Massive Dirac fields in naked and in black hole Reissner-Nordström manifolds

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The problem of the electrodynamical stability of the RN naked singularities is analyzed by studying the qualitative spectral properties of the Dirac equation. A comparison with the RN black hole cases is made. The Dirac vacuum appears to be stable in the case of the naked RN geometries, whereas in the case of the RN black holes the same mathematical approach confirms the existence of a discharge mechanism related with the Klein effect. [S0556-2821(98)06116-5]

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I. INTRODUCTION

According to the cosmic censorship Conjecture (CCC) naked singularities are forbidden due to the expected unphysical behavior associated with them. See, e.g., [1]. No theorem implementing CCC exists, neither at the classical level nor at the quantum one. In the case of charged singularities, one could expect that electrodynamical effects could be enough to dress the singularities and eventually turn them into a charged Reissner-Nordström (RN) black hole. This expectation arises from the studies carried at the classical level by Cohen and Gautreau in [2]. Therein it is shown that classical capture trajectories exist for test particles with charge to mass ratio greater than one (in natural units) and charge opposite to the one of the singular manifold and this fact allows a picture in which RN naked singularities are turned into RN black holes. At the quantum level, Damour and Deruelle [3] studied the case of a charged scalar field in a naked RN background and their semiquantitative analysis showed that a quantum dressing of the naked singularity takes place because of a particle creation related with the Klein effect.

In this work we try to carry an analogous analysis for the quantum Dirac equation on the classical naked RN background. We underline the following very important points of our study: We consider an ''eternal'' (that is, existing at all times) RN naked singularity; we look for quantum electrodynamical instabilities for the ground state of massive half-spin particles. The classical electromagnetic field associated with the given geometry is the only source of instability we will take into account.

We start by discussing some mechanisms allowing quantum electrodynamical instability in flat spacetime, because they can give some hints about the instability phenomenon we are looking for. Then the Dirac one particle hamiltonian is studied and its qualitative spectral properties are analyzed. The aim is to understand if it is possible to get a particle creation process related with the Klein effect. The answer is negative.

Then, a detailed study of the RN black hole cases is made also in order to make a comparison. It is confirmed that, in the case of RN black holes, discharge processes related to the Klein effect are avoided only for very small black hole charge to black hole mass ratio if the Dirac field one considers is the electron field.

A further discussion is found in the conclusions.

II. THE KLEIN EFFECT AND QED IN STRONG FIELDS: POSSIBLE FLAT SPACETIME ANALOGIES

In this section we recall some general theorems that are strong enough to control the stability problem for the Dirac vacuum in presence of an external potential in flat spacetime.

We will follow ideas discussed in [4] and in [5]. We can start our analysis by considering the one particle Hamiltonian $h=h_0+V$, where h_0 is the free Hamiltonian and V is a static external potential. For some formal details, see [4]. One can choose the spectral decomposition of the (self-adjoint extension of) h into positive and negative ''energy'' states as a well defined basis for a second quantized theory, just in the same way one defines the electron and positron free states in the standard free theory. Note that no gap between positive and negative states is required and that the Fock space vacuum $|\Omega\rangle$ associated with such spectral decomposition is stable under the evolution generated by the second quantization Hamiltonian operator [4]. Then no particle creation can be expected if a static potential is introduced.

A second question that can be raised is if the given potential is a good scattering potential on the free field Fock space in the sense that quantum Møller wave operators exist. Consider a scattering operator $S:\mathcal{H}_{in}\to\mathcal{H}_{out}$, where $\mathcal{H}_{in},\mathcal{H}_{out}$ are the one particle asymptotic Hilbert spaces. Let Σ be an unitary implementation of S in the Fock space F with cyclic vector (vacuum) $|\Omega\rangle$. The probability of persistence of the vacuum is

$$|\langle \text{vacuum},t\to+\infty|\text{vacuum},t\to-\infty\rangle|^2=|(\Omega,\Sigma\Omega)|^2. \quad (1)$$

The pair creation probability is

$$p=1-|(\Omega,\Sigma\Omega)|^2. \quad (2)$$

In the case of Dirac particles, it holds theorem (10.10) of [5]: *If Dirac particles are affected by a static external potential such that the scattering operator S exists and it is unitary, then S is unitarily implementable in the Fock space F and it holds*

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$$\Sigma\Omega=\Omega$$

that is $p=0$.

