Chern-Simons state for the nondiagonal Bianchi type IX model

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The Bianchi type IX mixmaster model is quantized in its nondiagonal form, imposing spatial diffeomorphism, time reparametrization, and Lorentz invariance as constraints on physical state vectors *before* gauge fixing. The result turns out to be different from quantizing the diagonal model obtained by gauge fixing already on the classical level. For the nondiagonal model a generalized nine-dimensional Fourier transformation over a suitably chosen manifold connects the representations in metric variables and in Ashtekar variables. A space of five states in the metric representation is generated from the single physical Chern-Simons state in Ashtekar variables by choosing five different integration manifolds, which cannot be deformed into each other. For the case of a positive cosmological constant Λ we extend our previous study of these five states for the diagonal Bianchi type IX model to the nondiagonal case. It is shown that additional discrete (permutation) symmetries of physical states arise in the quantization of the *nondiagonal* model, which are satisfied by two of the five states connected to the Chern-Simons state. These have the characteristics of a wormhole ground state and a Hartle-Hawking ''no-boundary'' state, respectively. We also exhibit a special gauge fixing of the time reparametrization invariance of the quantized system and define an associated manifestly positive scalar product. Then the wormhole ground state is left as the only normalizable physical state connected to the Chern-Simons state. [S0556-2821(98)04316-1]

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I. INTRODUCTION

Quantum general relativity has advanced over the last decade at a remarkable and accelerating pace. The introduction of Ashtekar's new variables [1–3] to replace the earlier metric representation soon afterwards led to the discovery of a formally exact physical state of the canonically quantized theory with a nonvanishing cosmological constant, the Chern-Simons state [4,5]. Then the introduction of the loop representation [6] permitted us to reexpress the Chern-Simons state as a topological invariant of framed loops on three-space [7], the Kauffman brackets [8]. It also led to the discovery of further physical states lacking, however, one important general property of the Chern-Simons state, namely, a well-defined nondegenerate space-time in the classical limit. The Chern-Simons state semiclassically describes a de Sitter (or anti–de Sitter) space-time for a positive (negative) cosmological constant, respectively [4]. Subsequent important advances were the introduction of spin networks [9] and quantum spin networks [10] as a discretized description of three-space in which areas and volumes are quantized in Planck units and which furnish yet another representation of physical states.

The choice of different variables and representations has therefore played, and continues to play, a crucial role in the development of the theory. It is not always clear, however, to which extent the different representations are equivalent to each other. This question is particularly relevant for the connection between the metric representation and the representation in Ashtekar variables. In fact it was shown in [11] that the two representations are, in general, not equivalent. One may therefore wonder: Has the Chern-Simons state, the only known physical state with a well-defined classical limit, a counterpart in the metric representation and is it unique? In general, the answer to this question is still unknown.

Recently we examined this question for a spatially homogeneous minisuperspace model of Bianchi type IX with a diagonal metric tensor in the cases of a positive [12] and negative [13] cosmological constant Λ. A *diagonal* form of the metric tensor can be assumed in the *classical* theory without the restriction of generality, because it is a permissible gauge-fixing condition for the remnant of the spatial diffeomorphism group in the Bianchi type IX model. Taking also the matrix of Ashtekar variables as diagonal amounts in addition to a gauge fixing of the Lorentz gauge group. Quantizing such a diagonal model therefore means to apply gauge fixing of the diffeomorphism group and the Lorentz group *before* quantization. The result of our study of the diagonal Bianchi type IX model was that actually five distinct physical states in the metric representation are generated by transforming the Chern-Simons state from Ashtekar variables to metric variables. This change of representation takes the form of a generalized multidimensional Fourier transformation in the space spanned by the complex Ashtekar variables along arbitrary paths or, more precisely, integration manifolds with boundaries pinned by the condition that partial integration without boundary terms must be allowed.

In the present work we take our investigation of this basic question a step further and examine the metric representations of the Chern-Simons state for the *nondiagonal* Bianchi type IX model. Why is this step interesting, and why cannot the results of our study of the diagonal case just be taken over? Quantizing the nondiagonal model amounts to interchanging the steps of gauge fixing and quantizing: the gauge fixing of the spatial diffeomorphism group and Lorentz group is now done *after* the quantization. The result is, in general, not the same as before the exchange of these steps. In fact, in Sec. III A we give a very simple example from quantum mechanics which shows that gauge fixing *after* quantization is preferable, in general, because it takes all

symmetries into account in the quantization process in a natural manner. The example also makes clear that the result, in general, differs from that of *first* gauge fixing and *then* quantizing by (i) quantum corrections in the Hamiltonian, (ii) weight functions in the naturally defined scalar product, and (iii) additional discrete symmetry requirements to be satisfied by the solutions.

We shall see that all three differences also show up in our study of the nondiagonal Bianchi type IX model. In fact the noncommutativity of gauge fixing and quantization in the Bianchi type IX model was previously discussed by Major and Smolin within the framework of path integral quantization [16,17]. Our investigation here differs by the use of canonical quantization and, in particular, by studying a special quantum state, the Chern-Simons state. Comparing the two ways of quantization we find that on the leading semiclassical level the solutions for the nondiagonal model are the same as in the diagonal case. However, in the next to leading semiclassical order they already differ due to the quantum corrections in the Hamiltonian and due to changes in the naturally defined scalar product. Even more importantly, the additional discrete symmetry requirements are only met by two of the five linearly independent states, leaving just two physical states in the metric representation of the nondiagonal model which are generated by the Chern-Simons state, a generalized wormhole state and a Hartle-Hawking state [14,15]. It is remarkable that states of both kinds are related to the Chern-Simons state and that the two different semiclassical boundary conditions singling out either one of them can still be satisfied at this stage.

In a final step we also gauge-fix (after quantization) the time reparametrization invariance and introduce a manifestly positive scalar product on the space of physical states. As will be shown in a separate paper only the generalized wormhole state is normalizable in this scalar product. Gauge fixing of the time reparametrization invariance in the *classical* Bianchi type IX model was discussed in [18]. However, the particular class of gauge-fixing conditions considered there would not lead to a positive scalar product quantum mechanically.

The rest of this paper is organized as follows: In Sec. II we define our notation and set up the metric representation of the constraint equations of the nondiagonal model (Sec. II A), extract a well-known exact solution for $\Lambda=0$, the wormhole state (Sec. II B), and give the representation of the constraint equations in Ashtekar variables (Sec. II C). In Sec. III we discuss in detail the differences between the diagonal and the nondiagonal Bianchi type IX model. We first give a simple quantum mechanical example, the harmonic oscillator in two dimensions with a rotational gauge symmetry (Sec. III A). Then the corresponding comparison for the Bianchi type IX model is given (Sec. III B). We also mention here briefly a gauge fixing of the time reparametrization symmetry and an accompanying physical inner product on the space of physical states (Sec. III C) but a more detailed presentation of this point is beyond the scope of the present paper and will be given separately [19]. In Sec. IV the transformation of the Chern-Simons state of the nondiagonal Bianchi type IX model from the Ashtekar representation (Sec. IV A)

to the metric representation (Sec. IV B) is given, leading to a general integral representation over a suitably chosen three-dimensional manifold Σ^3. In Sec. V various asymptotic forms of this integral representation are evaluated. The limits considered are *first* $\hbar \to 0$ (Sec. V A) and *then* either $\Lambda \to 0$ (Sec. V A 1) or $\Lambda a^2 \to 0$ (Sec. V A 2), where a is the geometrical mean of the scale parameters, and $\Lambda \to 0$ (Sec. V B) without necessarily taking a second limit. In Sec. VI we exhibit for the general case, i.e., without taking asymptotic limits, five possible and topologically distinct choices of the integration manifolds Σ^3 leading to five exact solutions of the constraint equations in the metric representation. We also discuss their relation with the asymptotic results of Sec. V and their normalizability with respect to the inner product of Sec. III C. Our conclusions are then summarized in Sec. VII. Three appendixes deal with certain technical details. The limit $\Lambda \to 0$, the results of which are discussed in Sec. V B, is quite subtle and therefore done in some detail (Appendix A). It leads to a nice integral representation of the vacuum (i.e., $\Lambda = 0$) solutions (Appendix B). Finally we check certain required nontrivial continuity and differentiability properties of the integrand of the integral representation of Sec. IV B on the integration manifolds Σ^3 (Appendix C).

II. QUANTUM CONSTRAINT EQUATIONS

In this section we shall set up our notation and give a brief derivation of the quantum Einstein equations for the homogeneous Bianchi type IX model. While the classical constraint equations for this model are determined uniquely, the quantum operators associated with these constraints suffer from the well-known ambiguity of the factor ordering, in particular in the Hamiltonian constraint. The main purpose of the following will be the motivation of a special choice of factor ordering. Technical details of the derivation will be summarized rather briefly.

A. Metric representation of the constraint equations

Let us start with the Einstein Hilbert action for a gravitational field with a cosmological constant Λ:

$$\mathcal{S}_{EH}[g_{\mu\nu}] = \int_{\mathcal{M}} \mathrm{d}^4x \sqrt{-g}\,(^4\mathcal{R} - 2\Lambda). \qquad (2.1)$$

Here $\mathcal{M}$ is the space-time manifold, $\boldsymbol{g}=(g_{\mu\nu})$ the four-metric on $\mathcal{M}$, $g=\det(g_{\mu\nu})$, and $^4\mathcal{R}$ the curvature scalar of the four-metric. The common prefactor $1/16\pi G$ of Eq. (2.1) has been avoided by picking units with $G=(16\pi)^{-1}$. As is well known, the Arnowitt-Deser-Misner (ADM) space-time split and a subsequent Legendre transform with respect to $\dot{h}_{ij}$ yield the following equivalent expression for the Einstein-Hilbert action [20]:

$$\mathcal{S}_{EH} = \int \mathrm{d}t \int \mathrm{d}^3x\,(\tilde{\pi}^{ij}\dot{h}_{ij} - N\mathcal{H}_G - N_i\mathcal{H}^i), \qquad (2.2)$$

with

$$\mathcal{H}_G = G_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl} - \sqrt{h}(^3\mathcal{R} - 2\Lambda), \quad \mathcal{H}^i = -2\tilde{\pi}^{ij}{}_{|j},$$
$$(2.3)$$

where

$$G_{ijkl} := \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}). \qquad (2.4)$$

In the transition from Eq. (2.1) to Eq. (2.2) a surface term has been omitted, since it will have no effect on the constraint equations. N and N_i are the lapse function and the shift vector, respectively, and the three-metric (h_{ij}) of the spatial manifold $t = \mathrm{const}$ is used to raise and lower the spatial indices $i, j, k, \ldots$. A stroke denotes covariant derivatives with respect to the three-metric, $^3\mathcal{R}$ is the curvature scalar of the spatial manifold, and h denotes the determinant of the three-metric.

Let us now consider a *homogeneous* three-manifold of one of the Bianchi types; then there exists an invariant basis of one-forms $\boldsymbol{\omega}^p = \omega^p{}_i(x)\, dx^i$, such that any homogeneous tensor field on the manifold has spatially constant components when expanded in this basis [21,22]. In particular, we have

$$d\boldsymbol{\omega}^p = \frac{1}{2}m^{pq}\varepsilon_{qrs}\boldsymbol{\omega}^r \wedge \boldsymbol{\omega}^s, \qquad (2.5)$$

with a constant structure matrix m^{pq}. In the following we shall be interested in the Bianchi type IX case, where the structure matrix is of the simple form $m^{pq} = \delta^{pq}$. If all the tensor fields occurring in Eq. (2.2) are expanded in the invariant basis, we arrive at the following expression for the Einstein Hilbert action:[1]

$$\mathcal{S}_{EH} = \int dt\, \mathcal{L}_{EH} = \int dt \int d^3x\, \omega(\tilde{\pi}^{pq}\dot{h}_{pq} - N\mathcal{H}_0 - N^p\mathcal{H}_p),$$
$$(2.6)$$

$$\mathcal{H}_0 = G_{pqrs}\tilde{\pi}^{pq}\tilde{\pi}^{rs} - \sqrt{h}(^3\mathcal{R} - 2\Lambda),$$

$$\mathcal{H}_p = 2\varepsilon_{prq}m^{qn}h_{ns}\tilde{\pi}^{sr}, \qquad (2.7)$$

where

$$h\,^3\mathcal{R} = \frac{1}{2}(m^{ps}h_{sp})^2 - m^{ps}h_{sq}m^{qr}h_{rp}. \qquad (2.8)$$

Here $\omega = \det(\omega^p{}_i)$ in Eq. (2.6) contains the only spatial dependence of $\mathcal{S}_{EH}$, which therefore can be integrated out explicitly. Afterwards, the Lagrangian equations with respect to N and N^p imply the following set of first class constraints:

$$\mathcal{H}_0 = 0, \quad \mathcal{H}_p = 0, \qquad (2.9)$$

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[1]The following expressions are valid not only for the Bianchi type IX model with $m^{pq} = \delta^{pq}$, but even for a general Bianchi type A model characterized by $m^{pq} = m^{qp}$.

with $\mathcal{H}_0$ and $\mathcal{H}_p$ being phase space functions of the fundamental variables h_{pq} and the momenta $\tilde{\pi}^{pq}$ only.[2] If we make use of the remarkable identity

$$\sqrt{h}\,^3\mathcal{R} = -m^{pq}G_{pqrs}m^{rs}, \qquad (2.10)$$

which is valid for the general Bianchi type A case, the Hamiltonian constraint $\mathcal{H}_0$ may be written in one of the two following forms:

$$\mathcal{H}_0 = [\tilde{\pi}^{pq} \pm im^{pq}]G_{pqrs}[\tilde{\pi}^{rs} \mp im^{rs}] + 2\Lambda\sqrt{h}. \quad (2.11)$$

Let us first restrict ourselves to the kinetic part (which becomes the full Hamiltonian constraint for the Bianchi type I model)

$$T = \tilde{\pi}^{pq}G_{pqrs}\tilde{\pi}^{rs}, \qquad (2.12)$$

describing a particle moving freely on a six-dimensional manifold with coordinates h_{pq} and the (indefinite) supermetric G^{pqrs}, which is the inverse of G_{pqrs}.[3] To quantize such a system, we may employ new coordinates h'_{pq} of the minisuperspace such that, at least locally, the supermetric takes a diagonal form: i.e.,

$$T = \tilde{\pi}^{pq\prime}\eta_{pqrs}\tilde{\pi}^{rs\prime}, \qquad (2.13)$$

with

$$\eta_{pqrs} := \begin{cases} -1, & \text{if } p = q = r = s = 1, \\ \dfrac{1}{2}(\delta_{pr}\delta_{qs} + \delta_{ps}\delta_{qr}), & \text{otherwise.} \end{cases}$$

In these ''free falling'' superspace coordinates the associated quantum operator is expected to be

$$\hat{T} = -\hbar^2 \frac{\partial}{\partial h'_{pq}}\eta_{pqrs}\frac{\partial}{\partial h'_{rs}}, \qquad (2.14)$$

and, transforming back to the h_{pq} coordinates, we arrive at

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[2]According to Eq. (2.6) $\tilde{\pi}^{pq}$ are *not* the canonically conjugate momenta to h_{pq}; they differ from them by a rescaling factor $V := \int d^3x\,\omega$. Consequently, the canonical Poisson brackets read

$$\{\tilde{\pi}^{pq}, h_{rs}\} = -\frac{1}{V}\delta^{pq}_{rs}$$

with

$$\delta^{pq}_{rs} := \frac{1}{2}(\delta^p_r\delta^q_s + \delta^p_s\delta^q_r),$$

and the additional factor $1/V$ should be carried over to the canonical commutation relations of the quantized theory. There, it can be eliminated by a rescaling of Planck's constant $\hbar$. Since we are mainly interested in the quantized theory, we will use this freedom to set $V = 1$ in the following.

