Semiclassical gravitation and quantization for the Bianchi type-I universe with large anisotropy

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We use a perturbative method to evaluate the effective action of a free scalar field propagating in Bianchi type-I spacetime with large space anisotropy. The zeta-function regularization method is used to evaluate the action to second order in the Schwinger perturbative formula. As the quantum corrections contain a fourth derivative in the metric we apply the method of iterative reduction to reduce it to second-order form to obtain a self-consistent solution of the semiclassical gravity theory. The reduced Einstein equation shows that the space anisotropy, which will be smoothed out during the evolution of universe, may play an important role in the dynamics of early universe. We quantize the corresponding minisuperspace model to investigate the behavior of the space anisotropy in the initial epoch. From the wave function of the Wheeler-DeWitt equation we see that the probability for the Bianchi type-I spacetime with large anisotropy is less than that with small anisotropy. [S0556-2821(98)06418-2]

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I. INTRODUCTION

The effective action plays an important role in investigating the theory of quantum fields in curved spacetime. In the conformally flat spacetime the effective action can be completely determined by the local geometry [1]. If the spacetime is not conformally flat then the effective action could not be evaluated exactly and one can only evaluate it in perturbation. Many years ago, Hartle and Hu [2] used the dimensional regularization method to expand the effective action of a scalar field propagating in Bianchi type-I spacetime to the second order of space anisotropy. From the effective action they investigated the problem of dissipating space anisotropy by quantum field effects [3]. This method has also been used to investigate the quantum field in an inhomogeneous spacetime [4] and a cosmic string [5]. However, as these results are restricted to a spacetime with small anisotropy it is of value to investigate the case with large anisotropy.

In this paper a simple prescription for the expansion of the effective action in the Bianchi type-I universe with large anisotropy is described. We use the zeta-function regularization method to evaluate the renormalized effective action to second order in the Schwinger perturbative formula [6]. As the quantum corrections contain up to a fourth derivative in the metric the associated Einstein equation will suffer the problems of changing the Hamiltonian structure, lacking stability, and unphysical solutions appearing, etc. [7–9]. Therefore, we apply the method of iterative reduction to reduce it to a second-order equation [9]. We then investigate the reduced equation and see that the space anisotropy, which will be smoothed out during the evolution of universe, may play an important role in the early universe. To see the behavior of the space anisotropy in the initial epoch we quantize the corresponding minisuperspace model. We analyze the Wheeler-DeWitt equation and see that probability for the Bianchi type-I spacetime with large anisotropy is less than that with small anisotropy.

This paper is organized as follows. In Sec. II a simple prescription to expand effective action in the Bianchi type-I universe with large anisotropy is described. It then uses the zeta-function regularization method to evaluate the effective action to second order in the Schwinger perturbative formula. In Sec. III the method of iterative reduction is used to reduce the action to the second-order form. The solution of the reduced Einstein equation is then analyzed. In Sec. IV we quantize the reduced action and analyze the associated Wheeler-DeWitt equation. The last section is devoted to a short summary.

II. EXPANSION OF EFFECTIVE ACTION WITH LARGE ANISOTROPY

A. Method

We consider the Lagrangian describing a massless scalar field conformally coupling to the gravitational background:

$$L=-\frac{1}{2}\,g^{\mu\nu}\partial_\mu\Phi\,\partial_\nu\Phi-\frac{1}{12}\,R\Phi^2, \tag{2.1}$$

where R is the curvature scalar. We will calculate the renormalized effective action in the Bianchi type-I spacetime with the line element [10]

$$d^2s=g_{\mu\nu}dx^\mu dx^\nu=-dt^2+\Sigma_i e^{2\beta_i}dx_i{}^2, \tag{2.2}$$

where

$$\beta_1\equiv\beta_++\sqrt{3}\beta_-,\quad \beta_2\equiv\beta_+-\sqrt{3}\beta_-,\quad \beta_3\equiv-2\beta_+. \tag{2.3}$$

The effective action in the metric $\tilde{g}_{\mu\nu}=a(t)^2g_{\mu\nu}$ can be found through a conformal transformation formula, which will be described in Sec. II C.