This implies that no particle creation is possible in such a static external potential [5]. Particularly, vacuum stability for the Coulomb potential is predicted. In this case, an instability of the vacuum is naively expected for the Dirac Hamiltonian when the quantized electron field is in presence of the classical Coulomb field generated by an highly charged nucleus;[1] moreover, a consequent emission of pairs of positrons has been predicted. The above theorem shows that it is not possible to find out such discharge phenomenon unless some adiabatic time dependence for the potential is introduced. See also the final discussion. The so called Klein effect allows to recover in some sense a particle creation effect even in presence of a static potential. Let us consider a Dirac Hamiltonian defined on a interval[2] and characterized by two asymptotic regions, say, at $A\geqslant-\infty$ and at $B\leqslant\infty$. Roughly speaking, we can define as ''Klein region'' an overlap region between asymptotic positive (negative) continuum states at A and asymptotic negative (positive) continuum states at B. Compare also [5], Sec. 4.7. Usually, such phenomenon takes place when $A=-\infty$ and $B=+\infty$.[3] Then, e.g., an electron state at A is seen as a positron state at B if it belongs to the overlap region: One gets the so called Klein effect.[4] We underline that the Klein effect takes place because one assigns a physical relevance to the given asymptotic states; see also the discussion below. At the level of quantum field theory, the presence of level crossing between negative and positive energy asymptotic states of the Dirac particles gives rise to a nonzero particle current. For an extensive discussion about the Klein effect, also in curved spacetime cases, see [6].

We note that the above theorem is not violated by static potentials inducing a Klein effect, as steplike ones. Indeed, for these cases, Bongaarts and Ruijsenaars [7] show that S cannot be unitarily implemented in the free particle Fock space F. One can physically interpret [8] the presence of a Klein overlap region in terms of particle creation: $p>0$.

We think a further discussion is useful on this peculiar kind of ''instability.'' Indeed, as seen, one can adopt the above spectral decomposition of the one particle hamiltonian with real spectrum[5] in order to construct a well defined Fock space vacuum $|\Omega\rangle$ that is obviously stable [4]. Nevertheless, one could consider as physically interesting only a field theory having a satisfactory interpretation in terms of scattering states [7]. Then let us suppose that one has asymptotic Hamiltonian operators in the asymptotic regions such that a Klein region appears. This fact amounts to have a nonvanishing current particle occurring between the two asymptotic regions (Klein effect) [9] and could be interpreted as an instability of the physical system (external field, etc.). In other words, one chooses to privilege a particle interpretation in terms of asymptotic states and then the fact that no unitary scattering operator can be found is assumed as a signal of a particle creation process. In our case, the presence of a Klein region could be interpreted as an instability of the RN solution with respect to the ordinary electron field that could be defined in the asymptotically flat region of RN spacetime.

In the following, we will look for the Klein effect for Dirac particles both in the naked geometry and in the RN black hole case.

III. THE DIRAC HAMILTONIAN ON NAKED RN MANIFOLDS

Let us define the one particle Hamiltonian for Dirac massive particles on the naked RN geometry. The metric of the background RN manifold is

$$\begin{gathered} ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2d\Omega^2, \\ f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2}, \end{gathered} \tag{3}$$

where M is the mass[6] and Q is the charge, and it holds $Q^2>M^2$. The manifold $[t\in\mathbb{R};r\in(0,\infty);\Omega\in S^2]$ cannot be extended.

Because of the spherical symmetry of the problem, we can separate the variables and study a reduced problem on a fixed eigenvalue sector of the angular momentum operator. Since the treatment is standard [10], we limit ourselves to write the reduced radial Hamiltonian and to study its qualitative spectral properties (localization of the essential spectrum).

Some remarks are necessary before we start our analysis.

The manifold we are considering is not globally hyperbolic.[7] This means that, rigorously speaking, it is not available the standard approach to quantum field theory on a given geometrical background. It is anyway somehow tempting to approach the problem by means of the standard tools of quantum theory. For an attempt of rigorous approach to the problem of quantum field theory (QFT) on a nonglobally hyperbolic spacetime see [12].

A further remark is that there is a choice of the physical ''vacuum'' to be made. As it is usual in curved spacetime, the existence of a static Killing vector is not a sufficient condition in order to get the corresponding vacuum as a physical state.[8] The requirement of the physical state to be an Hadamard state is in general considered to be a good crite-

[1]In the case of a pointlike nucleus is required for the atomic number Z to be greater than 137.

