both gauge invariant and defined on Σ, by considering a dynamical system described by a difference of two simple WZNW models.

Our gauge group is $H_+ \times H_-$, a four-parameter subgroup of $SL(2,R)\times SL(2,R)$. Two of the gauge fields, $A_+^{(+)}$ and $B_-^{(-)}$, ensure the equality of currents in two sectors, in accordance with the results of the Hamiltonian analysis of the WZNW system [10]. The remaining two fields, $A_+^{(0)}$ and $B_-^{(0)}$, become components of the connection of the induced gravity action. In this way, the Riemannian structure on Σ is seen to be closely related to the $SL(2,R)$ gauge fields.

The results obtained here can be used to clarify the connection between globally regular solutions of the equations of motion for the WZNW system, and the related singular solutions (in the coordinate sense) of the induced gravity [8,9]. In particular, it will be interesting to improve our understanding of the WZNW black hole solutions in the context of induced gravity [14].

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APPENDIX A: RIEMANNIAN STRUCTURE OF Σ

In this Appendix we present some formulas on the Riemannian structure of 2D manifold Σ, which are used in the paper.

Coordinates of points in two-dimensional manifold Σ are denoted by ξ^μ $(\mu=0,1)$. Basic tensorial objects in the coordinate basis are

$$\text{vectors:}\quad e_\mu=\partial_\mu;\quad \text{1-forms:}\quad \theta^\mu=d\xi^\mu;$$
$$\text{metric:}\quad e_\mu\cdot e_\nu=g_{\mu\nu};\quad \varepsilon^{01}=1.$$

Another useful basis is the local light-cone basis:

$$\text{vectors:}\quad e_i=\partial_i=e_i{}^\mu e_\mu;$$
$$\text{1-forms:}\quad \theta^i=d\xi^i=e^i{}_\mu d\xi^\mu\quad (i=+,-);$$
$$\text{metric:}\quad e_i\cdot e_j=e_i{}^\mu e_j{}^\nu g_{\mu\nu}=\eta_{ij},$$
$$\eta_{+-}=\eta_{-+}=1;\quad \varepsilon^{-+}=1.$$

Connection and curvature. Riemannian connection on Σ, $\omega^i{}_j=\varepsilon^i{}_j\omega$, is defined by the first structural equation:

$$d\theta^i+\omega^i{}_j\theta^j=0. \tag{A1}$$

The exterior derivative of a 1-form $u=u_k\theta^k$ can be written as

$$du=(du_i)\theta^i+u_i d\theta^i=(du_i-u_s\omega^s{}_i)\theta^i=(\nabla_k u_i)\theta^k\theta^i,$$

where $\nabla_k u_i$ is the covariant derivative of a 1-form:

$$\nabla_k u_i=\partial_k u_i-\varepsilon^s{}_i\omega_k u_s\qquad \nabla_k u_\mp=(\partial_k\mp\omega_k)u_\mp .$$

By noting that $u^\pm=u_\mp$, one easily finds the covariant derivative of a vector.

The curvature is defined by the second structural equation:

$$d\omega^i{}_j=\frac{1}{2}R^i{}_{jkl}\theta^k\theta^l. \tag{A2}$$

Using $d\omega=(\nabla_k\omega_l)\theta^k\theta^l$, one finds

$$R^i{}_{jkl}=\varepsilon^i{}_j(\nabla_k\omega_l-\nabla_l\omega_k),$$
$$R=2R_{-+}=2(\nabla_-\omega_+-\nabla_+\omega_-). \tag{A3}$$

Conformal rescaling. Let us now derive the transformation law of the connection under conformal rescaling of the metric. The relation $g_{\mu\nu}=e^{2F}\hat{g}_{\mu\nu}$ implies $\theta^i=e^F\hat{\theta}^i$. Replacing this into the first structural equation gives

$$d\hat{\theta}^i+dF\hat{\theta}^i+\omega^i{}_k\hat{\theta}^k=0\Rightarrow\hat{\omega}^i{}_j\hat{\theta}^j=dF\hat{\theta}^i+\omega^i{}_k\hat{\theta}^k.$$

Since $\hat{\omega}^i{}_j=\varepsilon^i{}_j\hat{\omega}$, we easily obtain

$$\hat{\omega}_+-\hat{\partial}_+F=\omega_+e^F,\quad \hat{\omega}_-+\hat{\partial}_-F=\omega_-e^F. \tag{A4}$$