The Hamiltonian associated with the Lagrangian described in Eq. (2.1) can be written as

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$$H = H_0 + D + (Q - Q_0), \tag{2.4}$$

where

$$H_0 \equiv \partial_t^{\,2} - \Sigma_i \partial_{x_i}^{\;2} + Q_0, \tag{2.5}$$

$$D \equiv \Sigma_j D_j \partial_{x_j}^{\;2} = \Sigma_j (1 - e^{-2\beta_j}) \partial_{x_i}^{\;2}, \tag{2.6}$$

and

$$Q \equiv \dot{\beta}_+^2 + \dot{\beta}_-^2 \tag{2.7}$$

denotes the quantity of the space anisotropy.

In this paper we will investigate the back-reaction effect only for the universe near the initial epoch. If we denote a large space anisotropy at initial time $t=0$ by Q_0 then the quantity $(Q - Q_0)$ in Eq. (2.4) will be very small if the universe is sufficiently near the initial epoch. From Eqs. (3.11) and (3.13) we see that for the universe near the initial epoch the anisotropy Q_0 can be very large but the function β_j and the quantity D_j are very small. Thus the effective action can be evaluated in perturbation by regarding D_j and $(Q - Q_0)$ as the small quantities. The fact that D_j is small is very crucial for the perturbation expansion used in this paper. In short, the limitation of the computation in this paper is that it is valid only for a very short time; indeed, for a parametrically short time. It is in this limitation that the β_j function (and thus D_j) are small and the derivative of β_j functions (and thus the anisotropy Q_0) are large at the initial time. Thus we can regard D_j and $(Q - Q_0)$ as the small quantities and the effective action can be evaluated in perturbation.

B. Calculations

We will evaluate the renormalized effective action by the ζ-function regularization [1,11] method:

$$W = -\frac{i}{2} \ln[\mathrm{Det}(H)] = -\frac{i}{2} [\zeta'(0) + \zeta(0) \ln \mu^2]. \tag{2.8}$$

In the proper-time formalism, the ζ function can be found from the relation

$$\zeta(\nu) = [\Gamma(\nu)]^{-1} \int d^4x [-g(x)]^{1/2} \times \int_0^\infty i ds (is)^{\nu-1} \langle x | e^{isH} | x \rangle, \tag{2.9}$$

where the operator H is defined in Eq. (2.4) and it is understood that $H \to H - i\varepsilon$, with ε a small positive quantity. As the quantities $(Q - Q_0)$ and D are small the ζ function can be calculated by the Schwinger perturbative formula [6]

$$\begin{aligned} \mathrm{Tr}\, e^{-isH} = \mathrm{Tr}\Bigg[& e^{-isH_0} - is\, e^{-isH_0}(Q - Q_0) - is\, e^{-isH_0} D + \frac{s^2}{2} \int_0^1 du\, e^{-is(1-u)H_0}(Q - Q_0) e^{-isuH_0}(Q - Q_0) \\ & + \frac{s^2}{2} \int_0^1 du\, e^{-is(1-u)H_0}(Q - Q_0) e^{-isuH_0} D + \frac{s^2}{2} \int_0^1 du\, e^{-is(1-u)H_0} D e^{-isuH_0}(Q - Q_0) \\ & + \frac{s^2}{2} \int_0^1 du\, e^{-is(1-u)H_0} D e^{-isuH_0} D + \cdots \Bigg]. \end{aligned} \tag{2.10}$$

Then the ζ function can be expressed as

$$\zeta(\nu) = \zeta_0(\nu) + \zeta_Q(\nu) + \zeta_D(\nu) + \zeta_{QQ}(\nu) + \zeta_{QD}(\nu) + \zeta_{DQ}(\nu) + \zeta_{DD}(\nu) + \cdots, \tag{2.11}$$

where $\zeta_i(\nu)$ and $\zeta_{ij}(\nu)$ are defined and calculated below.