[2]It can be thought as obtained from a variable separation process. See also the following sections.

[3]An example is discussed in Sec. IV.

[4]In literature it is also known as Klein's paradox.

[5]There is a remarkable difference with respect to the scalar charged particle case, that is associated with the fact that for strong enough static fields the Klein-Gordon Hamiltonian gets complex eigenvalues, whereas the Dirac hamiltonian spectrum is real.

[6]Here we use natural units $\hbar=c=G=1$. See also the Appendix.

[7]See [1] and especially [11], Chap. 6.

[8]See the section on RN black holes.

rion for selecting physically acceptable states. In the case of a naked singularity, characterized by divergences of curvature invariants at the singularity, it is not clear if it is possible to get an Hadamard state.

In the following, we will assume an heuristic point of view. We will limit ourselves to study the electrodynamical instability problem by choosing a vacuum state associated with the static Killing vector ∂_t. It will result that, given the lack of self-adjointness of the Hamiltonian operator, it is *a priori* possible to define an infinity of candidate vacuum states. Between them, we have no reason to exclude the existence of a (unique?) Hadamard state. But, given our limited interest into an electrodynamical stability problem, it will be not necessary to find out explicitly such a state, as it will result from the next sections.

The reduced Hamiltonian is

$$H_{red}=\begin{bmatrix} \sqrt{f}m-eV & -f\partial_r+k\dfrac{\sqrt{f}}{r} \\ f\partial_r+k\dfrac{\sqrt{f}}{r} & -\sqrt{f}m-eV \end{bmatrix},$$

where

$$f(r)=1-\frac{2M}{r}+\frac{Q^2}{r^2},$$

$$k=\text{angular momentum eigenvalue},$$

$$V(r)=\frac{Q}{r}.$$

It is the one particle Hamiltonian operator projected on angular momentum eigenstates. For more details see [10]. The charge of the RN solution is chosen to be positive $Q>0$.

We want to see if the given reduced Hamiltonian is essentially self-adjoint on $C_0^\infty(0,\infty)^2$ or not, that is, if a boundary condition at $r=0$ has to be imposed. We will use some known theorems about first order ordinary differential equations systems.

We start by defining the following smooth change of variable (tortoiselike)

$$\frac{dx}{dr}=\frac{1}{f(r)},$$

$$x=r+M\log\left(\frac{r^2-2Mr+Q^2}{Q^2}\right)+(2M^2-Q^2)\frac{1}{\sqrt{Q^2-M^2}}\arctan\left(\frac{r-M}{\sqrt{Q^2-M^2}}\right)+C$$

and we choose the arbitrary integration constant C in such a way that $x\in(0,\infty)$. This allows us to write

$$H_{red}=H_0+V(x), \qquad (4)$$

where

$$H_0=\begin{bmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{bmatrix}$$

and

$$V(r(x))=\begin{bmatrix} \sqrt{f}m-eV & +k\dfrac{\sqrt{f}}{r} \\ +k\dfrac{\sqrt{f}}{r} & -\sqrt{f}m-eV \end{bmatrix}.$$

In what follows, we rewrite suitably the reduced hamiltonian in order to get a form allowing us to use theorem 4.16 of [5].[9] Then we rewrite the potential term as follows:

$$V(r(x))=\begin{bmatrix} m+\phi_{sc}(x)+\phi_{el}(x) & +k\dfrac{1}{x}+\phi_{am}(x) \\ +k\dfrac{1}{x}+\phi_{am}(x) & -m-\phi_{sc}(x)+\phi_{el}(x) \end{bmatrix},$$

where

$$\phi_{sc}(x)=(\sqrt{f}-1)m,$$

$$\phi_{el}(x)=-\frac{eQ}{r(x)},$$

$$\phi_{am}(x)=k\left(\frac{\sqrt{f}}{r(x)}-\frac{1}{x}\right).$$

Theorem 4.16 of [5] states that, given an Hamiltonian operator H_{red} as in Eq. (4) and if the potential terms $\phi_{sc}(x),\phi_{el}(x),\phi_{am}(x)$ are locally integrable functions in $(0,\infty)$, then H_{red} is essentially self-adjoint if and only if there exists a $\lambda\in C$ such that the equation