[3]More precisely, we require $G^{pqmn}G_{mnrs} = \delta^{pq}_{rs}$.

$$\hat{T} = -\hbar^2 \frac{1}{\sqrt{G}} \frac{\partial}{\partial h_{pq}} \sqrt{G} G_{pqrs} \frac{\partial}{\partial h_{rs}}, \qquad (2.15)$$

which is easily recognized as the invariant Laplace-Beltrami operator on minisuperspace. Here G is the absolute value of the determinant of the supermetric G^{pqrs}, which can be shown to be proportional to h^{-1}.

These considerations suggest the following factor ordering for the quantum version of the Hamiltonian constraint (2.11):

$$\hat{\mathcal{H}}_0 = \frac{1}{\sqrt{G}} \left[-i\hbar \frac{\partial}{\partial h_{pq}} + im^{pq} \right] \sqrt{G} G_{pqrs}$$

$$\times \left[-i\hbar \frac{\partial}{\partial h_{rs}} - im^{rs} \right] + 2\Lambda \sqrt{h}$$

$$= -\hbar^2 \sqrt{h} \frac{\partial}{\partial h_{pq}} \frac{1}{\sqrt{h}} G_{pqrs} \frac{\partial}{\partial h_{rs}}$$

$$- \hbar \sqrt{h} \frac{\partial}{\partial h_{pq}} \left(\frac{1}{\sqrt{h}} G_{pqrs} m^{rs} \right) - \sqrt{h} (^3\mathcal{R} - 2\Lambda),$$

$$\qquad (2.16)$$

where we have chosen the signs of im^{pq} in the first line to make the special physical state defined below in Eq. (2.24) become an exponentially decaying solution. We end up with a Wheeler-DeWitt operator, which, apart from a special factor ordering in the kinetic term, contains a quantum correction to the potential, due to the action of the derivative operators on the supermetric. Such a term is well known from the diagonal model; cf. [12]. The factor ordering chosen in Eq. (2.16) has the nice implication that the Chern-Simons state becomes an exact solution to all quantum constraints in the Ashtekar representation, as will be shown in Sec. IV A.

To put the diffeomorphism constraint in more concrete terms, we should be further interested in a h_{pq} representation of $\tilde{\pi}^{pq}$, obeying the canonical commutation relations

$$[\tilde{\pi}^{pq}, h_{rs}] = -i\hbar \ \delta^{pq}_{rs} . \qquad (2.17)$$

Moreover, we wish $\tilde{\pi}^{pq}$ to be a Hermitian operator with respect to the natural auxiliary inner product on the extanded Hilbert space, in which the Wheeler-DeWitt operator (2.16) is Hermitian:

$$\langle \Psi | \Phi \rangle = \int d^9 h_{pq} \prod_{p>q} \delta(h_{pq} - h_{qp}) \sqrt{G} \Psi(h_{rs}) \Phi(h_{rs}).$$

$$\qquad (2.18)$$

Here the components of h_{pq} range over all real values which

are consistent with (h_{pq}) having positive eigenvalues.[4] As is known from the quantization on a curved manifold, the h_{pq} representation of the momentum operator has to be corrected with a relative weight factor $G^{1/4}$, leading to

$$\tilde{\pi}^{pq} = -i\hbar \ G^{-1/4} \frac{1}{2} \left(\frac{\partial}{\partial h_{pq}} + \frac{\partial}{\partial h_{qp}} \right) G^{1/4}$$

$$= -i\hbar \ h^{1/4} \frac{\partial}{\partial h_{(pq)}} h^{-1/4}. \qquad (2.19)$$

Using this h_{pq} representation, we may rewrite $\hat{\mathcal{H}}_0$ and $\hat{\mathcal{H}}_p$ in the form

$$\hat{\mathcal{H}}_0 = (a^{pq})^\dagger G_{pqrs} a^{rs} + 2\Lambda \sqrt{h} = \hat{\mathcal{H}}_0^\dagger, \qquad (2.20)$$

with

$$a^{pq} := h^{-1/4} [\tilde{\pi}^{pq} - im^{pq}] h^{1/4},$$

and

$$\hat{\mathcal{H}}_p = 2\varepsilon_{prq} m^{qn} h_{ns} \tilde{\pi}^{sr} = 2\varepsilon_{prq} m^{qn} h_{ns} a^{sr}$$

$$= -2i\hbar \ \varepsilon_{prq} m^{qn} h_{ns} \frac{\partial}{\partial h_{(sr)}} = \hat{\mathcal{H}}_p^\dagger. \qquad (2.21)$$

That the diffeomorphism constraints can indeed be written in the form of Eq. (2.21) is checked easily by using the identity

$$h^{1/4} \frac{\partial}{\partial h_{(pq)}} h^{-1/4} = -\frac{1}{4} h^{pq}, \qquad (2.22)$$

which implies that there is no additional contribution to $\hat{\mathcal{H}}_p$ arising from the determinant factor, as one might expect. With Eqs. (2.20) and (2.21) we have now nice, self-adjoint operators, which, moreover, form a closed algebra, as we will see in Eq. (2.38) below.

B. Wormhole state

As is immediately seen from Eq. (2.16), the wave function

$$\Psi_{WH} := \exp \left[-\frac{1}{\hbar} m^{pq} h_{pq} \right] =: \exp \left[-\frac{\Phi}{\hbar} \right] \qquad (2.23)$$

is a solution to the Hamiltonian constraint for $\Lambda = 0$, which moreover satisfies $a^{pq} \Psi_{WH} = 0$ and therefore solves the diffeomorphism constraint (2.21) as well. For the Bianchi type IX model, Eq. (2.23) may also be written in the form

$$\Psi_{WH} = \exp \left[-\frac{1}{\hbar} (\lambda_1 + \lambda_2 + \lambda_3) \right], \qquad (2.24)$$

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[4] The so-defined integration regime for the auxiliary inner product has a nontrivial boundary at $h=0$, where at least one of the three eigenvalues of (h_{pq}) vanishes. Consequently, we have to restrict ourselves to states Ψ, which vanish for $h \to 0$ in a suitable manner to assure Hermiticity of the differential operators occurring in Eq. (2.16).

where λ_p are the eigenvalues of the three-metric h_{pq}, and this state is known as the "wormhole" state from the diagonal model.[5] The fact that it will occur as a prefactor to all further wave functions discussed within this paper suggests to perform the following similarity transformation:

$$\Psi = e^{-\Phi/\hbar}\Psi', \quad \hat{\mathcal{H}}_p = e^{-\Phi/\hbar}\hat{\mathcal{H}}_p'e^{\Phi/\hbar}, \quad \hat{\mathcal{H}}_0 = e^{-\Phi/\hbar}\hat{\mathcal{H}}_0'e^{\Phi/\hbar}. \tag{2.25}$$

In this new representation, the transformed operators take the form

$$\hat{\mathcal{H}}_p' = -2i\hbar\varepsilon_{prq}m^{qn}h_{ns}\frac{\partial}{\partial h_{(sr)}}, \tag{2.26}$$

$$\hat{\mathcal{H}}_0' = \frac{1}{\sqrt{G}}\left[i\hbar\frac{\partial}{\partial h_{pq}} - 2im^{pq}\right]\sqrt{G}G_{pqrs}i\hbar\frac{\partial}{\partial h_{rs}} + 2\Lambda\sqrt{h}, \tag{2.27}$$

which will be recovered from a very different approach in the next section.

C. Representation of the constraint equations in Ashtekar variables

In this section we want to derive the so-called Ashtekar representation [1,2] of the quantized nondiagonal Bianchi type IX model using the inverse densitized triad of the three-metric h_{pq} and complexified canonically conjugate variables thereof. The first step in this direction has already been performed in the last section by splitting off the wormhole state (2.23). What remains to be done now is to introduce the inverse, densitized triad of the three-metric h_{pq}, defined via

$$h \cdot h^{pq} = \widetilde{e}^p_a \cdot \widetilde{e}^q_a. \tag{2.28}$$

Here and in the following $a, b, c, \ldots$ are flat, internal indices running from 1 to 3; they are raised and lowered with the flat metric δ_{ab} and will therefore always be chosen to be lower indices without any restriction. The introduction of a triad has the great advantage that the three-metric defined via Eq. (2.28) is automatically positive definite, at least as long as the triad is real valued, and this is favorable for a definition of an inner product on the space of wave functions. However, as is well known, we gain three additional degrees of freedom by introducing such a triad, corresponding to the three possible rotations in the flat local tangent space. These redundant rotational degrees of freedom are accompanied by three additional first class constraints referred to as the Gauss constraints.

To construct the generators of these constraints we observe that Eq. (2.28) is invariant under rotations of $\widetilde{e}^p_a$ in the local tangent space, generated by

$$J_a' = i\varepsilon_{abc}\mathcal{A}_{pb}\widetilde{e}^p_c, \tag{2.29}$$

with

$$\mathcal{A}_{qa} := -\frac{\hbar}{2}\frac{\partial}{\partial\widetilde{e}^q_a}.$$

The J_a' satisfy the angular momentum algebra

$$[J_a', J_b'] = i\hbar\varepsilon_{abc}J_c' \tag{2.30}$$

and commute with $\hat{\mathcal{H}}_p'$ and $\hat{\mathcal{H}}_0'$. They are the generators of the Gauss constraints. The general solution of the Gauss constraints $J_a'\psi' = 0$ is obviously given by wave functions of the form

$$\psi'(\widetilde{e}^p_a) = \Psi'(h_{pq}), \tag{2.31}$$

where h_{pq} is a function of the $\widetilde{e}^p_a$ via Eq. (2.28). Acting on solutions of the Gauss constraints the operator $\partial/\partial\widetilde{e}^p_a$ can be written as

$$\mathcal{A}_{pa} = -\frac{\hbar}{2}\frac{\partial}{\partial\widetilde{e}^p_a} = \frac{\hbar}{e}G_{mnpq}\widetilde{e}^q_a\frac{\partial}{\partial h_{mn}}, \tag{2.32}$$

where e is the square root of the determinant of $\widetilde{e}^p_a$. This permits us to rewrite the constraint operators (2.26) and (2.27) in terms of the operators $\widetilde{e}^p_a$ and $\mathcal{A}_{pa}$ as[6]

$$\mathcal{H}_p' := 2i\varepsilon_{pqr}m^{rs}\widetilde{e}^q_a\mathcal{A}_{sa}, \tag{2.33}$$

$$\mathcal{H}_0' := e^{-1}\varepsilon_{abc}\varepsilon_{pqr}\widetilde{e}^p_a\widetilde{e}^q_b\mathcal{Q}^r_c, \tag{2.34}$$

with the operators

$$\mathcal{Q}^p_a := \mathcal{F}^p_a + \frac{\Lambda}{3}\widetilde{e}^p_a := m^{pq}\mathcal{A}_{qa} + \frac{1}{2}\varepsilon^{pqr}\varepsilon_{abc}\mathcal{A}_{qb}\mathcal{A}_{rc} + \frac{\Lambda}{3}\widetilde{e}^p_a. \tag{2.35}$$

The operators $\mathcal{Q}^p_a$ are very convenient and can also be used to reexpress the J_a' and $\mathcal{H}_p'$ as

$$J_a' := \frac{3i}{\Lambda}\varepsilon_{abc}\mathcal{A}_{pb}\mathcal{Q}^p_c, \tag{2.36}$$

$$\mathcal{H}_p' = 2i(\varepsilon_{pqr}\widetilde{e}^q_a\mathcal{Q}^r_a - \mathcal{A}_{pa}J_a'). \tag{2.37}$$

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[5]This name derives from the fact that Φ solves the Euclideanized Hamilton-Jacobi equation with Euclidean four-geometry at a large scale parameter. We shall use this name in the following to refer to the state (2.23) or (2.24) for $\Lambda = 0$ or its generalization for $\Lambda \neq 0$, without entering a discussion of wormholes, however.

[6]In Eq. (2.34) we have to order the field-strength operators $\mathcal{Q}^p_a$ to the right; otherwise $\hat{\mathcal{H}}_0'$ would not give Eq. (2.27) when acting on Gauss invariant states.

By construction $\mathcal{H}'_p$ and $\mathcal{H}'_0$ coincide with $\hat{\mathcal{H}}'_p$ and $\hat{\mathcal{H}}'_0$ defined in Eqs. (2.26) and (2.27) when acting on the invariant subspace spanned by the solutions of the Gauss constraints (but they extend these operators also to nonphysical states outside this subspace, which are of no interest to us, however). The commutation relations of $\mathcal{H}'_p$ and $\mathcal{H}'_0$ are particularly easy to evaluate in the representation (2.33),(2.34):

$$[\mathcal{H}'_p,\mathcal{H}'_q]=i\hbar\varepsilon_{pqr}m^{rs}\mathcal{H}'_s, \quad [\mathcal{H}'_0,\mathcal{H}'_p]=0. \quad (2.38)$$

They imply the same commutator algebra for the $\hat{\mathcal{H}}'_p$, $\hat{\mathcal{H}}'_0$, and, via the similarity transformation (2.25), also for the $\hat{\mathcal{H}}_p$, $\hat{\mathcal{H}}_0$.

Summarizing our results so far, we have shown that, assuming that $\psi'(\widetilde{e}^p_a)$ is a solution to the constraints (2.29), (2.33), and (2.34), the transformed wave function $\Psi(h_{pq})$, connected with Ψ' via Eq. (2.25), is a solution to the constraints (2.20) and (2.21) in the metric representation. In particular, if we are able to solve the more restrictive but simpler set of equations

$$\mathcal{Q}^p_a\psi'(\widetilde{e}^p_a)=0, \quad (2.39)$$

it is clear from the definitions given in Eqs. (2.34), (2.36), and (2.37) that we have also found a quantum state of the full, nondiagonal Bianchi type IX model with a nonvanishing cosmological constant in the metric representation. This will only be a special class of solutions, however, because Eq. (2.39) represents nine conditions, while the Hamiltonian, diffeomorphism, and Gauss constraints together constitute seven conditions only.