Acting on these equations with $\hat{\nabla}_-$ and $\hat{\nabla}_+$, respectively, and using Eq. (A4) again, one finds

$$\hat{\nabla}_-\hat{\omega}_+-\hat{\nabla}_-\hat{\partial}_+F=e^{2F}\nabla_-\omega_+,$$
$$\hat{\nabla}_+\hat{\omega}_-+\hat{\nabla}_+\hat{\partial}_-F=e^{2F}\nabla_+\omega_-.$$

Subtracting two equations one finds the effect of conformal rescaling on the curvature:

$$R(\hat{g})-2\eta^{ij}\hat{\nabla}_i\hat{\nabla}_jF=e^{2F}R(g). \tag{A5}$$

We display here some useful formulas:

$$\theta^-\theta^+=d^2\xi\sqrt{-g},\quad *\theta^i=\varepsilon^i{}_k\theta^k,$$
$$\theta^i\theta^j=\varepsilon^{ij}d^2\xi\sqrt{-g},\quad *\theta^i\theta^j=\eta^{ij}d^2\xi\sqrt{-g},$$
$$A\pm *A=2A_\mp\theta^\mp,$$

$$(*A \pm A)B = 2A_{\mp}B_{\pm}d^2\xi\sqrt{-g}. \quad \text{(A6)}$$

APPENDIX B: ON THE GEOMETRY OF $SL(2,R)$ MANIFOLD

Here we present some useful results concerning the Riemannian structure of the group manifold $SL(2,R)$.

If the generators of the group $SL(2,R)$ are chosen as $t_{(\pm)} = 1/2(\sigma_1 \pm i\sigma_2)$, $t_{(0)} = (1/2)\sigma_3$, where σ_k are the Pauli matrices, the Lie algebra $[t_a, t_b] = f_{ab}{}^c t_c$ takes the form

$$[t_{(+)}, t_{(-)}] = 2t_{(0)}, \quad [t_{(\pm)}, t_{(0)}] = \mp t_{(\pm)}. \quad \text{(B1)}$$

Then, the calculation of the Cartan metric $\gamma_{ab} = (t_a, t_b) = 1/2\, f_{ac}{}^d f_{bd}{}^c$ yields

$$\gamma_{ab} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad a,b = (+),(0),(-). \quad \text{(B2)}$$

Raising and lowering of the tangent space indices $(a,b,\ldots)$ are performed with γ_{ab} and its inverse γ^{ab}.

The group $SL(2,R)$ has the property that any element g in a neighborhood of identity admits the Gauss decomposition:

$$g = g_+(x)g_0(\varphi)g_-(y),$$

$$g_+ = e^{xt_{(+)}} = 1 + xt_{(+)}, \quad g_- = e^{yt_{(-)}} = 1 + yt_{(-)},$$

$$g_0 = e^{\varphi t_{(0)}} = \cos(\varphi/2) + 2t_{(0)}\sin(\varphi/2). \quad \text{(B3a)}$$

where $q^\alpha = (x,\varphi,y)$ are group coordinates. In this parametrization we have

$$g = \begin{pmatrix} e^{\varphi/2} + xye^{-\varphi/2} & xe^{-\varphi/2} \\ ye^{-\varphi/2} & e^{-\varphi/2} \end{pmatrix}. \quad \text{(B3b)}$$

Now, we can write $v = g^{-1}dg = E^a t_a = t_a E^a{}_\alpha dq^\alpha$, where the quantities $E^a{}_\alpha$ serve as the vielbein on the group manifold. The above expression for g leads to

$$E^{(+)} = e^{-\varphi}dx,$$

$$E^{(0)} = 2ye^{-\varphi}dx + d\varphi,$$

$$E^{(-)} = -y^2e^{-\varphi}dx - yd\varphi + dy,$$

so that the vielbein $E^a{}_\alpha$ and its inverse $E^\alpha{}_a$ are given as

$$E^a{}_\alpha = \begin{pmatrix} e^{-\varphi} & 0 & 0 \\ 2ye^{-\varphi} & 1 & 0 \\ -y^2e^{-\varphi} & -y & 1 \end{pmatrix}, \quad E^\alpha{}_a = \begin{pmatrix} e^{\varphi} & 0 & 0 \\ -2y & 1 & 0 \\ -y^2 & y & 1 \end{pmatrix}. \quad \text{(B4)}$$