$$H_{red}g=\lambda g \qquad (5)$$

admits a solution $g\notin L^2[(0,R),dx]^2$ for $R>0$, i.e. a solution not square integrable in a right neighbor of $x=0$. The hypothesis of local integrability is clearly satisfied by the potentials given above. About the existence or not of a non square integrable solution for a given λ, the so called Weyl alternative generalized to a system of first order ordinary equations[10] ensures that, if the integrability condition is verified for all the solutions corresponding to a given value of λ, then it is verified for every $\lambda\in C$. The analysis of this topic shows that all the solutions are square integrable in a right neighbor of $x=0$. Indeed, working in the r variable, one gets the following system of first order equations:

$$\partial_r g_1+\frac{k}{\sqrt{f}r}g_1+\left[-\frac{m}{\sqrt{f}}+\frac{1}{f}(-eV-\lambda)\right]g_2=0,$$

[9] It is equivalent to theorems appearing in [13].

[10] See, e.g., [14], theorem 5.6.

$$-\partial_r g_2 + \frac{k}{\sqrt{f}r} g_2 + \left[\frac{m}{\sqrt{f}} + \frac{1}{f}(-eV - \lambda) \right] g_1 = 0.$$

Note that the coefficients appearing in the above equations are regular in the limit $r \to 0$. We get the following asymptotic limit for $r \to 0 \Leftrightarrow x \to 0$ for the eigenvalue equation (5):

$$\partial_r g_1 + \frac{k}{Q} g_1 = O(r),$$

$$\partial_r g_2 - \frac{k}{Q} g_2 = O(r).$$

Near $r = 0$ one can limit to consider a (regular) series expansion of the solution and then locally in a right neighbor of $r = 0$ it holds

$$g_1(r) = \exp\left(-\frac{k}{Q} r \right) (a_1 + O(r)),$$

$$g_2(r) = \exp\left(\frac{k}{Q} r \right) (b_1 + O(r)).$$

So it results that solutions of Eq. (5) belong to $L^2[(0,R), [1/f(r)]\, dr]^2$ for $R > 0$, that is they are square integrable in the x variable. This holds for every choice of λ and of the parameters e, m, Q, M entering the eigenvalue equation (but, obviously, with $Q^2 > M^2$). As a consequence, the reduced Hamiltonian is not essentially self-adjoint, particularly its lack of self-adjointness arises near $r = 0$. According to theorem 5.7 of [14], the deficiency indices of H_{red} are (1,1). We recall that an analogous problem of lack of essential self-adjointness in $r = 0$ was found also for the case of uncharged scalar field in [15].

The spacetime singularity at $r = 0$ is so also related to a nontrivial problem to define a physically meaningful self-adjoint extension of the reduced Hamiltonian (if any).

We stress that there are timelike singularities allowing a well defined quantum evolution for test particles in the sense that the one particle Hamiltonian results to be essentially self-adjoint even in presence of a singular manifold. Such problem is analyzed in [16], where some examples of well-behaved quantum evolution for the case of scalar test particles in presence of timelike singularities are given. The naked RN manifold does not belong to this special class of timelike singularities, because boundary conditions at $r = 0$ are required for scalar test particles [15,16]. Our result in particular shows that the naked RN manifold does not allow a one particle well defined evolution also in the case of a charged spin $\frac{1}{2}$ field minimally coupled to the electromagnetic (external) field of the singularity.

The standard separation of variables we implicitly used in order to get H_{red} allows to write the total Hamiltonian H as it follows:

$$H = \oplus_{j=1/2, 3/2, \ldots}^{\infty} \oplus_{m_j=-j}^{j} \oplus_{k_j=\pm(j+1/2)} h_{m_j,k_j};$$

here h_{m_j,k_j} is H_{red} and the relation between the quantum numbers k_j, m_j and j is shown in the notation.

As a further remark, we can choose to restrict our study to the self-adjoint extensions of the total Hamiltonian that can be obtained by selecting a self-adjoint extension for each h_{m_j,k_j} and we can also impose the same boundary condition for all the h_{m_j,k_j}. From a physical point of view, this means preserving the commutativity of the angular momentum operator with the self-adjoint extension of the total Hamiltonian. Other choices could be allowed but we adopt this physically reasonable strategy.