We first calculate $\zeta_0(\nu)$:

$$\begin{aligned} \zeta_0(\nu) & \equiv [\Gamma(\nu)]^{-1} \int d^4x [-g(x)]^{1/2} \int_0^\infty i ds (is)^{\nu-1} \langle x | e^{-isH_0} | x \rangle \\ & = i [\Gamma(\nu)]^{-1} \int d^4x \int \frac{d^4p}{(2\pi)^4} \int_0^\infty ds s^{\nu-1} e^{-s(p^2 + Q_0)} \\ & = i \int d^4x \int \frac{d^4p}{(2\pi)^4} (p^2 + Q_0)^{-\nu} \\ & = i (32\pi^2)^{-1} \int d^4x Q_0^{\,2} \left[1 + \nu \left(\frac{3}{2} - \ln Q_0 \right) + O(\nu^2) \right]. \end{aligned} \tag{2.12}$$

To obtain the above result we have carried the integration by letting $s \to is$ and rotating p_0 through $\pi/2$ in the complex plane. This procedure will also be used in the following.

Next, we calculate $\zeta_Q(\nu)$:

$$\zeta_Q(\nu)\equiv[\Gamma(\nu)]^{-1}\int d^4x[-g(x)]^{1/2}\int ids(is)^{\nu-1}(-is)\langle x|e^{-isH_0}(Q-Q_0)|x\rangle$$
$$=-i[\Gamma(\nu)]^{-1}\int d^4x(Q-Q_0)\int \frac{d^4p}{(2\pi)^4}\int_0^\infty ds s^\nu e^{-s(p^2+Q_0)}$$
$$=i(16\pi^2)^{-1}\int d^4x(Q-Q_0)Q_0[1+\nu(1-\ln Q_0)+O(\nu^2)]. \tag{2.13}$$

In the same way we have

$$\zeta_D(\nu)\equiv[\Gamma(\nu)]^{-1}\int d^4x[-g(x)]^{1/2}\int ids(is)^{\nu-1}(-is)\langle x|e^{-isH_0}D|x\rangle$$
$$=i(64\pi^2)^{-1}\int d^4x(\Sigma_jD_j)Q_0{}^2\left[1+\nu\left(\frac{3}{2}-\ln Q_0\right)+O(\nu^2)\right]. \tag{2.14}$$

We now turn to evaluate the functions $\zeta_{ij}(\nu)$ which involves more calculations. From the definition we have

$$\zeta_{QQ}(\nu)\equiv[\Gamma(\nu)]^{-1}\int d^4x[-g(x)]^{1/2}\int ids(is)^{\nu-1}\left(-\frac{s^2}{2}\right)\int_0^1 du\langle x|e^{-is(1-u)H_0}(Q-Q_0)e^{-isuH_0}(Q-Q_0)|x\rangle$$
$$=-[2\Gamma(\nu)]^{-1}\int d^4x[Q(x_0)-Q_0]\int d^4y[Q(y_0)-Q_0]\int \frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\int_0^\infty ds(s)^{\nu+1}$$
$$\times\int_0^1 du\; e^{-\{s(1-u)[k^2+Q_0(x_0)]+su[p^2Q_0(y_0)]\}}e^{i(p-k)\cdot(x-y)},$$

in which we have inserted the complete set $\Sigma_k|k\rangle\langle k|$ or $\Sigma_p|p\rangle\langle p|$ before the operator H_0. After integrating the variables u, s, $\mathbf{y}$, $\mathbf{p}$, and $\mathbf{k}$, then shifting $k_0\rightarrow k_0+p_0$ and performing the p_0, k_0 and y_0 integrations we have the final result:

$$\zeta_{QQ}(\nu)=-i\,\frac{\pi}{4}\int d^4x(Q-Q_0)^2[1-\nu\ln Q_0+O(Q_0^{-1})+O(\nu^2)]. \tag{2.15}$$

Through the same manipulation we have

$$\zeta_{QD}(\nu)\equiv[\Gamma(\nu)]^{-1}\int d^4x[-g(x)]^{1/2}\int ids(is)^{\nu-1}\left(-\frac{s^2}{2}\right)\int_0^1 du\langle x|e^{-is(1-u)H_0}(Q-Q_0)e^{-isuH_0}D|x\rangle$$
$$=-i\,\frac{3\pi}{24}\int d^4x(Q-Q_0)Q_0\Sigma_jD_j\{1+\nu[1-\ln Q_0]+O(Q_0^{-1})+O(\nu^2)\}, \tag{2.16}$$