III. COMPARISON BETWEEN THE DIAGONAL AND NONDIAGONAL BIANCHI TYPE IX MODEL

We shall now adress the interesting question of whether the nondiagonal Bianchi type IX model presented in Sec. II and the diagonal Bianchi type IX model discussed in [12] have the same physical content. On the classical level, it is of course unnecessary to distinguish between these two models, because we can use the gauge freedom of the diffeomorphism constraints to transform the nondiagonal Bianchi type IX model to the diagonal one. When discussing the *quantized* Bianchi type IX model, most authors restricted themselves to the diagonal case; i.e., they solved for the diffeomorphism constraints on the classical level, and performed the canonical quantization procedure for the effective three-dimensional system. This approach immediately suggests itself, and one may hope that a quantization of the full Bianchi type IX model (on a six-dimensional configuration space) with diffeomorphism constraints imposed on the quantum mechanical level should physically lead to the same results.

In the following, we shall first discuss a very simple example, which immediately shows that this belief is in general not true; we will then show in Sec. III B that the two quantization procedures used for the diagonal and nondiagonal Bianchi type IX model indeed differ drastically.

A. Example: The two-dimensional harmonic oscillator with $L=0$

Let us consider a well-known example, the two-dimensional harmonic oscillator with unit mass and unit frequency, but with the additional constraint that the angular momentum should vanish:

$$\mathcal{H}=\mathcal{H}_0+NL,$$

$$\mathcal{H}_0=\tfrac{1}{2}(p_1^2+p_2^2)+\tfrac{1}{2}(q_1^2+q_2^2), \quad L=q_1p_2-q_2p_1. \quad (3.1)$$

Here $\mathcal{H}_0$ is the Hamiltonian of the unconstrained system and N is a Lagrangian multiplier. As for the Bianchi type IX model, we have the nice property that the constraint is given by a conserved quantity, but L generates *gauge* transformations only if we identify all the directions, in which the effective one-dimensional oscillator can move.

If we first quantize this system similarly to the diagonal Bianchi type IX case, we have to solve $L=0$ on the classical level, which is done by

$$q_2=0\Rightarrow p_2=\dot{q}_2=0. \quad (3.2)$$

In this gauge we arrive at the effective Hamiltonian

$$\mathcal{H}_{eff}=\frac{1}{2}p^2+\frac{1}{2}q^2, \quad q\equiv q_1, \quad (3.3)$$

which is easily quantized because it simply describes a one-dimensional harmonic oscillator:

$$\hat{\mathcal{H}}_{eff}=-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2}+\frac{1}{2}q^2, \quad \langle\Psi|\Phi\rangle=\int_{-\infty}^{+\infty}dq\ \Psi^*(q)\Phi(q). \quad (3.4)$$

Let us now, second, proceed in analogy to the nondiagonal Bianchi type IX case. Then we have to quantize first, and obtain

$$\hat{\mathcal{H}}=\hat{\mathcal{H}}_0+N\hat{L}, \quad \langle\psi|\phi\rangle=\int_{\mathbb{R}^2}d^2q\,\psi^*(q_1,q_2)\phi(q_1,q_2), \quad (3.5)$$

with

$$\hat{\mathcal{H}}_0=-\frac{\hbar^2}{2}\left(\frac{\partial^2}{\partial q_1^2}+\frac{\partial^2}{\partial q_2^2}\right)+\frac{1}{2}(q_1^2+q_2^2),$$

$$\hat{L}=-i\hbar\left(q_1\frac{\partial}{\partial q_2}-q_2\frac{\partial}{\partial q_1}\right). \quad (3.6)$$

Solving $\hat{L}\psi=0$, the wave function must be of the form

$$\psi(q_1,q_2)=\Psi(q), \quad q=\sqrt{q_1^2+q_2^2}. \quad (3.7)$$

However, to ensure that ψ is a differentiable function with respect to q_1 and q_2, we must require $\Psi(q)$ to be an even function in q:

$$\Psi(q) = \Psi(-q). \tag{3.8}$$

Furthermore, if we compute the action of $\hat{\mathcal{H}}$ on such a solution $\Psi(q)$, we effectively have

$$\hat{\mathcal{H}}_{eff} = -\frac{\hbar^2}{2}\frac{1}{q}\frac{\partial}{\partial q}q\frac{\partial}{\partial q} + \frac{1}{2}q^2 = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} - \frac{\hbar^2}{2q}\frac{\partial}{\partial q} + \frac{1}{2}q^2, \tag{3.9}$$

and the effective scalar product for solutions to the angular momentum constraint becomes

$$\langle\psi|\phi\rangle = \int_{R^2}d^2q\,\psi^*(q_1,q_2)\phi(q_1,q_2)$$

$$= 2\pi\int_0^\infty dq\,q\Psi^*(q)\Phi(q) = \langle\Psi|\Phi\rangle_{eff}. \tag{3.10}$$

As a result, we have three important differences between the quantization procedures pointed out above: (i) The effective Hamilton operators (3.4) and (3.9) differ by a factor ordering term $-(\hbar^2/2q)\partial/\partial q$, which becomes singular where the coordinate transformation from q_1,q_2 to polar coordinates with radius q is not invertible; (ii) the scalar products (3.4) and (3.10) contain different weight functions and different integration regimes; (iii) for the second quantization procedure, we get a parity requirement (3.8) being absent in the first case, because we originally start with differentiable functions on a higher dimensional configuration space.

All three differences will now be recovered when comparing the diagonal and nondiagonal Bianchi type IX models.

B. Three differences between the quantized diagonal and nondiagonal Bianchi type IX models

To compare our results of Sec. II for the nondiagonal Bianchi type IX model to the diagonal case discussed in [12], let us try to solve the diffeomorphism constraints (2.21) on the quantum mechanical level. Three special, regular solutions to these constraint equations $\hat{\mathcal{H}}_p\Psi(h_{pq})=0$ are the invariants of the three-metric, which read

$$T := \mathrm{Tr}\,(h_{pq}), \quad Q := \delta_{mn}\varepsilon^{mpq}\varepsilon^{nrs}h_{pr}h_{qs}, \quad h = \det\,(h_{pq}). \tag{3.11}$$

Therefore the general solution to $\hat{\mathcal{H}}_p\Psi=0$ is any function of these three invariants:

$$\Psi(h_{pq}) = \chi(T,Q,h), \tag{3.12}$$

where $\chi(T,Q,h)$ is a differentiable function with respect to its three arguments T,Q,h. We may now express this general solution in terms of the three eigenvalues λ_p of the three-metric, which, however, are *not* $C^1(\boldsymbol{R}^6)$ functions of h_{pq}, because of the cubic roots which are needed to express the λ_p in terms of h_{pq}. Nevertheless, a function $\Psi(h_{pq})$ solving $\hat{\mathcal{H}}_p\Psi=0$ according to Eq. (3.12) actually *is* $C^1(\boldsymbol{R}^6)$ with respect to h_{pq}, and so

$$\psi(\lambda_p) = \Psi(h_{pq})$$

$$= \chi(\lambda_1+\lambda_2+\lambda_3,\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1,\lambda_1\lambda_2\lambda_3) \tag{3.13}$$

is a regular function in h_{pq}, too. We observe that, as a consequence, $\psi(\lambda_p)$ is symmetric under arbitrary permutations of the λ_p, while any wave function which is not symmetric under these permutations will not be differentiable with respect to h_{pq}. This symmetry requirement is the analogue to the parity requirement (3.8) of our example in Sec. III A.

It will be convenient to introduce new variables

$$\sigma_p := \frac{2}{\hbar}\sqrt{\lambda_q\lambda_r} > 0, \tag{3.14}$$

with

$$\varepsilon_{pqr} = 1,$$

instead of the eigenvalues λ_p. In the diagonal gauge they play the role of the inverse densitized triad. The inverse transformation reads

$$\lambda_p = \frac{\hbar}{2}\frac{\sigma_q\sigma_r}{\sigma_p}, \tag{3.15}$$

and we arrive at the representation

$$\Psi(\sigma_p) = \psi(\lambda_p) = \psi\left(\frac{\hbar}{2}\frac{\sigma_q\sigma_r}{\sigma_p}\right); \tag{3.16}$$

so the wave function $\Psi(\sigma_p)$ is not only invariant under arbitrary permutations of the σ_p, but in addition invariant under reflections $\sigma_p\to-\sigma_p,\sigma_q\to-\sigma_q,\sigma_r\to\sigma_r$. These are *necessary* symmetry requirements for the wave functions in the σ_p representation, which were absent in the diagonal case; cf. [12]. The result is independent of the factor ordering chosen for the Hamiltonian constraint, since we only made use of the diffeomorphism constraints, which have a unique, geometrically fixed factor ordering.

The effective Hamiltonian constraint in the σ_p representation becomes

$$\mathcal{H}_0'\Psi' \propto \sqrt{\sigma_1\sigma_2\sigma_3}\left[\frac{1}{\sigma_1}\mathcal{Q}_1 + \frac{1}{\sigma_2}\mathcal{Q}_2 + \frac{1}{\sigma_3}\mathcal{Q}_3\right]\Psi' = 0, \tag{3.17}$$

where

$$\mathcal{Q}_p := \partial_q\partial_r - \partial_p + \frac{\sigma_q\partial_r - \sigma_r\partial_q}{\sigma_q^2 - \sigma_r^2} + \frac{\lambda}{2}\sigma_p, \quad \varepsilon_{pqr} = 1, \tag{3.18}$$

where

$$\lambda := \hbar\frac{\Lambda}{3}, \quad \partial_q := \frac{\partial}{\partial\sigma_q}. \tag{3.19}$$

It is possible to show that the very restrictive condition (2.39) now implies that each operator $\mathcal{Q}_p$ annihilates Ψ' separately. A comparison with the corresponding result for the diagonal model [cf. Eq. (2.23) of [12]] reveals that, apart from a global factor $\sqrt{\sigma_1 \sigma_2 \sigma_3}$, we have an additional factor ordering term in Eq. (3.18), which becomes singular where two of the σ_p become identical.[7] The occurrence of the singularities in these terms and the zeros in the measure of Eq. (3.21) (see below) reflect the fact that the transformation (3.20) between the variables h_{pq} in which the quantization is performed and the variables $\{\sigma_p, \varphi_a\}$, which separate the gauge degrees of freedom from the physical degrees of freedom, is not invertible whenever two of the σ_p take the same value. However, this factor ordering term is just a quantum correction, which does not affect the semiclassical limit $\hbar \to 0$, and should be expected to appear; cf. (i) of Sec. III A.

To conclude this discussion, we should remark on the inner product on the Hilbert space of wave functions, belonging to the two different models. In a first step we want to compute the auxiliary inner product (2.18) for the nondiagonal model, which remains for diffeomorphism invariant wave functions. Let us therefore rewrite the fundamental variables h_{pq} in the form

$$h_{pq} = \sum_{r=1}^{3} \Omega_{pr} \lambda_r \Omega_{qr}, \tag{3.20}$$

where the SO(3) matrix (Ω_{pr}) depends on three Euler angles φ_a. After a subsequent transformation to the σ_p defined in Eq. (3.14) the auxiliary scalar product (2.18) transforms into

$$\langle \Psi | \Phi \rangle \propto \int d\Omega(\varphi_a) \int d^3 \sigma_p \frac{|(\sigma_1^2 - \sigma_2^2)(\sigma_2^2 - \sigma_3^2)(\sigma_3^2 - \sigma_1^2)|}{(\sigma_1 \sigma_2 \sigma_3)^{3/2}}$$

$$\times \Psi^*(\sigma_p) \Phi(\sigma_p), \tag{3.21}$$

where $d\Omega(\varphi_a)$ is the invariant Haar measure on the group manifold of SO(3) and the variables σ_p must be positive and of a fixed order, e.g., $\sigma_3 > \sigma_2 > \sigma_1 > 0$.[8] The Euler angles are pure gauge degrees of freedom and have to be gauge fixed at some arbitrary value in the physical scalar product, which is simply done by dropping the factor $\int d\Omega$ in Eq. (3.21), which is just a constant prefactor, anyway.

Remarkably, we get the same result for the effective inner product, if we start from an auxiliary inner product in the triad variables e_{pa}, which are defined via

$$e_{pa} \cdot e_{qa} = h_{pq}. \tag{3.22}$$

If we impose the simple inner product

$$\langle \Psi | \Phi \rangle = \int_{R^9} d^9 e_{pa} \psi^*(e_{pa}) \phi(e_{pa}) \tag{3.23}$$

in these triad coordinates, we have to eliminate the gauge freedoms with respect to the diffeomorphism *and* the Gauss constraints. To fix all these gauge freedoms, we may impose the six gauge conditions

$$\chi^p_a = \tilde{e}^p_a = 0, \quad p \neq a, \tag{3.24}$$

which require the triad to be diagonal. Then the effective measure in the scalar product (3.23) can be calculated as a Faddeev-Popov determinant of the commutators between the generators $\mathcal{H}_p, J'_a$ of the gauge transformations and the gauge conditions (3.24). If we denote the remaining diagonal elements of $\tilde{e}^p_a$ by σ_p, and, moreover, take into consideration all the symmetries of the wave functions with respect to permutations of the σ_p, we arrive exactly at the scalar product given in Eq. (3.21).

In the diagonal Bianchi type IX model a natural scalar product is

$$\langle \Psi | \Phi \rangle \propto \int_{R^3} d\alpha \, d\beta_+ \, d\beta_- \Psi^*(\alpha, \beta_\pm) \Phi(\alpha, \beta_\pm), \tag{3.25}$$

because the variables α, β_+, and β_- introduced in [12] are the free falling coordinates on minisuperspace for the diagonal model. Transforming this to the σ_p representation, we arrive at

$$\langle \Psi | \Phi \rangle \propto \int_{\sigma_p > 0} \frac{d\sigma_1 \, d\sigma_2 \, d\sigma_3}{\sigma_1 \sigma_2 \sigma_3} \Psi^*(\sigma_p) \Phi(\sigma_p), \tag{3.26}$$

where the three σ_p integrals have to be performed along the positive real axes now. Obviously, the additional factors $\sigma_p^2 - \sigma_q^2$ occurring in Eq. (3.21) are absent in the diagonal case; i.e., the measures in the scalar product of the diagonal and the nondiagonal model differ, as was already expected when discussing point (ii) in Sec. III A.

We conclude that quantization and gauge fixing are essentially not interchangeable for the model under investigation. The quantization procedure followed in Sec. II seems preferable because it seems much nearer to a quantization of the full, inhomogeneous field than the methods used for the diagonal case. Moreover, all four constraints $\mathcal{H}_\mu$ are quantized and in this sense treated in a similar way, as one would expect from the viewpoint of general covariance.

To complete our discussion of the inner product on the space of wave functions, we should give the *physical* inner product, which is obtained from the *auxiliary* inner product (3.21) or (3.26) by eliminating the last gauge degree of freedom corresponding to the Hamiltonian constraint $\mathcal{H}_0$.

C. Physical inner product on the Hilbert space of wave functions

To shorten the present paper, we want to postpone a detailed motivation and derivation of the physical inner product

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[7]Such a factor ordering term is well known from earlier work; cf. [23].