A comment has to be done about curvature invariants divergences at the singularity. Indeed, they can actually represent a strong drawback for the external field approximation implicitly adopted for the gravitational field: the well known divergences of the curvature invariants suggest that classical general relativity (GR) should fail at the singularity [3].[11] In fact, it lacks a clear criterion allowing to check if a naked RN singularity could be realized as a classical solution of GR. Cf. also the Appendix. It is puzzling that such a possible breakdown of the external field approximation cannot be inferred in a straightforward way from the behavior of the wave equations (for the Dirac equation as well as for the Klein-Gordon one) near the singularity. In order to try to overcome these problems, one could adopt the proposal contained in [3]: From a qualitative point of view, given the repulsive character of the singularity and given the curvature invariants explosion near the singularity, it seems reasonable to impose on the wave function a vanishing boundary condition near $r = 0$. Anyway, the above repulsive character is operative only for geodetic timelike motions and for nonradial null ones. For nongeodetic ones (e.g., for radially infalling charged particles) and for null radial motions it is possible to hit the singularity, so that it is quite unclear which boundary condition could be physically meaningful. Nevertheless, as it will result from the following subsection, our analysis of the electrodynamical stability features for the Dirac equation in the given naked geometry can be considered boundary conditions independent.

A. Essential spectrum and the Klein effect

Essential spectrum features are the same for every self-adjoint extension of the reduced Hamiltonian, because the reduced Hamiltonian has finite and equal deficiency indices. Then the positive and negative continuum energy states (e.g., electron and positron states) can be found even without discussing the highly nontrivial problem of choosing the boundary conditions on the singularity.

We can find the essential spectrum of the reduced Hamiltonian as follows. We use the decomposition method (see [14], chapter 11). Let us split the interval $(0,\infty)$ as $(0,d] \cup [d,\infty)$ and define H_0 and H_+ the restriction of the reduced Hamiltonian to the former and to the latter interval,

[11]This problem is related with the existence problem for Hadamard-like states.

respectively. Then, according to theorem 11.5 of [14], for any $d\in(0,\infty)$ it holds

$$\sigma_e(H_{red})=\sigma_e(H_0)\cup\sigma_e(H_+).$$

We note that the operator H_{red} is regular at $x=d$ and so its restrictions above are regular too. It follows that H_0 is characterized by the limit circle case at both the extremes. Then its self-adjoint extensions have discrete spectrum[12] and $\sigma_e(H_0)=\emptyset$. The essential spectrum of H_{red} can be contributed only by H_+.

According to theorem 16.5 of [14], given that

$$\lim_{x\to\infty}V(x)=\begin{bmatrix} m & 0\\ 0 & -m\end{bmatrix}$$

then for every self-adjoint extension of H_+ it holds $\sigma_e(H_+)\cap(-m,m)=\emptyset$.

Moreover, it is easy to show that the hypothesis of theorem 16.6 of [14] is satisfied, so it also holds $(-m,m)^c=(-\infty,-m]\cup[m,\infty)\subset\sigma_e(H_+)$.[13] The above results allow us to write

$$\sigma_e(H_{red})=(-\infty,-m]\cup[m,\infty).$$

As a final remark, given that the potential $V(x)$ has components of class C^1 (at least) in $[d,\infty)$ for any $d>0$ and given the above value of $\lim_{x\to\infty}V(x)$, it results, according to theorem 16.7 of [14] that every self-adjoint extension of H_+ has purely absolutely continuous spectrum in $(-\infty,-m)\cup(m,\infty)$ and the same is true for $H_{\rm red}$.[14]

We can now discuss the problem of the existence of a Klein region [5,6] according to the ideas exposed in the preceding section. In our case, $A=0$ and $B=\infty$ and, as it results from the above calculation, no such overlap exists.

The static nature of the geometry and of the classical electromagnetic field associated with it allows us to conclude that pair creation probability is zero. Indeed a static external gravitational field cannot give rise to a particle creation process and this holds also for a purely electric static potential as the one we have. We can construct a Fock space based on the spectral decomposition of the one particle reduced Hamiltonian. Then the Dirac Hamiltonian operator generates the time evolution of a linear system and its ground state is stable. In this case we have excluded the possibility of a Klein effect,[15] so we can also reasonably exclude particle creation from an ''eternal'' naked RN singularity.

Concluding this section, we note that, from the point of view of qualitative spectral properties of the reduced Hamiltonian, the behavior in the naked background is analogous to the behavior in the external Coulomb field of a charged (pointlike) nucleus. Indeed, in the latter case one can verify by means of the decomposition method the absence of a continuum spectrum contribution from the region near the source [14]. From this point of view, there is so a strong similarity with a ''standard'' scattering center.