$$\zeta_{DQ}(\nu)\equiv[\Gamma(\nu)]^{-1}\int d^4x[-g(x)]^{1/2}\int ids(is)^{\nu-1}\left(-\frac{s^2}{2}\right)\int_0^1 du\langle x|e^{-is(1-u)H_0}De^{-isuH_0}(Q-Q_0)|x\rangle$$
$$=-i\,\frac{3\pi}{24}\int d^4x(Q-Q_0)Q_0\Sigma_jD_j\{1+\nu[1-\ln Q_0]+O(Q_0^{-1})+O(\nu^2)\}, \tag{2.17}$$

$$\zeta_{DD}(\nu)\equiv[\Gamma(\nu)]^{-1}\int d^4x[-g(x)]^{1/2}\int ids(is)^{\nu-1}\left(-\frac{s^2}{2}\right)\int_0^1 du\langle x|e^{-is(1-u)H_0}De^{-isuH_0}D|x\rangle$$
$$=-i\,\frac{\pi}{16}\int d^4xQ_0^2[2(\Sigma_jD_j)^2+\Sigma_jD_j^2]\left\{1+\nu\left[\frac{3}{2}-\ln Q_0\right]+O(Q_0^{-1})+O(\nu^2)\right\}. \tag{2.18}$$

Now, substituting the calculated ζ functions into Eq. (2.8) we finally obtain the effective action $W[g_{\mu\nu}]$ for the free scalar field propagating in the metric (2.2):

$$W[g_{\mu\nu}]=\int d^4x\bigg[-\frac{\pi^2}{64}Q_0^2\ln Q_0-\frac{\pi^2}{32}(Q-Q_0)Q_0\ln Q_0+\frac{\pi^2}{128}(\Sigma_j D_j)Q_0^2\ln Q_0\bigg], \tag{2.19}$$

in which, for convenience, we have let the parameter $\mu=1$. The effective action in Eq. (2.19) contains only these from ζ_0, ζ_Q, and ζ_D. Those from ζ_{ij} are smaller than those from ζ_i and are therefore neglected. The Lagrangian density coming from ζ_0 is a constant and can be absorbed by the renormalization of a cosmological constant.

Note that the effective action in Eq. (2.19) dose not have an imaginary part and thus there has been no particle produced [1]. This is because in our approximation we assume that both $(Q-Q_0)$ and D are small. This means that we constrain the system near the initial epoch; thus the universe has yet not evolved too much and the particle has yet not been produced. This behavior had also been found in our previous investigation about a rotational spacetime [12]. Following the above calculations we can see that if we let the universe evolve, thus Q approaches zero, then the effective action will have an imaginary part and a particle has been produced. However, in this case the universe will be far from the initial stage. However, as will be seen in the next section, what we are concerned with is the state near the initial epoch, not far from the initial epoch. Thus, the effective action in Eq. (2.19) is a suitable one which will be used to analyze in the next section. Note also that the imaginary part, if it appears, may possibly be shown in ζ_{ij}, as those in [12,13]. Thus, in this section we calculate the zeta function to the second order in the Schwinger perturbative to see whether the imaginary part will be appear. However, our calculation shows that it does not appear, at least to the order we have evaluated.

C. Conformal transformation

The more cosmologically interesting spacetime is that described by the metric

$$d^2s=\tilde{g}_{\mu\nu}dx^\mu dx^\nu=a(t)^2g_{\mu\nu}dx^\mu dx^\nu=a(t)^2[-dt^2+\Sigma_i e^{2\beta_i}dx^2]. \tag{2.20}$$

Now the effective action in the metric $\tilde{g}_{\mu\nu}\equiv a(t)^2g_{\mu\nu}$, denoted as $W[\tilde{g}_{\mu\nu}]$, can be easily found by the conformal transformation formula [13]

$$\begin{aligned}W[\tilde{g}_{\mu\nu}]=W[g_{\mu\nu}]+\frac{1}{16\pi^2}\hbar\int d^4x[-g(x)]^{1/2}\bigg\{&A[U(R_{\mu\nu\lambda\delta}R^{\mu\nu\lambda\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2)+2RU,_{\mu}U^{,\mu}-4R_{\mu\nu}U^{,\mu}U^{,\nu}\\&-4U,_{\mu}^{;\mu}U,_{\lambda}U^{,\lambda}-2(U,_{\lambda}U^{,\lambda})^2]+B\bigg[U\bigg(R_{\mu\nu\lambda\delta}R^{\mu\nu\lambda\delta}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2\bigg)+\frac{2}{3}R(U,_{\mu}^{;\mu}+U,_{\mu}U^{,\mu})\\&-2U,_{\mu}U^{,\mu}-4U,_{\mu}^{;\mu}U,_{\lambda}U^{,\lambda}-2(U,_{\lambda}U^{,\lambda})^2\bigg]\bigg\},\end{aligned} \tag{2.21}$$

