Initial singularity free quantum cosmology in two-dimensional Brans-Dicke theory

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We consider two-dimensional Brans-Dicke theory to study the initial singularity problem. It turns out that the initial curvature singularity can be finite for a certain Brans-Dicke constant ω by considering the quantum back reaction of the geometry. For $\omega=1$, the universe starts with the finite curvature scalar and evolves into flat spacetime. Furthermore the divergent gravitational coupling at initial time can be finite effectively with the help of quantum correction. The other type of universe is studied for the case of $0<\omega<1$. [S0556-2821(98)02218-8]

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I. INTRODUCTION

The classical solution of general relativity may yield a curvature singularity in black hole physics and cosmology. This divergent physical quantity seems to be mild with the help of quantum gravity. However, a consistent quantum theory of gravity has not been established. The perturbation theory of Einstein gravity is invalid in higher loops. As far as a consistent and renormalizable quantum gravity does not appear, it seems to be difficult to resolve the singularity problem.

On the other hand, in two dimensions there exists some renormalizable gravity such as the Callan-Giddings–Harvey-Strominger (CGHS) model [1] and the Russo-Susskind-Thorlacius (RST) model [2] including soluble one-loop quantum effect. And various aspects of quantum cosmology in two-dimensional gravity have been studied in Refs. [3–5,12]. The two-dimensional Brans-Dicke theory is also a good candidate to study the initial singularity problem on the basis of conventional quantum field theory without encountering the four-dimensional complexity. If the model gives the initial singularity classically, it would be interesting to study how to modify the classical singularity through the quantum back reaction of the geometry. If quantum corrections drastically modify the classical theory and the curvature singularity does not appear, then the initial curvature singularity may be ascribed to the classical concept.

In this paper, we shall study the classical curvature singularity of expanding universe in the two-dimensional Brans-Dicke model which exhibits the initial singularity of curvature scalar and the divergent coupling in Sec. II. This initial singularity can be shown to be finite for some special value of Brans-Dicke constant by considering the quantum back reaction of the geometry in Sec. III. For $\omega=1$ the model exhibits the finite curvature scalar at initial comoving time whereas for $0<\omega<1$ the curvature is bounded and the extremum exists at finite comoving time. And the gravitational coupling in both cases becomes finite. If $\omega>1$, the curvature scalar is not bounded. Finally a discussion is given in Sec. IV.

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II. CLASSICAL BRANS-DICKE THEORY

We start with two-dimensional Brans-Dicke action

$$S_{\rm BD}=\frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\phi}[R-4\omega(\nabla\phi)^2], \tag{1}$$

where R and ϕ are curvature scalar and redefined Brans-Dicke scalar field, respectively, and ω is an arbitrary positive constant. We write down $\psi=e^{-2\phi}$, then the conventional Brans-Dicke form is recovered. In our model, the large ω limit does not give the locally nontrivial gravity in contrast to the four-dimensional Brans-Dicke theory since in two dimensions the Einstein-Hilbert action is proportional to the Euler characteristic.

We now consider the classical conformal matter fields given by

$$S_{\rm Cl}=-\frac{1}{2\pi}\int d^2x\sqrt{-g}\,\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2, \tag{2}$$

where $i=1,2,\dots,N$ and N is the number of conformal matter fields. Then the actions (1) and (2) lead to classical equations of motion with respect to $g_{\mu\nu}$ and ϕ,

$$G_{\mu\nu}=T^{\rm Cl}_{\mu\nu}, \tag{3}$$

$$R+4\omega(\nabla\phi)^2-4\omega\Box\phi=0, \tag{4}$$

where

$$\begin{aligned}G_{\mu\nu}&=\frac{2\pi}{\sqrt{-g}}\frac{\delta S_{\rm BD}}{\delta g^{\mu\nu}}\\&=2e^{-2\phi}\{\nabla_\mu\nabla_\nu\phi-2(1+\omega)\nabla_\mu\phi\nabla_\nu\phi\\&\quad+g_{\mu\nu}[(2+\omega)(\nabla\phi)^2-\Box\phi]\}\end{aligned} \tag{5}$$

and classical energy-momentum tensor $T^{\rm Cl}_{\mu\nu}$ is set to zero for simplicity.