[12] Cf. [14], p. 123.

[13] The notation I^c means the complementary set of the set I.

[14] Cf. also theorem (4.18) in [5].

[15] Cf. [3] for the scalar field case.

IV. RN BLACK HOLES

In this section we deal with the problem of quantum electrodynamical instability in the case of the RN black holes. It is known that it is operative a Klein effect that causes the discharge of the black hole [17,18,6]. The charged scalar field case was extensively studied in [17]. Nevertheless, a nonapproximate study of the spectral properties of the Dirac Hamiltonian is still lacking.[16] The reduced Hamiltonian has the same form as in the naked case but for the fact that now there exist real zeroes $r_+\geqslant r_->0$ of the function $f(r)$ corresponding to the event horizon and to the Cauchy horizon respectively and we consider only the external region $r\in(r_+,\infty)$. As a consequence, the explicit expression of the tortoise coordinate x changes. Indeed, in the non extremal case one gets

$$x=r+\frac{r_+^2}{r_+-r_-}\log\left(\frac{r-r_+}{r_+}\right)-\frac{r_-^2}{r_+-r_-}\log\left(\frac{r-r_-}{r_-}\right)$$

and in the extremal one ($r_+=r_-$)

$$x=r+2r_+\ \log\left(\frac{r-r_+}{r_+}\right)-\frac{r_+^2}{r-r_+}.$$

In both cases, it holds $x\in(-\infty,+\infty)$. This means that the reduced problem is equivalent to a one dimensional problem on the whole real line.

There is a further difference to be taken into account with respect to the naked case. Indeed, the choice of the physical state, i.e., of the positive and negative frequency solutions, cannot trivially be given by the positive and negative frequency solutions associated with the static Killing vector characterizing the geometry. The request of regularity (Hadamard condition) selects the ''Hartle-Hawking'' state [17,19] if an eternal nonextremal black hole is considered.[17] The extremal case is more puzzling because there is no definitive notion of Hadamard state in the case of the extremal geometry and we choose for it the standard (''Boulware'') vacuum in agreement with the lack of a geometrical temperature of the extremal black hole. In the nonextremal case we privilege the ''Hartle-Hawking'' state, that could be obtained also by generating a finite temperature KMS state by means of ''heating up'' at the black hole temperature the ''Boulware'' vacuum, as, e.g., it results from a rigorous approach [20] for

[16] There are in literature some approximate studies due to Soffel *et al.* [10], and the so called constant electric field approximation [18], that allows an explicit computation of the discharge rate.

[17] The ''Unruh'' vacuum is selected if a collapse-generated black hole is taken into account. We will consider only the eternal case for simplicity.

the scalar field case on a Schwarzschild background.[18] The charged field contribution to the corresponding partition function gives rise to a chemical potential associated with the charge [19]. The effect of the vacuum choice affects the charge decay rate in the following sense: a thermal state decay rate is found, such that for very low temperature the ''vacuum one '' is retrieved [17]. This means that, at the level of actual decay rate calculations, a very massive black hole (that is an astrophysical one) will not contribute a sizeable thermal effect on the discharge rate till its mass becomes very small [17].

As a final remark, we recall the well known fact that in the case of nonextremal RN black holes, in order get a finite electromagnetic potential on the event horizon, the choice $A_0 = Q/r$; $A_i = 0$, $i = 1,2,3$ we used in the case of naked RN manifolds is not a good one; a gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, with $\Lambda = Q/r_h = V_h$, is required [19]. Then wave function time dependence gets the shift $\lambda \to \lambda - eV_h$ and the potential in the Hamiltonian becomes $eV \to e(V - V_h)$. We note that the given gauge choice can be used also in the extremal case.

In any case, it is important to study the (would be) ''Boulware'' vacuum, and to verify the existence of a Klein region.