where $U\equiv\ln(a)$, $A=-1/360$, and $B=1/120$. This formula also appears in [14] in which the other applications were discussed. The explicit expression of Eq. (2.21) will be shown in the next section.

III. REDUCED ACTION AND SEMICLASSICAL GRAVITATION

From the effective actions shown in Eqs. (2.19) and (2.21) we see that they contain a fourth derivative in the metric. The associated system will thus suffer the problems of changing the Hamiltonian structure, lacking the stability, and unphysical solutions appearing, etc. [7–9]. Therefore, we will apply the method of iterative reduction to reduce it to a second-order equation. We will follow the method which was used by Parker and Simon [9] to find the reduced system for the spacetime without space anisotropy.

The method of iterative reduction is as follows. The semiclassical gravitation theory is that described by the generalized Einstein equation with quantum corrections

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G[T^c_{\mu\nu}+T^q_{\mu\nu}], \tag{3.1}$$

where $T^c_{\mu\nu}$ is the stress tensor of the classical radiation which is included to support the expansion of the Universe at the late epoch compared with the Planck time. Other matter is neglected because it becomes negligible [9]. The quantity $T^q_{\mu\nu}$ is the renormalized stress tensor of the quantum field, which can be found from the renormalized effective action $W[\tilde{g}_{\mu\nu}]$. Because $T^q_{\mu\nu}$ is the quantum correction term, it will be first order in $\hbar$. Thus, to zeroth order of $\hbar$, we can neglect $T^q_{\mu\nu}$.

(I) Therefore, from the Bianchi identity $T^{c\ \ ;\nu}_{\mu\nu}=0$ we have a relation

$$\varepsilon \cong \varepsilon_0 a^{-4}, \tag{3.2}$$

where ε is the energy density of classical radiation and ε_0 an integration constant.

(II) The spatial-spatial component Einstein equation leads to

$$a^{-2}(\dot{a}^2 a^{-2}+\ddot{a}a^{-1}+\ddot{\beta}_1+2\dot{\beta}_i\dot{a}a^{-1})=\frac{8\pi G}{3}\,\varepsilon. \tag{3.3}$$

It then implies two useful relations

$$a^{-2}(\dot{a}^2 a^{-2}+\ddot{a}a^{-1})=\frac{8\pi G}{3}\,\varepsilon, \tag{3.4}$$

$$\ddot{\beta}_\pm+2\dot{\beta}_\pm a^{-1}=0,\Rightarrow \quad \dot{\beta}_\pm \cong c_\pm a^{-2}, \tag{3.5}$$

where $c_\pm$ are the integration constants.

(III) The time-time component Einstein equation leads to

$$\dot{a}^2 a^{-2}-Q=\frac{8\pi G}{3}\,a^2\varepsilon. \tag{3.6}$$

Then from Eqs. (3.2), (3.5), and (3.6) we have the relation

$$\dot{a}^2 \cong \frac{8\pi G}{3}\,\varepsilon_0+Aa^{-2}, \tag{3.7}$$

where

$$A\equiv(c_+^2+c_-^2). \tag{3.8}$$

Differentiating Eq. (3.7) leads to

$$\ddot{a}=-Aa^{-3}. \tag{3.9}$$

And differentiating Eq. (3.9) leads to

$$\dddot{a}=3Aa^{-4}\dot{a}. \tag{3.10}$$

These relations will be substituted into the renormalized effective actions (2.19) and (2.21) to reduce them to second-order equations. Before doing this we will see two useful results.