The Cartan metric in the coordinate basis, $\gamma_{\alpha\beta} = E^a{}_\alpha E^b{}_\beta \gamma_{ab}$, has the form

$$\gamma_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 2e^{-\varphi} \\ 0 & 1 & 0 \\ 2e^{-\varphi} & 0 & 0 \end{pmatrix}, \quad \alpha,\beta = x,\varphi,y. \quad \text{(B5)}$$

From the relation $(v,v^2) = -6d\tau$ we obtain

$$d\tau = E^{(+)}E^{(0)}E^{(-)} = d(e^{-\varphi}dxdy). \quad \text{(B6)}$$

Similarly, the calculation of $\bar{v} = gdg^{-1} = t_a\bar{E}^a = t_a\bar{E}^a{}_\alpha dq^\alpha$ leads to

$$\bar{E}^{(+)} = -dx + xd\varphi + x^2e^{-\varphi}dy,$$

$$\bar{E}^{(0)} = -d\varphi - 2xe^{-\varphi}dy,$$

$$\bar{E}^{(-)} = -e^{-\varphi}dy,$$

or

$$\bar{E}^a{}_\alpha = \begin{pmatrix} -1 & x & x^2e^{-\varphi} \\ 0 & -1 & -2xe^{-\varphi} \\ 0 & 0 & -e^{-\varphi} \end{pmatrix},$$

$$\bar{E}^\alpha{}_a = \begin{pmatrix} -1 & -x & x^2 \\ 0 & -1 & 2x \\ 0 & 0 & -e^{\varphi} \end{pmatrix}. \quad \text{(B7)}$$

The metric $\bar{\gamma}_{\alpha\beta}$ is the same as $\gamma_{\alpha\beta}$.

Also, we shall be making use of the matrix $R^a{}_b$, defined by $g^{-1}t_b g = -t_a R_a{}^b$. Starting from the identity $g^{-1}\bar{v}g = -v$, which can be written in the form $\bar{E}^a{}_\alpha g^{-1}t_a g = -E^a{}_\alpha t_a$, one finds $R^a{}_b = E^a{}_\alpha \bar{E}^\alpha{}_b$. The calculation of $R_{ab} = \gamma_{ac}R^c{}_b$ yields

$$R_{ab}(g) = \begin{pmatrix} 2y^2e^{-\varphi} & 2xy^2e^{-\varphi} + 2y & -2x^2y^2e^{-\varphi} - 4xy - 2e^{\varphi} \\ -2ye^{-\varphi} & -2xye^{-\varphi} - 1 & 2x^2ye^{-\varphi} + 2x \\ -2e^{-\varphi} & -2xe^{-\varphi} & 2x^2e^{-\varphi} \end{pmatrix}. \quad \text{(B8)}$$

APPENDIX C: COVARIANT DERIVATIVE AND GAUGE TRANSFORMATIONS

In this appendix we exhibit gauge properties of the WZNW system in some detail.

(1) Writing the expression (3.4) for the covariant 1-form V in group coordinates q^α, one can obtain coordinate expression for the covariant derivative Dq^α on the group manifold:

$$t_a E^a{}_\alpha Dq^\alpha = t_a E^a{}_\alpha dq^\alpha - t_a R^a{}_b A^b - t_a B^a,$$

$$Dq^\alpha = dq^\alpha - \bar{E}^\alpha{}_a A^a - E^\alpha{}_a B^a. \tag{C1}$$

Effectively, the components A_- and B_+ are absent, Eq. (4.2), so that

$$D_+ q^\alpha = \partial_+ q^\alpha - \bar{E}^\alpha{}_a A^a_+ , \quad D_- q^\alpha = \partial_- q^\alpha - E^\alpha{}_b B^b_- .$$