The reduced Hamiltonian operator results to be essentially self-adjoint, being the so called limit point case verified at $-\infty$. This fact implies also the essential self-adjointness of the total Dirac Hamiltonian in the RN black hole case. In order to locate its essential spectrum we use again the decomposition method. We call H_- and H_+ self-adjoint extensions of the reduced Hamiltonian restricted to the intervals $(-\infty,0]$ and $[0,+\infty)$ respectively. Then, according to theorem 11.5 of [14], it holds

$$\sigma_e(H_{red}) = \sigma_e(H_-) \cup \sigma_e(H_+).$$

Then theorems 16.5 and 16.6 of [14] allow to conclude that, in the nonextremal case as well as in the extremal one,

$$\sigma_e(H_+) = (-m + eV_h, m + eV_h)^c,$$

$$\sigma_e(H_-) = \mathbb{R}.$$

We deduce that if $eV_h < m$ then there is no overlap of the asymptotic negative energy states at $+\infty$ and the positive energy states at $-\infty$; if $eV_h > m$ then there is an overlap region of the asymptotic negative energy states at $+\infty$ and the positive energy states at $-\infty$. Note also that $\sigma_e(H_{red}) = \mathbb{R}$ in both cases. This means that, in the latter case, there is a Klein region allowing the predicted discharge process.[19] Defining

$$\alpha \equiv \frac{Q}{M} \leqslant 1,$$

$$\gamma \equiv \frac{e}{m}$$

it is straightforward to see that the Klein effect condition $eV_h > m$ is equivalent to the condition

$$1 + \sqrt{1-\alpha^2} < \alpha\gamma.$$

If $\gamma \leqslant 1$, the above inequality cannot be satisfied and the Klein effect cannot take place. If $\gamma > 1$, then there exists a Klein region for $2\gamma/(1+\gamma^2) < \alpha \leqslant 1$, and no Klein region can exist for $\alpha < 2\gamma/(1+\gamma^2) < 1$. Note that, for $\gamma > 1$, the extremal black hole $\alpha = 1$ is always Klein instable.

If one considers the electron field, then $\gamma \sim 2 \times 10^{21}$ and $2\gamma/(1+\gamma^2) \sim 10^{-21}$, so Klein stable non extremal black holes should be characterized by a charge to mass ratio $\alpha < 10^{-21}$ and this bound is of the same order of the one deduced for the classical stability against classical accretion of oppositely charged dust. The latter is derived by comparing Newtonian force and Coulomb force for the electron [1,21]. Newtonian accretion wins over opposite charge accretion if $Mm \geqslant eQ$, that is for $\alpha \leqslant 1/\gamma$. The bound is lower than the one estimated [1] by considering the proton mass and it is found that classically the upper bound for α in order to get a stable charged black hole is $\alpha \sim 10^{-18}$. Such estimate is lowered by quantum electrodynamical instability considerations [17].

It is interesting to underline that, given the actual values of α necessary for a nonextremal black hole to be out of the Klein discharge region, and given the conditions $Q \gg e$ and $M \gg m$ necessary in order to get a sensible external field approximation, it is impossible to prepare even a Gedanken quantum scattering experiment allowing to transform such a black hole into a naked singularity (with $\alpha > 1$). Indeed, a violation of the external field approximation would be required.

As a concluding remark, we note that the quantum mechanical instability condition $\gamma > 1$ allowing a Klein discharge of RN black holes resembles the classical instability condition [2] $\gamma > 1$ allowing classical charged particles to dress a naked RN singularity by hitting it along classical radial trajectories. In a somehow unclear way, the instability of RN solution is related to the existence of charged particles with a charge to mass ratio greater than 1.

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[18] For the details of the above construction see [20].

[19] In the case one chooses, e.g., in the extremal case the gauge $A_0 = Q/r$ one finds

$$\sigma_e(H_+) = (-m, m)^c,$$
$$\sigma_e(H_-) = \mathbb{R}.$$

Obviously, both the spectrum and the Klein region conditions remain unaltered.

V. CONCLUSIONS

In a naive level discussion about quantum electrodynamical stability of the Dirac vacuum, the naked singularity appears to be characterized by analogous problems with respect to the ones affecting the flat spacetime Dirac equation in the Coulomb field of an highly charged pointlike nucleus ($Z > 137$). Indeed, in both cases there is a lack of self-adjointness, with the difference that in the naked RN manifold, as seen, this lack does not depend on the parameters, i.e., there is no possibility to get a set of parameters allowing an essentially self-adjoint reduced Hamiltonian.