First, from Eq. (3.5) we see that the space anisotropy defined in Eq. (2.7) becomes

$$Q\equiv Aa^{-4}. \tag{3.11}$$

Thus the anisotropy will be smoothed out during the evolution of expanding universe (i.e., if ''a'' becomes large.). However, if the value of ''A,'' which is proportional to the space anisotropy Q, was not too small then the anisotropy may be very large when ''a'' is small (i.e., near the initial epoch). This means that the anisotropy may affect the evolution of the Universe in the early epoch. However, it is known that the determination of the constant ''A'' in a theory is the question of the initial problem and it can only be solved by the theory of the quantum cosmology [15,16]. Thus, we will quantize the reduced semiclassical gravitation theory to investigate this problem in the next section.

Next, from Eq. (3.7) it is seen that when ''a'' is small then we can neglect the ε_0 term and it has a solution

$$a=(2At+a_0^2)^{1/2}. \tag{3.12}$$

Substituting this relation into Eq. (3.5) we find that

$$\beta_j=c_j(2A)^{-1}\ln[(2At+a_0^2)a_0^{-2}]. \tag{3.13a}$$

Thus β_j becomes small as $t\rightarrow 0$. In this case we see that the functions D_j defined in Eq. (2.6) become

$$D_j=2c_j(2A)^{-1}\ln[(2At+a_0^2)a_0^{-2}], \tag{3.13b}$$

which can be regarded as a small function as $t\rightarrow 0$. This proves the crucial assumption adopted in Sec. II A. Let us emphasize that the relations of Eqs. (3.12) and (3.13) can only be used for the universe near the initial epoch and our calculation in Sec. II B is a good approximation only for the universe near the initial time.

Before entering into the next section we will mention that the constant a_0 in Eq. (3.12) will not be zero. This is because as $a\rightarrow 0$ the time-time component of the reduced Einstein equation becomes

$$a^{-2}\dot{a}^2+\frac{4}{9}A^2a^{-6}-Aa^{-4}-\frac{8\pi G}{3}\,\varepsilon_0 a^{-2}=0, \tag{3.14}$$

in which the first term is coming from the classically Einstein action, the second and third terms are coming from the leading part of the quantum corrections in Eqs. (2.21) and (2.19) respectively, while the last term is that from the classical radiation. Equation (3.14) shows an interesting fact that the classical radiation does not affect the initial state of the universe much and it is the conformal part that will dominate the contribution. Now Eq. (3.14) could be regarded as an equation describing a particle with a unit mass in a potential $U(a)$ which becomes positively infinite as $a\rightarrow 0$ and thus the universe will be bounced at a positive radial a_0. Note that the radial a_0 must be a small value because the conformal part is a quantum correction which will contain $\hbar$. Such a bounce solution is also shown in the semiclassical gravitation model in the spacetime without anisotropy, as analyzed by Parker and Simon [9].

IV. QUANTIZATION OF THE REDUCED ACTION

Before performing the quantization we need some manipulations. First, for the later convenience we will change the metric to be

$$d^2s=-dt^2+e^{2\alpha}\Sigma_i e^{2\beta_i}dx^2, \tag{4.1}$$

which is a conventional form used in the Bianchi type-I quantum cosmology [17,18]. In this metric form, after the calculation, we can find, from Eqs. (2.19) and (2.21) that

$$W[g_{\mu\nu}]=\int d^4xe^{-\alpha}\left[-\frac{\pi^2}{64}Q_0^2\ln Q_0-\frac{\pi^2}{32}(Q-Q_0)Q_0\ln Q_0+\frac{\pi^2}{128}(\Sigma_jD_j)Q_0^2\ln Q_0\right], \tag{4.2}$$

$$W[\tilde{g}_{\mu\nu}]=W[g_{\mu\nu}]+\frac{1}{1920\pi^2}\hbar\int d^4xe^{3\alpha}\left\{\frac{2}{3}\dot{\alpha}^4-2(\ddot{\alpha}+2\dot{\alpha}^2)^2+48\alpha Q^2-4(\ddot{\alpha}+3\dot{\alpha}^2)Q\right.$$
$$\left.+12\alpha[\ddot{\beta}_+^2+\ddot{\beta}_-^2+2\dot{\alpha}(\ddot{\beta}_+\dot{\beta}_++\ddot{\beta}_-\dot{\beta}_-)+\dot{\alpha}^2Q]+\frac{32}{3}\dot{\alpha}\dot{\beta}_+^3-32\dot{\alpha}\dot{\beta}_+\dot{\beta}_-^2\right\}. \tag{4.3}$$