[8]The transformation (3.20) turns out to be invertible only for a fixed order of the λ_p (or, eqivalently, of the σ_p), because permutations of the λ_p may be realized by suitable rotations with SO(3) matrices Ω_{pr}.

to a future paper [19]. There, we will also investigate the normalizability of the wave functions constructed in Sec. IV, and it will turn out that the physical inner product uniquely determines one normalizable state in both of the two cases $\Lambda=0$ and $\Lambda>0$. In this subsection we only want to give the final form of the physical inner product on the space of wave functions and some main ideas how to arrive there [24–26].

The effective inner product for a constrained system with constraint $T=0$ is generally obtained by first choosing a suitable gauge condition χ and, second, calculating the following expectation value with respect to the auxiliary (or "kinematic") inner product:

$$\langle\Psi||\Phi\rangle_{phys}=\langle\Psi||J|\delta(\chi)|\Phi\rangle_{aux}, \qquad (3.27)$$

with

$$J:=i[T,\chi].$$

Here J is the so-called Faddeev-Popov determinant. This definition makes sense whenever T is a first order derivative operator on configuration space, because then $|J|$ will be a positive real number and Eq. (3.27) defines a positive Hermitian product. However, in the present case $T=\hat{\mathcal{H}}_0$ is a second order derivative operator, and we have to deal with two problems.

(i) At first sight, $|J|$ seems to be a mysterious quantity, because J itself is a first order derivative operator on configuration space. However, since J is a self-adjoint operator, we may decompose any state Ψ in terms of eigenstates of J to arrive at a very natural notion of $|J|$. This operator $|J|$ then is itself self-adjoint and, moreover, *positive*.

(ii) In general, $|J|$ and $\delta(\chi)$ will not commute, and so $\langle\Psi||\Phi\rangle_{phys}$ will neither be Hermitian nor positive. This would give a rather miserable physical inner product. Surprisingly, there exists a small set of gauge conditons χ such that the gauge condition commutes with the Faddeev-Popov determinant, i.e., such that χ solves

$$[[T,\chi],\chi]=0. \qquad (3.28)$$

It seems that this set of gauge conditions does not only exist for the case under study, $T=\hat{\mathcal{H}}_0$, but for any physically interesting case of constraints which are described by second order derivative operators. Using these special gauge conditions the physical inner product (3.27) obviously *is* positive and Hermitian.

Having solved these two problems, we are now ready to perform the last step in the construction of the physical inner product for the nondiagonal Bianchi type IX model. It is easily seen that the gauge condition

$$\chi(\sigma_1,\sigma_2,\sigma_3)=\frac{2}{\hbar}\ln\frac{\sigma_3}{\sigma_{3,0}} \qquad (3.29)$$

is a solution to Eq. (3.28). It fixes the variable σ_3 to have a specific value $\sigma_3\equiv\sigma_{3,0}$. Performing suitable coordinate transformations the associated inner product finally takes the following form:

$$\langle\Psi||\Phi\rangle_{\sigma_3}=\int_0^\infty d\beta_-\int_0^\infty dv\,\bar{\Psi}^*\left|i\hbar\frac{\partial}{\partial v}\right|\bar{\Phi}, \qquad (3.30)$$

with

$$\bar{\Psi}=\left[\frac{(\sigma_1^2-\sigma_2^2)(\sigma_2^2-\sigma_3^2)(\sigma_3^2-\sigma_1^2)}{\sigma_1\sigma_2\sigma_3}\right]^{1/2}\Psi, \qquad (3.31)$$

$$v=a^3=\left(\frac{\hbar}{2}\right)^{3/2}\sqrt{\sigma_1\sigma_2\sigma_3}, \qquad \beta_-=\frac{1}{2\sqrt{3}}\ln\frac{\sigma_2}{\sigma_1}.$$

This inner product has now an intrinsic "time" parameter σ_3, which must be given to compute probability distributions with respect to β_- and v, the remaining physical degrees of freedom.

IV. SOLUTIONS GENERATED FROM THE CHERN-SIMONS FUNCTIONAL

A. Ashtekar representation of the Chern-Simons state

As pointed out in Sec. II, we can find a special exact quantum state for the nondiagonal Bianchi type IX model with a nonvanishing cosmological constant, if we are able to determine a function ψ' lying in the kernel of $\mathcal{Q}^p_a$ defined in Eq. (2.35). To proceed in this way, let us consider the Fourier transformed version of Eq. (2.39), where it is essential to perform not a customary, but a generalized Fourier transform introduced in [12]:

$$\psi'(\widetilde{e}^p_a)\propto\int_{\Sigma^9}d^9A_{pa}\,\exp\left[\frac{2}{\hbar}A_{pa}\widetilde{e}^p_a\right]\tilde{\Psi}(A_{pa}). \qquad (4.1)$$

Here $\Sigma^9\subset\mathbf{C}^9$ is any 9-dimensional manifold in the 18-dimensional space of the complex A_{pa}, which allows for partial integration without getting any boundary terms. Then $\psi'(\widetilde{e}^p_a)$ is a solution to Eq. (2.39), if $\tilde{\Psi}(A_{pa})$ solves the following set of equations:

$$\left[m^{pq}A_{qa}-\frac{1}{2}\varepsilon^{pqr}\varepsilon_{abc}A_{qb}A_{rc}+\frac{\lambda}{2}\frac{\partial}{\partial A_{pa}}\right]\tilde{\Psi}(A_{pa})=0. \qquad (4.2)$$

A special solution to Eq. (4.2) is the Bianchi type IX restriction of the well-known Chern-Simons functional:

$$\tilde{\Psi}_{CS}=\exp\left[\frac{1}{\lambda}[-A_{pa}m^{pq}A_{qa}+2\,\det(A_{pa})]\right]. \qquad (4.3)$$

In the following we shall be interested in the transformation of this state to the metric representation, i.e., in the evaluation of integrals of the form (4.1). Topologically different choices of Σ^9 (i.e., choices which cannot be deformed into each other without passing a singularity) will lead to different states in the metric representation.

B. Metric representation of the Chern-Simons state

It will be convenient to introduce new variables

$$\kappa^p_a := \frac{\lambda}{\hbar}\widetilde{e}^p_a = \frac{\Lambda}{3}\widetilde{e}^p_a. \tag{4.4}$$

Then the wave function in the metric representation takes the form

$$\Psi'_{CS} \propto \int_{\Sigma^9} d^9 A_{pa}\, \exp\!\left[\frac{2}{\lambda}\!\left(\kappa^p_a A_{pa} - \frac{1}{2} A_{pa} m^{pq} A_{qa}\right.\right.$$
$$\left.\left. + \det(A_{pa})\right)\right], \tag{4.5}$$

where suitable integration manifolds Σ^9 remain to be determined. Here and in the following, constant prefactors, and also prefactors depending on λ, are absorbed in the proportionality sign ''$\propto$,'' since such a factor will always remain as a freedom of the wave function. Surprisingly, six of the nine A_{pa} integrals turn out to be Gaussian integrals and can be evaluated analytically, namely, the integrals with respect to A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, and A_{23}. Lengthy calculations finally give the result

$$\Psi'_{CS} \propto \int_{\Sigma^3} \frac{dx\, dy\, dz}{1-\vec{r}^2}\exp\!\left[\frac{1}{\lambda}\!\left(-\vec{r}^2 + 2\vec{\kappa}_3\cdot\vec{r}\right.\right.$$
$$\left.\left. + \frac{\vec{\kappa}_1^2 + \vec{\kappa}_2^2 + 2\vec{\kappa}_1\times\vec{\kappa}_2\cdot\vec{r} - (\vec{r}\cdot\vec{\kappa}_1)^2 - (\vec{r}\cdot\vec{\kappa}_2)^2}{1-\vec{r}^2}\right)\right], \tag{4.6}$$

where we have introduced the abbreviations $\vec{r} = (x,y,z)$ $:= (A_{31}, A_{32}, A_{33})$ and $\vec{\kappa}_p := (\kappa^p_1, \kappa^p_2, \kappa^p_3)$. It is clear that the three-direction is distinguished by the order in which the integrals are performed. As we know from Sec. II, Ψ'_{CS} according to Eq. (4.6) is automatically Gauss and diffeomorphism invariant, because it is a solution to Eq. (2.39). This means that Ψ'_{CS} can only depend on the eigenvalues λ_p of the three-metric h_{pq}, or, equivalently, on the σ_p introduced in Eq. (3.14). So we are free to choose a diagonal gauge for the triad in Eq. (4.6); i.e., we may take

$$(\kappa^p_a) = \mathrm{diag}\,(\kappa_1, \kappa_2, \kappa_3), \tag{4.7}$$

with

$$\kappa_3 \geqslant \kappa_2 \geqslant \kappa_1,$$

where

$$\kappa_p := \frac{\lambda}{2}\sigma_p = \frac{\Lambda}{12}\sqrt{\lambda_q\lambda_r}, \quad \varepsilon_{pqr}=1. \tag{4.8}$$

This special ordering of the three κ_p (or, equivalently, of the λ_p) is always possible, because permutations of the diagonal elements of a diagonal metric h_{pq} correspond to diffeomorphisms, which leave the wave functions invariant. Moreover,

such an ordering is even necessary to have a unique mapping $(\kappa^p_a) \rightarrow (\kappa_1, \kappa_2, \kappa_3)$ under Gauss and diffeomorphism transformations, and we will adopt the convention (4.7) throughout the following. In the gauge (4.7) the integral (4.6) takes the form

$$\Psi'_{CS} \propto \int_{\Sigma^3} \frac{dx\, dy\, dz}{1-r^2}\exp\!\left[\frac{1}{\lambda}\!\left(-r^2 + 2\kappa_3 z\right.\right.$$
$$\left.\left. + \frac{\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2 z - \kappa_1^2 x^2 - \kappa_2^2 y^2}{1-r^2}\right)\right], \tag{4.9}$$

where $r^2 := x^2 + y^2 + z^2$. We will see in Secs. V and VI that there are choices for Σ^3 in Eq. (4.9) leading to asymmetric wave functions under (formal) permutations of the κ_p.[9] However, as we know from Sec. III [cf. after Eq. (3.16) with $\kappa_p = (\lambda/2)\sigma_p$], only wave functions which are symmetric under permutations of the κ_p and under reflections $\kappa_p \rightarrow -\kappa_p, \kappa_q \rightarrow -\kappa_q, \kappa_r \rightarrow \kappa_r$ are of interest to us, and this will be an important restriction to select the physically interesting wave functions.

V. ASYMPTOTIC FORMS OF THE CHERN-SIMONS INTEGRAL

In order to get information about possible integration manifolds Σ^3 which can be used in Eq. (4.9) we begin by discussing the asymptotic behavior of the Chern-Simons integral in several asymptotic regimes. It will thereby become obvious that we deal with five linearly independent solutions, and the integration contours will be given in Sec. VI afterwards.

A. Semiclassical limit $\hbar \rightarrow 0$

Let us first of all examine the semiclassical behavior of the solutions described by Eq. (4.9). Surely, the semiclassical limit could have also been discussed by starting from the nine-dimensional integral (4.5), but, of course, an expansion of the three-dimensional integral (4.9) is much simpler. The saddle point form of Eq. (4.9) is displayed nicely by writing it in the form

$$\Psi'_{CS} \propto \int \frac{dx\, dy\, dz}{1-r^2}\exp\!\left[\frac{3F}{\hbar\Lambda}\right], \tag{5.1}$$

with

$$F := -r^2 + 2\kappa_3 z + \frac{\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2 z - \kappa_1^2 x^2 - \kappa_2^2 y^2}{1-r^2}, \tag{5.2}$$

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[9]In complete analogy there are eigenstates of the effective Hamiltonian (3.9) of a two-dimensional harmonic oscillator with $L=0$ which do *not* obey $\Psi(q) = \Psi(-q)$, but these are of no physical interest.

where we have inserted λ according to Eq. (3.19). Since the κ_p defined in Eq. (4.8) do not depend on $\hbar$, we approach a Gaussian integral in the limit $\hbar \to 0$, Λ fixed, which has to be evaluated at one of the saddle points of the exponent. However, *which* of the possible saddle point contributions arises for the integral under consideration is determined by the integration surface Σ^3 and requires a detailed discussion of the contours of steepest descent. Here we only want to give the *possible* asymptotic results for $\hbar \to 0$ which may be realized by suitable choices of the integration contours. The integration contours are discussed in Sec. VI.

The saddle points of the exponent are obtained by solving the equations

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0. \tag{5.3}$$

One can show that for $\sigma_1, \sigma_2, \sigma_3$ pairwise different, i.e., in particular on the sector $\sigma_3 > \sigma_2 > \sigma_1 > 0$ of interest, the only solution to Eq. (5.3) is given by

$$x = y = 0, \quad (z - \kappa_3)(1 - z^2)^2 = (\kappa_1 + \kappa_2 z)(\kappa_2 + \kappa_1 z). \tag{5.4}$$

A comparison with the diagonal model [12] reveals that for $x = y = 0$ the exponents of Eq. (5.1) and of the one-dimensional integral (4.17) of [12] become identical. Consequently, also the saddle point equations with respect to z coincide; cf. Eqs. (5.4) and (4.7) of [12]. As a result, the semiclassical actions of the wave functions approached in the limit $\hbar \to 0$ must be the same, since they are given as the saddle point values of the integrand's exponents. This is of course exactly what we expected to happen, because the Wheeler-DeWitt equations for the classically equivalent diagonal and nondiagonal Bianchi type IX models give rise to the same Hamilton-Jacobi equation in the limit $\hbar \to 0$. However, the quantum corrections to this leading order behavior are rather different, first because of the different dimensionality of the integrals, and second because of a new prefactor to the exponential function. Performing a Gaussian saddle point approximation, we get, for the nondiagonal case,

$$\Psi'_{CS} \overset{\hbar \to 0}{\propto} \left[\frac{(-2\pi\lambda)^3}{\det\left(\frac{\partial^2 F}{\partial r_a \partial r_b} \right)} \right]^{1/2} \frac{e^{F/\lambda}}{1 - r^2} \Bigg|_{\vec{r} = \vec{r}_s}$$

$$\propto \left[\frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 F}{\partial z^2} \right]^{-1/2} \frac{e^{F/\lambda}}{1 - r^2} \Bigg|_{\vec{r} = \vec{r}_s}. \tag{5.5}$$

Here the second proportionality stems from a special property of the exponent F, namely, the fact that the mixed derivatives of F vanish for $x = y = 0$. Comparing this with the asymptotic results obtained in the diagonal case [12], we find that corresponding results differ only by a factor

$$\gamma := \left[(1 - z^2) \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 F}{\partial y^2} \right]^{-1/2}_{x = y = 0, z = z_s}. \tag{5.6}$$

Since we already know the asymptotic results for the diagonal Bianchi type IX model, we only have to expand this additional factor at the saddle points to get the quantum corrections for the nondiagonal Bianchi type IX model. As in the diagonal case, we therefore have to deal with a saddle point equation of fifth order, and so analytical results are only available in additional asymptotic regimes. Because of the existence of five saddle points, one has five linearly independent solutions.