We now choose a conformal gauge such as $g_{\pm\mp}=-\frac{1}{2}e^{2\rho}$ and $g_{\pm\pm}=0$. From Eq. (5), we obtain the conformal gauge fixed forms

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$$G_{\pm\pm}=2e^{-2\phi}[\partial_\pm^2\phi-2(1+\omega)(\partial_\pm\phi)^2-2\partial_\pm\rho\partial_\pm\phi], \tag{6}$$

$$G_{+-}=2e^{-2\phi}[2\partial_+\phi\partial_-\phi-\partial_+\partial_-\phi], \tag{7}$$

and the equation of motion for ϕ is given by

$$\partial_+\partial_-\rho-2\omega\partial_+\phi\partial_-\phi+2\omega\partial_+\partial_-\phi=0, \tag{8}$$

where $x^\pm=t\pm x$. The classical solutions in the homogeneous space are given by

$$\rho=-\omega\phi, \tag{9}$$

$$e^{-2\phi}=Mt, \tag{10}$$

where M is an integration constant and we take M as a positive constant to obtain positive time t. The other two integration constants are chosen to be zero since they have no important role in our case. On the other hand, in the comoving coordinates defined by $ds^2=-d\tau^2+a^2(\tau)dx^2$, the curvature scalar $R(\tau)$ and scale factor $a(\tau)$ are written as

$$R(\tau)=-\frac{4\omega}{(2+\omega)^2}\frac{1}{\tau^2}, \tag{11}$$

$$a(\tau)=\left[\left(1+\frac{\omega}{2}\right)M\tau\right]^{\omega/(2+\omega)}, \tag{12}$$

where $M\tau=2/(2+\omega)(Mt)^{1+(\omega/2)}$ ($\tau>0$ and $t>0$). It is clear that the curvature scalar has an initial singularity with a power-law inflation. Since the behaviors of the scale factor are $\dot{a}(\tau)\sim\tau^{-2/(2+\omega)}>0$ and $\ddot{a}(\tau)\sim-\tau^{-(4+\omega)/(2+\omega)}<0$, the universe shows decelerating expansion. It is natural to obtain the negative curvature scalar in Eq. (11) corresponding to the desirable decelerating universe since in two dimensions R is directly proportional to $\ddot{a}(\tau)/a(\tau)$.

Note that the gravitational coupling can be defined by

$$g_N^2=e^{2\phi} \tag{13}$$

and from Eqs. (9) and (12), it yields

$$g_N^2=\left[\left(1+\frac{\omega}{2}\right)M\tau\right]^{-2/(2+\omega)}. \tag{14}$$

This shows that the divergent coupling at the initial time decreases and goes to zero in the asymptotically flat spacetime.

In the next section, we study whether the initial curvature singularity and the divergent coupling can be modified in the quantized theory or not. We hope the resulting quantum theory gives the singularity free cosmology with finite coupling.

III. QUANTUM BACK REACTION

We consider the one-loop effective action from the conformal matter fields which is given by

$$S_{Qt}=\frac{\kappa}{2\pi}\int d^2x\sqrt{-g}\left[-\frac{1}{4}R\frac{1}{\Box}R-\omega(\nabla\phi)^2-\frac{1-\omega}{2}\phi R\right], \tag{15}$$

where $\kappa=\hbar(N-24)/12$. The first term in Eq. (15) is due to the induced gravity from the conformal matter fields. The other two terms are regarded as local regularization ambiguities of conformal anomaly. A similar effective action was already treated for the purpose of studying the graceful exit problem of string cosmology in the large N limit [6,7]. The higher order of quantum correction beyond one loop is negligible in the large N approximation where $N\to\infty$ and $\hbar\to 0$ so that κ is assumed to be positive finite quantity.