Next, we express the reduction relations found in Eqs. (3.2), (3.5), (3.7), (3.8), (3.9), and (3.10) in the metric form (4.1), and then substitute these new relations into Eqs. (4.2) and (4.3). The results are

$$W[g_{\mu\nu}]=\int d^4xe^{-\alpha}\left[-\frac{\pi^2}{64}A^2a_0^{-8}\ln(Aa_0^{-4})-\frac{\pi^2}{32}A^2(a^{-4}-a_0^{-4})a_0^{-4}\ln(Aa_0^{-4})+\frac{\pi^2}{128}(\Sigma_jD_j)A^2a_0^{-8}\ln(Aa_0^{-4})\right], \tag{4.4}$$

$$W[\tilde{g}_{\mu\nu}]=W[g_{\mu\nu}]+\frac{1}{1920\pi^2}\int d^4xe^{3\alpha}\left\{\frac{2}{3}\left(\frac{8\pi G}{3}e^{-4\alpha}+Ae^{-6\alpha}\right)^2-2\left(\frac{32\pi G}{3}e^{-4\alpha}-Ae^{-6\alpha}\right)^2\right.$$
$$\left.+96\alpha A^2e^{-12\alpha}-\frac{160\pi G}{3}Ae^{10\alpha}+32\alpha e^{-6\alpha}(2c_+^2A+4c_+c_-^3)\left(\frac{8\pi G}{3}e^{-4\alpha}+Ae^{-6\alpha}\right)^{1/2}\right\}. \tag{4.5}$$

Finally, the classical action $W_c[\tilde{g}_{\mu\nu}]$ which shall be added before analyzing the quantization of the reduced system is

$$W_c[\tilde{g}_{\mu\nu}]=\frac{1}{16\pi G}\int d^4xe^{3\alpha}[-6\dot{\alpha}^2+6(\dot{\beta}_+^2+\dot{\beta}_-^2)], \tag{4.6}$$

where G denotes the gravitational constant.

The total action to be quantized is the summation of Eqs. (4.4), (4.5), and (4.6). From Eqs. (4.4) and (4.5) we see that after using the method of iterative reduction the quantum correction dose not change the canonical momentums used in the classical action. Thus the canonical momentums are

$$\Pi_\alpha=\delta W_c[\tilde{g}_{\mu\nu}]/\delta\alpha=-6\dot{\alpha}e^{3\alpha}, \tag{4.7}$$

$$\Pi_{\beta_\pm}=\delta W_c[\tilde{g}_{\mu\nu}]/\delta\beta_\pm=6\dot{\beta}_\pm e^{3\alpha}, \tag{4.8}$$

which are those used in the quantum cosmology model without the quantum-field effects [17,18]. Then, the Hamiltonian is defined by

$$H=\Pi_\alpha\dot{\alpha}+\Pi_{\beta_+}\dot{\beta}_++\Pi_{\beta_-}\dot{\beta}_--L, \tag{4.9}$$

in which the Lagrangian density L can be found from the total action.

Now, through the canonically substituting

$$\Pi_\alpha\to-i\partial/\partial\alpha, \tag{4.10}$$

$$\Pi_{\beta_\pm}\to-i\partial/\partial\beta_\pm, \tag{4.11}$$

the Wheeler-DeWitt equation associated with the reduced semiclassical gravitational equation becomes

$$\frac{d^2\Psi}{d\alpha^2}+\frac{d^2\Psi}{d\beta_+^2}+\frac{d^2\Psi}{d\beta_-^2}+U(\alpha,A,\beta_+,\beta_-)\Psi=0, \tag{4.12}$$

where the potential U is defined by

$$U(\alpha,A,\beta_+,\beta_-)=U_\alpha(\alpha)+U_\beta(\alpha,\beta_+,\beta_-), \tag{4.13}$$

in which

$$U_\alpha=\frac{3}{5\pi^2}A^2e^{-6\alpha}|\alpha|,$$

$$U_\beta=-\frac{9}{8\pi^2}e^{2\alpha}(\beta_+^2+\beta_-^2)A^2e^{-8\alpha_0}\ln(Ae^{-4\alpha_0}). \tag{4.14}$$

To obtain the above equation we have let $8\pi G=1$ and considered only the initial epoch, $\alpha\to-\infty$, i.e., $a\to 0$. The Wheeler-DeWitt equation (4.12) is too complex to be solved exactly and some approximations shall be adopted.