1. Limit $\Lambda \to 0$

One asymptotic regime which allows for an expansion of the five saddle points is that of small cosmological constant $\Lambda \to 0$. Since we are in the additional limit $\hbar \to 0$, the quantity $\lambda = \hbar \Lambda / 3$ defined in Eq. (3.19) tends to zero, too. In this limit one saddle point is given by

$$z \overset{\lambda \to 0}{\sim} \frac{1}{2} \sigma_3 \lambda \overset{\lambda \to 0}{\to} 0. \tag{5.7}$$

The contribution from this saddle point yields

$$\Psi'_{CS} \overset{\lambda \to 0}{\propto} \text{const}, \tag{5.8}$$

and therefore approaches the well-known wormhole state.[10]

Four further saddle points asymptotically lie at[11]

$$z \overset{\lambda \to 0}{\sim} -1 \pm \frac{\lambda}{4}(\sigma_2 - \sigma_1), \quad z \overset{\lambda \to 0}{\sim} 1 \pm \frac{\lambda}{4}(\sigma_2 + \sigma_1), \tag{5.9}$$

giving rise to an asymptotic behavior

$$\Psi'_{CS} \overset{\lambda \to 0}{\propto} \frac{e^{-\sigma_3 \mp (\sigma_2 - \sigma_1)}}{\sqrt{(\sigma_2 - \sigma_1)(\sigma_3 \pm \sigma_2)(\sigma_3 \mp \sigma_1)}},$$

$$\Psi'_{CS} \overset{\lambda \to 0}{\propto} \frac{e^{\sigma_3 \mp (\sigma_2 + \sigma_1)}}{\sqrt{(\sigma_1 + \sigma_2)(\sigma_3 \mp \sigma_1)(\sigma_3 \mp \sigma_2)}}, \tag{5.10}$$

respectively. These solutions are the generalizations of the diagonal analogues (2.25) and (2.26) given in [12]. The singularities which occur in the denominators of the asymmetric solutions of Eq. (5.10) are cancelled by the weight function of the inner product (3.21) and therefore do not constitute any physical problems.

2. Limit $\kappa \to \infty$

A second regime which allows for an analytical expansion of the semiclassical limit $\hbar \to 0$ is the case $\kappa \to \infty$, where the "mean" κ is understood as $\kappa := (\kappa_1 \kappa_2 \kappa_3)^{1/3}$. The three κ_p were defined in Eq. (4.8). For a fixed cosmological constant,

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[10] We remind the reader that the prime at Ψ'_{CS} denotes the fact that this wave function has to be mutliplied by Ψ_{WH} defined in Eq. (2.24) to become a solution of the Wheeler-DeWitt equation in the metric representation.

[11] The following expansions are only valid for $\sigma_2 > \sigma_1$, but we restricted ourselves to this case anyway.

this is the limit of the large overall scale paramter. In this limit, two of the saddle points behave according to

$$z \overset{\kappa \to \infty}{\sim} \pm \sqrt{-\frac{\kappa_1 \kappa_2}{\kappa_3}}, \qquad (5.11)$$

and turn out to be complex valued. This gives rise to a complex action of the wave function, describing a universe with a Lorentzian signature of the four-metric; cf. [12]. The leading order behavior for these solutions is given by

$$\Psi'_{CS} \overset{\kappa \to \infty}{\propto} \kappa^{-9/4} \exp\left[\pm \frac{4i\kappa^{3/2}}{\lambda}\right]; \qquad (5.12)$$

for a more detailed asymptotic result we refer to [12]. Three further saddle points have an asymptotic expansion

$$z \overset{\kappa \to \infty}{\sim} -\frac{\kappa_1}{\kappa_2}, \quad z \overset{\kappa \to \infty}{\sim} -\frac{\kappa_2}{\kappa_1}, \quad z \overset{\kappa \to \infty}{\sim} 1 + \kappa_3, \quad (5.13)$$

and lead to wave functions of the form

$$\Psi'_{CS} \overset{\kappa \to \infty}{\propto} [(\kappa_2^2 - \kappa_1^2)(\kappa_3^2 - \kappa_2^2)]^{-1/2} e^{\kappa_2^2/\lambda},$$

$$\Psi'_{CS} \overset{\kappa \to \infty}{\propto} [(\kappa_2^2 - \kappa_1^2)(\kappa_3^2 - \kappa_1^2)]^{-1/2} e^{\kappa_1^2/\lambda}, \qquad (5.14)$$

$$\Psi'_{CS} \overset{\kappa \to \infty}{\propto} [(\kappa_3^2 - \kappa_1^2)(\kappa_3^2 - \kappa_2^2)]^{-1/2} e^{\kappa_3^2/\lambda},$$

respectively. These saddle point contributions are known from the diagonal model for the "asymmetric" solutions introduced there; however, here we have additional prefactors, which become divergent on the symmetry lines $\sigma_p = \sigma_q, p \neq q$, but again these singular terms are cancelled by the measure of the scalar product (3.21).

B. Solutions to the vacuum model approached in the limit $\Lambda \to 0$

In order to get solutions of the nondiagonal Bianchi type IX model for $\Lambda \to 0$ (without taking the semiclassical limit $\hbar \to 0$ first) we now want to discuss the behavior of the Chern-Simons integral in this limit. While one of the solutions approached for $\Lambda \to 0$ again turns out to be the wormhole state (2.24), the other four solutions are *not* given by the asymptotic results (5.10), as one might think at first sight. The reason for this is the fact that we are not allowed to perform the usual Gaussian saddle point expansion, since the exponent and, in particular, the prefactor to the exponential function become singular in the limit $\Lambda \to 0$, where r approaches ± 1. Comparing the situation with the diagonal model, where a saddle point expansion in the same limit actually *was* allowed, one might ask for a difference between the two models that forbids an analogous expansion in the present case. The answer to this question lies hidden in the prefactor to the exponential function: while we dealt with an integrable square-root singularity in the diagonal case, we here have to integrate into a singularity of first order, even in the case $\Lambda = 0$, and this requires much caution. In fact, it turns out that the calculation of the limit $\Lambda \to 0$ of the Chern-Simons integral is very subtle and worth discussing in a

separate section in Appendix A. Moreover, we show in Appendix B that the solutions derived in Appendix A can be written in another, very nice and compact form, which reads

$$\Psi'_\varrho \overset{\lambda \to 0}{\sim} = \int_{\mathcal{C}_\varrho} ds \, e^s \prod_{\nu=0}^3 (s - s_\nu)^{-1/2}. \qquad (5.15)$$

Here the integration contour $\mathcal{C}_\varrho$ is one of the four curves shown in Fig. 1, and the s_ν are special sums of the σ_p, defined by

$$s_0 = \sigma_1 + \sigma_2 + \sigma_3, \quad s_1 = \sigma_1 - \sigma_2 - \sigma_3,$$

$$\qquad\qquad (5.16)$$

$$s_2 = -\sigma_1 + \sigma_2 - \sigma_3, \quad s_3 = -\sigma_1 - \sigma_2 + \sigma_3.$$

For a detailed discussion of these wave functions and a comment on their symmetries with respect to permutations of the σ_p we refer to Appendixes A and B.

Altogether, we find in the limit $\Lambda \to 0$ again five linearly independent solutions associated with the Chern-Simons wave function (4.5).

VI. THREE-DIMENSIONAL INTEGRATION MANIFOLDS FOR FIVE EXACT SOLUTIONS

In this section we want to define explicitly five integration contours Σ^3 for the integral (4.9), leading to five linearly independent solutions of the Wheeler-DeWitt equation (3.17). To have a better view of the symmetries of the solutions, let us, for a moment, go back to the nine-dimensional integral representation (4.5). It can be verified easily that the two different integration surfaces

$$\Sigma^9_\pm := \{(A_{pa})\epsilon \mathbf{C}^9 | A_{pa} \in \mathbf{R} e^{\pm i\pi/6}\} \qquad (6.1)$$

lead to a finite integral, because the cubic terms in the exponent of Eq. (4.5) remain purely imaginary, while all the quadratic terms have a negative real part. Moreover, since the
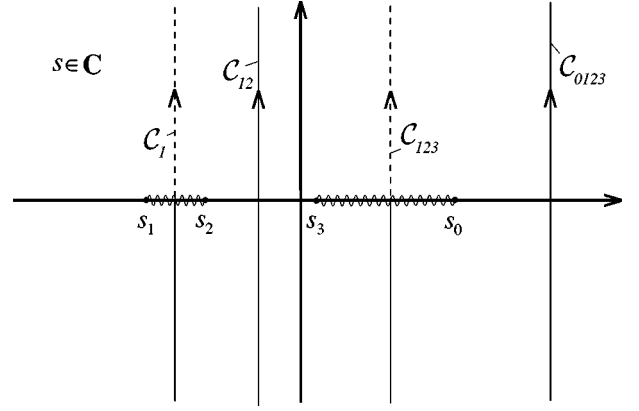

FIG. 1. Integration curves in the complex s plane which lead to linearly independent vacuum solutions. The dashed line indicates that one has to evaluate the integrand in the second Riemannian branch after having crossed one of the cuts. The two cuts are represented by wavy lines.

only border of this surface lies at infinity, it is clear that the integral (4.5), performed over $\Sigma_\pm^9$, must lead to a solution of the nondiagonal Bianchi type IX model. In particular, we then know that these wave functions are Gauss and diffeomorphism invariant, and we may again choose the diagonal gauge (4.7) for the matrix (κ_a^p). Then Eq. (4.5) takes the form

$$\Psi'_{CS,\pm} \propto \int_{\Sigma_\pm^9} d^9 A_{pa} \exp\left[\frac{2}{\lambda}\left(\kappa_1 A_{11} + \kappa_2 A_{22} + \kappa_3 A_{33}\right.\right.$$
$$\left.\left. -\frac{1}{2} A_{pa} m^{pq} A_{qa} + \det(A_{pa})\right)\right]. \quad (6.2)$$

If we now consider a formal permutation $\kappa_1 \leftrightarrow \kappa_2$ in Eq. (6.2), we can reestablish the original integral by a suitable transformation in the A_{pa} space, namely, $A_{11} \leftrightarrow A_{22}, A_{32} \leftrightarrow A_{13}, A_{23} \leftrightarrow A_{31}$. Since all integration variables A_{pa} are integrated along the same axes in the complex A_{pa} planes, the integration contour remains unchanged under this coordinate transformation and we regain the integral (6.2). Furthermore, the two integrals (6.2) are invariant under a substitution $\kappa_1 \rightarrow -\kappa_1, \kappa_2 \rightarrow -\kappa_2$, which may be seen by a transformation $A_{1a} \rightarrow -A_{1a}, A_{2a} \rightarrow -A_{2a}$ in the A_{pa} space. Thus we conclude that both wave functions are completely symmetric under arbitrary permuations of the κ_p, and also under reflections $\kappa_p \rightarrow -\kappa_p, \kappa_q \rightarrow -\kappa_q, \kappa_r \rightarrow \kappa_r$, a symmetry property which is possessed only by two of the five solutions which are generated by the Chern-Simons functional.

Performing the six Gaussian integrations which lead from Eq. (4.5) to Eq. (4.9), we reach the following three-dimensional integration surface for the integral (4.9):

$$\Sigma_\pm^3 := \{\vec{r} \in \mathbf{C}^3 | r_a \in \mathbf{R} e^{\pm i\pi/6}\}. \quad (6.3)$$

In the following it will be convenient to introduce the two superpositions

$$\Psi'_{WH,\Lambda} \propto \Psi'_{CS,+} + \Psi'_{CS,-}, \quad \Psi'_{0123,\Lambda} \propto \Psi'_{CS,+} - \Psi'_{CS,-}, \quad (6.4)$$

and also new integration variables

$$x = r\sqrt{1-\eta^2}\cos\varphi, \quad y = r\sqrt{1-\eta^2}\sin\varphi, \quad z = r\eta, \quad (6.5)$$

for the x, y, z integral, which may be understood as spherical coordinates with $\eta = \cos\vartheta$. For the transformation (6.5) the volume element simply transforms like $dx\,dy\,dz = r^2 dr\,d\eta\,d\varphi$, and the condition $r^2 = x^2 + y^2 + z^2$ is obeyed even for complex values of r, η, φ. With this new parametrization, the integration surfaces for the two solutions defined in Eq. (6.4) can now be written in the following form:

$$\Sigma_{WH}^3 = \{\vec{r} \in \mathbf{C}^3 | -1 < \eta < 1, 0 < \varphi < 2\pi, r \in \mathbf{R} + i\varepsilon\}, \quad \varepsilon > 0, \quad (6.6)$$

$$\Sigma_{0123}^3 = \{\vec{r} \in \mathbf{C}^3 | -1 < \eta < 1, 0 < \varphi < 2\pi, r \in K(1, R)\},$$
$$R < 2. \quad (6.7)$$

In Eq. (6.7) $K(1, R)$ denotes a pole integral around $r = 1$ with radius R; cf. Appendix A. Let us try to interpret these integration surfaces: in the first case (6.6) we simply integrate over a sphere with radius r in the complex $\vec{r}$ space; however, this radius is not chosen to be real valued as usual, but slightly imaginary to avoid the singularities of the integrand lying at $r = \pm 1$. The integration manifold Σ_{0123}^3 also describes an integral over a full sphere, but now the radius describes itself as a circle around the singularity at $r = +1$ in the complex r plane. While the two parametrizations given in Eqs. (6.6) and (6.7) will prove useful for an asymptotic expansion in the limit $\Lambda \rightarrow 0$, which is performed in Appendix A, they now allow for a further study of the semiclassical behavior of the corresponding solutions.

If we consider the wave function $\Psi'_{0123,\Lambda}$ in the semiclassical limit $\hbar \rightarrow 0$, we get the dominant contributions to the integral for $\eta = \pm 1$, so that the integrand becomes independent of φ. The remaining r variable then coincides with the $\pm z$ variable, which was the only integration variable in the diagonal Bianchi type IX model discussed in [12]. For a discussion of the curves of steepest descent in the z plane we refer to this earlier work. However, if one has finally deformed a desired integration contour into the curves of steepest descent, there remains one remarkable difference between the two models: while in the diagonal case we had a cut in the complex z plane for $|\mathrm{Re}\,z| > 1$, this cut is absent in the present nondiagonal case, owing to the different prefactors to the exponential function. So integrals along parts of the real axis of the z plane, which cancelled in the diagonal case, may now be different from zero, and vice versa. A detailed discussion of the curves of steepest descent and the resulting asymptotic contributions finally gives the following result for the wave functions Ψ, being connected with Ψ' via Eq. (2.25).

In the semiclassical limit $\hbar \rightarrow 0$ the wave function $\Psi_{0123,\Lambda}$ shows a no-boundary behavior for $a \rightarrow 0$; i.e., the four-manifolds which correspond to the semiclassical trajectories in minisuperspace are regular at the point $a = 0$. For $a \rightarrow \infty$, the integration surface Σ_{0123}^3 picks up several saddle point contributions: the two complex conjugate saddle points (5.11) contribute as well as the two real-valued, negative saddle points in Eq. (5.13). These latter two saddle points are the reason why $\Psi_{0123,\Lambda}$ is *not* normalizable in the physical inner product (3.30), as will be shown in a future paper [19].