By introducing an auxiliary field ψ, Eq. (15) may be written in the local form

$$S_{Qt}=\frac{\kappa}{2\pi}\int d^2x\sqrt{-g}\left[\frac{1}{4}R\psi-\frac{1}{16}(\nabla\psi)^2-\omega(\nabla\phi)^2-\frac{1-\omega}{2}\phi R\right]. \tag{16}$$

Now the total effective action is defined by

$$S_T=S_{\rm BD}+S_M, \tag{17}$$

where the matter part is formally composed of two pieces of $S_M=S_{Cl}+S_{Qt}$, however, we neglect the conformal matter fields as in the classical cosmology for simplicity. The equations of motion with respect to $g_{\mu\nu}$, ϕ, and ψ from the action (17) are

$$G_{\mu\nu}=T^M_{\mu\nu}, \tag{18}$$

$$e^{-2\phi}[R-4\omega\Box\phi+4\omega(\nabla\phi)^2]=-\frac{\kappa}{4}(1-\omega)R+\omega\kappa\Box\phi, \tag{19}$$

$$\Box\psi=-2R, \tag{20}$$

where $T^M_{\mu\nu}$ is the energy-momentum tensor, which is given by

$$\begin{aligned}T^M_{\mu\nu}&=-\frac{2\pi}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\\&=\frac{\kappa}{4}\left[\nabla_\mu\nabla_\nu\psi+\frac{1}{4}\nabla_\mu\psi\nabla_\nu\psi-g_{\mu\nu}\left(\Box\psi+\frac{1}{8}(\nabla\psi)^2\right)\right]\\&\quad-\frac{\kappa}{2}(1-\omega)[\nabla_\mu\nabla_\nu\phi-g_{\mu\nu}\Box\phi]\\&\quad+\kappa\omega\left[\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right].\end{aligned} \tag{21}$$

In conformal gauge fixing, the energy-momentum tensor is written as

$$T^M_{\pm\pm}=\kappa[\partial^2_\pm\rho-(\partial_\pm\rho)^2]-\frac{\kappa}{2}(1-\omega)[\partial^2_\pm\phi-2\partial_\pm\rho\partial_\pm\phi]+\kappa\omega(\partial_\pm\phi)^2-\kappa t_\pm, \tag{22}$$

$$T^M_{+-}=-\kappa\partial_+\partial_-\rho+\frac{\kappa}{2}(1-\omega)\partial_+\partial_-\phi, \tag{23}$$

where $t_\pm$ arises from elimination of auxiliary field and reflects the nonlocality of induced gravity. Defining new fields as [7–9]

$$\Omega=-\frac{\kappa}{2}(1+\omega)\phi+e^{-2\phi}, \tag{24}$$

$$\chi=\kappa\rho-\frac{\kappa}{2}(1-\omega)\phi+e^{-2\phi}, \tag{25}$$

the equations of motion (18)–(20) are obtained in the simple form

$$\partial_+\partial_-\Omega=0, \tag{26}$$

$$\partial_+\partial_-\chi=0, \tag{27}$$

and the constraints are given by

$$G_{\pm\pm}-T^M_{\pm\pm}=-\frac{1}{\kappa}(\partial_\pm\Omega)^2+\frac{1}{\kappa}(\partial_\pm\chi)^2-\partial^2_\pm\chi+\kappa t_\pm=0. \tag{28}$$

In the homogeneous spacetime, general solutions are

$$\chi=\chi_0t+A, \tag{29}$$

$$\Omega=\Omega_0t+B, \tag{30}$$

where Ω_0, χ_0, A, and B are constants. Choosing the quantum matter state as a vacuum [10], $t_\pm=0$, the constraint equation (28) results in $\Omega_0=\pm\chi_0$. Hereafter we consider only the case of $\Omega_0=+\chi_0\equiv M$ corresponding to the classical solution $(\rho=-\omega\phi)$ for $\kappa\to0$. Also we can take $A=B=0$ without loss of physical result. From the definitions of χ and Ω in Eqs. (24) and (25) and the general solutions of Eqs. (29) and (30), we obtain the following closed forms:

$$e^{(2/\omega)\rho}+\frac{1}{2\omega}(1+\omega)\kappa\rho=Mt, \tag{31}$$

$$e^{-2\phi}-\frac{1}{2}(1+\omega)\kappa\phi=Mt. \tag{32}$$

To study how the universe evolves as time goes on, we redefine time t as a comoving time τ defined by $\tau=\int^t dte^{\rho(t)}$, and then the metric can be expressed as $ds^2=-e^{\rho(t)}(dt^2-dx^2)=-d\tau^2+a^2(\tau)dx^2$, where $a(\tau)$ is a scale factor. In the case of $\tau\to+0$ $(t\to-\infty)$, the behavior of scale factor is approximately given by

$$a(\tau)\approx\frac{2\omega M}{\kappa(1+\omega)}\tau\to+0, \tag{33}$$

$$\dot a(\tau)\approx\frac{2\omega M}{\kappa(1+\omega)}>0, \tag{34}$$

$$\ddot a(\tau)\approx-\frac{\omega}{(1+\omega)^3}\frac{32}{\kappa^3}M^2\left(\frac{2\omega M}{\kappa(1+\omega)}\tau\right)^{(2-\omega)/\omega}<0, \tag{35}$$

where $\tau\approx[(1+\omega)\kappa/2\omega M]e^{[2\omega M/(1+\omega)\kappa]t}$. Hence in initial stage of inflation, the size of the universe approaches zero and the universe exhibits the decelerating expansion. And the asymptotic behavior of scale factor for $\tau\to+\infty$ (i.e., $t\to+\infty$) is given by

$$a(\tau)\approx\left[\left(1+\frac{\omega}{2}\right)M\tau\right]^{\omega/(2+\omega)}\to+\infty, \tag{36}$$

$$\dot a(\tau)\approx\frac{\omega}{2}M\left\{\left[\left(1+\frac{\omega}{2}\right)M\tau\right]^{2/(2+\omega)}+\frac{\kappa}{4}(1+\omega)\right\}^{-1}\to+0, \tag{37}$$

$$\ddot a(\tau)\approx-\frac{\omega}{2}M^2\left[\left(1+\frac{\omega}{2}\right)M\tau\right]^{-(4+\omega)/(2+\omega)}\to-0, \tag{38}$$

where $\tau\approx2/M(2+\omega)(Mt)^{(2+\omega)/2}$. This shows that the spacetime exhibits the decelerating expansion for $0<\tau<\infty$.

The exact expression of curvature scalar is written in the form

$$R(\tau)=-\omega M^2a^{2/\omega(1-\omega)}(\tau)\left[a^{2/\omega}(\tau)+\frac{\kappa}{4}(1+\omega)\right]^{-3}. \tag{39}$$

Note that the condition for boundedness of the scalar curvature is given by the range of $0<\omega\leqslant1$ as is easily seen from Eq. (39). Therefore we restrict the Brans-Dicke constant as $0<\omega\leqslant1$ to avoid curvature singularity. The initial curvature scalar approaches asymptotically zero for $0<\omega<1$. It is finite $R(0)=-8M^2/\kappa^3$ for $\omega=1$ (see Figs. 1 and 2).

In fact, we can obtain the closed form of scale factor. By differentiating Eq. (31) and transforming coordinates from (t,x) to (τ,x), we find the equation of scale factor $a(\tau)$

$$\dot a(\tau)=\frac{\omega}{2}M\left[a^{2/\omega}(\tau)+\frac{\kappa}{4}(1+\omega)\right]^{-1}. \tag{40}$$

Through the integration of Eq. (40), this leads to the closed form

$$[a(\tau)]^{1+2/\omega}+\frac{\kappa}{4}(1+\omega)\left(1+\frac{2}{\omega}\right)a(\tau)=\left(1+\frac{\omega}{2}\right)M\tau \tag{41}$$

with the initial condition $a(0)=0$. It is difficult to generically show the behavior of the scale factor, so the scale factor