The second analogy is that the static Coulomb potential seems to be not able to generate a quantum electrodynamical instability of the naked RN geometry, no matter how strongly charged the singularity could be. Moreover, there is no room for a Klein discharge mechanism. This means that the possible quantum electrodynamical instability could be then hidden in being the vacuum overcritical: As in flat spacetime Coulomb problem, one could suppose that one or more negative eigenvalues exist and give rise to the so called charged vacuum.[20] The overcriticality in flat spacetime is conjectured to be a condition ensuring the electrodynamical instability if it is coupled with an adiabatic time variation of the (would be static) external potential [5,23]. So, even if one could show that there are some negative eigenvalues for massive Dirac fields on a naked RN background, one should also consider a further problem of defining suitably an adiabatic time dependence of the geometry. Moreover, in order to find out the eigenvalues it is required an explicit definition of the self-adjoint extension of the reduced Hamiltonian. There is a 1-parameter family of possible self-adjoint extensions, and the possibility to select one of them on firm physical grounds lacks.

A possibility to overcome the above problem could be to remove the hypothesis of an ''eternal'' naked RN geometry and to consider a full backreaction problem [24], but we will not pursue it in this paper.

Finally, we cannot exclude that no acceptable physical state exists on a naked RN background (in this case self-consistency of the theory would lack) and that real physical quantum states associated with the RN naked singularity could be very different from ''Boulware'' states we discussed.

As far as the RN black hole case is considered, it is confirmed that a Klein region exists if $\alpha > 10^{-21}$.

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[20]See [5,22] and references therein.

APPENDIX

We recall that M, Q appearing in the expression of $f(r)$ are actually given by

$$M \to \frac{G}{c^2} M \equiv M^* = \frac{l_{pl}}{m_{pl}} M,$$

$$Q \to \sqrt{\frac{G}{c^4}} Q \equiv Q^* = \sqrt{\frac{l_{pl}}{m_{pl} c^2}} Q,$$

where the electrical units are unrationalized. Here, we will indicate with M^*, Q^* the lengths associated with the mass M and the charge Q respectively as in the above formula, in order not to use the same graphical symbols (in the paper we avoided this explicit distinction in order to simplify the notation). Given that $l_{pl} m_{pl} c^2 = \hbar c$ and recalling the definition of the fine structure constant $\alpha_e = e^2/\hbar c$ we can also write

$$\frac{G}{c^2} M = l_{pl} \frac{M}{m_{pl}},$$

$$\sqrt{\frac{G}{c^4}} Q = l_{pl} \sqrt{\alpha_e} \frac{Q}{e}.$$

We can also pose $Q = Ze$, as usual in atomic physics. We get

$$\frac{Q^*}{M^*} = \frac{\sqrt{\alpha_e}(Q/e)}{M/m_{pl}} = \sqrt{\alpha_e} Z \frac{m_{pl}}{M}.$$

In the case of the electron, one gets

$$e^* = 8.54 \times 10^{-2} l_{pl},$$

$$m^* = 4.18 \times 10^{-23} l_{pl}.$$

Note that the extremal black hole with $Q = e$ should get an horizon radius $r_{bh} = e^* \sim 0.1 l_{pl} < l_{pl}$, so it cannot be considered a reasonable solution of General Relativity. As a criterion for the validity of a solution of General Relativity can be assumed the following one [25]

$$\lambda_{\text{compton}} \ll R_{\text{Schwarzschild}}, \qquad \text{(A1)}$$

where $\lambda_{\text{compton}}, R_{\text{Schwarzschild}}$ are respectively the compton wavelength and the Schwarzschild radius of the solution. Clearly, such inequality is not satisfied by an extremal RN black hole with $Q = e$, that appears to be in a full quantum gravity regime.

Note that the above criterion (A1) does not seem to be useful in the case of a naked RN solution; indeed, it has no meaningful Schwarzschild radius, and it lacks a natural geometric scale for this kind of solution of GR.

We limit ourselves to note that one can naively define the analogous of the classical electron radius in the case of the naked singularity, that is, one introduces a classical length scale

$$r_{\text{classical}}=\frac{Q^{*2}}{M^{*}},$$

such that the Coulomb field energy valued at $r=r_{\text{classical}}$ equates exactly the singularity mass M. Below this value, the effective mass of the given gravitational source becomes negative and this fact explains the repulsive character of the singularity [2]. Then one could impose

$$\frac{\lambda_{\text{compton}}}{2\pi}=\frac{l_{pl}m_{pl}}{M}=\frac{l_{pl}^{2}}{M^{*}}\ll r_{\text{classical}}=\frac{Q^{*2}}{M^{*}}$$

that is satisfied as far as $Q^{*2}\gg l_{pl}^{2}$. Note that this inequality is not satisfied by the electrons.

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