The solution $\Psi_{WH,\Lambda}$ is a generalization of the wormhole state (2.23) for a positive cosmological constant $\Lambda > 0$. For $a \rightarrow \infty$, we again gather contributions from the two complex conjugate saddle points (5.11) and now, in addition, from the real-valued *positive* saddle point $z \sim 1 + \kappa_3$ of Eq. (5.13). In [19] we will see that these latter three saddle points give rise to finite contributions in the physical norm of the wave function. We will show this by evaluating the physical norm according to Eq. (3.30) in the semiclassical limit $\hbar \rightarrow 0$. As a result, $\Psi_{WH,\Lambda}$ turns out to be the only Chern-Simons-like quantum state, which is normalizable in the physical inner product (3.30) *and* symmetric under permutations and reflec-

tions of the σ_p. The existence of a normalizable Chern-Simons-like state is in remarkable contrast to earlier work on special spatially homogeneous models discussed by Marugán [27].

Finally, we still have to define three further integration contours Σ^3 to obtain the complete set of Chern-Simons-like solutions. The construction of the vacuum solutions in Appendix A suggests three new possibilities to create linearly independent integrals. One of these further integration surfaces can simply be written in the form

$$\Sigma_{12}^3 = \{\vec{r} \in \mathbf{C}^3 \,|\, -\infty < \eta < -1, 0 < \varphi < 2\pi, r \in K(1,R)\},$$
(6.8)

while the other two have a very complicated form and may be read off from the representations (A20) and (A21) in Appendix A. That these latter two integration manifolds, despite their strange η junctions at $-\sigma_1/\sigma_2$ and $-\sigma_2/\sigma_1$, give rise to analytical solutions of our model even for $\Lambda \neq 0$ is shown in Appendix C. However, these solutions do not obey the necessary symmetry properties which were derived in Sec. III, and are therefore of no further interest to us.

One may now finally ask how the four solutions defined in Eqs. (A18)–(A21) of Appendix A are connected to the "asymmetric" solutions Ψ_ϱ' of the diagonal model discussed in [12]. An investigation of the semiclassical behavior, together with a discussion of the singularities on the lines $\sigma_p = \sigma_q, p \neq q$, shows that the solutions Ψ_1', Ψ_{12}', Ψ_{123}', and Ψ_{0123}' are actually the *sums* of these states as indicated through the choice of their indices; i.e., we have

$$\Psi_{12}' = \Psi_1' + \Psi_2', \qquad \Psi_{123}' = \Psi_1' + \Psi_2' + \Psi_3',$$

$$\Psi_{0123}' = \Psi_0' + \Psi_1' + \Psi_2' + \Psi_3', \qquad \sigma_3 > \sigma_2 > \sigma_1.$$
(6.9)

By inverting these relations with respect to Ψ_ϱ' one may further define asymmetric solutions corresponding to those of the diagonal model, with the same symmetry properties that have been pointed out in [12].

VII. DISCUSSION AND CONCLUSION

In this paper we have examined the transformation connecting the representations of quantum general relativity in metric variables and in Ashtekar's variables for the special case of spatially homogeneous but anisotropic space-times of a Bianchi type IX model with a *nondiagonal* metric tensor. While classically the nondiagonal case and the diagonal case are equivalent (in the absence of matter) due to the freedom of gauge fixing, there is a subtle difference quantum mechanically, because the steps of gauge fixing and quantization, in general, do not commute. This was explained via a simple example in Sec. III A. The example also made clear that gauge fixing *after* quantization is preferable because all symmetries are then implemented automatically and democratically. Once the two steps have been completed in this order it is then also clear *a posteriori* how to proceed correctly in the quantization after the gauge fixing has been

performed on the classical level. Performed in this way quantization and gauge fixing of course commute by construction.

For the nondiagonal Bianchi type IX model we have considered in detail the transformation of the Chern-Simons state from Ashtekar variables to metric variables. The Chern-Simons state, including its limit for vanishing cosmological constant, deserves a thorough study because it is undoubtedly the most important exact solution of all constraints of quantum general relativity found up until now. This is so because, unlike all other exact solutions, it describes a well-defined nondegenerate space-time in its classical limit and also because it makes an obvious connection between quantum general relativity and topological field theory [7,8]. Indeed it is remarkable that the general Chern-Simons state in Ashtekar variables makes no reference to metric concepts on the spacelike three-manifold. It should be noted, however, that writing down a physical state in Ashtekar variables does not yet define it completely, because "reality conditions" must still be imposed before a physical interpretation can be attempted. Imposing the reality conditions is a very nontrivial task and, as our results indicate, might not have a *unique* solution. We circumvent this problem completely by transforming back to the real metric representation before applying a gauge fixing and giving a physical interpretation.

The Ashtekar variables and the densitized inverse triad form canonically conjugate pairs. Hence, for our Bianchi type IX model the generalized multidimensional Fourier transformation we discussed in Secs. IV, V, and VI can be used to transform the Chern-Simons state to the metric representation. This is a *generalized* Fourier transformation because neither the integration contours, which are here nine-dimensional manifolds, nor their boundaries are fixed *a priori*, except for the condition that partial integration with the Chern-Simons state under the integral must be permitted without contribution from the boundaries. The boundaries are therefore determined entirely by the *singularities* of the Chern-Simons state. On the other hand, for fixed boundaries the different integration manifolds one can find may be deformed without changing the result, or may be combined by first running through one integration manifold and then through other inequivalent ones, leading to linear combinations of the physical states defined by each integration manifold separately.

In Eq. (4.9), with the five integration manifolds Σ^3 given by Eqs. (6.6), (6.7), (6.8), (A20), and (A21), we have obtained integral representations of five *exact* solutions to all constraints of quantum general relativity and we studied various limits of these solutions in Sec. V. In the leading semiclassical order the result for the nondiagonal and the diagonal model is the same, as one would expect from the classical equivalence of both cases. However, differences appear already in the next to leading order, obtained in the semiclassical expansion of our results in Sec. V A. It should be noted that even studying the next to leading semiclassical order of a physical state in quantum general relativity is rather nontrivial. It requires one to address operator ordering ambiguities in the Hamiltonian, which we have done in Sec. II A, to take proper account of quantum corrections in the

Hamiltonian implied by symmetries, which we have done in Sec. III B, and to apply a complete set of gauge-fixing conditions including a fixing of the time reparametrization symmetry, which we have done in Sec. III C. Only then is the next to leading semiclassical order of a physical state unambiguously defined.

Only two of the five solutions we constructed satisfy the complete permutation symmetry between the three main axes of the Bianchi type IX three-geometry, which, as shown in Sec. III B, is implied by the quantization of the *nondiagonal* model. These are the states corresponding to the integration contours (6.6) and (6.7). Their semiclassical limits identify them as a wormhole state and a Hartle-Hawking ''no-boundary'' state, respectively. It is remarkable that the *same* Chern-Simons state in Ashtekar variables can yield such diverse states depending on the choice of the integration manifold. This is a striking example that a physical state in Ashtekar variables is not yet defined before further conditions (here the choice of integration contours) are specified.

In the Hartle-Hawking proposal for the semiclassical initial condition of the classical evolution of the Universe it is not required that the ''no-boundary'' state be a normalizable vector in a Hilbert space. Rather the condition by which it is defined at least semiclassically is that the Euclidean four-geometries defined by the semiclassical wave function filling in the three-geometries on a spacelike slice are regular for $a \to 0$. A further requirement for the Hartle-Hawking state is to give a well-defined probability distribution at the semiclassical caustic surface where the semiclassical evolution, given by the wave function, switches from a Riemannian (''Euclidean'') to a pseudo-Riemannian (''Lorentzian'') space-time. This requirement is met by the Hartle-Hawking ''no-boundary'' state obtained here, and only by this state among the five Chern-Simons-like solutions, as was shown already in [12].

The wormhole state, on the other hand, one expects to be a normalizable vector in a Hilbert space with a well-defined scalar product, in which other state vectors describe, e.g., excited states of the wormhole. Without going into details, which will be given elsewhere [19], we have defined such a scalar product in Sec. III C by gauge fixing all gauge symmetries. It turns out that the wormhole state is, in fact, normalizable in this scalar product, while the ''no-boundary'' state is not. Thus imposing either the ''no-boundary'' condition or normalizability in the Hilbert space of physical states equipped with a fully gauge-fixed scalar product, one finds in each case a *unique* but mathematically and physically vastly different state as a metric representation of the Chern-Simons state.

One may hope that the nondiagonal quantization procedure presented here might be applicable even to the full inhomogeneous case, because all constraints are treated quantum mechanically in a similar way, and a solution to the quantum constraints is available in terms of the general Chern-Simons functional. Work in this direction is in progress.

ACKNOWLEDGMENTS

APPENDIX A: THE CHERN-SIMONS INTEGRAL IN THE LIMIT $\Lambda \to 0$

In the following we want to examine the possible limits $\Lambda \to 0$ of the Chern-Simons integral given in Eq. (4.9). First of all, we shall briefly recover the wormhole state, which is approached for the choice $\Sigma^3 = \Sigma^3_{WH}$ given in Eq. (6.6). Using this integration manifold, we may write

$$\Psi'_{WH} \propto \int_{-1}^{+1} d\eta \int_0^{2\pi} d\varphi \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \frac{r^2 \, dr}{1-r^2} \exp\left(\frac{F}{\lambda}\right), \quad \text{(A1)}$$

where we made use of the spherical coordinates in the (x,y,z) space, which were introduced in Eq. (6.5). Easy estimates show that for $\Lambda \to 0$ just an infinitesimal region around $r=0$ contributes to the inner r integral, and so we have the following asymptotic behavior:[12]

$$\Psi'_{WH} \overset{\lambda \to 0}{\propto} \int_{-1}^{+1} d\eta \int_0^{2\pi} d\varphi \int_{-\varepsilon}^{+\varepsilon} \frac{r^2 \, dr}{1-r^2} \exp\left(\frac{-r^2}{\lambda}\right)$$

$$\overset{\lambda \to 0}{\sim} 4\pi \lambda^{3/2} \int_{-\infty}^{+\infty} d\xi \, \xi^2 e^{-\xi^2} = 2(\pi\lambda)^{3/2}$$

$$\propto \text{const}, \quad r = \sqrt{\lambda}\xi. \quad \text{(A2)}$$

According to the transformation rule (2.25) we obviously approach the wormhole state Ψ_{WH} in the limit $\Lambda \to 0$, which was defined in Eq. (2.24).

Let us now turn to the much more complicated cases, for which the ''wormhole saddle point'' $x=y=z=0$ is *not* passed through by the integration surface Σ^3. For such solutions, we shall try to set not all borders of Σ^3 at infinity; instead we shall make use of the existence of two further singularities of the integrand with respect to r, namely, those at $r = \pm 1$. If we integrate into these singularities in a suitable manner, we expect to create further solutions of our Wheeler-DeWitt equation (3.17), because boundary terms generated by partial integrations will vanish at these borders. In the following it will be in fact sufficient to consider only one of the two singularities, say, $r = +1$, since any r integral in the neighborhood of $r = -1$ can be mapped onto a region around $r = +1$ by a transform $\eta \to -\eta$ in the η integral. This is due to the fact that the η, r dependence of the integrand in Eq. (4.9) is given in terms of r^2, $r\eta$, and η^2 only; cf. Eq. (A4) below. Thus we will be interested in integration contours for the r integral, which have one end point at $r = 1$. It turns out that, for a positive cosmological constant Λ, there are only two curves of interest: first, we may perform our r integration along the real axis from $r = 1$ to $r = +\infty$; second, we can also consider a pole integral, encircling the singularity at $r = 1$ in the mathematically positive sense.

To shorten our notation, we should establish the following nomenclature: a circle with radius R, which is centered at r_0 and passed through in the mathematically positive sense, will be denoted by $K(r_0,R)$. Then we can write the integrals of interest in the form

with

$$\Psi'_{CS} = \int d\eta \int d\varphi \int_C \frac{r^2\, dr}{1-r^2} \exp\left[\frac{G}{\lambda}\right], \tag{A3}$$

$$G := 1 - r^2 + 2\kappa_3 r\eta + \frac{1}{2}(\kappa_1^2 + \kappa_2^2) + \frac{1}{2}\frac{(\kappa_1^2+\kappa_2^2)(1+r^2\eta^2) + 4\kappa_1\kappa_2 r\eta + r^2(\kappa_1^2-\kappa_2^2)(\eta^2-1)\cos2\varphi}{1-r^2}, \tag{A4}$$

where C is one of the two contours $]1,+\infty[$ or $K(1,R)$, $0 < R < 2$. The equality sign in Eq. (A3) indicates that, from now on, all prefactors of the integral will be taken into account. To proceed in our calculation of the limit $\Lambda \to 0$, we can uniformly estimate those parts of the integral (A3), which lie outside an ε neighborhood of $r=1$ by a function, which vanishes exponentially for $\Lambda \to 0$. Therefore, just an ε neighborhood of $r=1$ contributes to the integral (A3), where ε may be chosen arbitrarily small. As a consequence, all terms of the integrand, which remain regular at $r=1$, can be substituted by their value taken at $r=1$, and only the r dependence of singular parts remains to be taken into consideration. Thus we arrive at once at

$$\Psi'_{CS} \overset{\lambda \to 0}{\sim} \frac{1}{2}\int d\eta\, e^{\sigma_3 \eta} \int d\varphi \int_{C_\varepsilon} \frac{dr}{1-r}\exp\left[\frac{2}{\lambda}(1-r) + \frac{\lambda}{8}\frac{a^2 - 2ab\cos2\varphi + b^2}{1-r}\right], \tag{A5}$$

where we made use of the definitions

$$a := \frac{\sigma_1 + \sigma_2}{2}(1+\eta), \quad b := \frac{\sigma_1 - \sigma_2}{2}(1-\eta). \tag{A6}$$

Here C_ε is either equal to $]1,1+\varepsilon[$ or given by $K(1,\varepsilon)$. After the transformation $\xi = (4/\lambda)(r-1)$ we arrive at

$$\Psi'_{CS} \overset{\lambda \to 0}{\sim} -\frac{1}{2}\int d\eta\, e^{\sigma_3 \eta}\int d\varphi \int_{C'}\frac{d\xi}{\xi}\exp\left[-\frac{\xi}{2} - \frac{a^2 - 2ab\cos2\varphi + b^2}{2\xi}\right], \tag{A7}$$

where now C' is the positive real axis or $K(0,R)$. The two possible ξ integrals can now easily be performed, leading to

$$\Psi'_{CS} \overset{\lambda \to 0}{\sim} \int d\eta\, e^{\sigma_3 \eta}\int d\varphi \begin{cases} -i\pi I_0[\sqrt{a^2 - 2ab\cos2\varphi + b^2}] & , \quad C' = K(0,R), \\ -K_0[\sqrt{a^2 - 2ab\cos2\varphi + b^2}] & , \quad C' = [0,+\infty]. \end{cases} \tag{A8}$$

Up to now we have taken into acount all freedom in the choice of the integration contours; any other choice for the r-integration contour would have given a linear combination of the two results given in Eq. (A8) or a constant as in Eq. (A2). While the η dependence of Eq. (A8), hidden in a and b, is too complicated to be integrated out analytically, there are several possibilities to evaluate the φ integral, which become more transparent in the new variable $u = \cos2\varphi$. With the abbreviation $X := a^2 - 2abu + b^2$ we find

$$\int_{-1}^{+1}\frac{du}{\sqrt{1-u^2}}K_0(\sqrt{X}) = \pi I_0(a)K_0(b), \quad b > a > 0, \tag{A9}$$

$$\int_{-1}^{+1}\frac{du}{\sqrt{1-u^2}}I_0(\sqrt{X}) = \pi I_0(a)I_0(b), \tag{A10}$$

$$\int_{-\infty}^{-1}\frac{du}{\sqrt{u^2-1}}K_0(\sqrt{X}) = K_0(a)K_0(b), \quad a,b > 0, \tag{A11}$$

$$\int_{+1}^{+\infty}\frac{du}{\sqrt{u^2-1}}I_0(\sqrt{X}) = I_0(a)K_0(b)$$
$$+ K_0(a)I_0(b), \quad a,b > 0. \tag{A12}$$

Obviously, the first two integrals correspond to an integration over real angles φ, while the latter two integrals require for φ values to lie on the imaginary axis. The first three integrals are tabulated in mathematical tables like [28], but the integral (A12) seems to be a new result. Thus we shall shortly comment on a proof of this formula, which, by the way, also holds for the other integrals (A9)–(A11): The idea is to show that the integral on the left hand side solves the differential equation of the modified Bessel functions with index zero with respect to a and b. After some partial integrations it turns out that this is indeed the case; however, we thereby gather a boundary term, which exclusively vanishes for the boundaries chosen in Eqs. (A9) to (A12). For these choices we then know that the integrals must be equal to a linear combination of the four possible products of I_0 and K_0

at arguments a and b. The coefficients of these possible four contributions can then be determined by evaluating the integrals at $a = b$ and, if necessary, by an additional investigation of the limits $a \to 0$ and $b \to 0$.