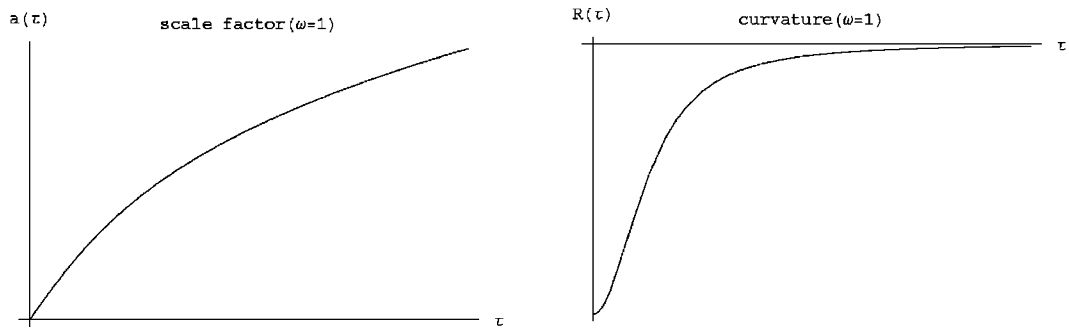

FIG. 1. For $\omega=1$ the curvature scalar in the beginning of inflation approaches a constant value and spacetime is flat in the far future.

and curvature scalar are depicted for the special values $\omega=1$ and $\omega=\frac{2}{3}$ in Figs. 1 and 2.

As for the gravitational coupling, it is easily seen from the string theoretic point of view by using the nonlinear σ model in Ref. [11]. The total action (17) can be written in the form

$$S_{\mathrm{T}}=\int d^2\sigma G_{ij}(X)\partial X^i\partial X^j, \tag{42}$$

where the target metric is given by

$$G_{ij}=\begin{pmatrix} 2\omega\left(e^{-2\phi}+\frac{\kappa}{4}\right) & e^{-2\phi}+\frac{\kappa}{4}(1-\omega) \\ e^{-2\phi}+\frac{\kappa}{4}(1-\omega) & -\frac{\kappa}{2} \end{pmatrix} \tag{43}$$

and the target coordinate is $X^i=(\phi,\rho)$. Therefore we obtain the effective coupling g_{eff} as

$$g_{\mathrm{eff}}^2=\frac{g_N^2}{1+\frac{\kappa}{4}(1+\omega)g_N^2}. \tag{44}$$

In classical theory such as $\kappa=0$ ($g_{\mathrm{eff}}=g_N$), the coupling diverges at the initial time of inflation as seen from Eq. (14) and approaches zero in the far future. However, after taking into account the quantum back reaction, the coupling is interestingly given by the finite value $4/\kappa(1+\omega)$ in the beginning of inflation and decreases monotonically during the expansion of the universe.

Next we study what is the source of the decelerating expansion of universe. If we take the quantum mechanically induced matter as a perfect fluid, then we can realize the induced pressure is directly related to the curvature scalar and the induced energy is always zero from the constraint equation. To show this fact, Eqs. (5) and (21) are written in the comoving coordinate as

$$G_{\tau\tau}=T^M_{\tau\tau}=0, \tag{45}$$

$$G_{xx}=-\frac{2}{\omega}[a(\tau)]^{2+(2/\omega)}\left[\frac{\ddot{a}(\tau)}{a(\tau)}+\frac{2}{\omega}\left(\frac{\dot{a}(\tau)}{a(\tau)}\right)^2\right], \tag{46}$$

$$T^M_{xx}=\frac{\kappa}{2\omega}(1+\omega)a(\tau)\ddot{a}(\tau) \tag{47}$$

by using the solution $\phi(\tau)=-(1/\omega)\ln a(\tau)$. Then we obtain

$$T^M_{xx}=-\frac{\kappa}{4}(1+\omega)M^2a^{2/\omega}(\tau)\left[a^{2/\omega}(\tau)+\frac{\kappa}{4}(1+\omega)\right]^{-3} \tag{48}$$

after eliminating $\ddot{a}(\tau)$ by using Eq. (40). The pressure for the perfect fluid becomes

$$\begin{aligned} p&\equiv\frac{1}{a^2}T^M_{xx} \\ &=\frac{\kappa}{4}\left(1+\frac{1}{\omega}\right)R, \end{aligned} \tag{49}$$