When the universe is near initial epoch we can replace α in U_β by α_0. Then U_β is only a function of β_+ and β_- and thus Eq. (4.12) could be separated into two equations:

$$\frac{d^2\Psi_\alpha}{d\alpha^2}+\left(\frac{3}{5\pi^2}A^2e^{-6\alpha}|\alpha|-C\right)\Psi_\alpha=0, \tag{4.15}$$

$$\frac{d^2\Psi_\beta}{d\beta_+^2}+\frac{d^2\Psi_\beta}{d\beta_-^2}+\left[C-\frac{9}{8\pi^2}(\beta_+^2+\beta_-^2)A^2e^{-6\alpha_0}\ln(Ae^{-4\alpha_0})\right]\times\Psi_\beta=0, \tag{4.16}$$

if we take

$$\Psi=\Psi_\alpha\Psi_\beta. \tag{4.17}$$

Note that the number C appearing in Eqs. (4.15) and (4.16) can be neglected if $\alpha_0\to-\infty$.

Now, Eq. (4.15) can be solved in the WKB approximation and the solution is

$$\Psi_\alpha=|\alpha|^{-1/4}e^{-3|\alpha|/2}\exp\left[-i\,\frac{3A}{5\pi}\,|G(\alpha)|\right], \tag{4.18}$$

where

$$G(\alpha)\equiv\int d\alpha|\alpha|e^{-3\alpha}. \tag{4.19}$$

Equation (4.16) can be solved exactly [19] and we have

$$\Psi_\beta=K_s\left\{i\,\frac{3}{\pi}\,(8)^{-1/2}Ae^{-3\alpha_0}[(\beta_+^2+\beta_-^2)\ln(Ae^{-4\alpha_0})]^{1/2}\right\}, \tag{4.20}$$

in which s is an integration constant, which is irrelevant to our discussion below, and K_s is a Bessel function. From the asympotical behavior of the Bessel function [19]

$$K_s(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}[1+O(z^{-1})], \tag{4.21}$$

we see that Ψ_β is a decreasing function of A and $\Psi_\beta\to0$ if $A\to\infty$. Because the probability density of universe is proportional to $|\Psi_\alpha\Psi_\beta|^2$, we thus conclude that the Bianchi type-I universe is not likely to be in a state with large space anisotropy. This completes our investigations.

V. CONCLUSION

We have presented a prescription to expand the effective action in the Bianchi type-I spacetime with large anisotropy. We used the zeta-function regularization method to evaluate the renormalized effective action of a quantum scalar field to second order in the Schwinger perturbative formula. As the quantum corrections contain up to a fourth derivative in the metric, to obtain the self-consistent solutions of the semiclassical gravity theory, we apply the method of iterative reduction to reduce it to a second-order equation. The reduced equation shows that the space anisotropy, which may play an important role in the early Universe, will be smoothed out during the evolution of Universe. We thus quantize the corresponding minisuperspace model to investigate the behavior of space anisotropy in the initial epoch. We solve the Wheeler-DeWitt equation in an approximation. From the wave function of the Wheeler-DeWitt equation we see that the probability of the Bianchi type-I spacetime with large anisotropy is less than that with small anisotropy. Thus the Bianchi type-I universe is not likely to be in the state with large space anisotropy.

Finally, let us make two remarks.

(1) In a previous paper [20] we had quantized the effective action with small anisotropy (which was first evaluated by [2]) and saw that the universe is likely to be in the state with small anisotropy. In this paper we quantize the effective action with large anisotropy (which is first evaluated in this paper) and see that the universe is not likely to be in the state with large anisotropy.

(2) It can be seen that the prescription used in this paper may also be applied to other cosmologically interesting spacetimes. However, it shall be mentioned that, although in general we can absorb some large quantities (in this paper it is Q_0) in the initial epoch into H_0, this does not ensure that the remaining terms (in this paper they are D_j) are small. Only if the chosen H_0 was sufficiently simple and the remaining term was small could the perturbative method be useful. The investigations about other spacetime, such as the Bianchi type-IX rotating universe, will be discussed elsewhere.

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