With the integrals (A9)–(A12) we may now perform the φ integration in Eq. (A8) and get a result of the form

$$\Psi'_{CS} \overset{\lambda \to 0}{\sim} \int d\eta \, e^{\sigma_3 \eta} Z_0^{(1)}(a) Z_0^{(2)}(b), \qquad (A13)$$

with $Z_0^{(1)}$ and $Z_0^{(2)}$ being modified Bessel functions with index zero. However, not any combination of $Z_0^{(1)}$ and $Z_0^{(2)}$ with η borders at ± 1 or $\pm \infty$ will lead to a solution of the three equations

$$\mathcal{F}_p \Psi'_{CS} := \mathcal{Q}_p|_{\Lambda=0} \Psi'_{CS} = \left[\partial_q \partial_r - \partial_p + \frac{\sigma_q \partial_r - \sigma_r \partial_q}{\sigma_q^2 - \sigma_r^2} \right] \Psi'_{CS} = 0, \qquad (A14)$$

which necessarily should be the case for the limit $\Lambda \to 0$ of the Chern-Simons integral because of Eq. (3.18). The reason for this is that, due to the complexification of the spherical coordinates (6.5), we might have integrated effectively over a three-dimensional manifold Σ^3, which produces boundary terms for partial integrations. For example, there is no reason why a surface described by $r \in \,]1, +\infty[\,, \varphi \in i\mathbf{R}, \eta \in \,]-1, +1[$ should lead to an integral solving Eq. (A14), although this integration contour leads to a finite integral. It will be a nice explicit check of solvability to investigate for which choices of $Z_0^{(1)}$ and $Z_0^{(2)}$ and for which η borders Ψ'_{CS} according to Eq. (A13) becomes a solution to the three equations (A14):

While the operator $\mathcal{F}_3$ annihilates the integrand of Eq. (A13) for arbitrary values of η, and therefore Ψ'_{CS} for arbitrary borders of the η integral, we have to perform suitable partial integrations to show the vanishing of $\mathcal{F}_1 \Psi'_{CS}$ and $\mathcal{F}_2 \Psi'_{CS}$. Consequently, we get some boundary terms, which will not vanish for all possible forms of Eq. (A13). A detailed discussion of these boundary terms reveals that there are exactly four possiblities to arrive at solutions of Eq. (A14). For the adopted ordering of the three σ_p these solutions are given by the following integrals:

$$\Psi'_1 = \int_{-\infty}^{-1} d\eta \, e^{\sigma_3 \eta} I_0(a) K_0(|b|),$$
$$\Psi'_{12} = i\pi \int_{-\infty}^{-1} d\eta \, e^{\sigma_3 \eta} I_0(a) I_0(b), \qquad (A15)$$

$$\Psi'_{123} = \int_{-\infty}^{+1} d\eta \, e^{\sigma_3 \eta} K_0(|a|) I_0(b),$$
$$\Psi'_{0123} = i\pi \int_{-1}^{+1} d\eta \, e^{\sigma_3 \eta} I_0(a) I_0(b). \qquad (A16)$$

As an example, we want to discuss the action of the operator $\mathcal{F}_1$ on the wave function Ψ'_{12}, which can be written in the form

$$\mathcal{F}_1 \Psi'_{CS} = \frac{1}{2} \left[\frac{e^{\sigma_3 \eta} (\eta^2 - 1)}{\sigma_2^2 - \sigma_3^2} \{ \sigma_3 [I_1(a) I_0(b) + I_0(a) I_1(b)] \right.$$
$$\left. - \sigma_2 [I_1(a) I_1(b) + I_0(a) I_0(b)] \} \right]_{\eta \to -\infty}^{\eta = -1}. \qquad (A17)$$

That this is indeed equal to zero is easily seen for $\eta = -1$ and, after some asymptotic expansions of the Bessel functions, also for $\eta \to -\infty$. Similar results hold for the other solutions, where one of the two Bessel functions might be substituted by K_0 or $-K_1$. In such cases the vanishing of the boundary terms at $\eta = \pm 1$ becomes a nontrivial question, because the K_ν-Bessel function might become singular at these points. A detailed investigation shows that there are only the four possibilities given in Eqs. (A15) and (A16) to avoid contributions from these boundaries.

The question which remains to be answered at this point is which integration contours for the φ integral give rise to the integrands in Eqs. (A15) and (A16). For the wave functions Ψ'_{12} and Ψ'_{0123} the answer is easily found with help of the integral (A10), and we find the explicit representations

$$\Psi'_{12} = -\frac{1}{2\pi} \lim_{\Lambda \to 0} \int_{-\infty}^{-1} d\eta \int_0^{2\pi} d\varphi \int_{K(1,R)} \frac{r^2 \, dr}{1 - r^2} \exp\left[\frac{G}{\lambda} \right], \qquad (A18)$$

$$\Psi'_{0123} = -\frac{1}{2\pi} \lim_{\Lambda \to 0} \int_{-1}^{+1} d\eta \int_0^{2\pi} d\varphi \int_{K(1,R)} \frac{r^2 dr}{1 - r^2} \exp\left[\frac{G}{\lambda} \right], \qquad (A19)$$

where the radius R of the r integral has to be chosen with $R < 2$. However, for the other two wave functions the answer is more difficult, because, according to Eqs. (A9)–(A12), there is no easy possibility to generate their integrands uniformly for all η values of their integration regime. So we have to cut the η integral artificially at those η values, for which we get $a = \pm b$, which actually happens at $\eta = -\sigma_2/\sigma_1 < -1$ and $\eta = -\sigma_1/\sigma_2 > -1$. Suitable linear combinations of the results (A9) and (A12) then reveal the following integral representations:

$$\Psi'_1 = -\frac{1}{2\pi} \lim_{\Lambda \to 0} \left(\int_{-\infty}^{-1} d\eta \int_0^{2\pi} d\varphi \int_1^{+\infty} \right.$$
$$- 2 \int_{-\infty}^{-\sigma_2/\sigma_1} d\eta \int_0^{2\pi} d\varphi \int_1^{+\infty}$$
$$\left. - \int_{-\infty}^{-\sigma_2/\sigma_1} d\eta \int_{-i\infty}^{+i\infty} d\varphi \int_{K(1,R)} \right) \frac{r^2 dr}{1 - r^2} \exp\left[\frac{G}{\lambda} \right], \qquad (A20)$$

$$\Psi'_{123} = -\frac{1}{2\pi}\lim_{\Lambda\to 0}\left(\int_{-\infty}^{+1}d\eta\int_0^{2\pi}d\varphi\int_1^{+\infty}\right.$$

$$-2\int_{-\sigma_2/\sigma_1}^{-\sigma_1/\sigma_2}d\eta\int_0^{2\pi}d\varphi\int_1^{+\infty}$$

$$\left.-\int_{-\sigma_2/\sigma_1}^{-\sigma_1/\sigma_2}d\eta\int_{-i\infty}^{+i\infty}d\varphi\int_{K(1,R)}\right)\frac{r^2 dr}{1-r^2}\exp\left[\frac{G}{\lambda}\right]. \quad (A21)$$

Considering the representations (A18)–(A21) it is now possible to read off the integration manifolds Σ^3 which lead to solutions of the Wheeler-DeWitt equation even for a nonvanishing cosmological constant Λ. Although the representations (A20) and (A21) look very strange because of the η junctions at $-\sigma_2/\sigma_1$ and $-\sigma_1/\sigma_2$, they lead indeed to differentiable solutions. Since this is a nontrivial claim, it will be proved in Appendix C, where we show that the integrand of the outer η integral is a continuous and differentiable function at the junction points, even for $\Lambda\neq 0$.

APPENDIX B: ALTERNATIVE INTEGRAL REPRESENTATIONS FOR THE VACUUM SOLUTIONS

We will now bring the one-dimensional integral representations found in Appendix A into a new, unified form, which will display nicely the symmetries of these solutions. The motivation of this new representation arises from the fact that one of the η integrals, namely, Ψ'_{0123} given in Eq. (A15), turns out to be a convolution integral, and may therefore be simplified with the aid of Laplace's convolution theorem. To become more precise, let us write

$$\Psi'_{0123} = 2\pi i\int_0^1 d\tau e^{-\sigma_3\tau}I_0[(\sigma_1-\sigma_2)\tau]$$

$$\times e^{\sigma_3(1-\tau)}I_0[(\sigma_1+\sigma_2)(1-\tau)], \quad (B1)$$

where we have substituted $\tau=(1-\eta)/2$ in Eq. (A16). If we denote

$$g(\alpha,\beta;t):=e^{\alpha t}I_0[\beta t], \quad (B2)$$

we may also write

$$\Psi'_{0123} = 2\pi i g(-\sigma_3,\sigma_2-\sigma_1;t)\star g(\sigma_3,\sigma_2+\sigma_1;t)\big|_{t=1}; \quad (B3)$$

i.e., Eq. (B1) is a convolution integral, evaluated at $t=1$. Let us recall Laplace's convolution theorem, which states

$$\mathcal{L}[f_1\star f_2](s) = \mathcal{L}[f_1](s)\cdot\mathcal{L}[f_2](s), \quad (B4)$$

with

$$\mathcal{L}[f](s):=\int_0^\infty dt\, e^{-st}f(t).$$

Using the inverse Laplace transform

$$f(t) = \frac{1}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty}ds\, e^{st}\mathcal{L}[f](s), \quad (B5)$$

where $\hat{s}$ has to lie to the right of all singularities of the integrand in Eq. (B5), Laplace's convolution theorem can also be written in the form

$$(f_1\star f_2)(t) = \frac{1}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty}ds\, e^{st}\mathcal{L}[f_1](s)\cdot\mathcal{L}[f_2](s). \quad (B6)$$

With the aid of the Laplace transform of g defined in Eq. (B2),

$$\mathcal{L}[g](s) = \int_0^\infty dt\, e^{(\alpha-s)t}I_0[\beta t] = [(\alpha-s)^2-\beta^2]^{-1/2}, \quad (B7)$$

we arrive at once at the identity

$$\Psi'_{0123} = \int_{\hat{s}-i\infty}^{\hat{s}+i\infty}ds\, e^s\prod_{\nu=0}^3(s-s_\nu)^{-1/2}, \quad (B8)$$

if we employ Eq. (B6) with $t=1$. Here we have introduced the four quantities s_ν via Eq. (5.16), which are ordered according to $s_1<s_2<s_3<s_0$ for $\sigma_1<\sigma_2<\sigma_3$. Since the quantity $\hat{s}$ occurring in Eq. (B8) must be placed to the right of all the s_ν, it must be chosen as $\hat{s}>s_0$. With Eq. (B8) we have reached a very nice representation for Ψ'_{0123}, which immediately shows up the symmetry with respect to arbitrary permutations of the s_ν: As the integrand obviously obeys this symmetry, and the integration curve which is placed to the right of all the s_ν can be chosen to be the same after two of the s_ν have been permuted, the integral must be invariant under such permutations. Translated to the σ_p, we first have a symmetry under permutations of the σ_p themselves. Second, the additional symmetry under permutations of the form $s_0\leftrightarrow s_p$ implies that the wave function Ψ'_{0123} is invariant under a reflection $\sigma_1\to-\sigma_1$, $\sigma_2\to-\sigma_2$, $\sigma_3\to\sigma_3$, and cyclic permutations thereof. Thus we directly recover the symmetry properties of Ψ'_{0123}, which we already claimed in Sec. VI.