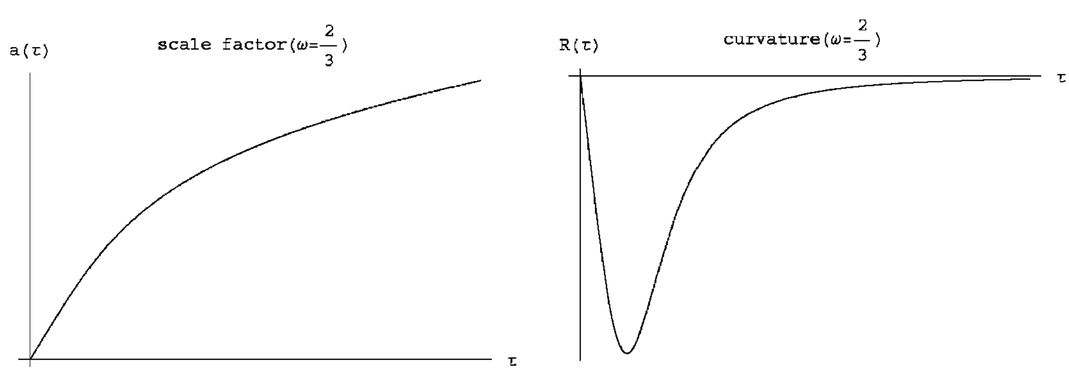

FIG. 2. For $\omega=\frac{2}{3}$ the curvature scalar in the beginning of inflation approaches zero and spacetime is flat in the far future.

where we used Eq. (39). Note that the curvature scalar which characterizes the geometry in two dimensions is of relevance to the pressure. This is a dynamical equation of motion while Eq. (45) is just a constraint equation in the comoving coordinate. Therefore, the source of dynamical evolution of the geometry is determined by the pressure. In our model, the induced energy always vanishes.

IV. DISCUSSION

The curvature scalar is defined by $R=2\ddot{a}(\tau)/a(\tau)$ in two-dimensional homogeneous space. In our model, the scale factor $a(\tau)$ vanishes at $\tau\to+0$. Therefore, one might wonder why the curvature scalar is finite for $(\omega=1)$ or zero for $(0<\omega\leqslant1)$ at $\tau\to+0$. If the scale factor is expanded as $a(\tau)=a_1\tau+\frac{1}{2}a_2\tau^2+O(\tau^3)$, then the scalar curvature diverges unless $a_2=0$. So the finiteness of curvature requires the absence of the order of τ^2 in the asymptotic expansion of $a(\tau)$ around $\tau=0$. Let us exhibit two special cases of $\omega=1$ and $\omega=\frac{2}{3}$ for simplicity. Here the scale factors around the initial comoving time are expanded as

$$a(\tau)=\frac{M}{\kappa}\tau-\frac{2M^3}{3\kappa^4}\tau^3+O(\tau^4)\quad \text{if}\quad \omega=1, \tag{50}$$

$$a(\tau)=\frac{4M}{5\kappa}\tau-\frac{3(4M)^4}{(5\kappa)^5}\tau^4+O(\tau^5)\quad \text{if}\quad \omega=\tfrac{2}{3}. \tag{51}$$

This is the reason why the curvature scalar is finite or zero near the origin although the scale factor goes to zero.

In summary, we have studied the curvature singularity problem using the conventional quantum field theory in the two-dimensional Brans-Dicke cosmology and obtained the bounded curvature scalar and the finite gravitational coupling for $0<\omega\leqslant1$. We hope that the consistent quantum gravity may solve the singularity problems in realistic cases in the future.

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