Taking into account additional conditions $A^{(-)}_+ = B^{(+)}_- = 0$, Eq. (4.4), one finds

$$D_+ x = [\partial_+ + A^{(0)}_+]x + A^{(+)}_+ ,$$

$$D_+ \varphi = \partial_+ \varphi + A^{(0)}_+ ,$$

$$D_+ y = \partial_+ y, \tag{C2a}$$

and

$$D_- x = \partial_- x,$$

$$D_- \varphi = \partial_- \varphi - B^{(0)}_- ,$$

$$D_- y = [\partial_- - B^{(0)}_-]y - B^{(-)}_- . \tag{C2b}$$

(2) Let us now consider $SL(2,R)\times SL(2,R)$ gauge transformations. Group elements transform according to $g' = \Omega g \bar{\Omega}^{-1}$, where $\Omega = e^{\varepsilon}$, $\bar{\Omega} = e^{\bar{\varepsilon}}$, and $\varepsilon = t_a \varepsilon^a$, $\bar{\varepsilon} = t_a \bar{\varepsilon}^a$. Infinitesimal transformations are

$$g^{-1}\delta g = g^{-1}\varepsilon g - \bar{\varepsilon},$$

$$\delta q^\alpha = -\bar{E}^\alpha{}_a \varepsilon^a - E^\alpha{}_a \bar{\varepsilon}^a, \tag{C3}$$

or, in components:

$$\delta x = \varepsilon^{(+)} + x\varepsilon^{(0)} - x^2 \varepsilon^{(-)} - e^{\varphi}\bar{\varepsilon}^{(+)},$$

$$\delta\varphi = \varepsilon^{(0)} - 2x\varepsilon^{(-)} + 2y\bar{\varepsilon}^{(+)} - \bar{\varepsilon}^{(0)},$$

$$\delta y = e^{\varphi}\varepsilon^{(-)} + y^2\bar{\varepsilon}^{(+)} - y\bar{\varepsilon}^{(0)} - \bar{\varepsilon}^{(-)}.$$

Infinitesimal transformations of gauge potentials are obtained from Eq. (3.3):

$$\delta A^a = -d\varepsilon^a - f_{bc}{}^a A^b \varepsilon^c, \quad \delta B^a = -d\bar{\varepsilon}^a - f_{bc}{}^a B^b \bar{\varepsilon}^c.$$

Gauge fixing (4.2) leads to

$$\delta A^a_+ = -\partial_+ \varepsilon^a - f_{bc}{}^a A^b_+ \varepsilon^c, \quad \delta B^a_- = -\partial_- \bar{\varepsilon}^a - f_{bc}{}^a B^b_- \bar{\varepsilon}^c.$$

(3) Now, restriction to $H_+ \times H_-$ is achieved by demanding $\varepsilon^{(-)} = \bar{\varepsilon}^{(+)} = 0$. The restricted transformations take the form

$$\delta x = \varepsilon^{(+)} + x\varepsilon^{(0)},$$

$$\delta\varphi = \varepsilon^{(0)} - \bar{\varepsilon}^{(0)},$$

$$\delta y = -y\bar{\varepsilon}^{(0)} - \bar{\varepsilon}^{(-)}, \tag{C4}$$

and

$$\delta A^{(0)}_+ = -\partial_+ \varepsilon^{(0)}, \quad \delta A^{(+)}_+ = -[\partial_+ + A^{(0)}_+]\varepsilon^{(+)} + A^{(+)}_+ \varepsilon^{(0)},$$

$$\delta B^{(0)}_- = -\partial_- \bar{\varepsilon}^{(0)}, \quad \delta B^{(-)}_- = -[\partial_- - B^{(0)}_-]\bar{\varepsilon}^{(-)} - B^{(-)}_- \bar{\varepsilon}^{(0)}. \tag{C5}$$

(4) Using $\delta V^a = f_{bc}{}^a \bar{\varepsilon}^b V^c$, where $V^a = E^a{}_\alpha Dq^\alpha$, one finds

$$\delta(Dq^\alpha) = [(E^\alpha{}_a f_{bc}{}^a E^c{}_\beta)\bar{\varepsilon}^b - E^\alpha{}_a(\delta E^a{}_\beta)]Dq^\beta.$$

Restriction to $H_+ \times H_- [\varepsilon^{(-)} = \bar{\varepsilon}^{(+)} = 0]$ yields

$$\delta(Dx) = \varepsilon^{(0)} Dx, \quad \delta(D\varphi) = 0, \quad \delta(Dy) = -\bar{\varepsilon}^{(0)} Dy.$$

Then, gauge transformations of auxiliary fields $f_\pm$ are $\delta f_+ = \bar{\varepsilon}^{(0)} f_+$, $\delta f_- = -\varepsilon^{(0)} f_-$. From this it follows $\delta(\varphi + \ln|f_- f_+|) = 0$.

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