Regarding the result (B8) in more detail, one may now ask if there could be further integration curves, apart from the one used in Eq. (B8), leading also to vacuum solutions. Investigating the action of the operators $\mathcal{F}_p$ defined in Eq. (A14) we find the following result:

$$\mathcal{F}_3\left[\int ds\, e^s \prod_{\nu=0}^{3} (s-s_\nu)^{-1/2}\right] = \frac{1}{2}\int ds\, \frac{d}{ds}\left[\left(-\frac{1}{s-s_0}+\frac{1}{s-s_1}+\frac{1}{s-s_2}-\frac{1}{s-s_3}\right)e^s \prod_{\nu=0}^{3} (s-s_\nu)^{-1/2}\right], \qquad (B9)$$

and similar results hold for the action of the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on the s integral. Thus we conclude that all four integration curves shown in Fig. 1 lead to solutions of $\mathcal{F}_p\Psi'=0$, because the boundary term (B9) vanishes at the end points of these curves. However, up to now it remains unclear how these vacuum solutions are connected to those which were found in Appendix A. In the following, we shall be interested in a direct transformation between the two representations, which is rather tricky and therefore should be presented in detail at least for one of the remaining solutions. In our calculations, we will need the following two integrals:

$$e^{\alpha t}I_0[\beta t]=\frac{1}{2\pi i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} \frac{ds\, e^{st}}{\sqrt{(\alpha-s)^2-\beta^2}}, \qquad \hat{s}>\alpha\pm\beta, \qquad (B10)$$

$$\int_{-\infty}^{+\infty} dt\, e^{\alpha t}K_0[\beta|t|]=\frac{\pi}{\sqrt{\beta^2-\alpha^2}}. \qquad (B11)$$

Let us now choose as an example the transformation of the wave function Ψ'_1:

$$\Psi'_1=2\int_0^{\infty} d\tau\, e^{-\sigma_3(\tau+1)}K_0[(\sigma_2-\sigma_1)(\tau+1)]e^{-\sigma_3\tau}I_0[(\sigma_1+\sigma_2)\tau] \; ; \tau=-\frac{1+\eta}{2}$$

$$\overset{(B10)}{=}\frac{1}{i\pi}\int_0^{\infty} d\tau\, e^{-\sigma_3(\tau+1)}K_0[(\sigma_2-\sigma_1)(\tau+1)]\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} \frac{ds'\, e^{s'\tau}}{\sqrt{(s'+s_0)(s'+s_3)}} \; ;\hat{s}>-s_0,-s_3$$

$$\overset{\tau'=1+\tau}{\underset{s=-s'}{=}}\frac{1}{i\pi}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, e^s\, [(s-s_0)(s-s_3)]^{-1/2}\int_1^{\infty} d\tau'\, e^{-(\sigma_3+s)\tau'}K_0[(\sigma_2-\sigma_1)|\tau'|] \; ;\hat{s}<s_0,s_3. \qquad (B12)$$

To allow for the interchange of the τ' and the s integral performed in the last two lines, we further have to make sure that $\hat{s}>s_1$; otherwise the τ' integral in the last line would not exist. If we further require $\hat{s}<s_2$, the τ' integral will exist even for an extension of the τ' integral to $-\infty$. Let us consider the additional contribution which we would get in case of this extension:

$$\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, e^s\, [(s-s_0)(s-s_3)]^{-\frac{1}{2}}\int_{-\infty}^{+1} d\tau'\, e^{-(\sigma_3+s)\tau'}K_0[(\sigma_2-\sigma_1)|\tau'|]$$

$$(B13)$$

$$=\int_{-\infty}^{+1} d\tau'\, e^{-\sigma_3\tau'}K_0[(\sigma_2-\sigma_1)|\tau'|]\underbrace{\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, e^{s\overbrace{(1-\tau')}^{>0}}[(s-s_0)(s-s_3)]^{-\frac{1}{2}}}_{0} = 0 \; .$$

Here the last s integral vanishes, because, without meeting any singularity of the integrand, the integration contour can be deformed to the negative real axis, where the s integrand vanishes exponentially. Thus, we still have a representation of Ψ'_1, if we extend the τ' integral in the last line of Eq. (B12) to the whole real axis. After employing the formula (B11) we arrive at

$$\Psi'_1=\frac{1}{i}\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, [(s-s_0)(s-s_3)]^{-1/2}$$

$$\times[(s_2-s)(s-s_1)]^{-1/2}$$

$$=\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, e^s \prod_{\nu=0}^{3} (s-s_\nu)^{-1/2}, \qquad s_1<\hat{s}<s_2.$$

$$(B14)$$

Similar calculations reveal the remaining two identities

$$\Psi'_{12}=\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, e^s \prod_{\nu=0}^{3} (s-s_\nu)^{-1/2}, \qquad s_2<\hat{s}<s_3, \qquad (B15)$$

$$\Psi'_{123}=\int_{\hat{s}-i\infty}^{\hat{s}+i\infty} ds\, e^s \prod_{\nu=0}^{3} (s-s_\nu)^{-1/2}, \qquad s_3<\hat{s}<s_0. \qquad (B16)$$

It is now easy to comment on the symmetries of the wave functions (B14)–(B16): While Ψ'_{123} obviously is symmetric under permutations of the s_p and therefore under permutations of the σ_p, it is not symmetric under a permutation $s_p\leftrightarrow s_0$ as the wave function Ψ'_{0123}. The vacuum state Ψ'_{12} is only symmetric under the permutation $s_1\leftrightarrow s_2$ and, consequently, under $\sigma_1\leftrightarrow\sigma_2$; the solution Ψ'_1 has no symmetry of this form at all.

Finally, we shall be interested in the limit $\hbar\to0$ of our vacuum solutions, where we should remind the reader that

the $\hbar$ dependence is hidden in the s_ν via the σ_p, which depend on $\hbar$ according to Eq. (3.14). As an example, let us compute the limit $\hbar \to 0$ of the wave function Ψ_1': First of all, the integration contour $\mathcal{C}_1$ can be deformed to the negative real axis, giving

$$\Psi_1' = -2 \int_{-\infty}^{s_1} ds \ e^s \prod_{\nu=0}^{3} (s - s_\nu)^{-1/2}. \qquad (B17)$$

Substituting $\bar{s}_\nu = \hbar s_\nu$ and $\bar{\xi} = \hbar(s_1 - s)$ we get

$$\Psi_1' = -2\hbar e^{s_1} \int_0^\infty \frac{d\bar{\xi}}{\sqrt{\bar{\xi}}} e^{-\bar{\xi}/\hbar}$$
$$\times [(\bar{\xi} - \bar{s}_1 + \bar{s}_2)(\bar{\xi} - \bar{s}_1 + \bar{s}_0)(\bar{\xi} - \bar{s}_1 + \bar{s}_3)]^{-1/2}. \qquad (B18)$$

In the limit $\hbar \to 0$ just an infinitesimal neighborhood around $\bar{\xi} = 0$ contributes to the $\bar{\xi}$ integral; thus we may expand

$$\Psi_1' \overset{\hbar \to 0}{\sim} -2\hbar \ e^{s_1} \int_0^\varepsilon \frac{d\bar{\xi}}{\sqrt{\bar{\xi}}} e^{-\bar{\xi}/\hbar} [(\bar{s}_1 - \bar{s}_2)(\bar{s}_1 - \bar{s}_0)$$
$$\times (\bar{s}_3 - \bar{s}_1)]^{-1/2}, \qquad (B19)$$

where we have substituted $\bar{\xi} = 0$ in those parts of the integrand, which remain regular for $\hbar \to 0, \bar{\xi} \to 0$, since we may choose ε arbitrarily small. In the new variable $\xi = \bar{\xi}/\hbar$, we finally arrive at

$$\lim_{\hbar \to 0} \Psi_1' = \frac{-e^{s_1}}{\sqrt{2(\sigma_2 - \sigma_1)(\sigma_2 + \sigma_3)(\sigma_3 - \sigma_1)}} \int_0^\infty \frac{d\xi}{\sqrt{\xi}} e^{-\xi}$$
$$= -\sqrt{\frac{\pi}{2}} \frac{e^{\sigma_1 - \sigma_2 - \sigma_3}}{\sqrt{(\sigma_2 - \sigma_1)(\sigma_2 + \sigma_3)(\sigma_3 - \sigma_1)}}. \qquad (B20)$$

The limit $\hbar \to 0$ of the other vacuum solutions may be calculated analogously. As a result, we obtain the same asymptotic behavior as in the limit $\hbar \to 0, \Lambda \to 0$, which we discussed in Eq. (5.10) of Sec. V; i.e., the result is independent of the order in which these two limits are taken.

APPENDIX C: CONTINUITY AND DIFFERENTIABILITY OF THE INTEGRAND ON THE ASYMMETRIC INTEGRATION SURFACES

We finally want to show that the two integrals (A20) and (A21) given in Appendix A give analytical solutions of the Wheeler-DeWitt equation (3.17) not only for $\Lambda \to 0$, but also for $\Lambda \neq 0$. To show this we have to prove that the effective integrand of the η integral is continuous and differentiable at the junctions $\eta = -\sigma_1/\sigma_2$ and $\eta = -\sigma_2/\sigma_1$; the Wheeler-DeWitt operator would otherwise produce boundary terms when acting on the wave functions Ψ_1' and Ψ_{123}', and they could not be solutions. In the following, we will restrict ourselves to the solution Ψ_1' in the case $\Lambda > 0$ defined in Eq. (A21), which can be written in the form

$$2\pi\Psi_1' = -\int_{-\infty}^{-1} d\eta \int_0^{2\pi} d\varphi \int_1^\infty \frac{r^2 \, dr}{1-r^2} \exp\left[\frac{G}{\lambda}\right]$$
$$+ \int_{-\infty}^{-\sigma_2/\sigma_1} d\eta \left(2 \int_0^{2\pi} d\varphi \int_1^\infty + \int_{-i\infty}^{+i\infty} d\varphi \int_{K(1,R)} \right)$$
$$\times \frac{r^2 \, dr}{1-r^2} \exp\left[\frac{G}{\lambda}\right]. \qquad (C1)$$

With the abbreviations P and Q defined via

$$P := \int_0^{2\pi} d\varphi \int_1^\infty \frac{r^2 \, dr}{1-r^2} \exp\left[\frac{G}{\lambda}\right],$$
$$Q := \frac{1}{2} \int_{-i\infty}^{+i\infty} d\varphi \int_{K(1,R)} \frac{r^2 \, dr}{1-r^2} \exp\left[\frac{G}{\lambda}\right], \qquad (C2)$$

we can rewrite Eq. (C1) as a piecewise defined η integral:

$$2\pi\Psi_1' = \int_{-\infty}^{-1} d\eta \begin{cases} P+2Q, & \eta < -\sigma_2/\sigma_1, \\ -P, & \eta > -\sigma_2/\sigma_1. \end{cases} \qquad (C3)$$

To show the continuity and differentiability of the integrand with respect to η at the η junction we then have to show the following two properties of P and Q:

$$(P+Q)\Big|_{\eta = -\sigma_2/\sigma_1} = 0, \qquad \frac{d}{d\eta}(P+Q)\Big|_{\eta = -\sigma_2/\sigma_1} = 0. \qquad (C4)$$

To proceed in this direction let us try to evaluate the φ integrals in P and Q. The φ integral for P can be interchanged with the r integral and is then performed easily, leading to

$$P = 2\pi \int_1^\infty \frac{r^2 \, dr}{1-r^2} I_0\left[\frac{\lambda}{2} ab \frac{r^2}{1-r^2}\right] \exp\left[\frac{G'}{\lambda}\right], \qquad (C5)$$

with

$$G' := 1 - r^2 + 2\kappa_3 r\eta + \frac{1}{2}(\kappa_1^2 + \kappa_2^2)(1 - \eta^2)$$
$$+ \frac{\lambda^2}{4} \frac{a^2 + b^2}{1-r^2} - \frac{2\kappa_1 \kappa_2 \eta}{1+r}. \qquad (C6)$$

The Q integral is more complicated, because it is not allowed to interchange the r and the φ integrals which have to be performed there: after such an exchange the φ integral would not exist. However, we are free to open the r-integration contour $K(1,R)$ at $r \to +\infty$, and the resulting integration curve can be deformed into a line integral $r_0 - i\mathbf{R}$, where we can choose $-1 < r_0 < 0$. For this r-integration contour we may then interchange the r and the φ integrals, because the φ integral exists for any r value along the contour. Thus we can write

$$2Q = \int_{r_0+i\infty}^{r_0-i\infty} \frac{r^2 \, dr}{1-r^2} \int_{-i\infty}^{+i\infty} d\varphi \, \exp\left[\frac{G}{\lambda}\right]$$

$$= 2i \int_{r_0+i\infty}^{r_0-i\infty} \frac{r^2 \, dr}{1-r^2} K_0\left[\frac{\lambda}{2} ab \frac{r^2}{1-r^2}\right] \exp\left[\frac{G'}{\lambda}\right], \quad (C7)$$

and in the last line the r integral can be deformed back to the positive real axis, giving an integral around the cut at $r>1$, which is generated by the K_0-Bessel function. Let us define this cut integral explicitly in terms of the contour $\mathcal{C}_{cut}$ via

$$\mathcal{C}_{cut} := K(1,R) \oplus \{r-i0 \,|\, 1+R < r < \infty\}$$

$$\ominus \{r+i0 \,|\, 1+R < r < \infty\}. \quad (C8)$$

Obviously, $\mathcal{C}_{cut}$ is a superposition of a pole integral and two line integrals along the cut, where one of these line integrals is performed in the upper half plane, while the other one has to be evaluated in the lower half plane. The integral over the integration curve (C8) is independent of the radius R, as long as this ranges between $R=2$ and $R=0$. Using the formulas

$$K_0(-t \pm i0) = \mp i\pi I_0(t) + K_0(t), \quad t>0, \quad (C9)$$

we can evaluate the two line integrals which are contained in Eq. (C8), where it turns out that the K_0 contributions cancel, while we get the I_0 contributions twice. Thus we can rewrite Eq. (C7) as

$$2Q = -4\pi \int_{1+R}^{\infty} \frac{r^2 \, dr}{1-r^2} I_0\left[\frac{\lambda}{2} ab \frac{r^2}{1-r^2}\right] \exp\left[\frac{G'}{\lambda}\right]$$

$$+ 2i \int_{K(1,R)} \frac{r^2 \, dr}{1-r^2} K_0\left[\frac{\lambda}{2} ab \frac{r^2}{1-r^2}\right] \exp\left[\frac{G'}{\lambda}\right]. \quad (C10)$$

If we now consider the limit $R \to 0$ in Eq. (C10), the first integral takes the form of the P integral [cf. Eq. (C5)], and we get

$$2Q = -2P + 2i \lim_{R \to 0} \int_{K(1,R)} \frac{r^2 \, dr}{1-r^2} K_0\left[\frac{\lambda}{2} ab \frac{r^2}{1-r^2}\right] \exp\left[\frac{G'}{\lambda}\right]. \quad (C11)$$

Comparing this result with the desired claim (C4), all that remains to be shown is that the pole integral in Eq. (C11) and its η derivative vanish in the limit $R \to 0$. In this limit $R \to 0$ the only contribution to the integral arises from an infinitesimal region around $r=1$, and so we can expand the K_0-Bessel function for large arguments; furthermore, parts of the integrand, which remain regular for $r \to 1$, may be substituted by their values taken at $r=1$. We then find, for the pole integral,

$$\int_{K(1,R)} \frac{r^2 \, dr}{1-r^2} K_0\left[\frac{\lambda}{2} ab \frac{r^2}{1-r^2}\right] \exp\left[\frac{G'}{\lambda}\right]$$

$$\overset{R \to 0}{\propto} \int_{K(1,R)} \frac{dr}{\sqrt{1-r}} \exp\left[\frac{\lambda}{8} \frac{(a-b)^2}{1-r}\right], \quad (C12)$$

where we have only taken into account the r-dependent parts of the integral, while a σ_p-dependent prefactor has been omitted. If we now consider the right hand side of Eq. (C12) at the η junction $\eta = -\sigma_2/\sigma_1$, a and b become coincident. Thus the essential singularity of the integrand disappears, and all that remains is an integrable square-root singularity, for which the pole integral vanishes in the limit $R \to 0$. Similar arguments show that also the η derivative of the pole integral in Eq. (C11) vanishes for $R \to 0$; so, after all, we have proved our claim (C4) and, therefore, the continuity and differentiability of the η integrand in Eq. (C1). Analogous calculations may be performed for the second nontrivial integration surface Σ_{123}^3, where we have to deal with two different η junctions.

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